特殊関数・初等関数の計算メモ

目次

所

２者間の勝敗数からレーティング差の区間推定を行う計算を、二項累積分布と正則ベータ関数の関係を用いようとした際のまとめ書きです。

指数関数・対数関数

- ja.wikipedia: 指数関数
- ja.wikipedia: 底に関する指数函数
- en.wikipedia: Exponential function
- ja.wikipedia: 冪乗
- en.wikipedia: Exponentiation
- ja.wikipedia: 対数
- en.wikipedia: Logarithm
- ja.wikipedia: 自然対数
- en.wikipedia: Natural logarithm

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
&= 1 + x\left(1 + \frac{x}{2}\left(1 + \frac{x}{3}\left(1 + \frac{x}{4}\left(1 + \cdots\right)\right)\right)\right) \\
e^x - 1 &= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
&= x\left(1 + \frac{x}{2}\left(1 + \frac{x}{3}\left(1 + \frac{x}{4}\left(1 + \cdots\right)\right)\right)\right) \\
e^{2x} - 1 &= (e^x - 1)^2 + 2(e^x - 1) \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \qquad (-1 < x \le 1) \\
\ln(1+x) &= \ln\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{1}{3}y^3 + \frac{1}{5}y^5 + \frac{1}{7}y^7 + \cdots\right) \qquad \left(-1 < y = \frac{x}{x+2} < 1\right) \\
\ln(2) &= 3\ \ln\left(1 + \frac{1}{80}\right) + 5\ \ln\left(1 + \frac{1}{24}\right) + 7\ \ln\left(1 + \frac{1}{15}\right) \\
&= 3\ \ln\left(\frac{1 + \frac{1}{161}}{1 - \frac{1}{161}}\right) + 5\ln\left(\frac{1 + \frac{1}{49}}{1 - \frac{1}{49}}\right) + 7\ \ln\left(\frac{1 + \frac{1}{31}}{1 - \frac{1}{31}}\right) \\
\ln\left(a \cdot 2^n\right) &= \ln(a) + \ln\left(2^n\right) = \ln(a) + n\ \ln(2) \\
\ln\left(e^a + e^b\right) &= a + \ln\left(1 + e^{b-a}\right) = b + \ln\left(1 + e^{a-b}\right) \\
\ln\left(e^a - e^b\right) &= a + \ln\left(1 - e^{b-a}\right)
\end{aligned}
$$

ロジット

- ja.wikipedia: ロジット
- en.wikipedia: Logit
- ja.wikipedia: ロジスティック方程式

- en.wikipedia: Logistic function

対数の底は 1 より大きければ何でも良いが、ここでは特に明示のない限り自然対数の底 e を用いることとする。

$$
\begin{aligned}
\operatorname{logit}(p) &= \log\left(\frac{p}{1-p}\right) = \log(p) - \log(1-p) = -\log\left(\frac{1}{p} - 1\right) \\
\operatorname{logit}^{-1}(\alpha) &= \operatorname{logistic}(\alpha) = \frac{1}{1+\exp(-\alpha)} = \frac{\exp(\alpha)}{\exp(\alpha)+1} \\
\log(R) &= \log\left(\frac{p_1/(1-p_1)}{p_2/(1-p_2)}\right) = \log\left(\frac{p_1}{1-p_1}\right) - \log\left(\frac{p_2}{1-p_2}\right) = \operatorname{logit}(p_1) - \operatorname{logit}(p_2)
\end{aligned}
$$

イロレーティング

- ja.wikipedia: イロレーティング
- en.wikipedia: Elo rating system

ロジスティック関数の一種を用いて定義されており、ロジットとイロレーティングの値は一次関数を用いて変換できる。

$$
\begin{aligned}
E_A &= \frac{1}{1+10^{(R_B-R_A)/400}} = \frac{Q_A}{Q_A+Q_B} \\
E_B &= \frac{1}{1+10^{(R_A-R_B)/400}} = \frac{Q_B}{Q_A+Q_B} \\
E_A + E_B &= 1 \\
Q_A &= 10^{R_A/400} \\
Q_B &= 10^{R_B/400}
\end{aligned}
$$

三角関数

- ja.wikipedia: 三角関数
- en.wikipedia: Trigonometric functions

$$
\begin{aligned}
\sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad \text{for all } z \\
\cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \quad \text{for all } z \\
\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\
\cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \\
\sin(2x) &= 2 \sin x \cos x \\
\cos(2x) &= \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - \sin^2 x \\
\sin(3x) &= 3 \sin x - 4 \sin^3 x \\
\cos(3x) &= 4 \cos^3 x - 3 \cos x \\
\sin^2 \frac{x}{2} &= \frac{1 - \cos x}{2} \\
\cos^2 \frac{x}{2} &= \frac{1 + \cos x}{2} \\
\tan^2 \frac{x}{2} &= \frac{1 - \cos x}{1 + \cos x}
\end{aligned}
$$

階乗

- ja.wikipedia: 階乗
- en.wikipedia: Factorial

$$
\begin{aligned}
&n! = \prod_{k=1}^{n} k = n(n-1)(n-2)\cdots 2 \cdot 1 \\
&0! = 1 \\
&1! = 1 \\
&2! = 2 \\
&3! = 6 \\
&4! = 24 \\
&5! = 120 \\
&6! = 720 \\
&7! = 5040
\end{aligned}
$$

