

Real Analysis and Calculus Pennote

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1 Sequence

Axiom of Completeness

A nonempty set of real numbers that is bounded above has a least upper bound, i.e. supremums of bounded sets are real numbers.

Monotone Subsequence

Every sequence contains a monotonic subsequence.

Monotone Convergence Theorem

Suppose that $\{x_n\}$ is a monotonic sequence. Then, $\{x_n\}$ is convergent if and only if $\{x_n\}$ is bounded.

Bolzano-Weierstrass

Every bounded sequence contains a convergent subsequence.

Cauchy Convergence Criterion

A sequence $\{x_n\}$ is convergent iff for each $\varepsilon > 0$ there exists an integer N with the property that

$$|x_n - x_m| \leq \varepsilon$$

for all $n \geq N$ and $m \geq N$.

Completeness Axiom of reals
 \implies Monotone Convergence Theorem
 \implies Bolzano-Weierstrass Theorem
 \implies Cauchy Convergence Criterion

2 Series

2.1 Conditions of convergence

If $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \rightarrow 0$ as $k \rightarrow \infty$.

If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then so does the series $\sum_{k=1}^{\infty} a_k$.

A series $\sum_{k=1}^{\infty} a_k$ is said to be absolutely convergent if $\sum_{k=1}^{\infty} |a_k|$ converges.

In order to distinguish convergence from absolute convergence, we refer to the former as non-absolutely convergence, or conditional convergence.

A series $\sum_{k=1}^{\infty} a_k$ is said to be non-absolutely (or conditional) convergent if it converges but the series $\sum_{k=1}^{\infty} |a_k|$ diverges.

2.2 Properties of Convergent Series

2.2.1 Dirichlet's Theorem: rearrangements of series

Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers. If $\{b_n\}$ is any rearrangement of $\{a_n\}$ then

1. $\sum_{n=1}^{\infty} b_n$ is an absolutely convergent series.
2. $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

2.2.2 Conditional convergence

If the series $\sum_{n=1}^{\infty} a_n$ converges but does *not* converge absolutely (i.e. $\sum_{n=1}^{\infty} a_n$ is conditionally convergent) and $\gamma \in \mathbb{R}$ is any real number, then there exists a rearrangement $\{b_n\}$ of the sequence $\{a_n\}$ so that

$$\sum_{n=1}^{\infty} b_n = \gamma$$

2.3 Converge Tests

2.3.1 Null test

If the terms of the series $\sum_{k=1}^{\infty} a_k$ do not converge to zero, then the series diverges.

2.3.2 Comparison test

Given two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ such that $0 \leq a_k \leq b_k$ for all k .

1. If the larger series converges, then so does the smaller series.
2. If the smaller series diverges, then so does the larger series.

2.3.3 Ratio test

If terms of the series $\sum_{k=1}^{\infty} a_k$ are all positive and the ratios

$$\lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} < 1$$

then the series is convergent.

2.3.4 Root test

If terms of the series $\sum_{k=1}^{\infty} a_k$ are all nonnegative and the roots

$$\lim_{k \rightarrow \infty} \sqrt[k]{a_k} < 1$$

then the series is convergent.

2.3.5 Integral test

Let f be a nonnegative decreasing function on $[1, \infty)$. Then

$$\lim_{X \rightarrow \infty} \int_1^X f(x) dx$$

converges if and only if the series $\sum_{k=1}^{\infty} f(k)$ converges.

Proof since f is decreasing we have

$$\int_k^{k+1} f(x) dx \leq f(k) \leq \int_{k-1}^k f(x) dx$$

Thus

$$\int_1^{n+1} f(x) dx \leq \sum_{k=1}^n f(k) \leq f(1) + \int_1^n f(x) dx$$

The series converges if and only if the partial sums are bounded.

2.3.6 Alternating Series test

The series

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k$$

where the terms alternate in sign, converges if the sequence $\{a_k\}$ decreases monotonically to zero.

3 Power Series

A power series

$$f(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} a_n x^n, x \in S$$

where S will make sense.

3.1 Radius of Convergence

Given a power series $\sum_{n=0}^{\infty} a_n x^n$, either it converges absolutely for all $x \in \mathbb{R}$, or there exists $R \in [0, \infty)$ such that

- (1) it converges absolutely when $|x| < R$
- (2) it diverges when $|x| > R$.

Remark We can restate the theorem as

$$(-R, R) \subseteq S \subseteq [-R, R]$$

and the power series converges absolutely in $(-R, R)$. In particular we see that S is always an interval.

