Mathematical Modeling HW1

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1 Automobile

1. An automobile manufacturer makes a profit of $\$1,500$ on the sale of a certain model. It is estimated that for every $100 of rebate, sales increase by 15%.

1.1 Question

(a) What amount of rebate will maximize profit? Use the five-step method, and model as a one-variable optimization problem.

Step 1. Ask the question.

$$\begin{aligned}
\text{Variables:}& \\
r &= \text{money on rebate(\$)} \\
P &= \text{profit(\$)} \\
A &= \text{the amount of automobile will be sold} \\
\text{Assumptions:}& \\
0 &\le r \le 1500 \\
A &\ge 0 \\
\text{Objective:}& \\
&\text{Maximize} \quad P
\end{aligned}$$

Step 2. Select the modeling approach.
We model this problem as a one-variable optimization problem.

Step 3. Formulate the model.

$$P(r) = (1500 - r) \times A(1 + 0.15(r/100))$$

Step 4. Solve the model.

$$P(r) = (1500 - r) \times A(1 + 0.15(r/100)) = A(-0.0015r^2 + 1.25r + 1500)$$

which is a quadratic function and it reaches the global maximum at $r = 416.667$, $P_{max} = 1760.42A$

Step 5. Answer the question.

A rebate of \$416.667 will make the profit maximum, which is \$1760.42 times the vehicles sold.

1.2 Question

(b) Compute the sensitivity of your answer to the 15% assumption. Consider both the amount of rebate and the resulting profit. Substitute 15% by $s(s > 0)$. Then $P(r)$ becomes

$$P(r) = (1500 - r)\left(\frac{rs}{100} + 1\right) = -\frac{r^2 s}{100} + 15rs - r + 1500,$$

which is a quadratic function. Hence $P_{max} = \frac{25(225s^2+30s+1)}{s}$ at $r = \frac{50(15s-1)}{s}$. Denote the sensitivity of P to s as $S(P, s)$, and sensitivity of r to s as $S(r, s)$,

$$S(P, s) = \frac{\mathrm{d}P}{\mathrm{d}s} \cdot \frac{s}{P} = \frac{\left(5625 - \frac{25}{s^2}\right) s}{P} = 0.384615$$

$$S(r, s) = \frac{\mathrm{d}r}{\mathrm{d}s} \cdot \frac{s}{r} = \frac{50}{rs} = 0.799999$$

1.3 Question

(c) Suppose that rebates actually generate only a 10% increase in sales per \$100. What is the effect? What if the response is somewhere between 10 and 15% per \$100 of rebate?

By the sensitivity computed in (b), if the rebate descend from 15% to 10%, i.e. a 1/3 reduction, the profit will decease by $S(P, s) \times 1/3 = 0.128205 \approx 12.82\%$ and the rebate value will decrease by $S(r, s) \times 1/3 = 0.266666 \approx 26.67\%$.

~~(d) Under what circumstances would a rebate offer cause a reduction in profit?~~

2 Pig

2. In the pig problem, perform a sensitivity analysis based on the cost per day of keeping the pig. Consider both the effect on the best time to sell and on the resulting profit. If a new feed costing 60 cents/day would let the pig grow at a rate of 7lbs/ day, would it be worth switching feed? What is the minimum improvement in growth rate that would make this new feed worthwhile?

2.1 Sensitivity Analysis

Sol. The objective function is

$$P = (0.65 - 0.01t)(200 + 5t) - ct = t(1.25\ - 1.c) - 0.05t^2 + 130.$$

which is a quadratic function, and its maximum value is $5c^2 - 12.5c + 137.8134$ at $t = -10(c - 1.25)$. The sensitivity of P to c is $S(P, c)$, and the sensitivity of t to c is $S(t, c)$.

$$S(P, c) = \frac{\mathrm{d}P}{\mathrm{d}c} \cdot \frac{c}{P} = \frac{c(10.c - 12.5)}{P} = -0.027027 \approx -2.7027\%$$

which means if the cost of keeping the pig per day grows 100% will result in a 2.7027% decrease of the profit.

2.2 change of method

If a new feed costing 60 cents/day would let the pig grow at a rate of 7lbs/ day, the objective function becomes

$$P(t) = (0.65 - 0.01t)(200 + 7t) - 0.6t = -0.07t^2 + 1.95t + 130,$$

which is a quadratic function and it reaches maximum at $t = 13.9286$, since $t \in \mathbb{Z}$, the maximum value is $P(14) = 143.58$ at $t = 14$. The new profit is greater than the original one, thus it worth to switch feed.

2.3 Minimum in rate of growth

Assume that the rate of growth is $g(g > 0)$, then the objective function is

$$P = (0.65 - 0.01t)(200 + gt) - 0.6t$$

and we hope $P_{max} \geq 133.2$. Notice that

$$P = (0.65 - 0.01t)(200 + gt) - 0.6t = -0.01gt^2 + (0.65g - 2.6)t + 130.$$

which is a quadratic function. Hence $P_{max} = 45.5\ + 10.5625g + \frac{169.}{g}$ at $t = \frac{32.5(g-4.)}{g}$. Then we only need to solve

$$45.5\ + 10.5625g + \frac{169.}{g} \geq 133.2.$$

The solution to the above inequality is $0 \leq g \leq 3.04027$(this case is obviously impossible) or $g \geq 5.26269$. Hence the minimum rate of growth is 5.26 pounds a day.