Exercise01: Numerical Exercise Solutions

1. Which of the following points written in homogeneous coordinats represent the same point in $\mathbb{P}^2$ as $[6, 3, 2]$?

   (a) $[18, 9, 6]$ (b) $[12, -6, 4]$ (c) $\left[1, \frac{1}{2}, \frac{1}{3}\right]$ (d) $[1, 2, 3]$

   Solution
   In order for two points to be equivalent in $\mathbb{P}^2$, there must be a $\lambda \in \mathbb{R}$, such that the first is written as a λ-multiple of the second.
   (a) $[18, 9, 6] = 3 * [6, 3, 2] \Rightarrow$ Points are equivalent
   (b) If there exists such λ, it should hold that: $12 = 6\lambda, -6 = 3\lambda, 4 = 2\lambda$. This system of equations is inconsistent $\Rightarrow$ Points are not equivalent
   (c) $\left[1, \frac{1}{2}, \frac{1}{3}\right] = \frac{1}{6} * [6, 3, 2] \Rightarrow$ Points are equivalent
   (d) The system of equations $1 = 6\lambda, 2 = 3\lambda, 3 = 2\lambda$ is inconsistent $\Rightarrow$ Points are not equivalent

2. Consider the following projective transformation $\lambda q = HP$, which maps a world point P to the image point q with

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Compute the vanishing points for lines in the XZ plane (1) parallel to the Z-axis and (2) at $45°$ to the Z axis.
   Solution
   As seen in the lecture, the coordinates of a 3D line with direction $[l, m, n]$ passing though the point $[X_0, Y_0, Z_0]$ can be parametrized with $s \in \mathbb{R}$

$$\begin{aligned} X &= X_0 + sl \\ Y &= Y_0 + sm \\ Z &= Z_0 + sn \end{aligned}$$

   Given the projection matrix H, a 3D line can be projected to the image coordinates u and v

$$\begin{aligned} \lambda u &= X \\ \lambda v &= Y \\ \lambda &= Z \end{aligned}$$

$$u = \frac{X}{Z}, \qquad v = \frac{Y}{Z}$$

   In order to find the vanishing points, we need to compute the limit for $s \to \infty$

$$\lim_{s\to\infty} \frac{X_0 + sl}{Z_0 + sn} = \frac{l}{n} \qquad \lim_{s\to\infty} \frac{Y_0 + sm}{Z_0 + sn} = \frac{m}{n}$$

   (1) A line in the XZ plane parallel to the Z-axis has the direction $[0, 0, 1]$. Thus, the corresponding vanishing points is $(0, 0)$
   (2) A line in the XZ plane at $45°$ to the Z axis has the direction $[1, 0, 1]$. Thus, the corresponding vanishing points is $(1, 0)$

3. Consider a camera with the following camera matrix

$$K = \begin{bmatrix} f & 0 & u_x \\ 0 & f & u_y \\ 0 & 0 & 1 \end{bmatrix}.$$

State the modified camera matrix $\hat{K}$ if the image resolution is increased by a factor of two.
Solution
An increased resolution affects focal length f and the principal points u_x and u_y.
Since the number of the pixels is doubled, the vertical and horizontal number of pixels is increased by a factor of $\sqrt{2}$. Thus, both offsets are multiplied by the same factor.
The modified focal length is multipled by the same factor of $\sqrt{2}$ as the conversion factor from meters to pixel is increased as well.
Thus, the modified camera matrix $\hat{K}$ is

$$\hat{K} = \begin{bmatrix} \sqrt{2}f & 0 & \sqrt{2}u_x \\ 0 & \sqrt{2}f & \sqrt{2}u_y \\ 0 & 0 & 1 \end{bmatrix}.$$

4. Compute the image point q in pixels corresponding to the point $P = [2, 5, 5]$ expressed in the world frame in meters. Consider the following camera extrinsics: rotation matrix R and translation vector T (in meters) and camera matrix K

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad K = \begin{bmatrix} 420 & 0 & 355 \\ 0 & 420 & 250 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution

$$\lambda q = K\,[R|T]\,P = \begin{bmatrix} 420 & 0 & 355 \\ 0 & 420 & 250 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4230 \\ 2760 \\ 6 \end{bmatrix} \quad \Rightarrow \quad q = \begin{bmatrix} 705 \\ 460 \\ 1 \end{bmatrix}$$