Representation and statistical analyse of signals

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1 Introduction and reminders on random variables

Definition 1.1. *Let (Ω, τ, P) be a probability space, where Ω is the sample space, τ is a σ-algebra and P is a probability measure, and the set of instants T ($\mathbb{R}$ or $\mathbb{Z}$). A (real) stochastic process is the application*

$$\begin{aligned} X :\ & T \times \Omega \to \mathbb{R} \\ & (t, \omega) \mapsto X(t, \omega). \end{aligned}$$

- For $\omega = \omega_0$ fixed, $X(t, \omega_0)$ is an ordinary function (trajectory or sample) and, for $t = t_0$ fixed, it is a random variable.
- If $T = \mathbb{R}$, the signal is time continuous and we denote the application $X(t, \omega)$. If $T = \mathbb{Z}$, we have a discrete time signal and we denote $X[k, \omega]$.

Definition 1.2. *A σ-algebra on a set Ω is a collection τ of subsets of Ω such that*

1. *It includes the empty set ($\emptyset \in \tau$);*
2. *It is closed under complement ($A \in \tau \Rightarrow \overline{A} \in \tau$);*
3. *It is closed under countable union ($A_n \in \tau \Rightarrow \cup_n A_n \in \tau$).*

The pair (Ω, τ) is said to be a measurable space or a Borel space.

Definition 1.3. *A probability measure P is an application $P : \tau \to \mathbb{R}$ which respects the Kolmogorov axioms:*

1. $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. $A_i \cap A_j \Rightarrow P(\cup_i A_i) = \sum_i P(A_i)$

Definition 1.4. *Let (Ω, τ, P) be a probability space and $(\mathbb{R}, \mathfrak{B})$ a measurable space, with $\mathfrak{B}$ the Borel σ-algebra of $\mathbb{R}$[1]. A real-valued random variable is a measurable function $X : \Omega \to \mathbb{R}$, which means that $X^{-1}(B) \in \tau\ \forall B \in \mathfrak{B}$ (it relates events with number values). We write the probability of the event x be in the interval B as $Pr\{x \in B\} = P_x(B) = P(X^{-1}(B))$.*

Definition 1.5. *The cumulative distribution function (CDF) of a real-valued random variable X is the function*

$$F_X(x) = Pr\{X(\omega) \leq x\} = P_x(]-\infty, x])) = P(X^{-1}(]-\infty, x])).$$

The probability density function (PDF) $p_X(x)$ is

$$p_X(x) = F'_X(x) \Leftrightarrow F_X(x) = \int_{-\infty}^{x} p_X(\xi) d\xi.$$

The expected value of a random variable X whose CDF admits a PDF $p_X(x)$ is

$$\mathbb{E}[X] = \int_{\mathbb{R}} x p_X(x) dx.$$

- Expected value of a function: $\mathbb{E}[f(X(\omega))] = \int f(x) p_X(x) dx = \int f(x) dF_X(x) = \int f(x) dP_X(i)$

Definition 1.6. *The characteristic function of a scalar random variable X is*[2]

$$\phi_X(u) = \mathbb{E}[e^{juX}] = \int e^{juX} p_X(x) dx$$

and the second characteristic function is

$$\psi_X(u) = \ln(\phi_X(u)).$$

[1] The Borel σ-algebra on $\mathbb{R}$ is the smallest σ-algebra containing all open sets on $\mathbb{R}$.
[2] It is the Fourier transform with sign reversal in the complex exponential.

Theorem 1.1. *Let $\mathfrak{B}$ be the Borel σ-algebra of $\mathbb{R}$ and ϕ a positive bounded measure on $(\mathbb{R}, \mathfrak{B})$. Then, there exists a unique positive integrable function defined in $g \in L^1$, up to a set of measure zero, and a unique singular measure ϕ_s on $(\mathbb{R}, \mathfrak{B})$ such that*

$$\phi(B) = \int_B g(x)dx + \phi_s(B), \ \forall B \in \mathfrak{B}.$$

- A measure ϕ_s is said to be singular if $\exists S \in \mathfrak{B}$ with $\mu(S) = 0$ and $\forall B \in \mathfrak{B}, \phi_s(B) = \phi_s(B \cap S)$.
- It is often possible to write

$$\phi(B) = \int \underbrace{g(x) \sum_k \mu_k \delta(x - s_k)}_{\text{density of the measure } \phi} dx.$$

- The probability density function is a particular case of such measure:

$$P(B) = \int \underbrace{g(x) \sum_k \mu_k \delta(x - s_k)}_{\text{PDF}} dx.$$

Definition 1.7. *A temporal distribution (temporal law) of a random variable X is*

$$F_X(x_1, ..., x_n, ..., t_1, ..., t_n, ...) = P(X(t_1) \leq x_1, ..., X(t_n) \leq x_n, ...)$$

Particular cases:

- An n-order distribution (n-order law) is $F_X(x_1, ..., x_n, ..., t_1, ..., t_n) = P(X(t_1) \leq x_1, ..., X(t_n) \leq x_n)$.
- A first-order distribution is $F_X(x, t) = P(X(t_1) \leq x_1) = F_{X(t,\omega)}(x_1)$.

Proposition 1.1. *Properties of temporal distributions:*

i. Symmetry: $F_X(x_1, ..., x_n, t_1, ..., t_n) = F_X(x_n, ..., x_1, t_n, ..., t_1)$.

ii. Consistency: $F_X(x_1, ..., x_n, ..., t_1, ..., t_n, ...) = F_X(x_1, ..., x_n, x_{n+1}, ..., x_m, t_1, ..., t_n, t_{n+1}, ..., t_m)$.

Definition 1.8. *Definitions of equivalence[3]:*

i. Two signals are said to be wide sense equivalent when they have the same temporal distribution.

ii. Two random signals $S_1(t, \omega)$ and $S_2(t, \omega)$ are said to be strictly equivalent if $P(S_1(t, \omega) = S_2(t, \omega)) = 1, \forall t$.

iii. If these two signals are such that $P(S_1(t, \omega) = S_2(t, \omega) \forall t \in T) = 1$, they are indistinguishable.

For example, consider the following signals:

$$S_1 \equiv 0, \quad S_2 = \begin{cases} 1 \text{ in a random value in } [0, 1] \\ 0 \text{ everywhere else.} \end{cases}, \quad S_3 = \begin{cases} 1 \text{ everywhere if a random value in } [0, 1] \text{ is } 0.5 \\ 0 \text{ everywhere, otherwise.} \end{cases}$$

S_1 and S_2 are strictly equivalent, but not indistinguishable. On the other hand, S_1 and S_3 are indistinguishable.

