

Representation and statistical analyse of signals

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1 Introduction and reminders on random variables

Definition 1.1. Let (Ω, τ, P) be a probability space, where Ω is the sample space, τ is a σ -algebra and P is a probability measure, and the set of instants T (\mathbb{R} or \mathbb{Z}). A (real) stochastic process is the application

$$X : T \times \Omega \rightarrow \mathbb{R} \\ (t, \omega) \mapsto X(t, \omega).$$

- For $\omega = \omega_0$ fixed, $X(t, \omega_0)$ is an ordinary function (trajectory or sample) and, for $t = t_0$ fixed, it is a random variable.
- If $T = \mathbb{R}$, the signal is time continuous and we denote the application $X(t, \omega)$. If $T = \mathbb{Z}$, we have a discrete time signal and we denote $X[k, \omega]$.

Definition 1.2. A σ -algebra on a set Ω is a collection τ of subsets of Ω such that

1. It includes the empty set ($\emptyset \in \tau$);
2. It is closed under complement ($A \in \tau \Rightarrow \bar{A} \in \tau$);
3. It is closed under countable union ($A_n \in \tau \Rightarrow \cup_n A_n \in \tau$).

The pair (Ω, τ) is said to be a measurable space or a Borel space.

Definition 1.3. A probability measure P is an application $P : \tau \rightarrow \mathbb{R}$ which respects the Kolmogorov axioms:

1. $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. $A_i \cap A_j \Rightarrow P(\cup_i A_i) = \sum_i P(A_i)$

Definition 1.4. Let (Ω, τ, P) be a probability space and $(\mathbb{R}, \mathfrak{B})$ a measurable space, with \mathfrak{B} the Borel σ -algebra of \mathbb{R}^1 . A real-valued random variable is a measurable function $X : \Omega \rightarrow \mathbb{R}$, which means that $X^{-1}(B) \in \tau \forall B \in \mathfrak{B}$ (it relates events with number values). We write the probability of the event x be in the interval B as $\Pr\{x \in B\} = P_x(B) = P(X^{-1}(B))$.

Definition 1.5. The cumulative distribution function (CDF) of a real-valued random variable X is the function

$$F_X(x) = \Pr\{X(\omega) \leq x\} = P_x([-\infty, x]) = P(X^{-1}([-\infty, x])).$$

The probability density function (PDF) $p_X(x)$ is

$$p_X(x) = F'_X(x) \Leftrightarrow F_X(x) = \int_{-\infty}^x p_X(\xi) d\xi.$$

The expected value of a random variable X whose CDF admits a PDF $p_X(x)$ is

$$\mathbb{E}[X] = \int_{\mathbb{R}} xp_X(x) dx.$$

- Expected value of a function: $\mathbb{E}[f(X(\omega))] = \int f(x)p_X(x)dx = \int f(x)dF_X(x) = \int f(x)dP_X(i)$

Definition 1.6. The characteristic function of a scalar random variable X is²

$$\phi_X(u) = \mathbb{E}[e^{juX}] = \int e^{juX} p_X(x) dx$$

and the second characteristic function is

$$\psi_X(u) = \ln(\phi_X(u)).$$

¹The Borel σ -algebra on \mathbb{R} is the smallest σ -algebra containing all open sets on \mathbb{R} .

²It is the Fourier transform with sign reversal in the complex exponential.

Theorem 1.1. Let \mathfrak{B} be the Borel σ -algebra of \mathbb{R} and ϕ a positive bounded measure on $(\mathbb{R}, \mathfrak{B})$. Then, there exists a unique positive integrable function defined in $g \in L^1$, up to a set of measure zero, and a unique singular measure ϕ_s on $(\mathbb{R}, \mathfrak{B})$ such that

$$\phi(B) = \int_B g(x)dx + \phi_s(B), \quad \forall B \in \mathfrak{B}.$$

- A measure ϕ_s is said to be singular if $\exists S \in \mathfrak{B}$ with $\mu(S) = 0$ and $\forall B \in \mathfrak{B}, \phi_s(B) = \phi_s(B \cap S)$.
- It is often possible to write

$$\phi(B) = \int \underbrace{g(x) \sum_k \mu_k \delta(x - s_k)}_{\text{density of the measure } \phi} dx.$$

- The probability density function is a particular case of such measure:

$$P(B) = \int \underbrace{g(x) \sum_k \mu_k \delta(x - s_k)}_{\text{PDF}} dx.$$

Definition 1.7. A temporal distribution (temporal law) of a random variable X is

$$F_X(x_1, \dots, x_n, \dots, t_1, \dots, t_n, \dots) = P(X(t_1) \leq x_1, \dots, X(t_n) \leq x_n, \dots)$$

Particular cases:

- An n -order distribution (n -order law) is $F_X(x_1, \dots, x_n, \dots, t_1, \dots, t_n) = P(X(t_1) \leq x_1, \dots, X(t_n) \leq x_n)$.
- A first-order distribution is $F_X(x, t) = P(X(t_1) \leq x_1) = F_{X(t, \omega)}(x_1)$.

Proposition 1.1. Properties of temporal distributions:

- Symmetry:* $F_X(x_1, \dots, x_n, t_1, \dots, t_n) = F_X(x_n, \dots, x_1, t_n, \dots, t_1)$.
- Consistency:* $F_X(x_1, \dots, x_n, \dots, t_1, \dots, t_n, \dots) = F_X(x_1, \dots, x_n, x_{n+1}, \dots, x_m, t_1, \dots, t_n, t_{n+1}, \dots, t_m)$.

Definition 1.8. Definitions of equivalence³:

- Two signals are said to be wide sense equivalent when they have the same temporal distribution.
- Two random signals $S_1(t, \omega)$ and $S_2(t, \omega)$ are said to be strictly equivalent if $P(S_1(t, \omega) = S_2(t, \omega)) = 1, \forall t$.
- If these two signals are such that $P(S_1(t, \omega) = S_2(t, \omega) \forall t \in T) = 1$, they are indistinguishable.

For example, consider the following signals:

$$S_1 \equiv 0, \quad S_2 = \begin{cases} 1 & \text{in a random value in } [0, 1] \\ 0 & \text{everywhere else.} \end{cases}, \quad S_3 = \begin{cases} 1 & \text{everywhere if a random value in } [0, 1] \text{ is } 0.5 \\ 0 & \text{everywhere, otherwise.} \end{cases}$$

S_1 and S_2 are strictly equivalent, but not indistinguishable. On the other hand, S_1 and S_3 are indistinguishable.

