Geometry Algebra Pen Notes

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1 The Algebra of vectors

1.1 Vectors and its Algebraic Operations

1.1.1 Vectors

Definition A quantity with both magnitude and direction is called a vector. e.g. Force, velocity, acceleration, displacement, etc.

Notation Directed line segment: $\rightarrow$. We could draw a graph to make our proof.

- A directed line segment has a Initial point and a Terminal point.
- $\overrightarrow{AB}$. An arrow upper the letters.
- $\boldsymbol{\alpha}$: bold and lower case Roman letter.
- Sometimes, $\vec{a}, \underline{u}$
- Magnitude, size, length: $|\boldsymbol{\alpha}|, |\overrightarrow{AB}|$

Relevent Concepts Two vectors are equal if and only if their magnitude and direction are the same. No matter where they start.

The vector with zero magnitude is called the **zero vector**, denoted by **0**. **0** is the only vector is the only vector with specific direction.We have:

$$\overrightarrow{AB} = \vec{0} \iff A = B$$

A vector with magnitude 1 is called **unit vector**.

A vector having the same length, but opposite direction of $\vec{a}$, is called the negative of $\vec{a}$, denoted by $-\vec{a}$. $-\vec{a}$. Its a whole notation, not an operation. Thus

$$\overrightarrow{AB} = -\overrightarrow{BA}$$

1.1.2 Operations

Addition of vectors The sum of two vectors $\vec{a}$ and $\vec{b}$ written as

$$\vec{a}+\vec{b}=\vec{c}$$

$\vec{c}$ is a vector.

Defined by **triangle method**, that is

$$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$$

Defined by **parallelogram method** ...

Proposition of Vectors For vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, the addition satisfies:

- $\vec{a}+\vec{b}=\vec{b}+\vec{a}$, commutative law,
- $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$, association law.
- $\vec{a}+\vec{0}=\vec{a}$
- $\vec{a}+(-\vec{a})=\vec{0}$
- Vectors with addition is an abel group.

First two we draw a graph to prove it. Then we prove the rest ones.
3. Set $\vec{a}=\overrightarrow{AB}$ and $\vec{0}=\overrightarrow{BB}$. Then $\vec{a}+\vec{0}=\overrightarrow{AB}+\overrightarrow{BB}=\overrightarrow{AB}=\vec{a}$
4. Set $\vec{a}=\overrightarrow{AB}$. Then $-\vec{a}=\vec{BA}$. Thus, $\vec{a}+(-\vec{a})=\vec{AB}+\vec{BA}=\vec{0}$

Definition We can define the difference of two vectors $\vec{a}$ and $\vec{b}$ to be $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$

Triangular Inequality For any vectors $\vec{a}$ and $\vec{b}$, we have

$$|\vec{a}+\vec{b}|\leq|\vec{a}|+|\vec{b}|$$

1.2 Scalar multiplication

Definition The product of vector $\vec{a}$ and a scalar λ, wirtten as $\lambda\vec{a}$, is a vector, defined by

$$|\lambda\vec{a}|=|\lambda||\vec{a}|$$

Direction

- $\lambda>0$, $\lambda\vec{a}$ has the same direction as $\vec{a}$
- $\lambda<0$, $\lambda\vec{a}$ has the opposite direction as $\vec{a}$

Proposition

- $\lambda\vec{a} = \vec{0} \iff \lambda = 0 \quad or \quad \vec{a} = \vec{0}$
- $1\ \vec{a} = \vec{a}, \quad (-1)\vec{a} = -\vec{a}$
- $\lambda\mu\vec{a}) = (\lambda\mu)\vec{a}$
- $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$ distributive law
- $\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$ distributive law

Proof

- $\lambda\vec{a} = \vec{0} \iff |\lambda\vec{a}| = |\vec{0}| = 0 \iff |\lambda||\vec{a}| = 0 \iff |\lambda| = 0 \quad or \quad |\vec{a}| = 0 \iff \lambda = 0 \quad \vec{a} = \vec{0}$.
-
- ...
- $|\lambda(\mu\vec{a})| = |\lambda||\mu\vec{a}|$
  $= |\lambda||\mu||\vec{a}|$
  $= (|\lambda||\mu|)|\vec{a}|$
  $= |(\lambda\mu)\vec{a}|$

  That is, $\lambda(\mu\vec{a})$ and $(\lambda\mu)\vec{a}$ has the same **length**.

