# Mathematical Modeling HW5

Miyasaka Kion

April 28, 2022

## 1 Task: Prove proposition 4.1

Proposition 4.1. The following are equivalent:
(i) $T$ is firmly nonexpansive.
(ii) $T$ is $\frac{1}{2}$-averaged nonexpansive.
(iii) $\mathcal{I} - T$ is firmly nonexpansive.
(iv) $2T - \mathcal{I}$ is nonexpansive.

**Proof**$_1$ We first prove (ii) and (i) are equivalent, i.e. $T$ is $\frac{1}{2}$-averaged nonexpansive if and only if $T$ is firmly nonexpansive. $T$ is $\frac{1}{2}$-averaged nonexpansive which means $T$ can be written as $\frac{\mathcal{I}+\mathcal{N}}{2}$, where $\mathcal{N}$ is a non-expansive operator:

$$\|\mathcal{N}\mathbf{x} + \mathcal{N}\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| \iff \|\mathcal{N}\mathbf{x} + \mathcal{N}\mathbf{y}\|^2 \leq \|\mathbf{x} - \mathbf{y}\|^2$$

add some components to construct the inequality above, we have

$$\|\mathbf{x} - \mathbf{y}\|^2 + 2\langle \mathbf{x} - \mathbf{y}, \mathcal{N}\mathbf{x} - \mathcal{N}\mathbf{y}\rangle + \|\mathcal{N}\mathbf{x} + \mathcal{N}\mathbf{y}\|^2 \leq 2\|\mathbf{x} - \mathbf{y}\|^2 + 2\langle \mathbf{x} - \mathbf{y}, \mathcal{N}\mathbf{x} - \mathcal{N}\mathbf{y}\rangle$$

where

$$\begin{aligned}
\text{LHS} &= \langle (\mathbf{x} - \mathbf{y}) + (\mathcal{N}\mathbf{x} - \mathcal{N}\mathbf{y}), (\mathbf{x} - \mathbf{y}) + (\mathcal{N}\mathbf{x} - \mathcal{N}\mathbf{y})\rangle \\
&= \|\mathbf{x} - \mathbf{y} + \mathcal{N}\mathbf{x} - \mathcal{N}\mathbf{y}\| \\
\text{RHS} &= 2\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y}\rangle + 2\langle \mathbf{x} - \mathbf{y}, \mathcal{N}\mathbf{x} - \mathcal{N}\mathbf{y}\rangle \\
&= 2\langle \mathbf{x} - \mathbf{y} + \mathcal{N}\mathbf{x} - \mathcal{N}\mathbf{y}, \mathbf{x} - \mathbf{y}\rangle
\end{aligned}$$

Since LHS $\leq$ RHS, we have

$$\left\|\frac{\mathbf{x} + \mathcal{N}\mathbf{x}}{2} - \frac{\mathbf{y} + \mathcal{N}\mathbf{y}}{2}\right\|^2 \leq \left\langle \frac{\mathbf{x} + \mathcal{N}\mathbf{x}}{2} - \frac{\mathbf{y} + \mathcal{N}\mathbf{y}}{2}, \mathbf{x} - \mathbf{y}\right\rangle$$

which is equivalent to

$$\|T\mathbf{x} - T\mathbf{y}\|^2 \leq \|\mathbf{x} - \mathbf{y}\|.$$

. ■

**Proof$_2$** Now we prove (iii) and (ii) are equivalent, i.e. $\mathcal{I}-T$ is firmly nonexpansive if and only if $T$ is $\frac{1}{2}$-averaged nonexpansive. $\mathcal{I}-T$ is firmly nonexpansive infers

$$\|(\mathbf{x}-T\mathbf{x})-(\mathbf{y}-T\mathbf{y})\|^2 \leq \langle(\mathbf{x}-T\mathbf{x})-(\mathbf{y}-T\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle$$

let $\mathcal{N}$ such that that $T=\frac{\mathcal{I}+\mathcal{N}}{2}$, we have

$$\begin{aligned}\text{LHS} &= \left\|\left(\mathbf{x}-\frac{\mathbf{x}+\mathcal{N}\mathbf{x}}{2}\right)-\left(\mathbf{y}-\frac{\mathbf{y}+\mathcal{N}\mathbf{y}}{2}\right)\right\|^2 \\ &= 1/4\cdot\|\mathbf{x}-\mathbf{y}-(\mathcal{N}\mathbf{x}-\mathcal{N}\mathbf{y})\| \\ \text{RHS} &= 1/2\cdot\langle\mathbf{x}-\mathbf{y}-(\mathcal{N}\mathbf{x}-\mathcal{N}\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle\end{aligned}$$