Point processes

Definition 1.9. A point process is a continuous-time distribution of points on T . A counting process $N(t, \omega)$ can be used to count point process as it avoids to be equivalent to a zero process.

Definition 1.10. A Poisson process of intensity λ is a point process which may be defined in several ways:

- It is a process that follows
 - The number of points in non-overlapping intervals are independent: $N(t_k, \omega) - N(t_{k-1}, \omega), \dots, N(t_1, \omega) - N(t_0, \omega)$ are independent $\forall t_0 < \dots < t_k$;
 - The probability of having exactly one event in a “small” interval is proportional to the length of the interval: $P(N(t+h) - N(t) = 1) = \lambda h + o(h)$;
 - The probability of having more than one event in a “small” interval is negligible: $P(N(t+h) - N(t) > 1) = o(h)$.
- It is a process that follows

³Presented in a particularity growing order.

(a) The number of points in non-overlapping intervals are independent: $N(t_k, \omega) - N(t_{k-1}, \omega), \dots, N(t_1, \omega) - N(t_0, \omega)$ are independent $\forall t_0 < \dots < t_k$;

(b) $P(N(t+T) - N(t) = k) = e^{-\lambda T} \frac{(\lambda T)^k}{k!}$

3. The intervals T_i between the occurrence of two events are i.i.d. (independent and identically distributed) with probability density function

$$p(t) = \lambda e^{-\lambda t}, \quad t \geq 0.$$

Reminder:

- Independence: $P(N_1 \cap N_2) = P(N_1)P(N_2)$
- Uncorrelation: $\mathbb{E}[N_1 \cap N_2] = \mathbb{E}[N_1]\mathbb{E}[N_2]$
- Independence \Rightarrow uncorrelation.

2 Partial characterisation of stochastic processes and temporal properties

Definition 2.1. Important definitions:

	Continuous-time	Discrete-time
First-order moment	$m(t) := \mathbb{E}[X(t, \omega)]$	$m[k] := \mathbb{E}[X[k, \omega]]$
Centred signal	$X_C(t, \omega) := X(t, \omega) - m(t)$	$X_C[k, \omega] := X[k, \omega] - m[k]$
Second-order moment ⁴ or (auto)correlation	$\gamma(t_1, t_2) := \mathbb{E}[X(t_1, \omega)X^*(t_2, \omega)]$	$\gamma[k_1, k_2] := \mathbb{E}[X[k_1, \omega]X^*[k_2, \omega]]$
Covariance	$c(t_1, t_2) := \gamma_{X_C}(t_1, t_2) = \mathbb{E}[X_C(t_1)X_C^*(t_2)]$	$c[k_1, k_2] := \gamma_{X_C}[k_1, k_2] = \mathbb{E}[X_C[k_1]X_C^*[k_2]]$
Variance	$\sigma(t) := \sqrt{c(t, t)} = \sqrt{\mathbb{E}[X_C(t) ^2]}$	$\sigma[k] := \sqrt{c[k, k]} = \sqrt{\mathbb{E}[X_C[k] ^2]}$
Power	$P(t) := \mathbb{E}[X(t) ^2]$	$P[k] := \mathbb{E}[X[k] ^2]$

Other types of (random) power:

- Random instant power: $|X(t, \omega)|^2$
- Random temporal mean power: $\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |X(t, \omega)|^2 dt$

Proposition 2.1. Properties:

- $c(t_1, t_2) = \gamma_X(t_1, t_2) - m(t_1)m^*(t_2)$
- $\sigma^2(t) = \mathbb{E}[|X(t, \omega)|^2] - |m(t)|^2$

Definition 2.2. In the vector case, we have the vector of random variables

$$\mathbf{X}(t, \omega) = \begin{bmatrix} X_1(t, \omega) \\ \vdots \\ X_n(t, \omega) \end{bmatrix},$$

the first-order moment

$$\mathbf{m}(t) = \mathbb{E}[\mathbf{X}(t, \omega)] = \begin{bmatrix} m_1(t) \\ \vdots \\ m_n(t) \end{bmatrix},$$

the second-order (correlation) matrix

$$\Gamma(t_1, t_2) = \mathbb{E}[\mathbf{X}(t_1, \omega)\mathbf{X}^\dagger(t_2, \omega)],$$

the cross-correlation

$$\Gamma_{X_1, X_2}(t_1, t_2) = \mathbb{E}[X_1(t_1)X_2^*(t_2)]$$

and the covariance matrix

$$\mathbf{c}(t_1, t_2) = \mathbb{E}[\mathbf{X}_C(t_1, \omega)\mathbf{X}_C^\dagger(t_2, \omega)] = \Gamma(t_1, t_2) - \mathbf{m}(t_1)\mathbf{m}^\dagger(t_2).$$

⁴Generalisation: n -order moment (it is not unique as there is no rule for the application of complex conjugate).

$$m(t_1, \dots, t_n) = \mathbb{E}[X(t_1), \dots, X(t_n)], \quad n \geq 3.$$

A fact about real random variables The moments are coefficients of a series expansion of the characteristic function ϕ_X . Let $m_n = \mathbb{E}[X^n]$:

$$\phi_X(u) = \mathbb{E}[e^{juX}] = \mathbb{E}\left[\sum_k j^k u^k \frac{X^k}{k!}\right] = \sum_k j^k \frac{u^k}{k!} \mathbb{E}[X^k] = \sum_k j^k \frac{m_n}{k!} u^k$$

Proposition 2.2. *Properties:*

- i. The autocorrelation $\gamma(t_1, t_2)$ exists $\forall (t_1, t_2) \in T^2$ if and only if $\gamma(t, t) = \mathbb{E}[|X(t)|^2]$ exists ($< \infty$) $\forall t \in T$.
- ii. The autocorrelation $\gamma(t_1, t_2) = \mathbb{E}[X(t_1)X^*(t_2)]$ defines a pseudo inner product.
- iii. The Cauchy-Schwarz inequality holds: $|\gamma(t_1, t_2)|^2 \leq \gamma(t_1, t_1)\gamma(t_2, t_2)$.
- iv. Existency of $\gamma(t_1, t_2)$ implies existency of $m(t) = \mathbb{E}[X(t)]$.
- v. $\gamma(t_1, t_2)$ is a non-negative definite function (NND), i.e., $\sum_i \sum_j \lambda_i \lambda_j^* \gamma(t_i, t_j) \geq 0$, $\forall \lambda_i, t_i \in \mathbb{C}^n \times T^n$.

Proof.