  Then we consider the **direction**.

  Case 1. $\lambda\mu = 0$, $\vec{0} = (\lambda\mu)\vec{a} = \lambda(\mu\vec{a})$

  Case 2. $\lambda\mu > 0$, Trivially, $\lambda(\mu\vec{a})$has the same direction as $\vec{a}$;

  $(\lambda\mu)\vec{a}$has the same direction as $\vec{a}$;

  Case 3. $\lambda\mu < 0$, Trivially, $\lambda(\mu\vec{a})$has the opposite direction of $\vec{a}$;

  $(\lambda\mu)\vec{a}$has the opposite direction of $\vec{a}$;

  Thus, we can see that $\lambda(\mu\vec{a})$ has the same direction as $(\lambda\mu)\vec{a}$.

  Hence,

  $$(-\lambda)\vec{a} = (-1)\lambda\vec{a} = (-1)(\lambda\vec{a}) = -\lambda\vec{a}$$

  If $\lambda = 0 \quad or \quad \mu = 0 \quad or \quad \vec{a} = \vec{0}$, it is easily to see that $(\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a}$.

  consider $\lambda \neq 0 \quad or \quad \mu \neq 0 \quad or \quad \vec{a} \neq \vec{0}$.

1° assume that $\lambda > 0, \mu > 0$ then,

$$|(\lambda+\mu)\vec{a}| = |\lambda+\mu||\vec{a}|$$
$$= (\lambda+\mu)|\vec{a}|$$
$$= \lambda|\vec{a}| + \mu|\vec{a}|$$
$$= |\lambda||\vec{a}| + |\mu||\vec{a}|$$
$$= |\lambda\vec{a}| + |\mu\vec{a}| \quad (for \quad \vec{a} \parallel \vec{a})$$
$$= |\lambda\vec{a} + \mu\vec{a}|$$

2° If one of λ, μ is negative, we can put the terms containing the negative scalars to the other side of the equation.

For example:

$\lambda > 0, \mu < 0, \lambda + \mu < 0$

$$(\lambda+\mu)\vec{a} = \lambda\vec{a} + \mu\vec{a} \iff -\mu\vec{a} = \lambda\vec{a} + [-(\lambda+\mu)\vec{a}]$$
$$\iff \lambda\vec{a} + (-(\lambda+\mu))\vec{a} = (-\mu)\vec{a}$$

(which back to the case of $\lambda > 0, \mu > 0, \lambda + \mu > 0$)

- Case 1: $\lambda = 0$,or $\vec{a} = \vec{0}$, or $\vec{b} = \vec{0}$

  Case 2:$\lambda \neq 0$,or $\vec{a} \neq \vec{0}$, or $\vec{b} \neq \vec{0}$

  $\lambda(\vec{a}+\vec{b}) = \lambda\vec{a} + \lambda\vec{b}$(Graph...(similarity of triangle))

1.3 Collinear and Coplanar Vectors

Definition When we move some vectors to the same initial point, if they are on the same line or plane, then we say these vectors are **collinear** or **coplanar**.

Trivially,

- $\vec{0}$ Is collinear with any vector.
- Collinear vectors must be coplanar.
- Any two vectors must be coplanar.

1.3.1 Collinear Vectors

Notation two vectors are collinear $\iff$ their directions are the same or opposite. We write $\vec{a} \parallel \vec{b}$

Proposition For vectors $\vec{a}$ and $\vec{b}$, if there exists a scalar λ $s.t.$ $\vec{b} = \lambda\vec{a}$, then $\vec{a}, \vec{b}$ are collinear.

Theorem (Existence): Assume that $\vec{a} \neq \vec{0}$. If $\vec{a}, \vec{b}$ collinear, then there exists scalar $\lambda \quad s.t. \quad \vec{b} = \lambda\vec{a}$.