Point processes

Definition 1.9. *A point process is a continuous-time distribution of points on T. A counting process $N(t, \omega)$ can be used to count point process as it avoids to be equivalent to a zero process.*

Definition 1.10. *A Poisson process of intensity λ is a point process which may be defined in several ways:*

1. *It is a process that follows*
   (a) *The number of points in non-overlapping intervals are independent: $N(t_k, \omega) - N(t_{k-1}, \omega), ..., N(t_1, \omega) - N(t_0, \omega)$ are independent $\forall t_0 < ... < t_k$;*
   (b) *The probability of having exactly one event in a "small" interval is proportional to the length of the interval: $P(N(t + h) - N(t) = 1) = \lambda h + o(h)$;*
   (c) *The probability of having more than one event in a "small" interval is negligible: $P(N(t + h) - N(t) > 1) = o(h)$.*
2. *It is a process that follows*

[3] Presented in a strength growing order.

(a) *The number of points in non-overlapping intervals are independent:* $N(t_k,\omega) - N(t_{k-1},\omega), .., .N(t_1,\omega) - N(t_0,\omega)$ *are independent* $\forall t_0 < ... < t_k$*;*

(b) $P(N(t+T) - N(t) = k) = e^{-\lambda T}\frac{(\lambda T)^k}{k!}$

3. *The intervals* T_i *between the occurrence of two events are i.i.d. (independent and identically distributed) with probability density function*

$$p(t) = \lambda e^{-\lambda t}, \quad t \geq 0.$$

Reminder:

- Independence: $P(N_1 \cap N_2) = P(N_1)P(N_2)$
- Uncorrelation: $\mathbb{E}[N_1 \cap N_2] = \mathbb{E}[N_1]\mathbb{E}[N_2]$
- Independence $\Rightarrow$ uncorrelation.

2 Partial characterisation of stochastic processes and temporal properties

Definition 2.1. *Important definitions:*

	Continuous-time	Discrete-time
First-order moment	$m(t) := \mathbb{E}[X(t,\omega)]$	$m[k] := \mathbb{E}[X[k,\omega]]$
Centred signal	$X_C(t,\omega) := X(t,\omega) - m(t)$	$X_C[k,\omega] := X[k,\omega] - m[k]$
Second-order moment[4] or (auto)correlation	$\gamma(t_1,t_2) := \mathbb{E}[X(t_1,\omega)X^*(t_2,\omega)]$	$\gamma[k_1,k_2] := \mathbb{E}[X[k_1,\omega]X^*[k_2,\omega]]$
Covariance	$c(t_1,t_2) := \gamma_{X_C}(t_1,t_2) = \mathbb{E}[X_C(t_1)X_C^*(t_2)]$	$c[k_1,t_2] := \gamma_{X_c}[k_1,k_2] = \mathbb{E}[X_C[k_1]X_C^*[k_2]]$
Variance	$\sigma(t) := \sqrt{c(t,t)} = \sqrt{\mathbb{E}[\lvert X_C(t)\rvert^2]}$	$\sigma[k] := \sqrt{c[k,k]} = \sqrt{\mathbb{E}[\lvert X_C[k]\rvert^2]}$
Power	$P(t) := \mathbb{E}[\lvert X(t)\rvert^2]$	$P[k] := \mathbb{E}[\lvert X[k]\rvert^2]$

Other types of (random) power:

- Random instant power: $|X(t,\omega)|^2$
- Random temporal mean power: $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|X(t,\omega)|^2dt$

Proposition 2.1. *Properties:*

i. $c(t_1,t_2) = \gamma_X(t_1,t_2) - m(t_1)m^*(t_2)$

ii. $\sigma^2(t) = \mathbb{E}[|X(t,\omega)|^2] - |m(t)|^2$

Definition 2.2. *In the vector case, we have the vector of random variables*

$$\mathbf{X}(t,\omega) = \begin{bmatrix} X_1(t,\omega) \\ \vdots \\ X_n(t,\omega) \end{bmatrix},$$

the first-order moment

$$\mathbf{m}(t) = \mathbb{E}[\mathbf{X}(t,\omega)] = \begin{bmatrix} m_1(t) \\ \vdots \\ m_n(t) \end{bmatrix},$$

the second-order (correlation) matrix

$$\Gamma(t_1,t_2) = \mathbb{E}[\mathbf{X}(t_1,\omega)\mathbf{X}^\dagger(t_2,\omega)],$$

the cross-correlation

$$\Gamma_{X_1,X_2}(t_1,t_2) = \mathbb{E}[X_1(t_1)X_2^*(t_2)]$$

and the covariance matrix

$$\mathbf{c}(t_1,t_2) = \mathbb{E}[\mathbf{X_C}(t_1,\omega)\mathbf{X_C^\dagger}(t_2,\omega)] = \Gamma(t_1,t_2) - \mathbf{m}(t_1)\mathbf{m}^\dagger(t_2).$$

[4]Generalisation: n-order moment (it is not unique as there is no rule for the application of complex conjugate).

$$m(t_1,...,t_n) = \mathbb{E}[X(t_1),...,X(t_n)], \quad n \geq 3.$$

A fact about real random variables The moments are coefficients of a series expansion of the characteristic function ϕ_X. Let $m_n = \mathbb{E}[X^n]$:

$$\phi_X(u) = \mathbb{E}[e^{juX}] = \mathbb{E}\left[\sum_k j^k u^k \frac{X^k}{k!}\right] = \sum_k j^k \frac{u^k}{k!}\mathbb{E}\left[X^k\right] = \sum_k j^k \frac{m_n}{k!} u^k$$

Proposition 2.2. *Properties:*

i. The autocorrelation $\gamma(t_1, t_2)$ exists $\forall (t_1, t_2) \in T^2$ if and only if $\gamma(t,t) = \mathbb{E}[|X(t)|^2]$ exists ($< \infty$) $\forall t \in T$.

ii. The autocorrelation $\gamma(t_1, t_2) = \mathbb{E}[X(t_1)X^*(t_2)]$ defines a pseudo inner product.

iii. The Cauchy-Schwarz inequality holds: $|\gamma(t_1,t_2)|^2 \leq \gamma(t_1,t_1)\gamma(t_2,t_2)$.

iv. Existency of $\gamma(t_1,t_2)$ implies existency of $m(t) = \mathbb{E}[X(t)]$.

v. $\gamma(t_1,t_2)$ is a non-negative definite function (NND), i.e., $\sum_i \sum_j \lambda_i \lambda_j^* \gamma(t_i,t_j) \geq 0,\ \forall (\lambda_i) \in \mathbb{C}^n, \forall (t_i) \in T^n$.