Then,

$$\begin{aligned}&\|\mathbf{x}-\mathbf{y}-(\mathcal{N}\mathbf{x}-\mathcal{N}\mathbf{y})\|^2 \leq 2\langle\mathbf{x}-\mathbf{y}-(\mathcal{N}\mathbf{x}-\mathcal{N}\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle \\ &\iff \\ &\|\mathbf{x}-\mathbf{y}\|^2+\|\mathcal{N}\mathbf{x}-\mathcal{N}\mathbf{y}\|^2-2\langle\mathbf{x}-\mathbf{y}-(\mathcal{N}\mathbf{x}-\mathcal{N}\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle \leq 2\langle\mathbf{x}-\mathbf{y}\rangle^2-2\langle\mathbf{x}-\mathbf{y}-(\mathcal{N}\mathbf{x}-\mathcal{N}\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle \\ &\iff \\ &\|\mathcal{N}\mathbf{x}-\mathcal{N}\mathbf{y}\|^2 \leq \|\mathbf{x}-\mathbf{y}\|^2\end{aligned}$$

If $\mathcal{I}-T$ is firmly nonexpansive, we can assume that $\mathcal{N}$ does not exist, the above inequality cannot satisfied, i.e. there exist a nonexpansive operator $\mathcal{N}$ s.t. $T=\frac{\mathcal{I}+\mathcal{N}}{2}$. If $T$ is $\frac{1}{2}$-averaged nonexpansive, then let $\mathcal{N}$ be a nonexpasive operator s.t. $T=\frac{\mathcal{I}+\mathcal{N}}{2}$, we can infer that $\mathcal{I}-T$ is firmly nonexpansive,

■

**Proof$_3$** Finally we prove (i) and (iv) are equivalent, i.e. $2T-\mathcal{I}$ is nonexpansive if and only if $T$ is firmly nonexpansive. $2T-\mathcal{I}$ is nonexpansive means

$$\begin{aligned}\|(2T\mathbf{x}-\mathbf{x})-(2T\mathbf{y}-\mathbf{y})\| &\leq \|\mathbf{x}-\mathbf{y}\| \\ \iff \|2(T\mathbf{x}-T\mathbf{y})-(\mathbf{x}-\mathbf{y})\|^2 &\leq \|\mathbf{x}-\mathbf{y}\|^2 \\ \iff 4\|T\mathbf{x}-T\mathbf{y}\|^2+\|\mathbf{x}-\mathbf{y}\|^2-4\langle T\mathbf{x}-T\mathbf{y},\mathbf{x}-\mathbf{y}\rangle &\leq \|\mathbf{x}-\mathbf{y}\|^2 \\ \iff \|T\mathbf{x}-T\mathbf{y}\|^2 &\leq \|\mathbf{x}-\mathbf{y}\|^2 \\ \iff \|T\mathbf{x}-T\mathbf{y}\| &\leq \|\mathbf{x}-\mathbf{y}\| \quad (T \text{ is nonexpansive})\end{aligned}$$

■

## 2 Task: $\text{prox}_{\|\cdot\|_1}$ is firmly nonexpansive

Prove that $\text{prox}_{\|\cdot\|_1}$ is firmly nonexpansive according to (3.7) and (3.8). (Don't use Theorem 4.1 !!!)

- One-dimensional case For $x \in \mathbb{R}$,

$$\operatorname{prox}_{|\cdot|}(x) = \max\{|x|-1,0\} \cdot \operatorname{sgn}(x)$$

- $n$-dimensional case For $\boldsymbol{x} \in \mathbb{R}^n$,

$$\operatorname{prox}_{\|\cdot\|_1}(\boldsymbol{x}) = \left(\operatorname{prox}_{|\cdot|}(x_1), \operatorname{prox}_{|\cdot|}(x_2), \ldots, \operatorname{prox}_{|\cdot|}(x_n)\right)^\top$$

**Proof** We prove by mathematical induction.

**Base case: 1-dimension case** $\operatorname{prox}_{\|\cdot\|_1}$ is defined by

$$\operatorname{prox}_{\|\cdot\|_1}(x) = \max\{|x|-1,0\} \cdot \operatorname{sgn}(x)$$