- ii. The expectation defines a pseudo inner product: $\langle X, Y \rangle := \mathbb{E}[XY^*]$, i.e., it is symmetric, bilinear and almost surely positive-definite: $\langle X, X \rangle = 0 \Leftrightarrow P(X = 0) = 1$. We can then define the autocorrelation inner product by setting $X = X(t_1)$ and $Y = X(t_2)$.
- iii. Apply the Cauchy-Schwarz inequality to the inner product defined above.

$$\mathbb{E}[XY^*] \leq \mathbb{E}[|X|^2] \mathbb{E}[|Y|^2]$$

- iv. Apply the expectation Cauchy-Schwarz inequality to $X = X(t)$ and $Y = 1$:

$$|\mathbb{E}[X(t) \cdot 1]|^2 \leq \mathbb{E}[|X(t)|^2] \mathbb{E}[1] = \gamma(t, t).$$

- v. Take $Z(t_i) = \sum_i \lambda_i X(t_i)$ and calculate

$$\mathbb{E}[|Z|^2] = \mathbb{E}\left[\left(\sum \lambda_i X(t_i)\right) \left(\sum \lambda_j X(t_j)\right)^*\right] = \sum \sum \lambda_i \lambda_j^* \mathbb{E}[X(t_i)X^*(t_j)] \geq 0.$$

□

Stationarity

Definition 2.3. A random process $X(t)$ is said to be strict sense stationary (SSS) if its temporal law is invariant by time shift, i.e.

$$F_X(x_1, \dots, x_n, t_1, \dots, t_n) = F_X(x_1, \dots, x_n, t_1 + h, \dots, t_n + h) \quad \forall n \forall x_i \forall t_i \forall h.$$

Definition 2.4. A random process is said to be stationary of order n if its moments up to order n are stationary.

- In particular, a stationary process of order 1 is such that their mean $m(t)$ is constant, i.e.,

$$m(t) = m(t + h) \quad \forall t \forall h.$$

- A stationary process of order 2 or wide sense stationary (WSS) has $m(t)$ constant and also

$$\gamma(t_1, t_2) = \gamma(t_1 + h, t_2 + h) \quad \forall t \forall h.$$

This means that $\gamma(t_1, t_2) = \gamma(t, t + \tau)$ depends only on $\tau = t_2 - t_1$. We can write

$$\gamma(\tau) = \mathbb{E}[X(t + \tau), X^*(t)] = \mathbb{E}[X(t), X^*(t - \tau)].$$

Definition 2.5. Two random processes $X(t, \omega)$ and $Y(t, \omega)$ are said to be jointly stationary (of order 2) if $\begin{bmatrix} X(t, \omega) \\ Y(t, \omega) \end{bmatrix}$ is stationary (of order 2). Then, $\gamma_X = \gamma_X(\tau)$, $\gamma_Y = \gamma_Y(\tau)$, $\gamma_{XY} = \gamma_{XY}(\tau)$, $m_Y(t) = m_Y$ and $m_X(t) = m_X$.

Definition 2.6. A random process $X(t, \omega)$ is said to be cyclostationary if there exists T such that

$$\mathbb{E}[X(t + kT)] = \mathbb{E}[X(t)] \quad \text{and} \quad \gamma(t_1 + kT, t_2 + kT) = \gamma(t_1, t_2) \quad \forall k.$$

- If $X(t)$ is cyclostationary, then $Y(t) = X(t + \mathcal{O}(\omega))$, with \mathcal{O} uniformly distributed over $[0, T]$ is stationary.

Properties of WSS signals⁵

Proposition 2.3. *Basic properties. Let $X(t, \omega)$ be a random process. Then:*

- i. $\gamma(0) \geq |\gamma(\tau)| \quad \forall \tau$
- ii. $\gamma(0) = P(t)$
- iii. $\gamma(\tau) = \gamma^*(-\tau)$

Proof.

- i. Use the Cauchy-Schwarz inequality for expectation inner product:

$$|\mathbb{E}[X(t+\tau)X^*(t)]|^2 \leq \mathbb{E}[|X(t+\tau)|^2] \mathbb{E}[|X(t)|^2] \Rightarrow |\gamma(\tau)|^2 \leq \gamma(0)\gamma(0) \Rightarrow |\gamma(\tau)| \leq \gamma(0).$$

- ii. By definition, $\gamma(0) = \mathbb{E}[X(t+0)X^*(t)] = \mathbb{E}[|X(t)|^2] = P(t) = P$.
- iii. $\gamma(\tau) = \mathbb{E}[X(t+\tau)X^*(t)] = \mathbb{E}[X(t)X^*(t-\tau)] = \mathbb{E}[X(t-\tau)X^*(t)]^* = \gamma^*(-\tau)$.

□

Proposition 2.4. *Basic properties for vector case:*

- i. $\Gamma(\tau) = \mathbb{E}[\mathbf{X}(t+\tau)\mathbf{X}^*(t)] = \Gamma^\dagger(-\tau)$
- ii. If $\Gamma(0)$ is hermitian, then it is orthogonally diagonalisable and has real eigenvalues.

Proposition 2.5. *Periodicity:*

- i. If $\gamma(0) = \gamma(\tau_1)$ with $\tau_1 \neq 0$, then γ is τ_1 -periodic.
- ii. $\gamma(\tau)$ is τ_1 -periodic $\Leftrightarrow \gamma(0) = \gamma(\tau_1) \Leftrightarrow P(X(t) = X(t+\tau_1)) = 1$.

Proposition 2.6. *If $\gamma(0) = \gamma(\tau_1) = \gamma(\tau_2)$ with $\frac{\tau_1}{\tau_2} \notin \mathbb{Q}$ and $\tau_1, \tau_2 \neq 0$, then $\gamma(\tau) = \gamma$ is constant (as long as γ is continuous).*

Proposition 2.7. *The autocorrelation $\gamma(\tau)$ is uniformly continuous if and only if $\gamma(\tau)$ is continuous at $\tau = 0$.*

Proposition 2.8. *For non-WSS case, $\gamma(t_1, t_2) \in C^0$ on the diagonal of T (i.e., $(t, t) \in T^2$) if and only if γ is continuous on every $(t_i, t_j) \in T^2$.*

Markov process

Definition 2.7. A Markov process is a stochastic process whose future probabilities are determined by its most recent values, i.e.,

$$P(X(t_n) \in B_n | X(t_{n-1}) \in B_{n-1}, \dots, X(t_1) \in B_1) = P(X(t_n) \in B_n | X(t_{n-1}) \in B_{n-1}), \quad t_n > t_{n-1} > \dots > t_1.$$

In a Markov process, the n -order distribution $P(x_1, \dots, x_n, t_1, \dots, t_n)$ depends only on the second order distribution $P(x_1, x_2, t_1, t_2)$.