3.2 Convergence Test (of Power Series)

3.2.1 Ratio test (of Power Series)

Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

Suppose that

$$\frac{|a_{n+1}|}{|a_n|} \rightarrow \ell, \quad \text{as } n \rightarrow \infty$$

Then

$$R = \begin{cases} 0 & \text{if } \ell = \infty \\ \frac{1}{\ell} & \text{if } \ell \in \mathbb{R} \setminus \{0\} \\ \infty & \text{if } \ell = 0 \end{cases}$$

3.2.2 Root test (of Power Series)

Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

Suppose that

$$|a_n|^{\frac{1}{n}} \rightarrow \ell, \quad \text{as } n \rightarrow \infty$$

Then

$$R = \begin{cases} 0 & \text{if } \ell = \infty \\ \frac{1}{\ell} & \text{if } \ell \in \mathbb{R} \setminus \{0\} \\ \infty & \text{if } \ell = 0 \end{cases}$$

4 Maclaurin and Taylor Series

Definition If the function f has a power series representation on the interval $(c - R, c + R)$, then the power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n \\ &= \frac{f(c)}{0!} + \frac{f'(c)(x - c)}{1!} + \frac{f''(c)(x - c)^2}{2!} + \frac{f'''(c)(x - c)^3}{3!} + \dots \end{aligned}$$

is called the **Taylor Series of the function f about c** . In the particular case that $c = 0$, then Taylor series of f is usually called the Maclaurin series of f :

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots \end{aligned}$$

4.1 Things must be Memorized

1. For any $x \in \mathbb{R}$

$$e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} x^n / n!$$

2. For any $x \in \mathbb{R}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n+1} / (2n+1)!$$

3. For any $x \in \mathbb{R}$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots = \sum_{n=0}^{\infty} (-1)^n x^{2n} / (2n)!$$

4. The Binomial Theorem: for any $\alpha \in \mathbb{R}$ and x such that $|x| < 1$

$$\begin{aligned} (1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n + \dots \\ &= \sum_{n \geq 0} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n \end{aligned}$$

5. From 4. we have, for any x such that $|x| < 1$,

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots + x^n + \dots = \sum_{n=0}^{\infty} x^n \\ \frac{1}{1+x} &= 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots = \sum_{n=0}^{\infty} (-x)^n \end{aligned} \tag{1}$$

6. For any x such that $|x| < 1$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \frac{x^n}{n} + \cdots = \sum_{n \geq 1} (-1)^{n+1} \frac{x^n}{n}$$

5 Things Must be Mentioned about Series

5.1 $\sum a_n$ converges $\iff \sum a_n^3$ converges

5.1.1 \nRightarrow

$\sum a_n$ converges **does not** imply $\sum a_n^3$ converges in general for non positive a_n . For an m , write as $m = 3n + k$ where $0 \leq k < 3$ and define $a_{3n+k} = \frac{b_k}{(n+1)^{1/3}}$ where $b_0 = 2$ and $b_1, b_2 = -1$. Then a_n^3 in general looks like

$$\frac{8}{1}, \frac{-1}{1}, \frac{-1}{1}, \frac{8}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{8}{3}, \frac{-1}{3}, \frac{-1}{3}, \dots$$

which has partial sums $S_{3n} = 6 \sum_{i=1}^{\infty} 1/i$ diverging.

5.1.2 \nRightarrow

$$\sum \frac{1}{k^3} \text{ converges } \nRightarrow \sum \frac{1}{k} \text{ converges}$$

6 Basic Integration Methods

6.1 Some Other Things

6.1.1 Hyperbolic Function

Definition

$$\sinh x \stackrel{\text{def}}{=} \frac{e^x - e^{-x}}{2} \quad \cosh x \stackrel{\text{def}}{=} \frac{e^x + e^{-x}}{2} \quad (2)$$

6.1.2 Identities of Trigonometric Functions

Pythagorean Identities

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \tan^2 \theta + 1 &= \sec^2 \theta \\ 1 + \cot^2 \theta &= \csc^2 \theta \end{aligned} \quad (3)$$

6.1.3 Recite these Equations

For integrate is the inverse of the derivative, we just have to recite some basic rules about the derivative. Equation below request to be recited.

$$\begin{aligned} (\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} \\ (\arccos x)' &= -\frac{1}{\sqrt{1-x^2}} \\ (\arctan x)' &= \frac{1}{1+x^2} \\ (\tan x)' &= \sec^2 x \\ (\cot x)' &= -\csc^2 x \\ (\sinh x)' &= \cosh x \\ (\cosh x)' &= \sinh x \end{aligned} \quad (4)$$

6.2 Some Classic Integrations

The following equation have some interesting conclusion.

$$\begin{aligned} \int \sec x \, dx &= \ln \left| \frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}} \right| + C \\ &= \ln \left| \frac{1 + \sin x}{\cos x} \right| + C \\ &= \ln |\sec x + \tan x| + C \end{aligned} \quad (5)$$

7 Riemannn Integrable

Continuous¹ \Rightarrow Riemann Integrable \Rightarrow Boundness

¹On closed set

\uparrow
Monotone

8 Some Particular Functions

8.1 Weierstrass function

the Weierstrass function is an example of a real-valued function that is continuous everywhere but differentiable nowhere. In Weierstrass's original paper, the function was defined as a Fourier series:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1$, b is a positive odd integer, and

$$ab > 1 + \frac{3}{2}\pi$$

8.2 Volterra's function

The function is defined by making use of the Smith Volterra Cantor set and "copies" of the function defined by $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.

- V is differentiable everywhere
- The derivative V' is bounded everywhere
- The derivative is **not Riemann-integrable**.