Proof.

ii. The expectation defines a pseudo inner product: $\langle X, Y\rangle := \mathbb{E}[XY^*]$, i.e., it is symmetric, bilinear and almost surely positive-definite: $\langle X, X\rangle = 0 \Leftrightarrow P(X=0) = 1$. We can then define the autocorrelation inner product by setting $X = X(t_1)$ and $Y = X(t_2)$.

iii. Apply the Cauchy-Schwarz inequality to the inner product defined above.

$$\mathbb{E}[XY^*] \leq \mathbb{E}[|X|^2]\mathbb{E}[|Y|^2]$$

iv. Apply the expectation Cauchy-Schwarz inequality to $X = X(t)$ and $Y = 1$:

$$|\mathbb{E}[X(t)\cdot 1]|^2 \leq \mathbb{E}[|X(t)^2|]\mathbb{E}[1] = \gamma(t,t).$$

v. Take $Z(t_i) = \sum_i \lambda_i X(t_i)$ and calculate

$$\mathbb{E}[|Z|^2] = \mathbb{E}\left[\left(\sum \lambda_i X(t_i)\right)\left(\sum \lambda_j X(t_j)\right)^*\right] = \sum\sum \lambda_i \lambda_j^* \mathbb{E}\left[X(t_i)X^*(t_j)\right] \geq 0.$$

□

Stationarity

Definition 2.3. *A random process $X(t)$ is said to be strict sense stationary (SSS) if its temporal law is invariant by time shift, i.e.*

$$F_X(x_1, ..., x_n, t_1, ..., t_n) = F_X(x_1, ..., x_n, t_1 + h, ..., t_n + h) \quad \forall n \forall x_i \forall t_i \forall h.$$

Definition 2.4. *A random process is said to be stationary of order n if its moments up to order n are stationary.*

- *In particular, a stationary process of order 1 is such that them mean $m(t)$ is constant, i.e.,*

$$m(t) = m(t+h)\ \forall t \forall h.$$

- *A stationary process of order 2 or wide sense stationary (WSS) has $m(t)$ constant and also*

$$\gamma(t_1,t_2) = \gamma(t_1+h, t_2+h)\ \forall t \forall h.$$

This means that $\gamma(t_1,t_2) = \gamma(t, t+\tau)$ depends only on $\tau = t_2 - t_1$. We can write

$$\gamma(\tau) = \mathbb{E}[X(t+\tau), X^*(t)] = \mathbb{E}[X(t), X^*(t-\tau)].$$

Definition 2.5. *Two random processes $X(t,\omega)$ and $Y(t,\omega)$ are said to be jointly stationary (of order 2) if $\begin{bmatrix} X(t,\omega) \\ Y(t,\omega)\end{bmatrix}$ is stationary (of order 2). Then, $\gamma_X = \gamma_X(\tau)$, $\gamma_Y = \gamma_Y(\tau)$, $\gamma_{XY} = \gamma_{XY}(\tau)$, $m_X(t) = m_X$ and $m_Y(t) = m_Y$.*

Definition 2.6. *A random process $X(t,\omega)$ is said to be cyclostationary if there exists T such that*

$$\mathbb{E}[X(t+kT) = \mathbb{E}[X(t)]] \text{ and } \gamma(t_1+kT, t_2+kT) = \gamma(t_1,t_2) \quad \forall k.$$

- If $X(t)$ is cyclostationary, then $Y(t) = X(t + \mathcal{O}(\omega))$, with $\mathcal{O}$ uniformly distributed over $[0,T]$ is stationary.

Properties of WSS signals[5]

Proposition 2.3. *Basic properties. Let $X(t,\omega)$ be a random process. Then:*

i. $\gamma(0) \geq |\gamma(\tau)|\ \forall \tau$

ii. $\gamma(0) = P(t)$

iii. $\gamma(\tau) = \gamma^*(-\tau)$

Proof.

i. Use the Cauchy-Schwarz inequality for expectation inner product:

$$|\mathbb{E}[X(t+\tau)X^*(t)]|^2 \leq \mathbb{E}[|X(t+\tau)|^2]\mathbb{E}[|X(t)|^2] \Rightarrow |\gamma(\tau)|^2 \leq \gamma(0)\gamma(0) \Rightarrow |\gamma(\tau)| \leq \gamma(0).$$

ii. By definition, $\gamma(0) = \mathbb{E}[X(t+0)X^*(t)] = \mathbb{E}[|X(t)|^2] = P(t) = P$.

iii. $\gamma(\tau) = \mathbb{E}[X(t+\tau)X^*(t)] = \mathbb{E}[X(t)X^*(t-\tau)] = \mathbb{E}[X(t-\tau)X^*(t)]^* = \gamma^*(-\tau)$.

□

Proposition 2.4. *Basic properties for vector case:*

i. $\Gamma(\tau) = \mathbb{E}[\mathbf{X}(t+\tau)\mathbf{X}^*(t)] = \Gamma^\dagger(-\tau)$

ii. If $\Gamma(0)$ is hermitian, then it is orthogonally diagonalisable and has real eigenvalues.

Proposition 2.5. *Periodicity:*

i. If $\gamma(0) = \gamma(\tau_1)$ with $\tau_1 \neq 0$, then γ is τ_1-periodic.

ii. $\gamma(\tau)$ is τ_1-periodic $\Leftrightarrow \gamma(0) = \gamma(\tau_1) \Leftrightarrow P(X(t) = X(t+\tau_1)) = 1$.

Proposition 2.6. *If $\gamma(0) = \gamma(\tau_1) = \gamma(\tau_2)$ with $\frac{\tau_1}{\tau_2} \notin \mathbb{Q}$ and $\tau_1, \tau_2 \neq 0$, then $\gamma(\tau) = \gamma$ is constant (as long as γ is continuous).*

Proposition 2.7. *The autocorrelation $\gamma(\tau)$ is uniformely continuous if and only if $\gamma(\tau)$ is continuous at $\tau = 0$.*

Proposition 2.8. *For non-WSS case, $\gamma(t_1, t_2) \in C^0$ on the diagonal of T (i.e., $(t,t) \in T^2$) if and only if γ is continuous on every $(t_i, t_j) \in T^2$.*

Markov process

Definition 2.7. *A Markov process is a stochastic process whose future probabilities are determined by its most recent values, i.e.,*

$$P(X(t_n) \in B_n | X(t_{n-1}) \in B_{n-1}, ..., X(t_1) \in B_1) = P(X(t_n) \in B_n | X(t_{n-1}) \in B_{n-1}), \quad t_n > t_{n-1} > \cdots > t_1.$$

In a Markov process, the n-order distribution $P(x_1, ..., x_n, t_1, ..., t_n)$ depends only on the second order distribution $P(x_1, x_2, t_1, t_2)$.

3 Power Spectral Density (PSD)

We shall consider WSS signals here.