3 Power Spectral Density (PSD)

We shall consider WSS signals here.

Continuous case

Theorem 3.1. (Bochner) *A function $\gamma : \mathbb{R} \rightarrow \mathbb{C}$ is continuous and non-negative definite if, and only if, there exists a positive bounded measure φ on $(\mathbb{R}, \mathfrak{B})$ such that*

$$\gamma(\tau) = \int_{-\infty}^{+\infty} e^{j2\pi f\tau} d\varphi(f) \quad \forall \tau \in \mathbb{R}.$$

Applying the Theorem 1.1 to autocorrelation, we have

$$\gamma(\tau) = \int_{-\infty}^{+\infty} e^{j2\pi f\tau} \Gamma(f) df$$

$$\text{and } P = \gamma(0) = \int_{-\infty}^{+\infty} d\varphi(f) = \int_{-\infty}^{+\infty} \Gamma(f) df.$$

⁵All signals considered in this section are WSS, unless explicitly indicated.

Discrete case

Theorem 3.2. A function $\gamma : \mathbb{Z} \rightarrow \mathbb{C}$ is non-negative definite if and only if there exists a positive bounded measure ϕ on $(\mathbb{R}, \mathfrak{B})$ such that

$$\gamma[m] = \int_{-1/2}^{+1/2} e^{j2\pi\nu m} d\phi(\nu) \quad \forall m \in \mathbb{Z}.$$

Applying an analogous theorem, we have

$$\gamma[m] = \int_{-1/2}^{+1/2} e^{j2\pi\nu m} \Gamma(\nu) d\nu$$

and $P = \gamma[0] = \int_{-1/2}^{+1/2} d\phi(\nu) = \int_{-1/2}^{+1/2} \Gamma(\nu) d\nu$.

Definition 3.1. (Short definition)

Continuous time:

$$\Gamma(f) = \mathcal{F}\{\gamma(\tau)\} = \int_{-\infty}^{+\infty} \gamma(\tau) e^{j2\pi f\tau} d\tau.$$

Discrete time:

$$\Gamma[\nu] = \mathcal{F}\{\gamma[k]\} = \sum_{k=-\infty}^{+\infty} \gamma[k] e^{j2\pi k\nu}.$$

Proposition 3.1. Properties of the PSD:

- i. $\Gamma(f)$ is real and positive⁶.
- ii. If the signal $X(t, \omega)$ is real, then its PSD $\Gamma(f)$ is even.
- iii. Some PSD examples:

$\gamma(\tau)$	$\Gamma(f)$
$\frac{N_0}{2} \delta(\tau)$	$\frac{N_0}{2}$
$N_0 B \text{sinc}(2\pi B\tau)$	$\frac{N_0}{2} \text{rect}_{[-B, B]}(f)$
$N_0 B \text{sinc}(2\pi B\tau) \cos(2\pi f_0\tau)$	$\frac{N_0}{4} [\text{rect}_{[-B, B]}(f - f_0) + \text{rect}_{[-B, B]}(f + f_0)]$

Sampling a signal Let the signal $X(t, \omega)$ sampled at a rate $f_s = 1/T_s$, then the result signal is $X[k, \omega] = X(kT_s, \omega)$.

4 White noise

Definition 4.1. A (wide sense) white signal is a signal for which the PSD is constant (WSS signal).

Definition 4.2. A (strict sense) white signal is a signal $X(t, \omega)$ such that

- i. $X(t, \omega)$ is centred and WSS
- ii. $X(t_1, \omega)$ and $X(t_2, \omega)$ are independent $\forall t_1 \neq t_2$

Definition 4.3. A band-limited white signal is a signal for which the PSD is constant on a finite support.

Example: $\Gamma(f) = \text{rect}_{[-B, B]}(f - f_0)$.

Difference between continuous and discrete time:

- Continuous: $P = \int_{-\infty}^{+\infty} \Gamma(f) df = +\infty = \gamma(0)$
- Discrete: $P = \int_{-1/2}^{+1/2} \Gamma(\nu) d\nu = K = \gamma[0]$

⁶The Fourier transform of non-negative definite function is non-negative. In addition, if it had a negative part, we could filter it and get a negative signal, but it would have negative power, which is impossible!

5 Gaussian signals

Definition 5.1. A vector of random variables $\mathbf{X} = [X_1 \ \cdots \ X_n]^T$ is said to be Gaussian if its PDF is

$$p_{\mathbf{X}}(x_1, \dots, x_n) = (2\pi)^{-n/2} \det(\mathbf{C}) \exp \left[-\frac{1}{2} (\mathbf{X} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{X} - \mathbf{m}) \right]$$

where $\mathbf{m} = \mathbb{E}[\mathbf{X}]$ and $\mathbf{C} = \mathbb{E}[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T]$, which is a real symmetric matrix such that $\mathbf{u}^T \mathbf{C} \mathbf{u} \geq 0$. In this case, X_1, \dots, X_n are said to be jointly Gaussian.

Equivalent definitions:

- $\phi_{\mathbf{X}}(\mathbf{u}) = \mathbb{E}[e^{j\mathbf{u}^T \mathbf{X}}] = e^{j\mathbf{u}^T \mathbf{m} - \frac{1}{2} \mathbf{u}^T \mathbf{C} \mathbf{u}}$
- $\psi_{\mathbf{X}}(\mathbf{u}) = j\mathbf{u}^T \mathbf{m} - \frac{1}{2} \mathbf{u}^T \mathbf{C} \mathbf{u}$
- $\forall (\lambda_i)_{i \in \llbracket 1, n \rrbracket} \in \mathbb{R}^n$, $\sum \lambda_i X_i(\omega)$ is a Gaussian random variable for X_i jointly Gaussian.

Proposition 5.1. Properties of jointly Gaussian random variables:

- Linear combination of jointly Gaussian R.V. is a jointly Gaussian R.V.
- Uncorrelated jointly Gaussian R.V. \Leftrightarrow independent jointly Gaussian R.V.
- If $\mathbf{X}(\omega)$ is Gaussian, then $\mathbf{Y}(\omega) = \mathbf{A}\mathbf{X}(\omega)$ is Gaussian $\forall \mathbf{A} \in \mathcal{M}_{n \times n}(\mathbb{R})$

Definition 5.2. A complex vector $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$ is said to be Gaussian if $\begin{bmatrix} \mathbf{X} & \mathbf{Y} \end{bmatrix}^T$ is a real Gaussian vector.

