Learning Objectives:

1. Compute matrix vector products.

2. Relate matrix equations to vector equations/systems of linear equations.

3. Determine consistency of matrix equations to pivots, span of columns, and linear combinations.

1 Matrix equations

We saw in the last section that we can multiply a scalar and a vector; in this section we learn how to multiply matrices and vectors.

If A is an $m \times n$ matrix (m rows, n columns) with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and $\mathbf{x} \in \mathbb{R}^n$, then

$$Ax = (\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n.$$

Example 1. *Multiply* $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix}$ *and* $\mathbf{x} = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$.

Solution. We have

$$Ax = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ -5 \end{pmatrix} + 7 \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

An alternate way to multiply a matrix and a vector is using the “row-vector rule,” see below.

Row-vector rule for multiplying a matrix and a vector.

To compute:

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$$

Step 1:

$$\begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix}\begin{pmatrix} \bullet \\ \end{pmatrix} \quad 1 \cdot 4 + 2 \cdot 3 + -1 \cdot 7 = 3 = \bullet$$

Step 2:

$$\begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ \bullet \end{pmatrix} \quad 0 \cdot 4 + -5 \cdot 3 + 3 \cdot 7 = 6 = \bullet$$

So,

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

Example 2. *If we have a* $1 \times n$ *vector* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *and an* $n \times 1$ *vector* $\mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$, *then compute* $\mathbf{vw}$.

Solution. We have

$$\mathbf{vw} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

This is also called the *dot product* or *inner product* of two vectors!

We list a few (non-surprising) properties of matrix-vector multiplication:

If A is an $m \times n$ matrix, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and c is a scalar, then

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$.
2. $A(c\mathbf{u}) = c(A\mathbf{u})$.

Proof. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then

$$\begin{aligned} A(\mathbf{u}+\mathbf{v}) &= (\mathbf{a}_1 \dots \mathbf{a}_n) \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} \\ &= (u_1+v_1)\mathbf{a}_1 + \cdots + (u_n+v_n)\mathbf{a}_n \\ &= u_1\mathbf{a}_1 + v_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n + v_n\mathbf{a}_n \\ &= (u_1\mathbf{a}_1 + \cdots + u_n\mathbf{a}_n) + (v_1\mathbf{a}_1 + \cdots + v_n\mathbf{a}_n) \\ &= A\mathbf{u} + A\mathbf{v}. \end{aligned}$$

□

2 Consistency of matrix equations

Example 3. *If* $A = \begin{pmatrix} 2 & 3 & 1 \\ -2 & -1 & 0 \\ 5 & 8 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, *and* $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ *then write* $A\mathbf{x} = \mathbf{b}$ *as a vector equation. What connections to previous concepts do you see?*

Solution. We have

$$A\mathbf{x} = x_1 \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},$$

so letting $\mathbf{a}_i$ be the columns of the matrix A we have

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

This is the same type of vector equation from last section. We can find the solutions $\mathbf{x}$ if we row reduce the augmented matrix.

Theorem. Let A be an $m \times n$ matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$. Then, the solution set of the vector equation

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

is the same as the solution set of the linear system with augmented matrix

$$(\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n \ \mathbf{b})$$

which, in turn, is the same as the solution set of the matrix equation

$$A\mathbf{x} = \mathbf{b}.$$

In particular, we have

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Example 4. *Let* $A = \begin{pmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{pmatrix}$ *and* $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. *Is the equation* $A\mathbf{x} = \mathbf{b}$ *consistent for all possible* b_1, b_2, b_3*? That is, for any* $\mathbf{b}$*, does there exist an* $\mathbf{x} \in \mathbb{R}^n$ *solving the matrix equation?*

Hint: *Row reduce the associated augmented matrix and consider what happens when plugging different values of* b_1, b_2, b_3.

Solution. The augmented matrix is

$$\begin{pmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & -2b_1 + b_2 - 2b_3 \end{pmatrix}. \tag{1}$$

Note that the final column has a pivot if $-2b_1 + b_2 - 2b_3 \neq 0$. That is, the system is consistent if and only if $-2b_1 + b_2 - 2b_3 = 0$. Since this is not always true, we see that there are some values of $\mathbf{b}$ for which the system is not consistent.

Example 5. *Thinking on the previous result, what does it mean if* A *has a pivot in every row of its coefficient matrix?*

A. The equation $A\mathbf{x} = \mathbf{b}$ *is solvable for all* $\mathbf{b} \in \mathbb{R}^m$.

B. Each $\mathbf{b} \in \mathbb{R}^m$ *can be written as a linear combination of the columns of* A.

C. The columns of A *span* $\mathbb{R}^m$.

D. All of the above.

Solution. The answer is D. In fact, all of these statements are equivalent! Either all are simultaneously true, or all are simultaneously false. To see why this is equivalent to the first option, suppose A has a pivot in every row. Then, the augmented matrix $(A\ \mathbf{b})$ can be row reduced and will have a pivot in the final row, so the last row looks like $(0 \cdots\ 0\ 1\ b_m)$. No matter what b_m is, this is consistent, so the system is solvable. If A does not have a pivot in every row, then the final row will have the form $(0 \cdots\ 0\ b_m)$, which is not solvable when $b_m \neq 0$.

Example 6. T/F: *A set of* 3 *vectors in* $\mathbb{R}^4$ *can span all of* $\mathbb{R}^4$.

Solution. False. If we consider the matrix of vectors, it is size 4×3. If we row reduce then A cannot be equivalent to a matrix with pivots in each row. Thus, the columns of A cannot span all of $\mathbb{R}^4$.