Proof If $\vec{b} = \vec{0}$, then $\lambda = 0$

Consider $\vec{b} \neq \vec{0}$, then

$$\vec{b} = |\vec{b}|\frac{\vec{b}}{|\vec{b}|} = \begin{cases} \frac{\vec{b}}{|\vec{a}|}, & \vec{a} \text{ and } \vec{b} \text{ have the same direction} \\ \frac{-|\vec{b}|}{|\vec{a}|}, & \vec{a} \text{ and } \vec{b} \text{ have the opposite direction} \end{cases}$$

(Uniqueness): Assume that $\vec{b} = \lambda'\vec{a}$ Then

$$(\lambda - \lambda')\vec{a} = \vec{0} For \quad \vec{a} \neq \vec{0} \Rightarrow \quad \lambda = \lambda'$$

1.3.2 Three coplanar Vectors

Proposition For vectors $\vec{a}, \vec{b}, \vec{c}$ if there exists scalar λ, μ, such that $\vec{c} = \lambda\vec{a} + \mu\vec{b}$, then $\vec{a}, \vec{b}, \vec{c}$ are coplanar. (Graph...)

Proof If $\vec{a} = \vec{0}$, then $\vec{c} = \lambda\vec{b}$, so vectors $\vec{b} \parallel \vec{c}$, then $\vec{a}, \vec{b}, \vec{c}$ are Coplanar;

If $\vec{a} \neq \vec{0}$, then consider two cases:

Case 1: $\vec{a} \parallel \vec{b}$. Thus, there exist k s.t. $\vec{b} = k\vec{a}$.

Hence, $\vec{c} = \lambda\vec{a} + \mu\vec{b} = \lambda\vec{a} + \mu k\vec{a} = (\lambda + \mu k)\vec{a}$.

Therefore, $\vec{c} \parallel \vec{a}$.

Case 2: $\vec{a} \not\parallel \vec{b}$, then $\vec{c}$ is the diagrod of the parallelogram formed by $\lambda\vec{a}, \mu\vec{b}$

Theorem Assume that $\vec{a}, \vec{b}$ are **not collinear**, then for any vector $\vec{c}$ on the plane determined by $\vec{a}$ and $\vec{b}$, there exist **unique** scalars $\lambda, \mu \quad s.t.$

$$\vec{c} = \lambda\vec{a} + \mu\vec{b}$$

Proof 1°We first prove the existence of λ, μ.

(Graph...)

We can wrote $\vec{c} = \vec{c}_1 + \vec{c}_2, \vec{c}_1 \parallel \vec{a}, \vec{c}_2 \parallel \vec{b}$,

Since $\vec{a}, \vec{b} \neq \vec{0}$,

$\exists \lambda, \mu, \quad \vec{c}_1 = \lambda\vec{a}, \vec{c}_2 = \mu\vec{b}$

Then, $\vec{c} = \lambda\vec{a} + \mu\vec{b}$

2°Suppose that $\vec{c} = \lambda'\vec{a} + \mu'\vec{b}$

We see that $(\lambda - \lambda')\vec{a} + (\mu - \mu')\vec{b} = \vec{0}$

If $\lambda \neq \lambda'$, then

$$\vec{a} = -\frac{\mu - \mu'}{\lambda - \lambda'}\vec{b}$$

then $\vec{a} \parallel \vec{b}$, which is a contradiction.

thus, $\lambda = \lambda'$, we have $(\mu - \mu')\vec{b} = \vec{0}$

Since $\vec{b} \neq \vec{0}, \mu = \mu'$

1.3.3 Three Non-coplanar Vectors

Theorem If $\vec{a}, \vec{b}, \vec{c}$ are not coplaner, then or any vector $\vec{u}$, there exist unique scalars λ, μ, ν,s.t.

$$\vec{u} = \lambda\vec{a} + \mu\vec{b} + \nu\vec{c}$$

Proof 1°Existence of λ, μ, ν

(Graph...)

