PMAX Comparisons

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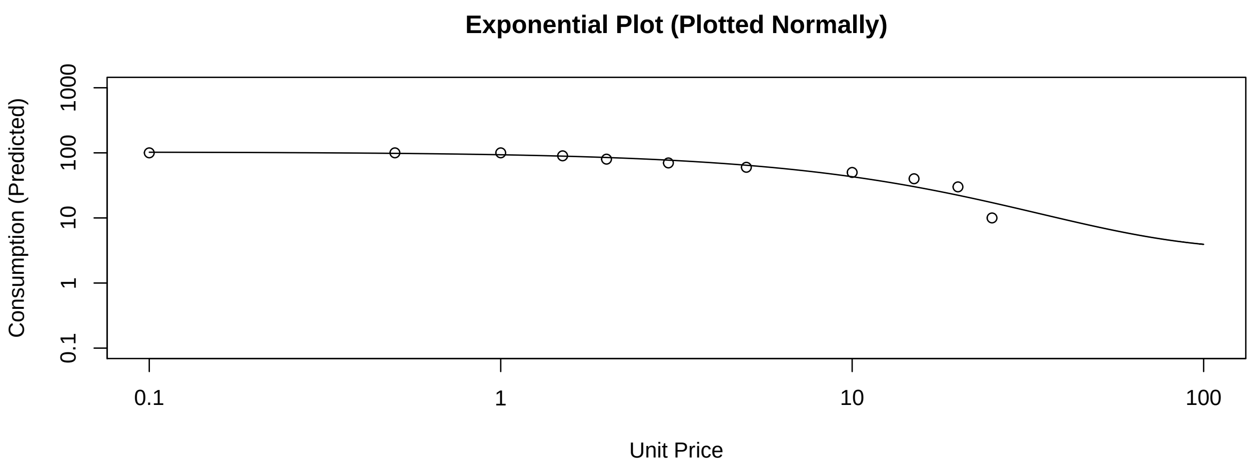
In this document, various methods for assessing indexing model slope are demonstrated and reviewed. Unless stated otherwise, each of these methods are presented and reviewed with respect to the Exponential model of demand. Individual methods are reviewed and evaluated here.

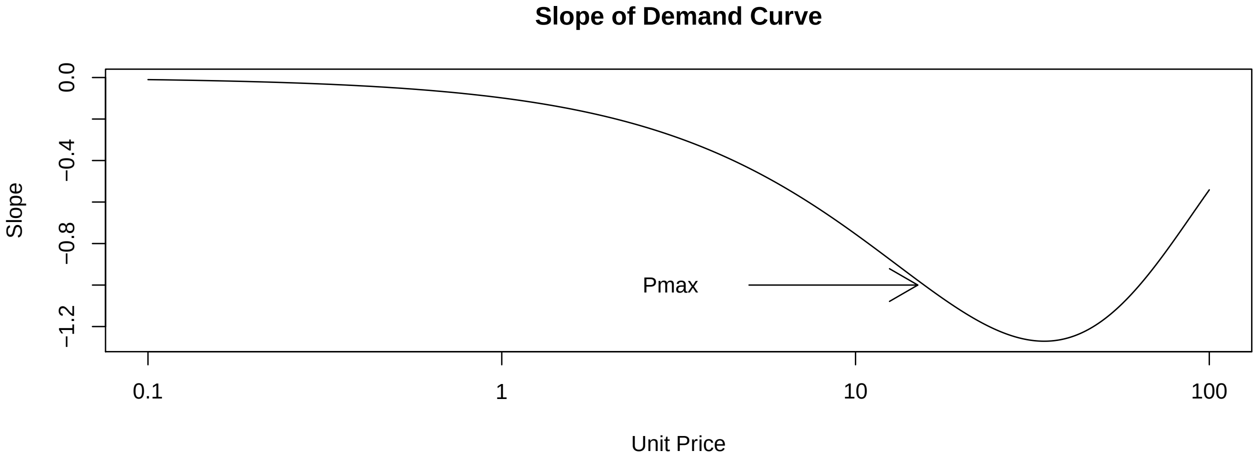
Plot of Demonstrated Demand Curve

For the case of this proposed example, the form of the demand curve discussed is illustrated below in log-log units (base 10 logarithm). The source code for producing this example, for the given model and parameters, is provided below:



Plot of Typical Predicted Slope (via Hursh's Derivative)

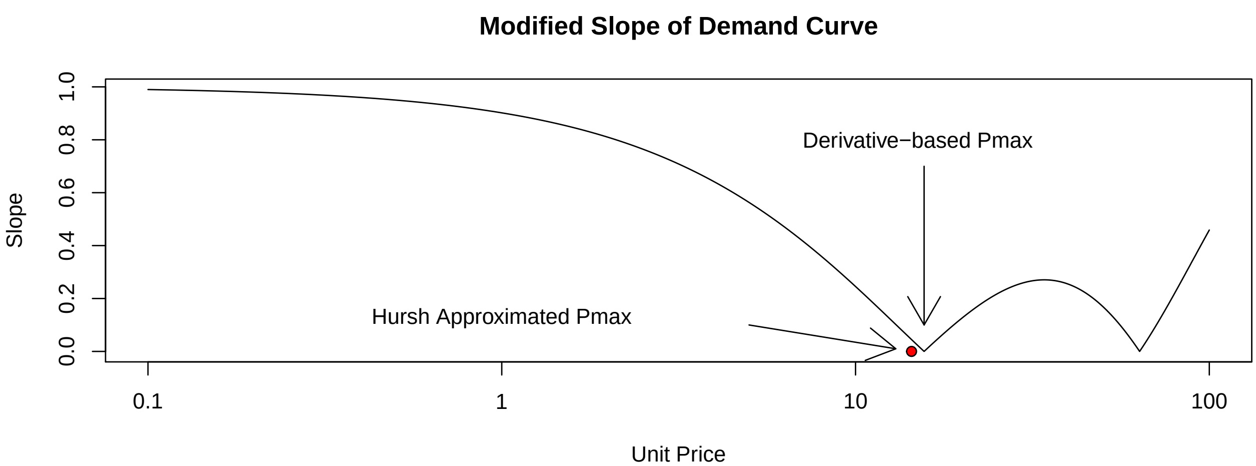
The slope of the demand curve is (as typically reported) interpreted in log-log units. The shared unit space is necessary to satisfy conditions of unit elasticity (i.e., PMAX), where a one Log Unit in increase in price is accompanied by a one Log Unit of consumption decrease. As projected by the first order derivative provided by Hursh, the slope of the demand curve takes the following form.



This figure represents the slope of the demand curve across some domain of prices. It warrants highlighting that the PMAX is not easily discriminable from this plot alone. For this reason, the slope prediction can be modified to PMAX more discriminable to the eye (and to optimization methods).

Plot of Modified Slope Prediction (via Hursh's Derivative)

The plot of the derivative of the demand curve provides the necessary information to determine where the slope of the curve reaches -1. However, these points can be made more visually apparent by plotting the absolute value of the derivative plus a constant of 1. By adding a constant of 1, the derivative with a value of -1 results in a value of zero. Also, by taking the absolute value, we can ensure that no other values can reach a point below zero. See below:



In this modified form, the same derivative is plotted in such a way that only the PMAX can take the value of zero. In this way, optimization can proceed in search of the lowest point (i.e., 0) where PMAX exists. Howvever, this figure also makes it clear that slope is -1 at more than 1 point in the curve (in many cases). It is for this reason that Hursh's approximated PMAX is used as a starting point, as this value is usually approximate to the desired PMAX value.