二重階乗

- 二重階乗 (double fractorial) / 半階乗 (semifactorial)
- ja.wikipedia: 二重階乗
- en.wikipedia: Double factorial

$$
\begin{aligned}
n!! &= \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n-2k) = n(n-2)(n-4)\cdots \\
n!! &= \prod_{k=1}^{\frac{n}{2}} (2k) = n(n-2)(n-4)\cdots 4 \cdot 2 \qquad (n \text{ is even}) \\
0!! &= 1 \\
2!! &= 2 \\
4!! &= 8 \\
6!! &= 48 \\
8!! &= 384 \\
10!! &= 3840 \\
12!! &= 46080 \\
14!! &= 645120 \\
n!! &= \prod_{k=1}^{\frac{n+1}{2}} (2k-1) = n(n-2)(n-4)\cdots 3 \cdot 1 \qquad (n \text{ is odd}) \\
1!! &= 1 \\
3!! &= 3 \\
5!! &= 15 \\
7!! &= 105 \\
9!! &= 945 \\
11!! &= 10395 \\
13!! &= 135135 \\
15!! &= 2027025
\end{aligned}
$$

ガンマ関数

- ja.wikipedia: ガンマ関数
- en.wikipedia: Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t}\, \mathrm{d}t \qquad (\Re z > 0)$$
$$\Gamma(z+1) = z\,\Gamma(z)$$
$$\Gamma\left(-\frac{3}{2}\right) = \frac{4\sqrt{\pi}}{3} \approx 2.363$$
$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi} \approx -3.545$$
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \approx 1.772$$
$$\Gamma(1) = 0! = 1$$
$$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \approx 0.886$$
$$\Gamma(2) = 1! = 1$$
$$\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4} \approx 1.329$$
$$\Gamma(3) = 2! = 2$$
$$\Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8} \approx 3.323$$
$$\Gamma(4) = 3! = 6$$
$$\Gamma(n+1) = n!$$
$$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \qquad (n \geq 1)$$
$$\Gamma\left(\frac{1}{2}-n\right) = \frac{(-2)^n}{(2n-1)!!}\sqrt{\pi} \qquad (n \geq 1)$$
$$\Gamma(z)\,\Gamma(1-z) = -z\,\Gamma(z)\,\Gamma(-z) = \frac{\pi}{\sin \pi z}$$

Lanczos 近似

- Lanczos 近似: en.wikipedia: Lanczos approximation
- パラメータ算出: https://mrob.com/pub/ries/lanczos-gamma.html

$$z! = \Gamma(z+1) \approx \sqrt{2\pi}\left(z+g+\frac{1}{2}\right)^{z+\frac{1}{2}} e^{-\left(z+g+\frac{1}{2}\right)} A_g(z)$$
$$A_g(z) \approx c_0 + \sum_{k=1}^{N-1} \frac{c_k}{z+k}$$

- $g = 5, N = 7$

$$\begin{aligned}
c_0 &= +1.000000000190015\\
c_1 &= +76.18009172947146\\
c_2 &= -86.50532032941677\\
c_3 &= +24.01409824083091\\
c_4 &= -1.231739572450155\\
c_5 &= +0.1208650973866179E-2\\
c_6 &= -0.5395239384953E-5
\end{aligned}$$

- $g = 7, N = 9$

$$
\begin{aligned}
c_0 &= +0.99999999999980993227684700473478 \\
c_1 &= +676.520368121885098567009190444019 \\
c_2 &= -1259.13921672240287047156078755283 \\
c_3 &= +771.3234287776530788486528258894 \\
c_4 &= -176.61502916214059906584551354 \\
c_5 &= +12.507343278686904814458936853 \\
c_6 &= -0.13857109526572011689554707 \\
c_7 &= +9.984369578019570859563E-6 \\
c_8 &= +1.50563273514931155834E-7
\end{aligned}
$$

- $g = 9, N = 11$

$$
\begin{aligned}
c_0 &= +1.000000000000000174663 \\
c_1 &= +5716.400188274341379136 \\
c_2 &= -14815.30426768413909044 \\
c_3 &= +14291.49277657478554025 \\
c_4 &= -6348.160217641458813289 \\
c_5 &= +1301.608286058321874105 \\
c_6 &= -108.1767053514369634679 \\
c_7 &= +2.605696505611755827729 \\
c_8 &= -0.7423452510201416151527E-2 \\
c_9 &= +0.5384136432509564062961E-7 \\
c_{10} &= -0.4023533141268236372067E-8
\end{aligned}
$$