Definition 5.3. A real/complex signal $X(t, \omega)$ is said to be Gaussian when $\begin{bmatrix} X(t_1, \omega) \\ \vdots \\ X(t_n, \omega) \end{bmatrix}$ is a real/complex Gaussian vector $\forall n \in \mathbb{N}$, $\forall (t_i)_{i \in \llbracket 1, n \rrbracket}$.

Proposition 5.2. Properties of Gaussian signals:

- Filtering Gaussian signals result in Gaussian channels.
- The mean $m(t)$ and the autocorrelation $\gamma(t_1, t_2)$ are enough to completely characterise the signal.
- If $X(t, \omega)$ is Gaussian, then:

- WSS \Leftrightarrow SSS
- uncorrelation \Leftrightarrow independent
- wide sense white noise \Leftrightarrow strict sense white noise

Special case: centred case

$$p_{\mathbf{Z}}(\mathbf{z}) = p_{X,Y}(\mathbf{x}, \mathbf{y}) = (2\pi)^{-n} \det(\tilde{\mathbf{C}}_{X,Y})^{-1/2} \exp \left[-\frac{1}{2} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}^T \tilde{\mathbf{C}}_{X,Y}^{-1} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right]$$

- $\mathbf{C}_Z = \mathbb{E}[(\mathbf{Z} - \mathbf{m}_Z)(\mathbf{Z} - \mathbf{m}_Z)^\dagger] = \mathbb{E}[\mathbf{Z}\mathbf{Z}^\dagger] = \mathbf{C}_X + \mathbf{C}_Y + j(-\mathbf{C}_{XY} + \mathbf{C}_{YX})$
- $\mathbf{C}_{XY} = \mathbf{C}_{YX}^T = \mathbb{E}[(\mathbf{X} - \mathbf{m}_X)(\mathbf{Y} - \mathbf{m}_Y)^T]$
- $\tilde{\mathbf{C}}_{X,Y} = \begin{bmatrix} \mathbf{C}_X & \mathbf{C}_{XY} \\ \mathbf{C}_{YX} & \mathbf{C}_Y \end{bmatrix}$
- $\mathbf{D}_Z = \mathbb{E}[\mathbf{Z}\mathbf{Z}^T] = \mathbf{C}_X - \mathbf{C}_Y + j(\mathbf{C}_{XY} + \mathbf{C}_{YX})$

Definition 5.4. A subclass of complex Gaussian signals is formed by circular Gaussian signals, which are characterised by zero relation matrix and zero mean, i.e., $\mathbf{m} = 0$ and $\mathbf{C} = 0$.

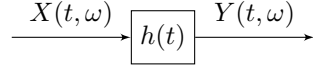
Centred Gaussian case is characterised by $\mathbf{D}_Z = 0$.

$$p_{\mathbf{Z}}(\mathbf{z}) = p_{X,Y}(\mathbf{x}, \mathbf{y}) = (\pi)^{-n} \det(\mathbf{C}_Z)^{-1} \exp[-(\mathbf{Z} - \mathbf{m}_Z)^\dagger \mathbf{C}_Z^{-1} (\mathbf{Z} - \mathbf{m}_Z)]$$

6 Filtering random signals

In this section, we will consider:

- WSS signals
- Time-invariant linear systems, i.e., $Y(t, \omega) = X(t, \omega) * h(t)$



- Bounded input-bounded output (BIBO) stable systems, i.e., the transfer function is absolutely integrable:

$$\int |h(t)| dt < \infty$$

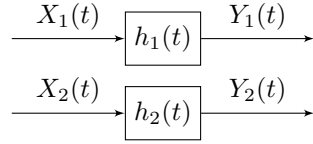
Proposition 6.1. $Y(t, \omega)$ exists almost surely, i.e., $P(Y(t) < \infty) = 1$.

Theorem 6.1. (Fubini) Let (E_1, T_1, m_1) and (E_2, T_2, m_2) be σ -finite measure spaces and a mapping $f : E_1 \times E_2 \rightarrow \mathbb{R}$. If $\int_{E_1} \int_{E_2} |f| dm_2 dm_1$ exists (i.e. converges), then

- $x \in E_1 \Rightarrow \int_{E_2} f dm_2$ exists almost everywhere (up to sets of measure zero)
- $x \in E_2 \Rightarrow \int_{E_1} f dm_1$ exists almost everywhere (up to sets of measure zero)

and $\int_{E_1} \int_{E_2} f dm_2 dm_1 = \int_{E_2} \int_{E_1} |f| dm_1 dm_2$.

Interference formula Let consider the following systems.



Case	Time domain	Frequency domain
General case	$\gamma_{Y_1 Y_2}(\tau) = \tilde{h}_2 * h_1 * \gamma_{X_1 X_2}(\tau)$	$\Gamma_{Y_1 Y_2}(f) = H_1 H_2^* \Gamma_{X_1 X_2}(f)$
$Y_i = X_i * h_i$	$\Gamma_Y(\tau) = h * h * \gamma_X(\tau)$	$\Gamma_Y(f) = H(f) ^2 \Gamma_X(f)$
$X_1 = X_2 = X$ and $h_2(t) = \delta(t)$	$\gamma_{YX}(\tau) = h * \gamma_X(\tau)$	$\Gamma_{YX}(f) = H(f) \Gamma_X(f)$

We define $\tilde{h}(t) := h^*(-t)$.

7 Narrow band signals

A narrow band is a signal for which the bandwidth B is much smaller than its central f_0 . In this chapter, we consider real, centred, WSS signals $X(t)$. So the PSD $\Gamma_X(f)$ is real and even.

Definition 7.1. The analytic signal $Z(t)$ associated to $X(t)$ is the canonical complex signal for which $X(t)$ is the real part, i.e.,

$$Z(t) = X(t) + jY(t)$$

where $\Re\{Z(t)\} = X(t)$ and $\Im\{Z(t)\} = Y(t) = h * X(t)$. We call $Y(t)$ the Hilbert transform.

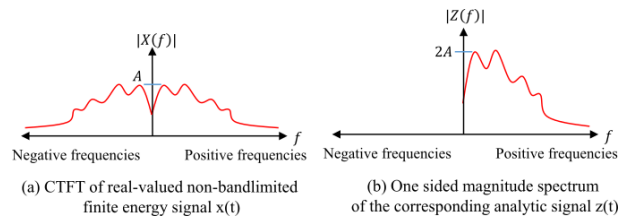


Figure 1: Example of analytic signal [1].