Continuous case

Theorem 3.1. *(Bochner) A function $\gamma : \mathbb{R} \to \mathbb{C}$ is continuous and non-negative definite if, and only if, there exists a positive bounded measure φ on $(\mathbb{R}, \mathfrak{B})$ such that*

$$\gamma(\tau) = \int_{-\infty}^{+\infty} e^{j2\pi f\tau} d\varphi(f) \quad \forall \tau \in \mathbb{R}.$$

Applying the Theorem 1.1 to autocorrelation, we have

$$\gamma(\tau) = \int_{-\infty}^{+\infty} e^{j2\pi f\tau} \Gamma(f) df$$

and $P = \gamma(0) = \int_{-\infty}^{+\infty} d\varphi(f) = \int_{-\infty}^{+\infty} \Gamma(f) df$.

[5] All signals considered in this section are WSS, unless explicitly indicated.

Discrete case

Theorem 3.2. *A function $\gamma : \mathbb{Z} \to \mathbb{C}$ is non-negative definite if and only if there exists a positive bounded measure ϕ on $(\mathbb{R}, \mathfrak{B})$ such that*

$$\gamma[m] = \int_{-1/2}^{1/2} e^{j2\pi\nu m} d\phi(\nu) \quad \forall m \in \mathbb{Z}.$$

Applying an analogous theorem, we have

$$\gamma[m] = \int_{-1/2}^{+1/2} e^{j2\pi\nu m}\Gamma(\nu)d\nu$$

and $P = \gamma[0] = \int_{-1/2}^{+1/2} d\phi(\nu) = \int_{-1/2}^{+1/2} \Gamma(\nu)d\nu$.

Definition 3.1. *(Short definition)*
Continuous time:

$$\Gamma(f) = \mathcal{F}\{\gamma(\tau)\} = \int_{-\infty}^{+\infty} \gamma(\tau)e^{j2\pi f\tau}d\tau.$$

Discrete time:

$$\Gamma[\nu] = \mathcal{F}\{\gamma[k]\} = \sum_{k=-\infty}^{+\infty} \gamma[k]e^{j2\pi k\nu}.$$

Proposition 3.1. *Properties of the PSD:*

i. $\Gamma(f)$ *is real and positive*[6].

ii. *If the signal $X(t,\omega)$ is real, then its PSD $\Gamma(f)$ is even.*

iii. *Some PSD examples:*

$\gamma(\tau)$	$\Gamma(f)$
$\frac{N_0}{2}\delta(\tau)$	$\frac{N_0}{2}$
$N_0 B \operatorname{sinc}(2\pi B\tau)$	$\frac{N_0}{2}\operatorname{rect}_{[-B,B]}(f)$
$N_0 B \operatorname{sinc}(2\pi B\tau)\cos(2\pi f_0\tau)$	$\frac{N_0}{4}\left[\operatorname{rect}_{[-B,B]}(f-f_0) + \operatorname{rect}_{[-B,B]}(f+f_0)\right]$

Sampling a signal Let the signal $X(t,\omega)$ sampled at a rate $f_s = 1/T_s$, then the result signal is $X[k,\omega] = X(kT_s,\omega)$.

4 White noise

Definition 4.1. *A (wide sense) white signal is a signal for which the PSD is constant (WSS signal).*

Definition 4.2. *A (strict sense) white signal is a signal $X(t,\omega)$ such that*

i. $X(t,\omega)$ *is centred and WSS*

ii. $X(t_1,\omega)$ *and* $X(t_2,\omega)$ *are independent* $\forall t_1 \neq t_2$

Definition 4.3. *A band-limited white signal is a signal for which the PSD is constant on a finite support.*

Example: $\Gamma(f) = \operatorname{rect}_{[-B,B]}(f - f_0)$.
Difference between continuous and discrete time:

- Continuous: $P = \int_{-\infty}^{+\infty} \Gamma(f)df = +\infty = \gamma(0)$
- Discrete: $P = \int_{-1/2}^{+1/2} \Gamma(\nu)d\nu = K = \gamma[0]$

[6] The Fourier transform of non-negative definite function in non-negative. In addition, if it had a negative part, we could filter it and get a negative signal, but it would have negative power, which is impossible!

5 Gaussian signals

Definition 5.1. *A vector of random variables* $\mathbf{X} = \begin{bmatrix} X_1 & \cdots & X_n \end{bmatrix}^T$ *is said to be Gaussian if its PDF is*

$$p_{\mathbf{X}}(x_1, ..., x_n) = (2\pi)^{-n/2} \det(\mathbf{C})^{-1/2} \exp\left[-\frac{1}{2}(\mathbf{X} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{X} - \mathbf{m})\right]$$

where $\mathbf{m} = \mathbb{E}[\mathbf{X}]$ *and* $\mathbf{C} = \mathbb{E}[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T]$, *which is a real symetric matrix such that* $\mathbf{u}^T\mathbf{C}\mathbf{u} \geq 0$. *In this case,* $X_1, .., X_n$ *are said to be jointly Gaussian.*

Equivalent definitions:

- $\phi_{\mathbf{X}}(\mathbf{u}) = \mathbb{E}[e^{j\mathbf{u}^T\mathbf{X}}] = e^{j\mathbf{u}^T\mathbf{m}}e^{-\frac{1}{2}\mathbf{u}^T\mathbf{C}\mathbf{u}}$
- $\psi_{\mathbf{X}}(\mathbf{u}) = j\mathbf{u}^T\mathbf{m} - \frac{1}{2}\mathbf{u}^T\mathbf{C}\mathbf{u}$
- $\forall (\lambda_i)_{i \in [\![1,n]\!]} \in \mathbb{R}^n$, $\sum \lambda_i X_i(\omega)$ is a Gaussian random variable for X_i jointly Gaussian.

Proposition 5.1. *Properties of jointly Gaussian random variables:*

i. Linear combination of jointly Gaussian R.V. is a jointly Gaussian R.V.

ii. Uncorrelated jointly Gaussian R.V. $\Leftrightarrow$ *independent jointly Gaussian R.V.*

iii. If $\mathbf{X}(\omega)$ *is Gaussian, then* $\mathbf{Y}(\omega) = \mathbf{A}\mathbf{X}(\omega)$ *is Gaussian* $\forall \mathbf{A} \in \mathcal{M}_{n \times n}(\mathbb{R})$

Definition 5.2. *A complex vector* $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$ *is said to be Gaussian if* $\begin{bmatrix} \mathbf{X} & \mathbf{Y} \end{bmatrix}^T$ *is a real Gaussian vector.*

Definition 5.3. *A real/complex signal* $X(t, \omega)$ *is said to be Gaussian when* $\begin{bmatrix} X(t_1, \omega) \\ \vdots \\ X(t_n, \omega) \end{bmatrix}$ *is a real/complex Gaussian vector* $\forall n \in \mathbb{N},\ \forall (t_i)_{i \in [\![1,n]\!]}$.