- $g = 607/128 = 4.7421875, N = 15$

$$
\begin{aligned}
c_0 &= +0.99999999999999709182 \\
c_1 &= +57.156235665862923517 \\
c_2 &= -59.597960355475491248 \\
c_3 &= +14.136097974741747174 \\
c_4 &= -0.49191381609762019978 \\
c_5 &= +0.33994649984811888699E-4 \\
c_6 &= +0.46523628927048575665E-4 \\
c_7 &= -0.98374475304879564677E-4 \\
c_8 &= +0.15808870322491248884E-3 \\
c_9 &= -0.21026444172410488319E-3 \\
c_{10} &= +0.21743961811521264320E-3 \\
c_{11} &= -0.16431810653676389022E-3 \\
c_{12} &= +0.84418223983852743293E-4 \\
c_{13} &= -0.26190838401581408670E-4 \\
c_{14} &= +0.36899182659531622704E-5
\end{aligned}
$$

大浦による実ガンマ関数近似

- 大浦拓哉, ガンマ関数および誤差関数の初等関数近似とその最適化 (ja) (en)

$$\Gamma(x) \approx \exp((x-0.5)\ln(x+V)-x)$$
$$\cdot((\cdots(A_n/(x+B_n)+\cdots A_1)/(x+B_1)+A_0)/x+A_r) \qquad (0 < x < \infty)$$

- $N=2$ のとき、相対誤差の上限 $=3.11289669E-8$

$$\begin{aligned}
V &= 2.102394798991390E+0\\
A_r &= 3.062185443705942E-1\\
A_0 &= 1.024166094985555E+0\\
A_1 &= 4.258010456317367E-1\\
B_1 &= 1.000008131602802E+0
\end{aligned}$$

- $N=6$ のとき、相対誤差の上限 $=2.09144255E-18$

$$\begin{aligned}
V &= 6.09750757539068576094375 58E+0\\
A_r &= 5.636065618975606496797 7564E-3\\
A_0 &= 1.224259773268799178464 5973E-1\\
A_1 &= 8.513708131650341831241 1656E-1\\
A_2 &= 2.250230475356181683669 5856E+0\\
A_3 &= 2.096295535389499773386 9983E+0\\
A_4 &= 5.021972270339209072588 4168E-1\\
A_5 &= 1.124058265716540738343 7999E-2\\
B_1 &= 1.000000000000655324317 0562E+0\\
B_2 &= 1.999999999620102305806 5171E+0\\
B_3 &= 3.000000046726524145843 1618E+0\\
B_4 &= 3.999996630000750893209 7016E+0\\
B_5 &= 5.000358988483192554161 3237E+0
\end{aligned}$$

- $N=13$ のとき、相対誤差の上限 $=8.43420741E-37$

$$
\begin{aligned}
V &= 1.3578122000703946473976913605273518882656614E+1\\
A_r &= 3.1782384299734898421289539143998119380977134 7E-6\\
A_0 &= 3.1482070283349300354582623623908339457199567 1E-4\\
A_1 &= 1.2793741608722984500693458490473661859807057 2E-2\\
A_2 &= 2.7874830306029980874434569055259616605925149 3E-1\\
A_3 &= 3.5748763958228570180758258557929027133608909 9E+0\\
A_4 &= 2.7927280421563325015666935178375281217565097 2E+1\\
A_5 &= 1.3321384650379738989446885832268784754972611 4E+2\\
A_6 &= 3.7950405192465422312792634449147935783985772 2E+2\\
A_7 &= 6.1562149993028259463346808196235292341274118 4E+2\\
A_8 &= 5.2400400869100650701118261358974985117143157 6E+2\\
A_9 &= 2.0418766202023711876168179075996496479906873 6E+2\\
A_{10} &= 2.8645619772729108683191342647193554200542230 7E+1\\
A_{11} &= 8.9507210141338984737305834751291040397994777 3E-1\\
A_{12} &= 1.8410863315761265630602733481713520754486215 7E-3\\
B_1 &= 9.9999999999999999999999999998291770674391866 49E-1\\
B_2 &= 2.0000000000000000000000000143725325109773686 179E+0\\
B_3 &= 2.9999999999999999999999878069870191969828857 942E+0\\
B_4 &= 4.0000000000000000000031822896056305388990947 492E+0\\
B_5 &= 4.9999999999999999603276569479688692879045960 6E+0\\
B_6 &= 6.0000000000000030034309156698097129603724912 1E+0\\
B_7 &= 6.9999999999998375247462698288225315905737578 8E+0\\
B_8 &= 8.0000000000071915518803021765161684809381068 9E+0\\
B_9 &= 8.9999999997000681861822653951282600848948454 0E+0\\
B_{10} &= 1.0000000014205005237309132429530491661283623 7E+1\\
B_{11} &= 1.0999998953920119680361242478373085333505653 4E+1\\
B_{12} &= 1.2000238108934194337280539725944422661290058 2E+1
\end{aligned}
$$