Definition 7.2. The Hilbert filter is $h(t)$ such that $Y(t) = h(t) * X(t)$. The analytical filter is $h_a(t)$ such that $Z(t) = h_a(t) * X(t)$.

The analytical filter is $h_a(t) = \delta(t) + jh(t)$, because

$$Z(t) = X(t) * h_a(t) = X(t) * [\delta(t) + jh(t)] = X(t) + jY(t)$$

Figure 2: Diagram of analytical and Hilbert filters.

What is the filter $h(t)$ that erases negative frequencies?

Let $Z(f) = X(f)H_a(f) = X_f[1 + H(f)]$. We search $H(f)$ such that $1 + jH(f) = 0$ for $f < 0$.

$$H(f) = \text{sgn}(f)^7 \xrightarrow{\mathcal{F}^{-1}} h(t) = \frac{1}{\pi t}$$

In this case, $H_a(f) = \begin{cases} 0, & f < 0 \\ 2, & f > 0 \end{cases}$.

Proposition 7.1. Statistical properties of $Y(t)$:

- $\mathbb{E}[Y(t)] = h(t) * \mathbb{E}[X(t)] = 0$
- $\Gamma_T(f) = \Gamma_X(f)|\text{sgn}(f)|^2 = \Gamma_X(f)$
- $\gamma_Y(\tau) = \mathcal{F}^{-1}\{\Gamma_Y(f)\} = \mathcal{F}^{-1}\{\Gamma_X(f)\} = \gamma_X(\tau)$
- $\Gamma_{YX}(f) = \Gamma_X(f)(j \text{sgn}(f)) = -\Gamma_X(f)(j \text{sgn}(f))^* = -\Gamma_{XY}(f)$
- $\gamma_{YX}(\tau) = \mathcal{F}^{-1}\{\Gamma_{YX}(f)\} = \mathcal{F}^{-1}\{-\Gamma_{XY}(f)\} = -\gamma_{XY}(\tau)$
- $\gamma_{XY}(0) = \mathbb{E}[X(t)Y(t)] = \gamma_{YX}(0) = -\gamma_{XY}(0) = 0$: X and Y are uncorrelated at the same instant.

Proposition 7.2. Statistical properties of $Z(t)$:

- $\mathbb{E}[Z(t)] = h_a(t) * \mathbb{E}[X(t)] = 0$
- $\Gamma_Z(f) = |H_a(f)|^2 \Gamma_X(f) = \begin{cases} 0, & f < 0 \\ 4\Gamma_X(f), & f > 0 \end{cases}$
- $\mathbb{E}[Z(t + \tau)Z(t)] = 0$: if $Z(t)$ is Gaussian, then it is circular Gaussian.

Definition 7.3. The baseband signal related to $Z(t)$ is $\alpha(t) = Z(t)e^{-j2\pi f_0 t}$. It corresponds to centring the PSD in the frequency domain.

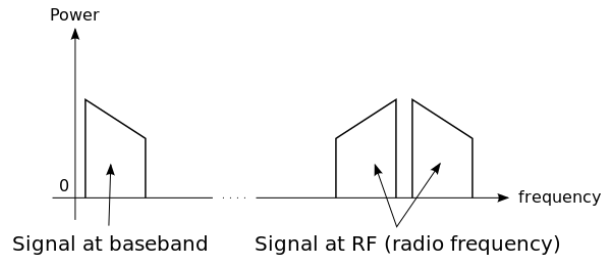


Figure 3: Example of baseband signal [2].

Proposition 7.3. Statistical properties of $\alpha(t)$:

- $\mathbb{E}[\alpha(t)] = e^{-j2\pi f_0 t} \mathbb{E}[Z(t)] = 0$
- $\gamma_\alpha(\tau) = \mathbb{E}[\alpha(t + \tau)\alpha^*(t)] = \mathbb{E}[Z(t + \tau)e^{-j2\pi f_0(t + \tau)}Z^*(t)e^{j2\pi f_0 t}] = \gamma_Z(\tau)e^{-j2\pi f_0 \tau}$
- $\Gamma_\alpha(f) = \mathcal{F}\{\gamma_Z(\tau)e^{-j2\pi f_0 \tau}\} = \Gamma_Z(f + f_0)$
- $\mathbb{E}[\alpha(t + \tau)\alpha(t)] = e^{j2\pi f_0 \tau} \mathbb{E}[Z(t + \tau)Z(t)] = 0$

$$\tau_{\text{sgn}}(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Definition 7.4. We can decompose the signal $\alpha(t) = p(t) + jq(t)$ in two components:

$p(t) = \Re\{\alpha(t)\}$ is called in-phase component

$q(t) = \Im\{\alpha(t)\}$ is called quadrature component

To develop properties of $p(t)$ and $q(t)$, it is useful to write them as

$$p(t) = \frac{\alpha(t) + \alpha^*(t)}{2} \quad \text{and} \quad q(t) = \frac{\alpha(t) - \alpha^*(t)}{2j}.$$

Proposition 7.4. Statistical properties of $p(t)$ and $q(t)$:

- $\mathbb{E}[p(t)] = \mathbb{E}[q(t)] = 0$
- $\gamma_p(\tau) = \gamma_q(\tau) = \frac{1}{4}(\gamma_\alpha(\tau) + \gamma_{\alpha^*}(\tau))$
- $\Gamma_p(f) = \Gamma_q(f) = \frac{1}{4}(\Gamma_\alpha(f) + \Gamma_\alpha(-f))$
- $\gamma_{pq}(\tau) = -\gamma_{qp}(\tau)$
- $\gamma_{pq}(0) = -\gamma_{qp}(0) = \gamma_{qp}(0) = 0$: $p(t)$ and $q(t)$ are uncorrelated at the same instant.
- $\Gamma_{pq}(f) = \mathcal{F}\{\gamma_{pq}(\tau)\} = \frac{1}{4j}(\Gamma_\alpha(f) - \Gamma_\alpha(-f))$
- If $\Gamma_\alpha(f)$ is symmetric, $\Gamma_{pq}(f) = 0$ and $\gamma_{pq}(\tau) = 0$: $p(t_1)$ and $q(t_2)$ are always uncorrelated.

