Learning Objectives:

1. Calculate eigenvalues, eigenvectors, and eigenspaces of a square matrix
2. Describe the geometric interpretation of eigenvectors under transformation

1 Eigenvectors

Definition: An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. The scalar λ is called the **eigenvalue** corresponding to $\mathbf{x}$.

Example 1. *Why is it important that we require* $\mathbf{x} \neq 0$*?*

Solution. If $\mathbf{x} = \mathbf{0}$ is allowed, then $A\mathbf{0} = \lambda\mathbf{0}$ no matter the value of λ, and so any real number would be an eigenvalue. Finding an eigenvector means that $A\mathbf{x} = \lambda\mathbf{x}$ has a *non-trivial* solution.

As an example:

Example 2. *Let* $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$ *and* $\mathbf{u} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. *Are* $\mathbf{u}$ *and* $\mathbf{v}$ *eigenvectors of* A*?*

Solution. We see that $A\mathbf{u} = -4\mathbf{u}$ but $A\mathbf{v} = \begin{pmatrix} -9 \\ 11 \end{pmatrix} \neq \lambda\mathbf{v}$. So, $\mathbf{u}$ is an eigenvector with eigenvalue -4. $\mathbf{v}$ is not an eigenvector.

Example 3. *What is the geometric interpretation of eigenvectors?*

Solution. Eigenvectors are vectors that do not change direction when the map A is applied.

Example 4. *Show that* 7 *is an eigenvalue of* $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$.

Solution. We wish to solve the equation $A\mathbf{x} = 7\mathbf{x}$. We cannot row reduce this equation as is, since the righthand side is not a constant. First re-write the equation as

$$A\mathbf{x} = (7I)\mathbf{x},$$

which means

$$A\mathbf{x} - 7I\mathbf{x} = \mathbf{0}.$$

Of course, this means

$$(A - 7I)\mathbf{x} = \mathbf{0}.$$

Now we can solve by row reduction (the right hand side is now constant). Importantly, we need to find a non-trivial solution to this equation.

First, we see that

$$A - 7I = \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix}.$$

Note here that the matrix $A - 7I$ is singular, since the columns are linearly dependent. This is a good thing because we are looking for non-trivial solutions of $(A - 7I)\mathbf{x} = \mathbf{0}$ in order to find an eigenvector. Row-reducing shows

$$\begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}.$$

So, the general solution of the homogeneous equation is $x_1 = x_2$, with x_2 free. Thus $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So, any non-zero vector which is a multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to eigenvalue 7.

Remark: In the next section we will see how to find the right λ without being told what it is.

2 Eigenspace

Example 5. *What connections are there between eigenvectors and subspaces?*

Solution. To find the eigenvalues in general, we must determine when

$$Ax = \lambda x$$

has a non-trivial solution. That is, when

$$(A - \lambda I)x = 0$$

has a non-trivial solution. The solution set of this system is a null space! Thus the set of solutions is a subspace. That means that the set of eigenvectors corresponding to λ form a subspace.

Definition: The **eigenspace** of A corresponding to the eigenvalue λ is the subspace corresponding to the set of all solutions to $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Example 6. *Let* $A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$. *An eigenvalue is* 2. *Find the dimension of the corresponding eigenspace.*

Note that $A - 2I = \begin{pmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{pmatrix}$. So, the general solution is

$$\mathbf{x} = \begin{pmatrix} 1/2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

So, a basis for the eigenspace is formed by those two vectors.

Example 7. *If* $A = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix}$, *then what are the eigenvalues of* A? **Hint:** *When does* $A - \lambda I$ *become singular?*

Solution. Since $A - \lambda I = \begin{pmatrix} 3-\lambda & 6 & -8 \\ 0 & -\lambda & 6 \\ 0 & 0 & 2-\lambda \end{pmatrix}$, there are fewer than 3 pivots when $\lambda = 3, 0, 2$. In fact, we see that since 0 is an eigenvalue then A is non-invertible!

Theorem: The eigenvalues of a triangular matrix are the entries on the main diagonal.

If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvalues corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

Proof. Suppose that the set is linearly dependent. Then, there is some $\mathbf{v}_j$ which is a linear combination of the previous vectors. Assume to take the smallest j index so that $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$ are linearly independent.

$$\mathbf{v}_j = \sum_{i=1}^{j-1} c_i \mathbf{v}_i.$$

We see that $\lambda_j \mathbf{v}_j = \sum_{i=1}^{j-1} c_i \lambda_j \mathbf{v}_i$. Also, applying A to both sides:

$$\lambda_j \mathbf{v}_j = \sum_{i=1}^{j-1} c_i \lambda_i \mathbf{v}_i.$$

So,

$$0 = \sum_{i=1}^{j-1} c_i (\lambda_i - \lambda_j) \mathbf{v}_i.$$

Since $\lambda_i - \lambda_j \neq 0$ and the $\mathbf{v}_i$ are assumed to be linearly independent, then $c_i = 0$. But this implies that $\mathbf{v}_j = 0$, a contradiction.

Example 8. T/F: *An $n \times n$ matrix can have at most n eigenvalues.*