不完全ガンマ関数

- 不完全ガンマ関数 (Incomplete gamma functions)
- ja.wikipedia: 不完全ガンマ関数
- en.wikipedia: Incomplete gamma function
- boost.org: math toolkit/sf gamma/igamma.html #math toolkit.sf gamma.igamma.implementation

$$
\begin{aligned}
\gamma(a,x) &= \int_0^x t^{a-1}\, e^{-t} \mathrm{d}t \qquad \text{(lower incomplete gamma function)} \\
\Gamma(a,x) &= \int_x^\infty t^{a-1}\, e^{-t} \mathrm{d}t \qquad \text{(upper incomplete gamma function)} \\
\Gamma(a) &= \gamma(a,x) + \Gamma(a,x) \\
\gamma(a+1,x) &= a\,\gamma(a,x) - x^a\, e^{-x} \\
\Gamma(a+1,x) &= a\,\Gamma(a,x) + x^a\, e^{-x} \\
\gamma(a,0) &= 0 \\
\Gamma(a,0) &= \Gamma(a) \quad (\Re(a) > 0) \\
\gamma(a,x) &\to \Gamma(a) \quad (x \to \infty) \\
\Gamma(0,x) &= -\operatorname{Ei}(-x) \quad \text{ for } x > 0 \\
\Gamma(1/2,x) &= \sqrt{\pi}\,\operatorname{erfc}\left(\sqrt{x}\right) \\
\gamma(1/2,x) &= \sqrt{\pi}\,\operatorname{erf}\left(\sqrt{x}\right) \\
\Gamma(1,x) &= e^{-x} \\
\gamma(1,x) &= 1 - e^{-x}
\end{aligned}
$$

正則化ガンマ関数

- 正則化ガンマ関数 (Regularized Gamma functions)
- en.wikipedia: Incomplete gamma function #Regularized Gamma functions and Poisson random variables

$$
\begin{aligned}
P(a,x) &= \int_0^x \frac{t^{a-1}\, e^{-t}}{\Gamma(a)} \mathrm{d}t = \frac{\gamma(a,x)}{\Gamma(a)} = 1 - Q(a,x) \\
Q(a,x) &= \int_x^\infty \frac{t^{a-1}\, e^{-t}}{\Gamma(a)} \mathrm{d}t = \frac{\Gamma(a,x)}{\Gamma(a)} = 1 - P(a,x) \\
P(a,x) + Q(a,x) &= 1 \\
P(a,0) &= 0 \\
Q(a,0) &= 1 \\
P(a,x) &\to 1 \quad (x \to \infty) \\
Q(a,x) &\to 0 \quad (x \to \infty)
\end{aligned}
$$

誤差関数

- ja.wikipedia: 誤差関数
- en.wikipedia: Error function
- 誤差関数 (error function)

$$
\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t
$$

- 相補誤差関数 (complementary error function)

$$
\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \mathrm{d}t = e^{-x^2} \operatorname{erfcx}(x)
$$

- スケーリング相補誤差関数 (scaled complementary error function)

$$\mathrm{erfcx}(x) = e^{x^2}\,\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} e^{x^2} \int_x^{\infty} e^{-t^2} \mathrm{d}t$$

大浦による実誤差関数近似

- 大浦拓哉, ガンマ関数および誤差関数の初等関数近似とその最適化 (ja) (en)

$$\mathrm{erf}(x) \approx \sum_{k=0}^{N-1} A_k \cdot x^{2k+1} \qquad (|x| \le 0.125)$$

- $N = 3$ のとき、相対誤差の上限 $= 3.1833E-9$

$$\begin{aligned}
A_0 &= +1.12837916390979E+0 \\
A_1 &= -3.76122716908058E-1 \\
A_2 &= +1.12210466658717E-1
\end{aligned}$$

- $N = 6$ のとき、相対誤差の上限 $= 8.5080E-19$

$$\begin{aligned}
A_0 &= +1.1283791670955125730449 43E+0 \\
A_1 &= -3.761263890318336015429662E-1 \\
A_2 &= +1.128379167066213010234749E-1 \\
A_3 &= -2.686616984476423776951305E-2 \\
A_4 &= +5.223878776856181012778436E-3 \\
A_5 &= -8.492024351869184700200701E-4
\end{aligned}$$

- $N = 11$ のとき、相対誤差の上限 $= 7.8857E-36$

$$\begin{aligned}
A_0 &= +1.128379167095512573896158903121545163800 03E+0 \\
A_1 &= -3.761263890318375246320529677070595464411 35E-1 \\
A_2 &= +1.128379167095512573896158899993588607653 93E-1 \\
A_3 &= -2.686617064513125175943204251496143231345 75E-2 \\
A_4 &= +5.223977625442187841951921590429374797334 06E-3 \\
A_5 &= -8.548327023450852354839323613401674725697 59E-4 \\
A_6 &= +1.205533298178877468761642502464909068793 64E-4 \\
A_7 &= -1.492565035734241848612754675986476535074 84E-5 \\
A_8 &= +1.646211354855869374637843152371971522895 16E-6 \\
A_9 &= -1.636545469246142882201252445966852025016 82E-7 \\
A_{10} &= +1.470196665080928459453403290898819569177 98E-8
\end{aligned}$$

大浦による実相補誤差関数近似

- 大浦拓哉, ガンマ関数および誤差関数の初等関数近似とその最適化 (ja) (en)

$$
\mathrm{erfc}(x) \approx E(x) + x \exp(-x^2) \sum_{k=0}^{N-1} \frac{A_k}{x^2 + B_k} \qquad (-\infty < x < \infty)
$$

$$
E(x) = \begin{cases} 2/(1+\exp(\alpha x)) & (x < \beta) \\ 0 & (x \geq \beta) \end{cases}
$$