With all these definitions, we can write the original signal as

$$X(t) = \Re\{\alpha(t)e^{j2\pi f_0 t}\} = p(t)\cos(2\pi f_0 t) - q(t)\sin(2\pi f_0 t).$$

A way to recover $p(t)$ and $q(t)$ from $X(t)$ is to multiply it and then filter using a low-pass ($f < 4\pi f_0$):

$$\begin{aligned} X(t)\cos(2\pi f_0 t) &= p(t) + \underbrace{p(t)\cos(4\pi f_0 t) - q(t)\sin(4\pi f_0 t)}_{\rightarrow 0} \\ X(t)[-2\sin(2\pi f_0 t)] &= q(t) - \underbrace{q(t)\cos(4\pi f_0 t) - p(t)\sin(4\pi f_0 t)}_{\rightarrow 0} \end{aligned}$$

Partiucular case: band-limited white noise

Let us consider the white noise $n(t)$. Its baseband signal is $\alpha_n(t) = p_n(t) + q_n(t)$.

Figure 4: PSD of white noise $\Gamma_n(f)$ and its baseband signal $\Gamma_{\alpha_n}(f)$.

- $\alpha_n(t)$ is circular Gaussian.
- The power of the baseband signal is $P = N_0 B$.
- As $\Gamma_{\alpha_n}(f)$ is symmetric, $p(t_1)$ and $q(t_2)$ are uncorrelated.
- Jointly Gaussian \Rightarrow independent.

Application: telecommunications

Figure 5: Application in telecommunications.

8 Mean square studies

Definition 8.1. The temporal mean of a random process is

$$M(\omega) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t, \omega) dt \quad \text{or} \quad M_T(\omega) = \frac{1}{T} \int_{-T/2}^{T/2} X(t, \omega) dt.$$

The statistical mean is

$$m(t) = \mathbb{E}[X(t, \omega)].$$

Ergodicity of a process is when the statistical mean is equal to the temporal mean, i.e.,

$$M(\omega) = m(t) =: m.$$

But we can define many types of “equality”. Here, we use the mean square equality:

$$M(\omega) = m(t) \Leftrightarrow \stackrel{\text{MS}}{=} \lim_{T \rightarrow \infty} \mathbb{E}[|M(\omega) - m(t)|^2] = 0$$

Moreover, we can define ergodicity to other objects. For example, for the correlation:

$$\gamma(\tau) = \mathbb{E}[X((t + \tau)X^*(t))] = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t + \tau, \omega) X^*(t, \omega) dt$$

When does ergodicity happen?

For WSS signals, a sufficient condition is

$$\int_{-\infty}^{+\infty} |\gamma_{X_C}(\tau)| d\tau < \infty.$$

Discrete case For discrete case, the temporal mean is defined as

$$M(\omega) = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{k=-N}^{+N} X[k, \omega]$$

and, for WSS signals, a sufficient condition is

$$\sum_{k=-\infty}^{+\infty} |\gamma_{X_C}[k]| < \infty.$$

Definition 8.2. Let (X_n) be a sequence of second order complex random variables. We say that X_n converges to X in mean square and we write $\lim_{n \rightarrow \infty} X_n \stackrel{\text{MS}}{=} X$ or $X_n \xrightarrow{\text{MS}} X$ if

$$\lim_{n \rightarrow \infty} \mathbb{E}[|X_n - X|^2] = 0.$$

Proposition 8.1. The Cauchy criterion says that

$$\lim_{n \rightarrow \infty} X_n \stackrel{\text{MS}}{=} X \Leftrightarrow \lim_{k \rightarrow \infty, l \rightarrow \infty} \mathbb{E}[|X_l - X_k|^2] = 0$$

Proposition 8.2. (Loeve lemma) Let (X_n) be a sequence of second order complex random variables. It converges (in MS) if and only if $\mathbb{E}[X_l X_k^*]$ has a finite limit when l and k tend independently to infinity.

Now we can apply those results to random signals theory.

Definition 8.3. Let $X(t)$ be a second order process. We say that $X(t)$ converges in mean square to the random variable X when $t \rightarrow t_0$ if the sequence of random variables $X_k(t_0) = X(t_0 + h_k) \xrightarrow{\text{MS}} X$ for all sequences $(h_k)_{k \in \mathbb{N}}$ of real numbers tending to 0.

Proposition 8.3. (MS limit existence criterion)

$$\lim_{t \rightarrow t_0} X(t) \text{ exists} \Leftrightarrow \lim_{t_1, t_2 \rightarrow t_0} \gamma_X(t_1, t_2) \text{ exists.}$$

Then $\gamma_X(t_1, t_2) \xrightarrow{\text{MS}} \mathbb{E}[|X|^2]$, $t_1, t_2 \rightarrow t_0$.

Definition 8.4. The second order signal $X(t)$ is mean square continuous at $t_0 \in T$ if

$$\lim_{t \rightarrow t_0} X(t) \stackrel{\text{MS}}{=} X(t_0), \quad t \in T.$$

Proposition 8.4. (MS continuity criterion)

$$X(t) \text{ is MS continuous at } t_0 \Leftrightarrow \gamma(t_1, t_2) \text{ is continuous at } (t_1, t_2) = (t_0, t_0).$$

Nevertheless, MS continuity in every point does not imply the continuity of the trajectories.

Definition 8.5. The second order signal $X(t)$ is mean square differentiable at $t_0 \in T$ if it exists the limit

$$X'(t_0) \stackrel{\text{MS}}{=} \lim_{h \rightarrow 0} \frac{X(t_0 + h) - X(t_0)}{h}.$$

Proposition 8.5. (*MS differentiability criterion*)

$X(t)$ MS differentiable at $t_0 \in T \Leftrightarrow \frac{\partial^2 \gamma(t_1, t_2)}{\partial t_1 \partial t_2}$ and $\frac{\partial^2 \gamma(t_1, t_2)}{\partial t_2 \partial t_1}$ exist in (t_0, t_0) and are equal.

Proposition 8.6. *The expectation of the derivative of a process is given by*

$$\mathbb{E}[X'(t)] = \frac{d}{dt} \mathbb{E}(X(t)).$$

Proposition 8.7. *By analogy with the usual definition, the mean square Riemann integral of the random signal $X(t)$ is*

$$I = \int_a^b X(t) dt \stackrel{\text{MS}}{=} \lim_{n \rightarrow \infty} \sum_{i=1}^n X(t_i)(t_{i+1} - t_i).$$

Proposition 8.8. (*MS integrability criterion*)

$$X(t) \text{ integrable in } T = [a, b] \Leftrightarrow \mathbb{E}[|I|^2] = \int_a^b \int_a^b \gamma_X(t, t') dt dt' \text{ exists.}$$

Particular case: For WSS signals, we have

$$X(t) \text{ has a MS limit at } t_0 \Leftrightarrow \gamma_X(\tau) \text{ has a limit at } \tau = 0$$

$$X(t) \text{ is MS continuous } t_0 \Leftrightarrow \gamma_X(\tau) \text{ is continuous at } \tau = 0$$

$$X(t) \text{ is MS differentiable at } t_0 \Leftrightarrow \gamma_X''(\tau) = \gamma_{X'}(\tau) \text{ exists at } \tau = 0$$

9 ARMA signals

We will consider a discrete time, WSS, centred signal $X[n]$.