2°Assume that

$$\vec{u} = (\lambda - \lambda^*)\vec{a} + (\mu - \mu^*)\vec{b} + (\nu - \nu^*)\vec{c}$$

Suppose $\lambda \neq \lambda^*$, then

$$\vec{a} = -\frac{\mu - \mu^*}{\lambda - \lambda^*}\vec{b} - \frac{\nu - \nu^*}{\lambda - \lambda^*}\vec{c}$$

Showing that $\vec{a}, \vec{b}\vec{c}.$ are coplaner. This is a contradiction.

Hence, $\lambda = \lambda^*$,We have

$$(\mu - \mu^*)\vec{b} + (\nu - \nu^*)\vec{c} = \vec{0}$$

Using the similar argument in the proof of the last theorem, we know that $\mu = \mu^*, \nu = \nu^*$ $(\vec{b} \not\parallel \vec{c})$

1.3.4 Three Points on the same line

Point C is on the line segment AB if and only if there exist scalar $\lambda, \mu \geq 0 \quad (\lambda + \mu = 1)$ s.t.

$$\overrightarrow{OC} = \lambda\overrightarrow{OA} + \mu\overrightarrow{OB}$$

for any point O.

Proof Point C is on $AB \iff \overrightarrow{AC} \parallel \overrightarrow{AB}$.

$$\iff |\overrightarrow{AC}| \leq |\overrightarrow{AB}|$$

$$\iff \exists \mu \in [0,1] \quad s.t. \quad \overrightarrow{AC} = \mu\overrightarrow{AB}$$

$$\iff \overrightarrow{OC} - \overrightarrow{OA} = \mu(\overrightarrow{OB} - \overrightarrow{OA})$$

$$\iff \overrightarrow{OC} = (1-\mu)\overrightarrow{OA} + \mu\overrightarrow{OB}$$

$$\iff \overrightarrow{OC} = \lambda\overrightarrow{OA} + \mu\overrightarrow{OB}, \quad (\lambda + \mu = 1)$$

1.4 Affine Coordinate System

1.4.1 Coordinate System

By a theorem of **1-2**, if $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are not coplanar, then for any vector $\vec{a}$, there exist unique scalar a_1, a_2, a_3 s.t.

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$$

The ordered tripe(a_1, a_2, a_3) is called the coordinate of $\vec{a}$.

It can be show that the mapping $\vec{a} \mapsto (a_1, a_2, a_3)$ is a one-to-one correspondence.

We simply write $\vec{a} = (a_1, a_2, a_3)$

Obviously, $\vec{e}_1 = (1,0,0), \quad \vec{e}_2 = (0,1,0), \quad \vec{e}_3 = (0,0,1)$

Origin + Basis = Coordinate System

$$[O, \vec{e}_1, \vec{e}_2, \vec{e}_3]$$

If $\vec{e}_1 \perp \vec{e}_2, \vec{e}_2 \perp \vec{e}_3, \vec{e}_1 \perp \vec{e}_3$ and $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are unit vectors, then $[O, \vec{e}_1, \vec{e}_2, \vec{e}_3]$ is called the Cartesian coordinate system.

1.4.2 Algebraic Operations Using Coordinate

Theorem In an affine coordinate system$[O, \vec{e}_1, \vec{e}_2, \vec{e}_3]$, assume that $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then,

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

Proof

$$\vec{a}+\vec{b}$$

$$=(a_1\vec{e}_1+a_2\vec{e}_2+a_3\vec{e}_3)+(b_1\vec{e}_1+b_e\vec{b}_2+b_3\vec{e}_3)$$

$$=(a_1+b_1)\vec{e}_1+(a_2+b_2)\vec{e}_2+(a_3+b_3)\vec{e}_3$$

$k\vec{a}=(ka_1,ka_2,ka_3)$

Proof

$$k\vec{a}=k(a_1\vec{e}_1+a_2\vec{e}_2+a_3\vec{e}_3)=ka_1\vec{e}_1+ka_2\vec{e}_2+ka_3\vec{e}_3$$

Corollary Coordinary of $\overrightarrow{AB}$ = Coordinate of B - Coordinate A

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$$

now let's assume that we have three vectors $\vec{a},\vec{b},\vec{c}$,

1.4.3 Scalar Products of Vectors

Definition The scalar product (or inner product, or dot product) of two vectors $\vec{a}$ and $\vec{b}$ is a scalar, denoted by $\vec{a}\cdot\vec{b}$, and is defined by