PMAX Method 1: Numerical Approximation

In the formula below, Hursh and colleagues proposed an approximated PMAX using fitted model parameters. This takes the following form:

$$P_{MAX} = (1/(Q0 * alpha * K^{1.5})) * (0.083 * K + 0.65)$$

```
## [1] 14.39964
```

PMAX Method 2: Slope Optimization (Hursh)

This method uses the first order derivative provided by Hursh and colleagues to represent the slope of the demand curve. The derivative provided is as follows:

$$ln10^K * (-A * Q * x * e^{-A*Q*x})$$

Using a the modified slope (Modified Slope Prediction) as a loss function, the value of x (i.e., unit price) can be optimized to yield a unit price with minimal loss (i.e., slope at the zero point, where the derivative is -1). Identical to that of the Modified Slope Prediction, the form of the loss function is as follows:

$$Loss = \mid ln10^K * (-A * Q * x * e^{-A*Q*x}) + 1 \mid$$

Using virtually any optimizer, with x (i.e., unit price) as the value to fit, the resulting parameter is the value at zero near the starting point. This is shown below:

```
##            p1        value fevals gevals niter convcode kkt1 kkt2 xtimes
## BFGS 15.62583 1.445072e-09     21     15    NA        0 TRUE TRUE  0.008
```

As a result, the result of optimization reveals a PMAX that is very near the approximated PMAX value.

PMAX Method 3: Modeled Slope Differentials

In a break from iterative methods altogether, slope differentials can be calculated numerically without using optimization methods. Using a "search"-type strategy, one can sample points along the demand curve and calculate a rough approximation of slope between sampled data points. However, it warrants strongly noting that even at high-resolution sampling (i.e., 0.0001 between points), this approach is fraught with possiblities for error and artifacts. An example how this can be done is performed below:

```
## [1] 15.62788
```

While approximate to other methods, it warrants noting that this approach is subject to many sources of error and bias, as the sampling resolution selected by the analyst can easily influence results.

PMAX Method 4: Analytical Solution

As an alternative to optimization and search, PMAX can be calculated analytically. This can be performed when the derivative of the Exponential model of demand are rearranged to the point where slope is equal to negative one. In this equation (where Alpha, Q, and K are known), solving for X (unit price) where Y == -1 is challenging for several reasons. Principal among them, the presence of X inside and outside of the exponent warrants methods beyond basic calculus and elementary functions.

To address this particular problem, solving for unit price in and out of the exponent can take place using the Lambert W function. This is also referred to as the omega function or the product logarithm, depending on the software you use. This logic can be applied to the first-order derivative provided by Hursh and colleagues, where unit price with a slope of -1 can be solve for. Using the principal branch of the W function, this calculation can be performed using the following calculation:

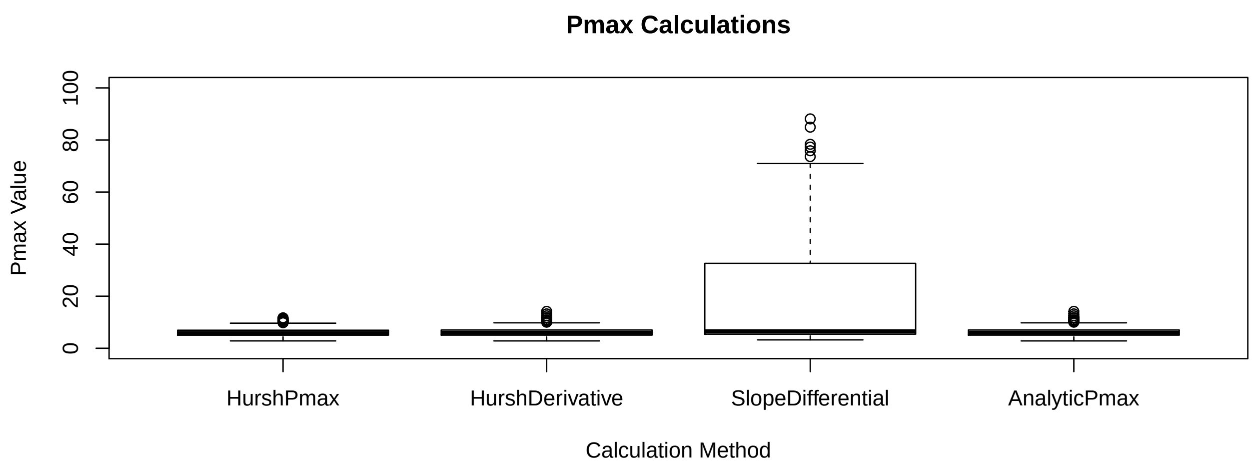
$$P_{MAX} = -W_0(-1/ln(10^k))/(A * Q_0)$$

```
## [1] 15.62583
```