- $N = 4$ のとき、相対誤差の上限 $= 7.07284210E-9$

$$
\begin{aligned}
\alpha &= 9.2088871045460211E+0 \\
\beta &= 5.0725473171624327E+0 \\
A_0 &= 3.8664221739686797E-1 \\
A_1 &= 1.5243017675919252E-1 \\
A_2 &= 2.3814912488843075E-2 \\
A_3 &= 1.3022729124288807E-3 \\
B_0 &= 1.1638196508217325E-1 \\
B_1 &= 1.0475380173789841E+0 \\
B_2 &= 2.9213215631713289E+0 \\
B_3 &= 6.0260843416158831E+0
\end{aligned}
$$

- $N = 8$ のとき、相対誤差の上限 $= 3.63856888E-17$

$$
\begin{aligned}
\alpha &= 1.26974899965115683898581 1E+1 \\
\beta &= 6.10399733098688199433433 8E+0 \\
A_0 &= 2.96316885199227377833635 7E-1 \\
A_1 &= 1.81581125134637069964095 5E-1 \\
A_2 &= 6.81866451424939493014828 2E-2 \\
A_3 &= 1.56907543161966709037809 2E-2 \\
A_4 &= 2.21290116681517572829152 2E-3 \\
A_5 &= 1.91395813098742864379169 7E-4 \\
A_6 &= 9.71013284010551623443484 1E-6 \\
A_7 &= 1.66642447174307752746801 0E-7 \\
B_0 &= 6.12158644495538758054929 4E-2 \\
B_1 &= 5.50942780056002084893683 1E-1 \\
B_2 &= 1.53039662058770396952752 7E+0 \\
B_3 &= 2.99957952311300634046573 9E+0 \\
B_4 &= 4.95867777128246701145053 3E+0 \\
B_5 &= 7.41471251099335406814757 5E+0 \\
B_6 &= 1.04765104356545237590143 5E+1 \\
B_7 &= 1.48455557345597956664618 5E+1
\end{aligned}
$$

- $N = 17$ のとき、相対誤差の上限 $= 7.45563242E-36$

$$
\begin{aligned}
\alpha &= 1.8296570980424689847157930974106706834567989E+1 \\
\beta &= 8.9588287394342176848213494031807385567358833E+0 \\
A_0 &= 2.1226887418241545314975570224238841543405658E-1 \\
A_1 &= 1.6766968820663231170102487414107148109808599E-1 \\
A_2 &= 1.0461429607758480243524362040994242136794358E-1 \\
A_3 &= 5.1557963860512142911764627378588661741526705E-2 \\
A_4 &= 2.0070986488528139460346647533434778000221814E-2 \\
A_5 &= 6.1717726506718148117513762897928828533989685E-3 \\
A_6 &= 1.4990611906920858646769185063310410160420122E-3 \\
A_7 &= 2.8760540416705806615617926157307107830366204E-4 \\
A_8 &= 4.3585593590380741491013549969419946961138883E-5 \\
A_9 &= 5.2174364856655433775383935118049845471172446E-6 \\
A_{10} &= 4.9333351722974670085736982894474122277208033E-7 \\
A_{11} &= 3.6846914376723888190666722894010079934846267E-8 \\
A_{12} &= 2.1729515092764086499231043367920037214553663E-9 \\
A_{13} &= 9.9870022842895735663712411206346261651079743E-11 \\
A_{14} &= 3.1775163189596489863458236395414830880404471E-12 \\
A_{15} &= 4.5657943993597540327708145643160878201453992E-14 \\
A_{16} &= 1.1940964427370412648558173558044106203476395E-16 \\
B_0 &= 2.9482230394292049252878077330764031336911886E-2 \\
B_1 &= 2.6534007354862844327590269604581049763591558E-1 \\
B_2 &= 7.3705575985730123132195272141160572531634811E-1 \\
B_3 &= 1.4446292893203104133929687855854497895783377E+0 \\
B_4 &= 2.3880606619376559912235584857800710490361124E+0 \\
B_5 &= 3.5673498777093386979273977202889759344704102E+0 \\
B_6 &= 4.9824969366355296879760903991854492761727217E+0 \\
B_7 &= 6.6335018387405633238409855625402006222862354E+0 \\
B_8 &= 8.5203645862651289478197632097553870198987021E+0 \\
B_9 &= 1.0643085317662274170216548777166393329207188E+1 \\
B_{10} &= 1.3001669850030489723387515813223808078289803E+1 \\
B_{11} &= 1.5596282517377690399267249728222735969695585E+1 \\
B_{12} &= 1.8429903207271748464995406180854691071748545E+1 \\
B_{13} &= 2.1533907893494593530979123915138686106579160E+1 \\
B_{14} &= 2.5076752889217226137869837117288885076749247E+1 \\
B_{15} &= 2.9515380437412601845256918753602002410389178E+1 \\
B_{16} &= 3.5792848810704122499184545805923520657604330E+1
\end{aligned}
$$