We hope $\operatorname{prox}_{\|\cdot\|_1}$ is firmly nonexpansive i.e. we hope that

$$\begin{aligned} & \left\|\operatorname{prox}_{\|\cdot\|_1} \mathbf{x} - \operatorname{prox}_{\|\cdot\|_1} \mathbf{y}\right\|^2 \leq \left\langle \operatorname{prox}_{\|\cdot\|_1} \mathbf{x} - \operatorname{prox}_{\|\cdot\|_1} \mathbf{y}, \mathbf{x}-\mathbf{y}\right\rangle \\ \iff & \left\langle \operatorname{prox}_{\|\cdot\|_1} \mathbf{x} - \operatorname{prox}_{\|\cdot\|_1} \mathbf{y}, \operatorname{prox}_{\|\cdot\|_1} \mathbf{x} - \operatorname{prox}_{\|\cdot\|_1} \mathbf{y} - (\mathbf{x}-\mathbf{y})\right\rangle \leq 0 \end{aligned}$$

We expand the LHS above,

$$\begin{aligned} & \text{LHS} \\ & = (\operatorname{sgn}(x)\max(0,|x|-1) - \operatorname{sgn}(y)\max(0,|y|-1))(\operatorname{sgn}(x)\max(0,|x|-1) - \operatorname{sgn}(y)\max(0,|y|-1) - x - y) \\ & = \begin{cases} (|x|-1)\operatorname{sgn}(x)((|x|-1)\operatorname{sgn}(x) - x + y) & |x|>1 \wedge |y| \leq 1 \\ ((|x|-1)\operatorname{sgn}(x) - (|y|-1)\operatorname{sgn}(y))((|x|-1)\operatorname{sgn}(x) - |y|\operatorname{sgn}(y) + \operatorname{sgn}(y) - x + y) & |x|>1 \wedge |y|>1 \\ (|y|-1)\operatorname{sgn}(y)((|y|-1)\operatorname{sgn}(y) + x - y) & |x| \leq 1 \wedge |y|>1 \end{cases} \\ & \leq 0 \quad \text{(Since the maximum of all cases are less or equal than 0 ).} \end{aligned}$$

Therefore, $\operatorname{prox}_{\|\cdot\|_1}$ is a firmly nonexpansive when dimension is 1.

**Induction step** if $\operatorname{prox}_{\|\cdot\|_1}$ is firmly nonexpensive when $\mathbf{x} \in \mathbb{R}^n$, i.e. by definition of $\operatorname{prox}_{\|\cdot\|_1}$, we have

$$\left\|\operatorname{prox}_{\|\cdot\|_1} \mathbf{x} - \operatorname{prox}_{\|\cdot\|_1} \mathbf{y}\right\|^2 \leq \left\langle \operatorname{prox}_{\|\cdot\|_1} \mathbf{x} - \operatorname{prox}_{\|\cdot\|_1} \mathbf{y}, \mathbf{x}-\mathbf{y}\right\rangle$$

which is equivalent to

$$\sum_{i=1}^n \left(\operatorname{prox}_{\|\cdot\|_1} x_i - \operatorname{prox}_{\|\cdot\|_1} y_i\right)^2 \leq \sum_{i=1}^n \left(\operatorname{prox}_{\|\cdot\|_1} x_i - \operatorname{prox}_{\|\cdot\|_1} y_i\right) \cdot (x_i - y_i)$$

for $\operatorname{prox}_{\|\cdot\|_1}$ is firmly nonexpansive when dimension is 1,

$$\left(\operatorname{prox}_{\|\cdot\|_1} x_{n+1} - \operatorname{prox}_{\|\cdot\|_1} y_{n+1}\right)^2 \leq \left(\operatorname{prox}_{\|\cdot\|_1} x_{n+1} - \operatorname{prox}_{\|\cdot\|_1} y_{n+1}\right)(x_{n+1} - y_{n+1})$$

by adding the 2 inequality above we get

$$\sum_{i=1}^{n+1} \left(\operatorname{prox}_{\|\cdot\|_1} x_i - \operatorname{prox}_{\|\cdot\|_1} y_i\right)^2 \leq \sum_{i=1}^{n+1} \left(\operatorname{prox}_{\|\cdot\|_1} x_i - \operatorname{prox}_{\|\cdot\|_1} y_i\right) \cdot (x_i - y_i)$$

which infers that $\left\|\operatorname{prox}_{\|\cdot\|_1} \mathbf{x} - \operatorname{prox}_{\|\cdot\|_1} \mathbf{y}\right\|^2 \leq \left\langle \operatorname{prox}_{\|\cdot\|_1} \mathbf{x} - \operatorname{prox}_{\|\cdot\|_1} \mathbf{y}, \mathbf{x} - \mathbf{y} \right\rangle$ holds when $\mathbf{x} \in \mathbb{R}^{n+1}$.

Hence $\operatorname{prox}_{\|\cdot\|_1}$ is firmly nonexpansive.

$\blacksquare$