Proposition 5.2. *Properties of Gaussian signals:*

i. Filtering Gaussian signals result in Gaussian channels.

ii. The mean $m(t)$ *and the autocorrelation* $\gamma(t_1, t_2)$ *are enough to completely characterise the signal.*

iii. If $X(t, \omega)$ *is Gaussian, then:*

- *WSS* $\Leftrightarrow$ *SSS*
- *uncorrelation* $\Leftrightarrow$ *independent*
- *wide sense white noise* $\Leftrightarrow$ *strict sense white noise*

Special case: Centred case

$$p_Z(\mathbf{z}) = p_{X,Y}(\mathbf{x}, \mathbf{y}) = (2\pi)^{-n} \det(\tilde{\mathbf{C}}_{X,Y})^{-1/2} \exp\left[-\frac{1}{2} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}^T \tilde{\mathbf{C}}_{X,Y}^{-1} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right]$$

- $\mathbf{C}_Z = \mathbb{E}[(\mathbf{Z} - \mathbf{m}_Z)(\mathbf{Z} - \mathbf{m}_Z)^\dagger] = \mathbb{E}[\mathbf{Z}\mathbf{Z}^\dagger] = \mathbf{C}_X + \mathbf{C}_Y + j(-\mathbf{C}_{XY} + \mathbf{C}_{YX})$
- $\mathbf{C}_{XY} = \mathbf{C}_{YX}^T = \mathbb{E}[(\mathbf{X} - \mathbf{m}_X)(\mathbf{Y} - \mathbf{m}_Y)^T]$
- $\tilde{\mathbf{C}}_{X,Y} = \begin{bmatrix} \mathbf{C}_X & \mathbf{C}_{XY} \\ \mathbf{C}_{YX} & \mathbf{C}_Y \end{bmatrix}$
- $\mathbf{D}_Z = \mathbb{E}[\mathbf{Z}\mathbf{Z}^T] = \mathbf{C}_X - \mathbf{C}_Y + j(\mathbf{C}_{XY} + \mathbf{C}_{YX})$

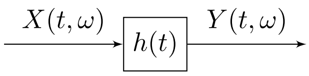
Definition 5.4. *A subclass of complex Gaussian signals is formed by circular Gaussian signals, which are characterised by zero relation matrix and zero mean, i.e.,* $\mathbf{m} = 0$ *and* $\mathbf{D}_Z = 0$.

$$p_Z(\mathbf{z}) = p_{X,Y}(\mathbf{x}, \mathbf{y}) = (\pi)^{-n} \det(\mathbf{C}_Z)^{-1} \exp[-(\mathbf{Z} - \mathbf{m}_Z)^\dagger \mathbf{C}_Z^{-1} (\mathbf{Z} - \mathbf{m}_Z)]$$

6 Filtering random signals

In this section, we will consider:

- WSS signals
- Time-invariant linear systems, i.e., $Y(t,\omega) = X(t,\omega) * h(t)$



- Bounded input-bounded output (BIBO) stable systems, i.e., the transfer function is absolutely integrable:

$$\int |h(t)|dt < \infty$$

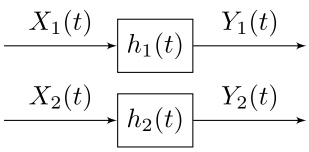
Proposition 6.1. *$Y(t,\omega)$ exists almost surely, i.e., $P(Y(t) < \infty) = 1$.*

Theorem 6.1. *(Fubini) Let (E_1, T_1, m_1) and (E_2, T_2, m_2) be σ-finite measure spaces and a mapping $f : E_1 \times E_2 \to \mathbb{R}$. If $\int_{E_1}\int_{E_2} |f|dm_2dm_1$ exists (i.e. converges), then*

- *$x \in E_1 \Rightarrow \int_{E_2} fdm_2$ exists almost everywhere (up to sets of measure zero)*
- *$x \in E_2 \Rightarrow \int_{E_1} fdm_1$ exists almost everywhere (up to sets of measure zero)*

and $\int_{E_1}\int_{E_2} fdm_2dm_1 = \int_{E_2}\int_{E_1} |f|dm_1dm_2$.

Interference formula Let consider the following systems.



Case	Time domain	Frequency domain
General case	$\gamma_{Y_1Y_2}(\tau) = \tilde{h}_2 * h_1 * \gamma_{X_1X_2}(\tau)$	$\Gamma_{Y_1Y_2}(f) = H_1H_2^*\Gamma_{X_1X_2}(f)$
$Y_i = X_i * h_i$	$\Gamma_Y(\tau) = \tilde{h} * h * \gamma_X(\tau)$	$\Gamma_Y(f) = \lvert H(f)\rvert^2\Gamma_X(f)$
$X_1 = X_2 = X$ and $h_2(t) = \delta(t)$	$\gamma_{YX}(\tau) = h * \gamma_X(\tau)$	$\Gamma_{YX}(f) = H(f)\Gamma_X(f)$

We define $\tilde{h}(t) := h^*(-t)$.

7 Narrow band signals

A narrow band is a signal for which the bandwidth B is much smaller than its central f_0. In this chapter, we consider real, centred, WSS signals $X(t)$. So the PSD $\Gamma_X(f)$ is real and even.

Definition 7.1. *The analytic signal $Z(t)$ associated to $X(t)$ is the canonical complex signal for which $X(t)$ is the real part, i.e.,*

$$Z(t) = X(t) + jY(t)$$

where $\Re\{Z(t)\} = X(t)$ and $\Im\{Z(t)\} = Y(t) = h * X(t)$. We call $Y(t)$ the Hilbert transform.

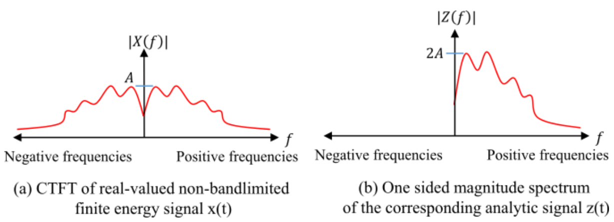


Figure 1: Example of analytic signal [1].

Definition 7.2. *The* <u>*Hilbert filter*</u> *is $h(t)$ such that $Y(t) = h(t) * X(t)$. The* <u>*analytical filter*</u> *is $h_a(t)$ such that $Z(t) = h_a(t) * X(t)$.*

The analytical filter is $h_a(t) = \delta(t) + jh(t)$, because

$$Z(t) = X(t) * h_a(t) = X(t) * [\delta(t) + jh(t)] = X(t) + jY(t)$$

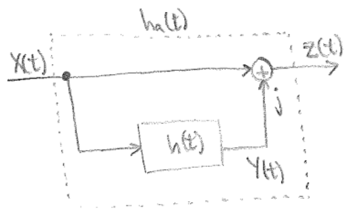


Figure 2: Diagram of analytical and Hilbert filters.