ベータ関数

- ja.wikipedia: ベータ関数
- en.wikipedia: Beta function

$$
\begin{aligned}
\mathrm{B}(x,y) &= \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\mathrm{d}t \\
\mathrm{B}(x,y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \\
\mathrm{B}(x,y) &= \mathrm{B}(y,x) \\
x\ \mathrm{B}(x,y+1) &= y\ \mathrm{B}(x+1,y) \\
\mathrm{B}(x,y) &= \mathrm{B}(x+1,y) + \mathrm{B}(x,y+1) \\
\mathrm{B}(x+1,y) &= \mathrm{B}(x,y)\cdot\frac{x}{x+y} \\
\mathrm{B}(x,y+1) &= \mathrm{B}(x,y)\cdot\frac{y}{x+y} \\
\mathrm{B}(x,y)\cdot\mathrm{B}(x+y,1-y) &= \frac{\pi}{x\sin(\pi y)} \\
\mathrm{B}(x,1-x) &= \frac{\pi}{\sin(\pi x)} \\
\mathrm{B}(1,x) &= \frac{1}{x}
\end{aligned}
$$

不完全ベータ関数

- 不完全ベータ関数 (incomplete beta function)
- ja.wikipedia: 不完全ベータ関数
- en.wikipedia: Beta function #Incomplete beta function

$$
\mathrm{B}_z(a,b) = \int_0^z t^{a-1}\,(1-t)^{b-1}\,\mathrm{d}t \qquad (0 \le \Re z \le 1)
$$

正則ベータ関数

- 正則ベータ関数 (regularized beta function) / 正則化不完全ベータ関数 (regularized incomplete beta function)
- Incomplete Beta Functions-Implementation
- Significant Digit Computation of the Incomplete Beta Function Ratios. AR DiDonato, AH Morris Jr-1988-dtic.mil
- Armido R. Didonato and Alfred H. Morris, Jr.. 1992. Algorithm 708: Significant digit computation of the incomplete beta function ratios. ACM Trans. Math. Softw. 18, 3(September 1992), 360-373.

$$
\begin{aligned}
I_z(a,b) &= \frac{\mathrm{B}_z(a,b)}{\mathrm{B}(a,b)} \qquad (0 \le \Re z \le 1) \\
p &= 1-q = \frac{a}{a+b} \\
q &= 1-p = \frac{b}{a+b} \\
y &= 1-x \\
I_0(a,b) &= 0 \\
I_1(a,b) &= 1 \\
I_x(a,1) &= x^a \\
I_x(1,b) &= 1-y^b \\
I_x(a,b) &= 1-I_y(b,a) \\
I_x(a,b) &= I_x(a+n,b) + \frac{x^a y^b}{aB(a,b)} \sum_{i=0}^{n-1} d_i x^i \\
I_x(a+n,b) &= I_x(a,b) - \frac{x^a y^b}{aB(a,b)} \sum_{i=0}^{n-1} d_i x^i \\
I_x(1+n,b) &= 1-y^b \left(1 + bx \sum_{i=0}^{n-1} d_i x^i\right) \\
I_x(a,b) &= I_x(a,b+n) - \frac{x^a y^b}{bB(a,b)} \sum_{j=0}^{n-1} d'_j y^j \\
I_x(a,b+n) &= I_x(a,b) + \frac{x^a y^b}{bB(a,b)} \sum_{j=0}^{n-1} d'_j y^j \\
I_x(a,1+n) &= x^a \left(1 + ay \sum_{j=0}^{n-1} d'_j y^j\right) \\
d_{i+1} &= \frac{a+b+i}{a+1+i} d_i,\ d_0 = 1 \\
d'_{j+1} &= \frac{a+b+j}{b+1+j} d'_j,\ d'_0 = 1
\end{aligned}
$$

BPSER

$$
I_x(a,b) = \frac{x^a}{a\,\mathrm{B}(a,b)} \left(1 + a \sum_{j=1}^{\infty} \frac{(1-b)(2-b)\cdots(j-b)}{j!(a+j)} x^j\right) \qquad (b \le 1,\ x \le 0.7) \mid (bx \le 0.7)
$$

BGRAT

$$
\begin{aligned}
I_x(a,b) &\approx M \sum_{n=0}^{\infty} p_n J_n(b,u) \qquad (a>b) \\
T &= a + \frac{b-1}{2} \\
u &= -T \ln x \\
H(c,u) &= \frac{e^{-u} u^c}{\Gamma(c)} \\
M &= \frac{H(b,u)\Gamma(a+b)}{\Gamma(a) T^b} \\
J_n(b,u) &= \left(\frac{u}{2T}\right)^{2n} \frac{Q(b+2n,u)}{H(b+2n,u)} = \left(\frac{\ln x}{2}\right)^{2n} \frac{Q(b+2n,u)}{H(b+2n,u)} \\
Q(b,u) &= \int_u^{\infty} \frac{e^{-t} t^{b-1}}{\Gamma(b)} \mathrm{d}t \qquad \text{(incomplete gamma function)} \\
J_{n+1}(b,u) &= \frac{(b+2n)(b+2n+1)}{4T^2} J_n(b,u) + \frac{u+b+2n+1}{4T^2} \left(\frac{\ln x}{2}\right)^{2n} \\
J_0(b,u) &= \frac{Q(b,u)}{H(b,u)}
\end{aligned}
$$