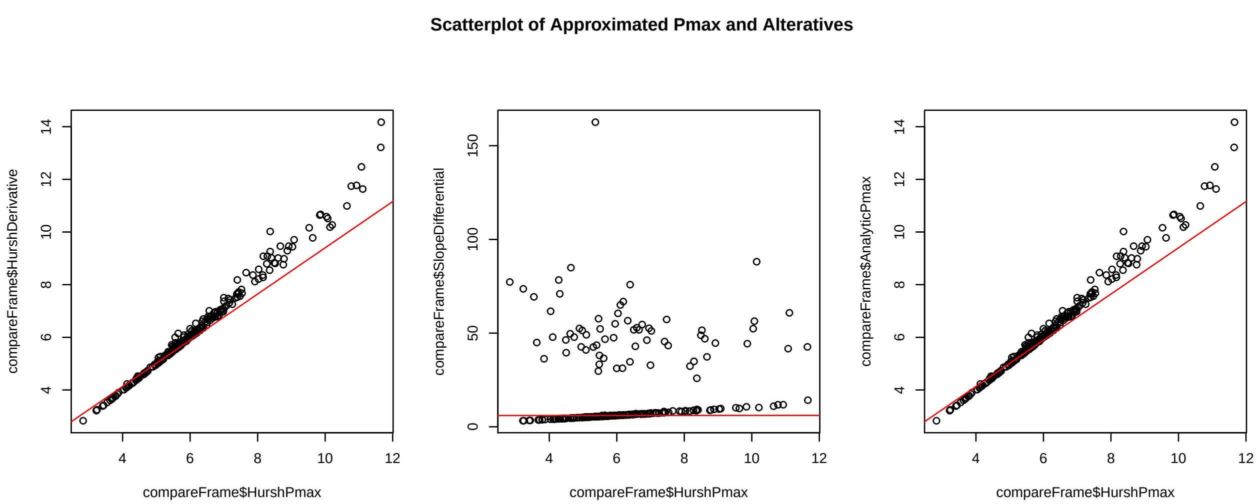
It warrants noting that this is essentially the exact solution for the derivative-based method (Method 2). This makes good sense from an analytic perspective, since this is just a solution to that problem without the need for optimization.