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos<\vec{a},\vec{b}>$$

where $<\vec{a},\vec{b}>\in[0,\pi]$

Proposition

- $\vec{a}\cdot\vec{a}=|\vec{a}|^2\geq 0\Rightarrow|\vec{a}|=\sqrt{\vec{a}\cdot\vec{a}}$
- $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ (commutative)
- $\vec{a}\cdot\vec{b}=0\iff\vec{a}\perp\vec{b}$ ($\vec{0}$ is perpendicular to any vectors)
- Cauchy-Schwarz inequality $|\vec{a}\cdot\vec{b}|\leq|\vec{a}||\vec{b}|$, the equation holds if and only if $\vec{a}$ is collinear with $\vec{b}$.

Theorem

- $(\lambda\vec{a})\cdot\vec{b}=\lambda(\vec{a}\cdot\vec{b})=\vec{a}\cdot(\lambda\vec{b})$
- $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

Example Law of Cosine

In triangle ABC

$$\vec{c}\cdot\vec{c}=(\vec{a}+\vec{b})^2=|\vec{a}|^2+|\vec{b}|^2+\vec{a}\cdot\vec{b}=|\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|\cos C$$

Example Three height of $\triangle ABC$ concurrent. Claim that $\overrightarrow{CO}\perp\overrightarrow{AB}$.

$$\overrightarrow{CO}\cdot\overrightarrow{AB}=...=0$$

Calculate $\vec{a}\cdot\vec{b}$ Using Coorinates In an affine coordinate system $[O,\vec{e}_1,\vec{e}_2,\vec{e}_3]$. Assume that $\vec{a}=(a_1,a_2,a_3),\vec{b}=(b_1,b_2,b_3)$, Then

$$\vec{a}\cdot\vec{b}=\sum a_i\vec{e}_i\cdot\sum b_i\vec{e}_i=...$$

In particular, in the Cartesian coordinate system,

$$\vec{a}\cdot\vec{b}=\sum a_ib_i$$

Moreover,

$$|\vec{a}|=\sqrt{\sum a_i^2}$$

$$\cos<\vec{a},\vec{b}>=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{\sum a_ib_i}{\sqrt{\sum a_i^2}\sqrt{\sum b_i^2}}$$

Example In a regular tentrahedron $ABCD$, E is the midpoint of AB and F is the midpoint of CD. Every length equals to a. Find $|\overrightarrow{EF}|$.

Consrruct an affine c.s. $[A,\overrightarrow{AB},\overrightarrow{AC},\overrightarrow{AD}]$. Write

$$\vec{e}_1=\overrightarrow{AB},\vec{e}_2=\overrightarrow{AC},\vec{e}_3=\overrightarrow{AD}$$

$$\overrightarrow{AF}=\frac{1}{2}(\vec{e}_2+\vec{e}_3)=(0,\frac{1}{2},\frac{1}{2})$$

$$\overrightarrow{AE}=...$$

Since $\vec{e}_i^2=a^2,\vec{e}_i\cdot\vec{e}_j=\frac{1}{2}a^2$

$$\overrightarrow{EF}^2=(-\frac{1}{2},\frac{1}{2},\frac{1}{2})^2=\frac{a^2}{2}$$

Thus $|\overrightarrow{EF}|=\frac{\sqrt{2}}{2}a$

1.4.4 Vector Product of Vectors

Definition The vector product (or cross product) of two vectors$\vec{a}$ and $\vec{b}$ is a vector, denoted by $\vec{a}\times\vec{b}$, and is defined by

$$\text{magnitude:}|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin<\vec{a},\vec{b}> \quad \text{direction: Right hand rule}$$

Note: $|\vec{a}\times\vec{b}|$ = area of parallelogram formed by $\vec{a},\vec{b}$

Proposition

- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times \vec{b} = \vec{0} \iff \vec{a} \parallel \vec{b}$
- $(\lambda \vec{a}) \times \vec{b} = \lambda(\vec{a} \times \vec{b}) = \vec{a} \times (\lambda \vec{b})$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Introdection to Determinant

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$$

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}$$