Definition 9.1. *The innovation of the process is the error of the infinite horizon prediction, i.e.,*

$$I[n] := X[n] - X[n]|\mathcal{H}_{X, n-1} = X[n] - \hat{X}[n]$$

where $\hat{X}[n]$ is the projection of X on \mathcal{H} the space spanned by its past, i.e., the set $\{X[n-1]\}_{i \in \mathbb{I}[1, +\infty]}$. It is calculated as

$$\hat{X}[n] = X[n]|\mathcal{H}_{X, n-1} = \sum_{i=1}^{\infty} \lambda_i X[n-i]$$

with $\lambda_i = \arg \min \|X[n] - \sum \lambda_i X[n-i]\| = \arg \min \mathbb{E}[|X[n] - \sum \lambda_i X[n-i]|^2]$.

Figure 6: Representation of orthogonal projection.

We could also perform a finite horizon linear prediction:

$$\hat{X}_N[n] = \sum_{i=1}^N \lambda_{i,N} X[n-i]$$

$$e_N[n] = X[n] - \hat{X}_N[n]$$

In general, $\lambda_{i,N} \neq \lambda_i$ and this prediction is less precise than the previous one. Applying the Z -transform on the innovation, we have

$$\mathcal{Z}\{I[n]\} = I(z) = X(z) - \sum_{i=1}^{\infty} \lambda_i z^{-i} X(z) = X(z) \underbrace{\left[1 - \sum_{i=1}^{\infty} \lambda_i z^{-i}\right]}_{G(z)}$$

If $I(z) = G(z)X(z)$, under right conditions we could obtain also $X(z) = \frac{1}{G(z)}I(z)$.

Theorem 9.1. (*Wold's theorem*) *Every WSS process $X[n]$ may be written as a sum $X[n] = X_r[n] + X_p[n]$, where*

- $X_p[n]$ is a predictable process, i.e., the error of prediction is zero ($\hat{X}_p[n] = X_p[n]$).

- $X_r[n]$ is a regular process, i.e., obtained by filtering a white noise with a stable and causal filter whose inverse is also stable and causal.

Definition 9.2. A (N, M) -ARMA (autoregressive moving average) process is a process $X[n]$ that is obtained by filtering a white noise $U(z)$ with a filter $H(z)$ of the following form:

$$X(z) = H(z)U(z) = \left(\frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}} \right) U(z).$$

A N -AR process is a $(N, 0)$ -ARMA process, i.e.,

$$H(z) = \frac{1}{1 + \sum_{k=1}^N a_k z^{-k}}.$$

A M -MA process is a $(0, M)$ -ARMA process, i.e.,

$$H(z) = \sum_{k=0}^M b_k z^{-k}.$$

Yule-Walker equation

$$\underbrace{\begin{bmatrix} \gamma[0] & \gamma[-1] & \cdots & \gamma[-N] \\ \gamma[1] & \gamma[0] & \cdots & \gamma[-N+1] \\ \vdots & \vdots & \ddots & \vdots \\ \gamma[N] & \gamma[N-1] & \cdots & \gamma[0] \end{bmatrix}}_{[\Gamma]} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \sigma_U^2 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$[\Gamma]^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{1}{\sigma_U^2} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

For AR case, it is common to write $a_k = -\lambda_k$.

The Yule-Walker equation can be used for:

1. Linear prediction

$$\sigma_U^2 = \mathbb{E}[|I[n]|^2] = \mathbb{E}[|e[n]|^2] = P_e$$

For any $X[n]$ WSS centred signal, for any N :

$$\begin{bmatrix} \gamma[0] & \cdots & \gamma[-N] \\ \gamma[1] & \cdots & \gamma[-N+1] \\ \vdots & \ddots & \vdots \\ \gamma[N] & \cdots & \gamma[0] \end{bmatrix} \begin{bmatrix} 1 \\ -\lambda_{1,N} \\ \vdots \\ -\lambda_{N,N} \end{bmatrix} = P_{e,N} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

2. Estimating the PSD

$$\Gamma(\nu) = \frac{\sigma_U^2}{|1 + \sum_{k=1}^N a_k e^{-j2\pi\nu k}|^2}$$

3. Modelling a signal

$$X[n] = \left(\frac{1}{1 + \sum_{k=1}^N a_k z^{-k}} \right) U[n]$$

10 Spectral analysis

Alternative PSD expression:

$$\Gamma(f) = T_e \sum_{k=-\infty}^{\infty} \gamma[k] e^{-j2\pi f k T_e} = \lim_{N \rightarrow \infty} \mathbb{E} \left[\frac{1}{(2N+1)T_e} T_e \sum_{k=-N}^N x[k] e^{-j2\pi f k T_e} \right]$$

Periodogram:

$$\hat{\Gamma}(f) = \frac{1}{NT_e} \left| T_e \sum_{k=0}^{N-1} x[k] e^{-j2\pi f k T_e} \right|^2 = \frac{1}{NT_e} |X_N(f)|^2$$

Bias:

$$\mathbb{E}[\hat{\Gamma}(f)] \neq \Gamma(f)$$

$$\mathbb{E}[\hat{\Gamma}(f)] = T_e \sum_{k=-(N-1)}^{N-1} \mathbb{E}[\hat{\gamma}[k]] e^{-j2\pi f k T_e} = T_e \sum_{k=-(N-1)}^{N-1} \frac{N-|k|}{N} \gamma[k] e^{-j2\pi f k T_e} = W_B(f) * \Gamma(f)$$

Variance (consistency):

$$\mathbb{E}[\hat{\Gamma}(f_1)\hat{\Gamma}(f_2)] = \left(\frac{T_e}{N}\right)^2 \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \mathbb{E}[x[k]x^*[l]x[m]x^*[n]] \exp[-j2\pi(f_1(k-l) + f_2(m-n))T_e]$$

Correlogram

$$\hat{\Gamma}_{corr}(f) = T_e \sum_{k=-M}^M \hat{\gamma}_k e^{-j2\pi f k T_e}$$

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