What is the filter $h(t)$ that erases negative frequencies?
Let $Z(f) = X(f)H_a(f) = X_f[1 + H(f)]$. We search $H(f)$ such that $1 + jH(f) = 0$ for $f < 0$.

$$jH(f) = \operatorname{sgn}(f)^7 \xrightarrow{\mathcal{F}^{-1}} h(t) = \frac{1}{\pi t}$$

In this case, $H_a(f) = \left\{ \begin{array}{ll} 0, & f < 0 \\ 2, & f > 0 \end{array} \right.$.

Proposition 7.1. *Statistical properties of $Y(t)$:*

- $\mathbb{E}[Y(t)] = h(t) * \mathbb{E}[X(t)] = 0$
- $\Gamma_T(f) = \Gamma_X(f)|\operatorname{sgn}(f)|^2 = \Gamma_X(f)$
- $\gamma_Y(\tau) = \mathcal{F}^{-1}\{\Gamma_Y(f)\} = \mathcal{F}^{-1}\{\Gamma_X(f)\} = \gamma_X(\tau)$
- $\Gamma_{YX}(f) = \Gamma_X(f)(j\operatorname{sgn}(f)) = -\Gamma_X(f)(j\operatorname{sgn}(f))^* = -\Gamma_{XY}(f)$
- $\gamma_{YX}(\tau) = \mathcal{F}^{-1}\{\Gamma_{YX}(f)\} = \mathcal{F}^{-1}\{-\Gamma_{XY}(f)\} = -\gamma_{XY}(\tau)$
- $\gamma_{XY}(0) = \mathbb{E}[X(t)Y(t)] = \gamma_{YX}(0) = -\gamma_{XY}(0) = 0$*: X and Y are uncorrelated at the same instant.*

Proposition 7.2. *Statistical properties of $Z(t)$:*

- $\mathbb{E}[Z(t)] = h_a(t) * \mathbb{E}[X(t)] = 0$
- $\Gamma_Z(f) = |H_a(f)|^2\Gamma_X(f) = \left\{ \begin{array}{ll} 0, & f < 0 \\ 4\Gamma_X(f), & f > 0 \end{array} \right.$
- $\mathbb{E}[Z(t+\tau)Z(t)] = 0$*: if $Z(t)$ is Gaussian, then it is circular Gaussian.*

Definition 7.3. *The* <u>*baseband signal*</u> *related to $Z(t)$ is $\alpha(t) = Z(t)e^{-j2\pi f_0 t}$. It corresponds to centring the PSD in the frequency domain.*

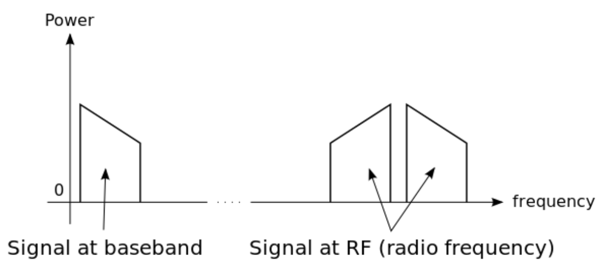


Figure 3: Example of baseband signal [2].

Proposition 7.3. *Statistical properties of $\alpha(t)$:*

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[7] $\operatorname{sgn}(x) = \left\{ \begin{array}{ll} -1, & x < 0 \\ 1, & x \geq 0 \end{array} \right.$.

- $\mathbb{E}[\alpha(t)] = e^{-j2\pi f_0 t}\mathbb{E}[Z(t)] = 0$
- $\gamma_\alpha(\tau) = \mathbb{E}[\alpha(t+\tau)\alpha^*(t)] = \mathbb{E}[Z(t+\tau)e^{-j2\pi(t+\tau)}Z^*(t)e^{j2\pi f_0 t}] = \gamma_Z(\tau)e^{-j2\pi f_0 \tau}$
- $\Gamma_\alpha(f) = \mathcal{F}\{\gamma_Z(\tau)e^{-j2\pi f_0\tau}\} = \Gamma_Z(f+f_0)$
- $\mathbb{E}[\alpha(t+\tau)\alpha(t)] = e^{j2\pi f_0 \tau}\mathbb{E}[Z(t+\tau)Z(t)] = 0$

Definition 7.4. *We can decompose the signal $\alpha(t) = p(t) + jq(t)$ in two components:*
$p(t) = \Re\{\alpha(t)\}$ *is called* *in-phase component*
$q(t) = \Im\{\alpha(t)\}$ *is called* *quadrature component*

To develop properties of $p(t)$ and $q(t)$, it is useful to write them as

$$p(t) = \frac{\alpha(t)+\alpha^*(t)}{2} \quad \text{and} \quad q(t) = \frac{\alpha(t)-\alpha^*(t)}{2j}.$$

Proposition 7.4. *Statistical properties of $p(t)$ and $q(t)$:*

- $\mathbb{E}[p(t)] = \mathbb{E}[q(t)] = 0$
- $\gamma_p(\tau) = \gamma_q(\tau) = \frac{1}{4}(\gamma_\alpha(\tau) + \gamma_{\alpha^*}(\tau))$
- $\Gamma_p(f) = \Gamma_q(f) = \frac{1}{4}(\Gamma_\alpha(f) + \Gamma_\alpha(-f))$
- $\gamma_{pq}(\tau) = -\gamma_{qp}(\tau)$
- $\gamma_{pq}(0) = -\gamma_{qp}(0) = \gamma_{qp}(0) = 0$*: $p(t)$ and $q(t)$ are uncorrelated at the same instant.*
- $\Gamma_{pq}(f) = \mathcal{F}\{\gamma_{pq}(\tau)\} = \frac{1}{4j}(\Gamma_\alpha(f) - \Gamma_\alpha(-f)$
- *If $\Gamma_\alpha(f)$ is symmetric, $\Gamma_{pq}(f) = 0$ and $\gamma_{pq}(\tau) = 0$: $p(t_1)$ and $q(t_2)$ are always uncorrelated.*

With all these definitions, we can write the original signal as

$$X(t) = \Re\{\alpha(t)e^{j2\pi f_0 t}\} = p(t)\cos(2\pi f_0 t) - q(t)\sin)(2\pi f_0 t).$$

A way to recover $p(t)$ and $q(t)$ from $X(t)$ is to multiply it and then filter using a low-pass ($f < 4\pi f_0$):

$$X(t)\cos(2\pi f_0 t) = p(t) + \underbrace{p(t)\cos(4\pi f_0 t) - q(t)\sin(4\pi f_0 t)}_{\to 0}$$
$$X(t)[-2\sin(2\pi f_0 t)] = q(t) - \underbrace{q(t)\cos(4\pi f_0 t) - p(t)\sin(4\pi f_0 t)}_{\to 0}$$

Partiuclar case: band-limited white noise
Let us consider the white noise $n(t)$. Its baseband signal is $\alpha_n(t) = p_n(t) + q_n(t)$.

