**Learning Objectives:**

1. Represent any linear transformation as a matrix multiplication.
2. Determine whether a given linear transformation is one-to-one and/or onto.

# 1 The matrix of a linear transformation

We previously saw that if $A$ is a matrix then the transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ is linear. What about the other direction? If $T$ is linear, can it always be represented by a matrix? The amazing fact is:

> **Every** linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ can be represented (uniquely) as a matrix transformation: $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix $A$.

The trick to understand this is to use the so-called *standard basis vectors.* In $\mathbb{R}^3$ they are

The *identity matrix* is the matrix whose columns are the standard basis vectors: $I_n = (\mathbf{e}_1 \ \ldots \ \mathbf{e}_n)$.

**Example 1.** *Suppose $T$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ which maps*

$$T(\mathbf{e}_1) = \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix}, \quad T(\mathbf{e}_2) = \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}.$$

*Determine a matrix $A$ which represents $T$.*

**Theorem:** Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, there exists a unique matrix $A$ of size $m \times n$ such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

The matrix $A$ is given by

where $\mathbf{e}_i$ is the $i$th basis vector. We call $A$ the *standard matrix of the linear transformation* $T$.

**Example 2.** *Write down the matrix representation of the rotation transformation* $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ *which rotates all vectors counterclockwise by* $90°$.

## 2 Transformation properties

We have seen a common theme of linear algebra is answering the question: "is $A\mathbf{x} = \mathbf{b}$ solvable? Are solutions unique?" We have also seen that the case $A\mathbf{x} = \mathbf{0}$ is particularly important.

We ask the same questions of transformations:

A mapping $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is **onto** or **surjective** if

That is, $T$ is onto if given any $\mathbf{b}$ there is

A mapping $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is **into** or **one-to-one** or **injective** if

That is, $T$ is one-to-one if whenever $T(\mathbf{u}) = T(\mathbf{v})$ then

**Example 3.** *Let $T$ be the linear transformation whose standard matrix row reduces to*

$$R = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}.$$

*Is $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ onto? into?*

**Theorem:** Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T$ is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

**Example 4.** *Assume $T$ is a linear transformation and $A$ is its standard matrix. Fill in the following table with the following concepts:*

*Columns of $A$ linearly independent, columns of $A$ span $\mathbb{R}^m$, pivot in each row of $A$, pivot in each column of $A$, row reducing never results in row $(0\ 0\ \ldots\ 0\ b)$, $A$ has no free variables, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution*

| *$T$ is one-to-one* | *$T$ is onto* |
|---|---|
| | |

**Example 5.** *Let* $T(x_1, x_2) = (3x_1+x_2, 5x_1+7x_2, x_1+3x_2)$*. Show that* $T$ *is injective. Is it surjective?*

**Example 6.** *Suppose that* $A$ *is a* $2 \times 3$ *matrix.* **T/F***: The transformation* $T(\mathbf{x}) = A\mathbf{x}$ *can be one-to-one.*