BFRAC(1)

$$
\begin{aligned}
I_x(a,b) &= \frac{x^a y^b}{\mathrm{B}(a,b)} \left(\frac{\alpha_1}{\beta_1 +} \frac{\alpha_2}{\beta_2 +} \cdots \right) \qquad \left(x \le p \equiv \frac{a}{a+b} \right) \\
y &= 1 - x \\
\alpha_1 &= 1 \\
\beta_1 &= \frac{a}{a+1}(\lambda + 1) \\
\lambda &= a - (a+b)x = (a+b)(p-x) \\
\alpha_{n+1} &= \frac{(a+n-1)(a+b+n-1)}{(a+2n-1)^2} n(b-n)x^2 \qquad (n \ge 1) \\
\beta_{n+1} &= n + \frac{n(b-n)x}{a+2n-1} + \frac{a+n}{a+2n+1}[\lambda + 1 + n(1+y)] \qquad (n \ge 0)
\end{aligned}
$$

BFRAC(2)

$$
\begin{aligned}
I_x(a,b) &= \frac{x^a y^b}{a\,\mathrm{B}(a,b)} \left(\frac{1}{1+} \frac{d_1}{1+} \frac{d_2}{1+} \cdots \right) \qquad \left(x \le p \equiv \frac{a}{a+b} \right) \\
y &= 1 - x \\
d_{2n} &= \frac{n(b-n)}{(a+2n-1)(a+2n)} x \qquad (n > 0) \\
d_{2n+1} &= \frac{(a+n)(a+b+n)}{(a+2n)(a+2n+1)} x \qquad (n \ge 0)
\end{aligned}
$$

BASYM

$$
\begin{aligned}
I_x(a,b) &\approx \frac{2}{\sqrt{\pi}} U e^{-z^2} \sum_{n=0}^{\infty} e_n L_n(z) (\beta\gamma)^n \qquad \left(x \le p \equiv \frac{a}{a+b}\right) \\
U &= \frac{p^a q^b}{\mathrm{B}(a,b)} \sqrt{\frac{2\pi(a+b)}{ab}} \\
z &= \sqrt{\varphi(x)} \\
\varphi(t) &= -\left(a \ln\frac{t}{p} + b \ln\frac{1-t}{q}\right) \ge 0 \qquad (0 < t < 1) \\
\beta\gamma &= \sqrt{q/a} \qquad (a \le b) \\
\beta\gamma &= \sqrt{p/b} \qquad (a \ge b) \\
a_n &= \frac{2}{n+2}\, q \left[1 + (-1)^n (a/b)^{n+1}\right] \qquad (a \le b) \\
a_n &= \frac{2}{n+2}\, p \left[(-1)^n + (b/a)^{n+1}\right] \qquad (a > b,\ n \ge 0) \\
p &= 1 - q = \frac{a}{a+b} \\
q &= 1 - p = \frac{b}{a+b} \\
b_0^{(r)} &= 1 \\
b_1^{(r)} &= r a_1 \qquad (r \ne 0) \\
b_n^{(r)} &= r a_n + \frac{1}{n} \sum_{i=1}^{n-1} [(n-i)r - i] b_i^{(r)} a_{n-i} \qquad (n = 2, 3, \dots) \\
c_n &= \frac{1}{n} b_{n-1}^{(-n/2)} \qquad (n \ge 1) \\
e_0 &= 1 \\
e_n &= -\sum_{i=0}^{n-1} e_i\, c_{n-i+1} \\
L_n(z) &= 2^{(n/2)-1} e^{z^2} \int_z^{\infty} e^{-u^2} u^n \mathrm{d}u \\
L_0(z) &= \frac{\sqrt{\pi}}{4} e^{z^2} \operatorname{erfc}(z) = \frac{\sqrt{\pi}}{4} \operatorname{erfcx}(z) \\
L_1(z) &= 2^{-3/2} \\
L_n(z) &= 2^{-3/2} \left(\sqrt{2} z\right)^{n-1} + (n-1) L_{n-2}(z) \qquad (n = 2, 3, \dots)
\end{aligned}
$$