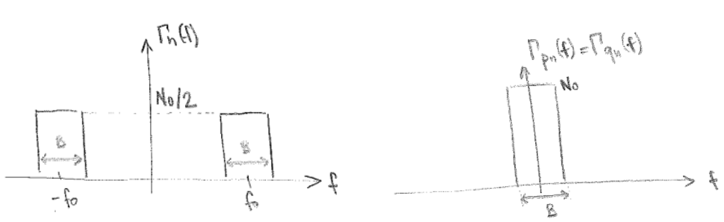

Figure 4: PSD of white noise $\Gamma_n(f)$ and its baseband signal $\Gamma_{\alpha_n}(f)$.

- $\alpha_n(t)$ is circular Gaussian.
- The power of the baseband signal is $P = N_0 B$.
- As $\Gamma_{\alpha_n}(f)$ is symmetric, $p(t_1)$ and $q(t_2)$ are uncorrelated.
- Jointly Gaussian $\Rightarrow$ independent.

Application: telecommunications

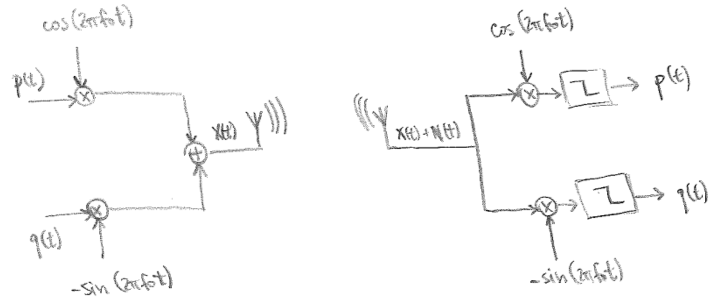

Figure 5: Application in telecommunications.

8 Mean square studies

Definition 8.1. *The temporal mean of a random process is*

$$M(\omega) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X(t,\omega)dt \text{ or } M_T(\omega) = \frac{1}{T}\int_{-T/2}^{T/2} X(t,\omega)dt.$$

The statistical mean is

$$m(t) = \mathbb{E}[X(t,\omega)].$$

Ergodicity of a process is when the statistical mean is equal to the temporal mean, i.e.,

$$M(\omega) = m(t) =: m.$$

But we can define many types of "equality". Here, we use the mean square equality:

$$M(\omega) \overset{\text{MS}}{=} m(t) \Leftrightarrow \lim_{T\to\infty} \mathbb{E}[|M(\omega) - m(t)|^2] = 0$$

Moreover, we can define ergodicity to other objects. For example, for the correlation:

$$\gamma(\tau) = \mathbb{E}[X((t+\tau)X^*(t)] = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} X(t+\tau,\omega)X^*(t,\omega)dt$$

When does ergodicity happen?
For WSS signals, a sufficient condition is

$$\int_{-\infty}^{+\infty} |\gamma_{X_C}(\tau)|d\tau < \infty.$$

Discrete case For discrete case, the temporal mean is defined as

$$M(\omega) = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{k=-N}^{+N} X[k,\omega]$$

and, for WSS signals, a sufficient condition is

$$\sum_{k=-\infty}^{+\infty} |\gamma_{X_C}[k]| < \infty.$$

Definition 8.2. *Let (X_n) be a sequence of second order complex random variables. We say that X_n converges to X in mean square and we write $\lim_{n\to\infty} X_n \overset{\text{MS}}{=} X$ or $X_n \overset{\text{MS}}{\to} X$ if*

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^2] = 0.$$

Proposition 8.1. *The Cauchy criterion says that*

$$\lim_{n\to\infty} X_n \overset{\text{MS}}{=} X \Leftrightarrow \lim_{k\to\infty, l\to\infty} \mathbb{E}[|X_l - X_k|^2] = 0$$

Proposition 8.2. *(Loeve lemma) Let (X_n) be a sequence of second order complex random variables. It converges (in MS) if and only if $\mathbb{E}[X_l X_k^*]$ has a finite limit when l and k tend independently to infinity.*

Now we can apply those results to random signals theory.

Definition 8.3. *Let $X(t)$ be a second order process. We say that $X(t)$ converges in mean square to the random variable X when $t \to t_0$ if the sequence fo random variables $X_k(t_0) = X(t_0 + h_k) \overset{\text{MS}}{\to} X$ for all sequences $(h_k)_{k\in\mathbb{N}}$ of real numbers tending to 0.*

Proposition 8.3. *(MS limit existence criterion)*

$$\lim_{t\to t_0} X(t) exists \Leftrightarrow \lim_{t_1,t_2\to t_0} \gamma_X(t_1,t_2) exists.$$

Then $\gamma_X(t_1,t_2) \overset{\text{MS}}{\to} \mathbb{E}[|X|^2],\ t_1,t_2 \to t_0$.

Definition 8.4. *The second order signal $X(t)$ is mean square continuous at $t_0 \in T$ if*

$$\lim_{t\to t_0} X(t) \overset{\text{MS}}{=} X(t_0),\ t \in T.$$

Proposition 8.4. *(MS continuity criterion)*

$$X(t) \text{ is MS continuous at } t_0 \Leftrightarrow \gamma(t_1,t_2) \text{ is continuous at } (t_1,t_2) = (t_0,t_0).$$

Nevertheless, MS continuity in every point does not imply the continuity of the trajectories.

Definition 8.5. *The second order signal $X(t)$ is mean square differentiable at $t_0 \in T$ if it exists the limit*

$$X'(t_0) :\overset{\text{MS}}{=} \lim_{h\to 0} \frac{X(t_0+h) - X(t_0)}{h}.$$

Proposition 8.5. *(MS differentiability criterion)*

$$X(t) \text{ MS differentiable at } t_0 \in T \Leftrightarrow \frac{\partial^2\gamma(t_1,t_2)}{\partial t_1 \partial t_2} \text{ and } \frac{\partial^2\gamma(t_1,t_2)}{\partial t_2 \partial t_1} \text{ exist in } (t_0,t_0) \text{ and are equal.}$$

Proposition 8.6. *The expectation of the derivative of a process is given by*

$$\mathbb{E}[X'(t)] = \frac{d}{dt}\mathbb{E}(X(t)).$$

Proposition 8.7. *By analogy with the usual definition, the mean square Riemann integral of the random signal $X(t)$ is*

$$I = \int_{a\ MS}^{b} X(t)dt :\overset{\text{MS}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} X(t_i)(t_{i+1} - t_i).$$

Proposition 8.8. *(MS integrability criterion)*

$$X(t) \text{ integrable in } T = [a,b] \Leftrightarrow \mathbb{E}[|I|^2] = \int_a^b \int_a^b \gamma_X(t,t')dtdt' exists.$$

Particular case: For WSS signals, we have

$$X(t) \text{ has a MS limit at } t_0 \Leftrightarrow \gamma_X(\tau) \text{ has a limit at } \tau = 0$$

$$X(t) \text{ is MS continuous } t_0 \Leftrightarrow \gamma_X(\tau) \text{ is continuous at } \tau = 0$$

$$X(t) \text{ is MS differentiable at } t_0 \Leftrightarrow \gamma_X''(\tau) = \gamma_{X'}(\tau) \text{ exists at } \tau = 0$$

9 ARMA signals

We will consider a discrete time, WSS, centred signal $X[n]$.

