Real Analysis and Calculus Note

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1 Sequence

Axiom of Completeness

A nonempty set of real numbers that is bounded above has a least upper bound, i.e. supremums of bounded sets are real numbers.

Monotone Subsequence

Every sequence contains a monotonic subsequence.

Monotone Convergence Theorem

Suppose that $\{x_n\}$ is a monotonic sequence. Then, $\{x_n\}$ is convergent if and only if $\{x_n\}$ is bounded.

Bolzano-Weierstrass

Every bounded sequence contains a convergent subsequence.

Cauchy Convergence Criterion

A sequence $\{x_n\}$ is convergent iff for each $\varepsilon > 0$ there exists an integer N with the property that

$$|x_n - x_m| \leq \varepsilon$$

for all $n \geq N$ and $m \geq N$.

$$\begin{aligned} &\text{Completeness Axiom of reals} \\ \Longrightarrow\ &\text{Monotonic Convergence Theorem} \\ \Longrightarrow\ &\text{Bolzano-Weierstrass Theorem} \\ \Longrightarrow\ &\text{Cauchy Convergence Criterion} \end{aligned}$$

2 Series

2.1 Conditions of convergence

If $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$ as $k \to \infty$.

If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then so does the series $\sum_{k=1}^{\infty} a_k$

A series $\sum_{k=1}^{\infty} a_k$ is said to be absolutely convergent if $\sum_{k=1}^{\infty} |a_k|$ converges.

In order to distinguish convergence from absolute convergence, we refer to the former as non-absolutely convergence, or conditional convergence.

A series $\sum_{k=1}^{\infty} a_k$ is said to be non-absolutely (or conditional) convergent if it converges but the series $\sum_{k=1}^{\infty} |a_k|$ diverges.

2.2 Properties of Convergent Series

2.2.1 Dirichlet's Theorem: rearrangements of series

Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers. If $\{b_n\}$ is any rearrangement of $\{a_n\}$ then

1. $\sum_{n=1}^{\infty} b_n$ is an absolutely convergent series.
2. $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

2.2.2 Conditional convergence

If the series $\sum_{n=1}^{\infty} a_n$ converges but does *not* converge absolutely (i.e. $\sum_{n=1}^{\infty} a_n$ is conditionally convergent) and $\gamma \in \mathbb{R}$ is any real number, then there exists a rearrangement $\{b_n\}$ of the sequence $\{a_n\}$ so that

$$\sum_{n=1}^{\infty} b_n = \gamma$$

2.3 Converge Tests

2.3.1 Null test

If the terms of the series $\sum_{k=1}^{\infty} a_k$ do not converge to zero, then the series diverges.

2.3.2 Comparison test

Given two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ such that $0 \leq a_k \leq b_k$ for all k.

1. If the larger series converges, then so does the smaller series.
2. If the smaller series diverges, then so does the larger series.

2.3.3 Ratio test

If terms of the series $\sum_{k=1}^{\infty} a_k$ are all positive and the ratios

$$\lim_{k \to \infty} \frac{a_{k+1}}{a_k} < 1$$

then the series is convergent.

2.3.4 Root test

If terms of the series $\sum_{k=1}^{\infty} a_k$ are all nonnegative and the roots

$$\lim_{k\to\infty} \sqrt[k]{a_k} < 1$$

then the series is convergent.

2.3.5 Integral test

Let f be a nonnegative decreasing function on $[1, \infty)$. Then

$$\lim_{X\to\infty} \int_1^X f(x)dx$$

converges if and only if the series $\sum_{k=1}^{\infty} f(k)$ converges.

Proof since f is decreasing we have

$$\int_k^{k+1} f(x)dx \leq f(k) \leq \int_{k-1}^{k} f(x)dx$$

Thus

$$\int_1^{n+1} f(x)dx \leq \sum_{k=1}^{n} f(k) \leq f(1) + \int_1^n f(x)dx$$

The series converges if and only if the partial sums are bounded.

2.3.6 Alternating Series test

The series

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

where the terms alternate in sign, converges if the sequence $\{a_k\}$ decreases monotonically to zero.

3 Power Series

A power series

$$f(x) \overset{\text{def}}{=} \sum_{n=0}^{\infty} a_n x^n, x \in S$$

where S will make sense.

3.1 Radius of Convergence

Given a power series $\sum_{n=0}^{\infty} a_n x^n$, either it converges absolutely for all $x \in \mathbb{R}$, or there exists $R \in [0, \infty)$ such that

(1) it converges absolutely when $|x| < R$

(2) it diverges when $|x| > R$.

Remark We can restate the theorem as

$$(-R, R) \subseteq S \subseteq [-R, R]$$

and the power series converges absolutely in $(-R, R)$. In particular we see that S is always an interval.