二項累積分布と正則ベータ関数との関係

- en.wikipedia: Beta function# Incomplete beta function

$$
F(k;\, n, p) = \Pr(X \le k) = I_{1-p}(n-k, k+1) = 1 - I_p(k+1, n-k)
$$

Maxima script

高精度の計算、自前実装ルーチンの検算用

- gist: イロレーティング計算用テストデータ

```
fpprec:64$
fpprintprec:36$
defstruct(wdl(w,d,l))$
defstruct(elorange(w,d,l,sr,r,low,high))$
normcdf(x):=erfc(-sqrt(5b-1)*x)*5b-1$
normcdfinv(r):=bf_find_root(normcdf(x)-r,x,-1b1,+1b1)$
elo(a,b,r):=bf_find_root(beta_incomplete_regularized(
  bfloat(a),bfloat(b),(1+10^(-25b-4*x))^-1)-r,x,-1b5,+1b5)$
elocalc(wdl,r):=new(elorange(wdl@w,wdl@d,wdl@l,-normcdfinv(r),r,
  elo(wdl@w+wdl@d*5b-1,wdl@l+wdl@d*5b-1+1,r),
  elo(wdl@w+wdl@d*5b-1+1,wdl@l+wdl@d*5b-1,1-r)))$
rlist:[2.5b-1,2b-1,normcdf(-1.0),1.5b-1,1.25b-1,1b-1,normcdf(-1.5),
  5b-2,2.5b-2,normcdf(-2.0),1b-2,normcdf(-2.5),5b-3,normcdf(-3.0),
  1b-3,5b-4,normcdf(-3.5),1b-4,5b-5,normcdf(-4.0),1b-5];
wlist:[
  new(wdl(0,1,0)),new(wdl(1,0,1)),new(wdl(1,1,1)),
  new(wdl(9,0,1)),new(wdl(9,1,0)),
  new(wdl(1e1,1,0)),new(wdl(1e1,0,1)),new(wdl(1e1,0,3)),
  new(wdl(1e1,1,10)),new(wdl(1e1,0,40)),new(wdl(1e1,1,40)),
  new(wdl(11,1,0)),new(wdl(12,1,0)),new(wdl(66,0,34)),
  new(wdl(1e2,1,0)),new(wdl(1e2,0,1)),new(wdl(1e2,0,3)),
  new(wdl(1e2,1,10)),new(wdl(1e2,0,40)),new(wdl(1e2,1,40)),
  new(wdl(550,0,450)),
  new(wdl(1e3,1,0)),new(wdl(1e3,0,1)),new(wdl(1e3,0,3)),
  new(wdl(1e3,1,10)),new(wdl(1e3,0,40)),new(wdl(1e3,1,40)),
  new(wdl(5155,0,4845)),
  new(wdl(1e4,1,0)),new(wdl(1e4,0,1)),new(wdl(1e4,0,3)),
  new(wdl(1e4,1,10)),new(wdl(1e4,0,40)),new(wdl(1e4,1,40)),
  new(wdl(50490,0,49510)),
  new(wdl(1e5,1,0)),new(wdl(1e5,0,1)),new(wdl(1e5,0,3)),
  new(wdl(1e5,1,10)),new(wdl(1e5,0,40)),new(wdl(1e5,1,40)),
  new(wdl(501546,0,498454)),
  new(wdl(1e6,1,0)),new(wdl(1e6,0,1)),new(wdl(1e6,0,3)),
  new(wdl(1e6,1,10)),new(wdl(1e6,0,40)),new(wdl(1e6,1,40)),
  new(wdl(5004887,0,4995113)),
  new(wdl(1e7,1,0)),new(wdl(1e7,0,1)),new(wdl(1e7,0,3)),
  new(wdl(1e7,1,10)),new(wdl(1e7,0,40)),new(wdl(1e7,1,40)),
```

```
  new(wdl(50015452,0,49984548)),
  new(wdl(1e8,1,0)),new(wdl(1e8,0,1)),new(wdl(1e8,0,3)),
  new(wdl(1e8,1,10)),new(wdl(1e8,0,40)),new(wdl(1e8,1,40)),
  new(wdl(500048862,0,499951138)),
  new(wdl(1e9,1,0)),new(wdl(1e9,0,1)),new(wdl(1e9,0,3)),
  new(wdl(1e9,1,10)),new(wdl(1e9,0,40)),new(wdl(1e9,1,40)),
  new(wdl(1e10,1,0)),new(wdl(1e10,0,1)),new(wdl(1e10,0,3)),
  new(wdl(1e10,1,10)),new(wdl(1e10,0,40)),new(wdl(1e10,1,40)),
  new(wdl(1e11,1,0)),new(wdl(1e11,0,1)),new(wdl(1e11,0,3)),
  new(wdl(1e11,1,10)),new(wdl(1e11,0,40)),new(wdl(1e11,1,40)),
  new(wdl(1e12,1,0)),new(wdl(1e12,0,1)),new(wdl(1e12,0,3)),
  new(wdl(1e12,1,10)),new(wdl(1e12,0,40)),new(wdl(1e12,1,40)),
  new(wdl(1e13,1,0)),new(wdl(1e13,0,1)),new(wdl(1e13,0,3)),
  new(wdl(1e13,1,10)),new(wdl(1e13,0,40)),new(wdl(1e13,1,40)),
  new(wdl(1e14,1,0)),new(wdl(1e14,0,1)),new(wdl(1e14,0,3)),
  new(wdl(1e14,1,10)),new(wdl(1e14,0,40)),new(wdl(1e14,1,40)),
  new(wdl(1e15,1,0)),new(wdl(1e15,0,1)),new(wdl(1e15,0,3)),
  new(wdl(1e15,1,10)),new(wdl(1e15,0,40)),new(wdl(1e15,1,40)),
  new(wdl(1e16,1,0)),new(wdl(1e16,0,1)),new(wdl(1e16,0,3)),
  new(wdl(1e16,1,10)),new(wdl(1e16,0,40)),new(wdl(1e16,1,40))
  ]$
for wi thru length(wlist) do
for ri thru length(rlist) do
  print(elocalc(wlist[wi],rlist[ri]))$
fpprec:40$
for ri thru length(rlist) do
  print(elocalc(new(wdl(5000154513,0,4999845487)),rlist[ri]))$
```