Simulated Demand and Comparisons

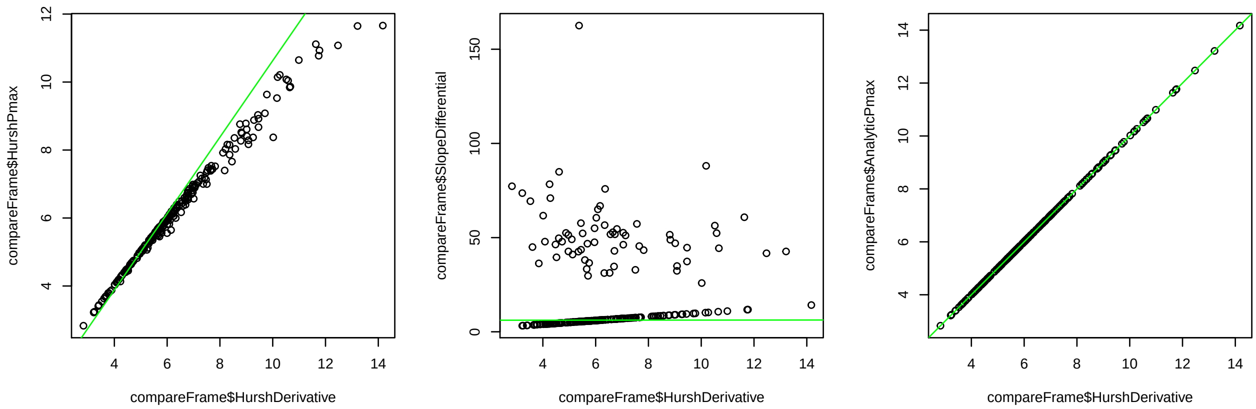
To evaluate these various methods, a brief simulation was conducted using data from an Alcohol Purchase Task. The source to run, and replicate, these simulations is provided below:



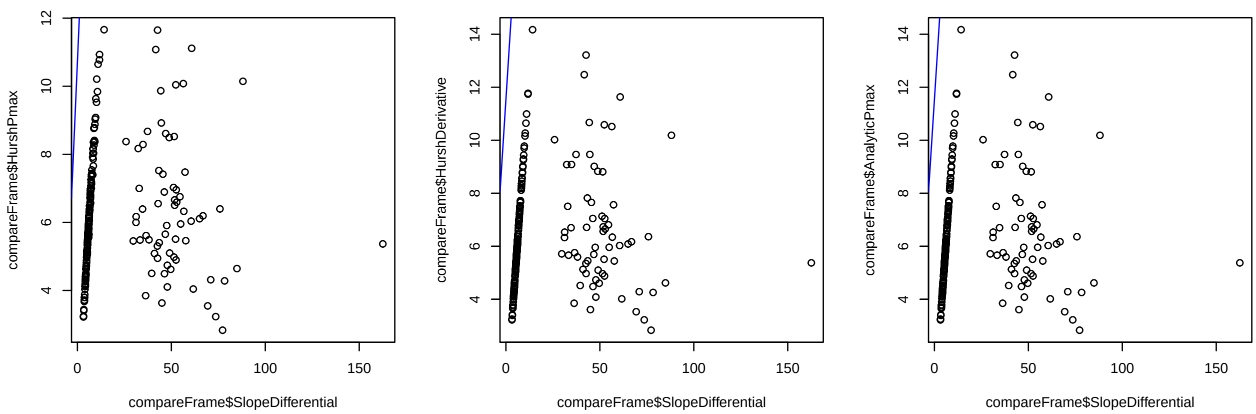

```
##                   HurshPmax HurshDerivative SlopeDifferential AnalyticPmax
## HurshPmax                           0.99371          0.095676      0.99371
## HurshDerivative                                      0.094611            1
## SlopeDifferential                                                 0.094611
## AnalyticPmax
```



Scatterplot of Derivative−based Pmax and Alteratives



Scatterplot of Slope−differential Pmax and Alteratives



Summary

In sum, there are several major takeaways from this review/simulation.

1. Hursh's approximated PMAX is nearly aways proximal to the value revealed from optimization using the derivative. This is good, as this value is easily computed with basic spreadsheet software.
2. Hursh's approximated PMAX is highly similar to the derivative-based approach, however it tended to overestimate at lower extremes and underestimate at the upper extremes. This is of course, compared to the derivative approach, which is assumed to be an exact assessment of model slope.
3. The analytical PMAX was correlated at 1 with the derivative-based approach, which required optimization. It is, for all intents and purposes, a suitable alternative for derivative-based optimization.
4. The slope-differential approach was wildly prone to error, even at extreme precision. This approach does provide an alternative to analytical and optimization-based methods, but the results derived from this way should be interpreted critically, as there would likely be may artifacts introduced (i.e., potentially many or no slopes at -1).