3.2 Convergence Test (of Power Series)

3.2.1 Ratio test (of Power Series)

Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

Suppose that

$$\frac{|a_{n+1}|}{|a_n|} \to \ell, \quad \text{as} \quad n \to \infty$$

Then

$$R = \begin{cases} 0 & \text{if} \quad \ell = \infty \\ \frac{1}{\ell} & \text{if} \quad \ell \in \mathbb{R}\backslash\{0\} \\ \infty & \text{if} \quad \ell = 0 \end{cases}$$

3.2.2 Root test (of Power Series)

Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

Suppose that

$$|a_n|^{\frac{1}{n}} \to \ell, \quad \text{as} \quad n \to \infty$$

Then

$$R = \begin{cases} 0 & \text{if} \quad \ell = \infty \\ \frac{1}{\ell} & \text{if} \quad \ell \in \mathbb{R}\backslash\{0\} \\ \infty & \text{if} \quad \ell = 0 \end{cases}$$

4 Maclaurin and Taylor Series

Definition If the function f has a power series representation on the interval $(c-R, c+R)$, then the power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= \frac{f(c)}{0!} + \frac{f'(c)(x-c)}{1!} + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \cdots \end{aligned}$$

is called the **Taylor Series of the function f about c.** In the particular case that $c = 0$, then Taylor series of f is usually called the Maclaurin series of f :

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots \end{aligned}$$

4.1 Things must be Memorized

1. For any $x \in \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} x^n/n!$$

2. For any $x \in \mathbb{R}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$$

3. For any $x \in \mathbb{R}$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)!$$

4. The Binomial Theorem: for any $\alpha \in \mathbb{R}$ and x such that $|x| < 1$

$$\begin{aligned} (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \cdots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n + \cdots \\ &= \sum_{n\geq 0} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n \end{aligned}$$

5. From 4. we have, for any x such that $|x| < 1$,

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots = \sum_{n=0}^{\infty} (-x)^n \end{aligned} \tag{1}$$

6. For any x such that $|x| < 1$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots = \sum_{n\geq 1}(-1)^{n+1}\frac{x^n}{n}$$

5 Things Must be Mentioned about Series

5.1 $\sum a_n$ converges $\iff$ $\sum a_n^3$ converges

5.1.1 $\not\Rightarrow$

$\sum a_n$ converges **does not** imply $\sum a_n^3$ converges in general for non positive a_n. For an m, write as $m = 3n + k$ where $0 \leq k < 3$ and define $a_{3n+k} = \frac{b_k}{(n+1)^{1/3}}$ where $b_0 = 2$ and $b_1, b_2 = -1$. Then a_n^3 in general looks like

$$\frac{8}{1}, \frac{-1}{1}, \frac{-1}{1}, \frac{8}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{8}{3}, \frac{-1}{3}, \frac{-1}{3}, \ldots$$

which has partial sums $S_{3n} = 6\sum_{i=1}^{\infty} 1/i$ diverging.

5.1.2 $\not\Leftarrow$

$$\sum \frac{1}{k^3} \quad \textit{converges} \not\Rightarrow \sum \frac{1}{k} \quad \textit{converges}$$

6 Basic Integration Methods

6.1 Some Other Things

6.1.1 Hyperbolic Function

Definition

$$\sinh x \overset{\text{def}}{=} \frac{e^x - e^{-x}}{2} \qquad \cosh x \overset{\text{def}}{=} \frac{e^x + e^{-x}}{2} \tag{2}$$

6.1.2 Identities of Trigonometric Functions

Pythagorean Identities

$$\begin{aligned} \sin^2\theta + \cos^2\theta =& 1 \\ \tan^2 + 1 =& \sec^2\theta \\ 1 + \cot^2 =& \csc^2\theta \end{aligned} \tag{3}$$

6.1.3 Recite these Equations

For integrate is the inverse of the derivative, we just have to recite some basic rules about the derivative. Equation below request to be recited.

$$\begin{aligned} (\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} \\ (\arccos x)' &= -\frac{1}{\sqrt{1-x^2}} \\ (\arctan x)' &= \frac{1}{1+x^2} \\ (\tan x)' &= \sec^2 x \\ (\cot x)' &= -\csc^2 x \\ (\sinh x)' &= \cosh x \\ (\cosh x)' &= \sinh x \end{aligned} \tag{4}$$

6.2 Some Classic Integrations

The following equation have some interesting conclusion.

$$\begin{aligned} \int \sec x \,\mathrm{d}x &= \ln\left|\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right| + C \\ &= \ln\left|\frac{1+\sin x}{\cos x}\right| + C \\ &= \ln|\sec x + \tan x| + C \end{aligned} \tag{5}$$

7 Riemannn Integrable

Continuous[1] $\Rightarrow$ Riemann Integrable $\Rightarrow$ Boundness

[1] On closed set

$\Uparrow$
Monotone

8 Some Particular Functions

8.1 Weierstrass function

the Weierstrass function is an example of a real-valued function that is continuous everywhere but differentiable nowhere. In Weierstrass's original paper, the function was defined as a Fourier series:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1, b$ is a positive odd integer, and

$$ab > 1 + \frac{3}{2}\pi$$

8.2 Volterra's function

The function is defined by making use of the Smith Volterra Cantor set and "copies" of the function defined by $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.

- V is differentiable everywhere
- The derivative V' is bounded everywhere
- The derivative is **not Riemann-integrable**.