Definition 9.1. *The innovation of the process is the error of the infinite horizon prediction, i.e.,*

$$I[n] := X[n] - X[n]|\mathcal{H}_{X,n-1} = X[n] - \hat{X}[n]$$

where $\hat{X}[n]$ is the projection of X on $\mathcal{H}$ the space spanned by its past, i.e., the set $\{X[n-1]\}_{i\in[\![1,+\infty]\!]}$. It is calculated as

$$\hat{X}[n] = X[n]|\mathcal{H}_{X,n-1} = \sum_{i=1}^{\infty} \lambda_i X[n-i]$$

with $\lambda_i = \arg\min \|X[n] - \sum \lambda_i X[n-i]\| = \arg\min \mathbb{E}[|X[n] - \sum \lambda_i X[n-i]|^2]$.

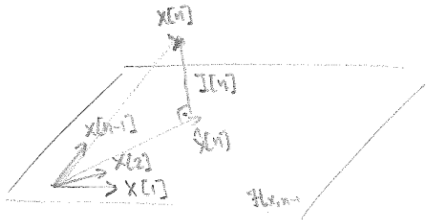

Figure 6: Representation of orthogonal projection.

We could also perform a finite horizon linear prediction:

$$\hat{X}_N[n] = \sum_{i=1}^{N} \lambda_{i,N} X[k-i]$$

$$e_N[n] = X[n] - \hat{X}[n]$$

In general, $\lambda_{i,N} \neq \lambda_i$ and this prediction is less precise than the previous one.
Applying the Z-transform on the innovation, we have

$$\mathcal{Z}\{I[n]\} = I(z) = X(z) - \sum_{i=1}^{\infty} \lambda_i z^{-i} X(z) = X(z) \underbrace{\left[1 - \sum \lambda_i z^{-i}\right]}_{G(z)}$$

If $I(z) = G(z)X(z)$, under right conditions we could obtain also $X(z) = \frac{1}{G(z)} I(z)$.

Theorem 9.1. *(Wold's theorem) Every WSS process $X[n]$ may be written as a sum $X[n] = X_r[n] + X_p[n]$, where*

- *$X_p[n]$ is a predictable process, i.e., the error of prediction is zero ($\hat{X}_p[n] = X_p[n]$).*
- *$X_r[n]$ is a regular process, i.e., obtained by filtering a white noise with a stable and causal filter whose inverse is also stable and causal.*

Definition 9.2. *A (N, M)-ARMA (autoregressive moving average) process is a process $X[n]$ that is obtained by filtering a white noise $U(z)$ with a filter $H(z)$ of the following form:*

$$X(z) = H(z)U(z) = \left(\frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \right) U(z).$$

A N-AR process is a $(N, 0)$-ARMA process, i.e.,

$$H(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}.$$

A M-MA process is a $(0, M)$-ARMA process, i.e.,

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}.$$

Yule-Walker equation For N-AR case:

$$\underbrace{\begin{bmatrix} \gamma[0] & \gamma[-1] & \cdots & \gamma[-N] \\ \gamma[1] & \gamma[0] & \cdots & \gamma[-N+1] \\ \vdots & \vdots & \ddots & \vdots \\ \gamma[N] & \gamma[N-1] & \cdots & \gamma[0] \end{bmatrix}}_{[\Gamma]} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \sigma_U^2 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$[\Gamma]^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{\sigma_U^2} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

For AR case, it is common to write $a_k = -\lambda_k$.
The Yule-Walker equation can be used for:

1. Linear prediction
$$\sigma_U^2 = \mathbb{E}[|I[n]|^2] = \mathbb{E}[|e[n]|^2] = P_e$$
For any $X[n]$ WSS centred signal, for any N:
$$\begin{bmatrix} \gamma[0] & \cdots & \gamma[-N] \\ \gamma[1] & \cdots & \gamma[-N+1] \\ \vdots & \ddots & \vdots \\ \gamma[N] & \cdots & \gamma[0] \end{bmatrix} \begin{bmatrix} 1 \\ -\lambda_{1,N} \\ \vdots \\ -\lambda_{N,N} \end{bmatrix} = P_{e,N} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

2. Estimating the PSD
$$\Gamma(\nu) = \frac{\sigma_U^2}{|1 + \sum_{k=1}^N a_k e^{-j2\pi\nu k}|^2}$$

3. Modelling a signal
$$X[n] = \left(\frac{1}{1 + \sum_{k=1}^N a_k z^{-k}} \right) U[n]$$

10 Spectral analysis

The objective is to estimate the PSD $\Gamma(f)$.
Alternative PSD expression:

$$\Gamma(f) = T_e \sum_{k=-\infty}^{\infty} \gamma[k] e^{-j2\pi f k T_e} = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{(2N+1)T_e} T_e \sum_{k=-N}^{N} x[k] e^{-j2\pi f k T_e} \right]$$

Periodogram:
$$\hat{\Gamma}(f) = \frac{1}{NT_e} \left| T_e \sum_{k=0}^{N-1} x[k] e^{-j2\pi f k T_e} \right|^2 = \frac{1}{NT_e} |X_N(f)|^2$$

- Based on alternative PSD expression.
- Biased and non-consistent.

Bias:
$$\mathbb{E}[\hat{\Gamma}(f)] \neq \Gamma(f)$$
$$\mathbb{E}[\hat{\Gamma}(f)] = T_e \sum_{k=-(N-1)}^{N-1} \mathbb{E}[\hat{\gamma}[k]] e^{-j2\pi f k T_e} = T_e \sum_{k=-(N-1)}^{N-1} \frac{N-|k|}{N} \gamma[k] e^{-j2\pi f k T_e} = W_B(f) * \Gamma(f)$$

Variance (consistency):
$$\mathbb{E}[\hat{\Gamma}(f_1)\hat{\Gamma}(f_2)] = \left(\frac{T_e}{N}\right)^2 \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \mathbb{E}[x[k]x^*[l]x[m]x^*[n]] \exp[-j2\pi(f_1(k-l) + f_2(m-n))T_e]$$

Correlogram:
$$\hat{\Gamma}_{corr}(f) = T_e \sum_{k=-M}^{M} \hat{\gamma}_k e^{-j2\pi f k T_e}$$

- Based on Fourier transform of autocorrelation.
- Non-biased and correlated.

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