At this point of the semester we take a moment to reflect. One of the most important lessons in linear algebra (and in fact a lot of math) is: *we can often use similar techniques to solve very different types of questions.* For example, calculating how many pivots a matrix has can tell us about existence of solutions, uniqueness of solutions, invertibility of the matrix, and more. Posing this mantra in a different way: *we can sometimes take what seem like very abstract or unrelated questions, and solve them using familiar techniques.*

So, even in very different situations we often up seeing very similar patterns. This leads us to ask:

When faced with very different looking questions, how do we know whether we can apply linear algebra techniques to solve them?

To begin answering this question, we can think about patterns and properties we have seen so far that have appeared in different contexts. For example, for vectors $\mathbf{x}$ and $\mathbf{y}$ we know

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}.$$

Also, for matrices A and B we saw

$$A + B = B + A.$$

We see that "commutative laws" (and many other algebraic laws) are common to both cases!

Example 1. *We remember that for functions $f(t)$ and $g(t)$ we also have*

$$f(t) + g(t) = g(t) + f(t).$$

So function addition is also commutative. This may make us wonder if linear algebra techniques could be used to study functions. (Spoiler: YES!)

Example 2. *Computer scientists often work in binary so that everything is represented as a vector of 0s and 1s. This comes with special rules for addition: for example it can be useful to define $0+1=1$ and $1+1=0$. One can then check that if $\mathbf{a}$ and $\mathbf{b}$ are two vectors in binary, say $\mathbf{a} = (1,0,1)$ and $\mathbf{b} = (0,1,1)$ then still*

$$\mathbf{a} + \mathbf{b} = (1,1,0) = \mathbf{b} + \mathbf{a}.$$

So this binary addition is also commutative. This may make us wonder if linear algebra techniques could be used for computer science in binary (Spoiler: Still YES!)

In all of the examples above, one could try to sit down and think about all of the properties that are similar between vectors, matrices, and functions, or all of the properties that are different. One is then faced with the challenge of identifying which parts *need* to be the same in order for use of linear algebra techniques... It turns out that **vector spaces** are the answer. Vector spaces capture the bare minimum requirements for a given set of objects to be amenable to our study.

Definition: A **vector space** is a nonempty set V of elements, called **vectors**, with two operations: *vector addition* and *scalar multiplication*. The vectors and scalars (real numbers) satisfy the following axioms: for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and for all scalars $c, d \in \mathbb{R}$,

i. The sum $\mathbf{u} + \mathbf{v} \in V$ (closure under addition)

ii. For each $\mathbf{u} \in V$ and scalar c, $c\mathbf{u} \in V$ (closure under scalar mult.)

iii. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity)

iv. There is a zero vector in V: $\mathbf{0} + \mathbf{u} = \mathbf{u}$. (additive identity)

v. For each $\mathbf{u} \in V$ there exists $\mathbf{v} \in V$ so that $\mathbf{u} + \mathbf{v} = \mathbf{0}$ (additive inverses, we often write $\mathbf{v} = -\mathbf{u}$.)

vi. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)

vii. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (scalar distribution)

viii. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (vector distribution)

ix. $c(d\mathbf{u}) = (cd)\mathbf{u}$ (compatibility of multiplication)

x. $1\mathbf{u} = \mathbf{u}$ (multiplicative identity)

Important: The word "vector" is a very loaded word here. It sometimes means vector in the way we have seen all semester: an ordered list of numbers from $\mathbb{R}^n$. However, mathematicians use the word vector to mean any element from a vector space. As long as the collection of objects we are studying satisfies all of the assumptions above, the elements are called vectors. For example functions can be called vectors, or polynomials can be called vectors. It turns out even matrices can be referred to as vectors! It all depends on context.

Also important: Vector spaces range from very familiar sets to very surprising and bizarre objects. It is very important to remember that "vector addition" and "scalar multiplication" have to be defined as part of the vector space. One could define "addition" of vectors $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ by

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 6 \end{pmatrix}.$$

Then it would be our job to go over each axiom of vector spaces to see if each rule holds. If they do, then this is a different type of vector space than the usual $\mathbb{R}^n$. If the rules do not hold, then it is not a vector space.

Example 3. *The spaces $\mathbb{R}^n$, $n \geq 1$ with regular vector addition and scalar multiplication are the classic examples of vector spaces.*

Example 4. *Let $\mathbb{P}_n$ denote the set of polynomials of degree $\leqslant n$:*

$$\mathbb{P}_n = \{\mathbf{p}(t)\colon \mathbf{p}(t) = a_0 + a_1 t + \cdots + a_n t^n\}.$$

Define addition by

$$\mathbf{p}(t) + \mathbf{q}(t) = (a_0 + a_1 t + \cdots + a_n t^n) + (b_0 + b_1 t + \cdots + b_n t^n) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n.$$

Notice that any or all coefficients are allowed to be zero here! Define scalar multiplication by

$$c\mathbf{p}(t) = ca_0 + ca_1 t + \cdots + ca_n t^n.$$

Then, the set $\mathbb{P}_n$ is a vector space! To be sure, we should check every single part of the definition. Let's check just a few:

- *(iv) The zero vector is a special element of the vector space that doesn't affect any other vectors. So, which polynomial do we need? The one whose coefficients are all zero:*

  $$\mathbf{0} = 0 + 0t + \cdots + 0t^n.$$

  It is slightly silly to write it this way, but I want to emphasize that it is not $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *or* $\mathbf{0} = 0$ *(a single number). The zero vector here is the polynomial whose values are zero for all t: we need to make sure $\mathbf{0} \in \mathbb{P}_n$. To rigorously check: let $\mathbf{p} \in \mathbb{P}_n$, so that $\mathbf{p}(t) = a_0 + \cdots + a_n t^n$. Then*

  $$\mathbf{0} + \mathbf{p} = (0 + a_0) + \cdots + (0 + a_n)t^n = a_0 + \cdots + a_n t^n = \mathbf{p}.$$

- *(v) Let* $\mathbf{p} \in \mathbb{P}_n$*. Then* $\mathbf{p}(t) = a_0 + \cdots + a_n t^n$*. We expect that the additive inverse should be the polynomial whose coefficients are all opposite sign of* $\mathbf{p}$*. To check let* $\mathbf{v} = -a_0 + \cdots + (-a_n)t^n$*. Then*
$$\mathbf{u} + \mathbf{v} = (a_0 - a_0) + \cdots + (a_n - a_n)t^n = 0 + \cdots + 0t^n = \mathbf{0}.$$

Feel free to check the other parts of the definition for yourself!

Example 5. *Let* $V = \mathbb{R}^+ = \{x : x > 0\}$*. Define vector addition by*

$$x \oplus y = xy$$

and scalar multiplication by
$$c \odot x = x^c.$$
Then V *forms a vector space! Check axioms (iii), (iv), (v), and (vii).*

Solution. (iii) We have

$$(x \oplus y) \oplus z = (xy)z = x(yz) = x \oplus (y \oplus z),$$

since regular multiplication of real numbers is associative.

(iv) Let $\mathbf{0} = 1$ (the zero vector is the real number 1). Then

$$1 \oplus x = 1x = x$$

so 1 is the zero vector.

(v) Given $x \in V$, take $v \in V$ to be $v = 1/x$. Then

$$x \oplus v = xv = x(1/x) = 1.$$

Since 1 is the zero vector we have shown that v is the inverse.

(vii) Let $x, y \in V$ and $c \in \mathbb{R}$. Then

$$c \odot (x \oplus y) = c \odot (xy) = (xy)^c = x^c y^c = (c \odot x) \oplus (c \odot y).$$

Example 6. *Let* H *be the set of points within the unit circle of the* xy *plane: so*
$$H = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x^2 + y^2 \leqslant 1 \right\}.$$
Equip H *with regular vector addition and scalar multiplication. Show that* H *is not a vector space.*

Solution. Elements are not closed under scalar multiplication: we know $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in H$ but $2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin H$.

1 Subspaces

Oftentimes a subset of a vector space is a vector space in its own right!

Definition: A **subspace** of a vector V is a subset H of V that has three properties

1. $0 \in H$
2. if $\mathbf{u}, \mathbf{v} \in H$ then $\mathbf{u} + \mathbf{v} \in H$
3. if $\mathbf{u} \in H$ and c is a scalar then $c\mathbf{u} \in H$.

If a subset H satisfies these three properties, then H automatically satisfies all of the axioms of a vector space (since all of the other axioms still hold automatically).

Example 7.

For any vector space V, the **zero subspace** *is the space consisting of exactly the zero vector $\{\mathbf{0}\}$.*

The space $\mathbb{P}$ consisting of all polynomials (of any degree) is a vector space. It has as subspaces $\mathbb{P}_n$ for any n.

Example 8. *Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?*

A. Yes, and I am confident

B. Yes, but I am not confident

C. No, but I am not confident

D. No, but I am not confident

Solution. No, the elements of $\mathbb{R}^2$ are vectors with two components and elements of $\mathbb{R}^3$ are vectors with three components, so it is not true that $\mathbb{R}^2$ is a subset of $\mathbb{R}^3$. However, there is a subspace of $\mathbb{R}^3$ which looks like $\mathbb{R}^2$:

$$H = \left\{ \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} : s, t \in \mathbb{R} \right\}$$

Example 9. *Show that* $H = \left\{ \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} : s, t \in \mathbb{R} \right\}$ *is a subspace of* $\mathbb{R}^3$.

Solution. First, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in H$ by taking $s = 0, t = 0$. Next, if $\mathbf{v}, \mathbf{w} \in H$ then $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ 0 \end{pmatrix}$. So $\mathbf{v} + \mathbf{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ 0 \end{pmatrix} \in H$. Finally, if $\mathbf{v} \in H$ and c is a scalar then $c\mathbf{v} = c\begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} cv_1 \\ cv_2 \\ 0 \end{pmatrix} \in H$. Thus H is a subspace of $\mathbb{R}^3$.

2 Spanning set as a subspace

The definitions of **linear combination** and **span** are the same as we saw before.

Definition: The vector $\mathbf{w} \in V$ is said to be a **linear combination** of the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ if there exist scalars $a_1, \ldots, a_n \in \mathbb{R}$ so that

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{w}.$$

The **span** of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ is the set of all their linear combinations, denoted $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Theorem: Given $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$, the set $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ forms a subspace of V.

Proof. (i). Since $0 = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n$ is a linear combination then $0 \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.
(ii). If $\mathbf{a}, \mathbf{b} \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ then there exist weights a_i and b_i so that

$$\mathbf{a} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n, \quad \mathbf{b} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n.$$

then

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\mathbf{v}_1 + \cdots + (a_n + b_n)\mathbf{v}_n \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

(iii). If $c \in \mathbb{R}$ and $\mathbf{a} \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ then

$$c\mathbf{a} = (ca_1)\mathbf{v}_1 + \cdots + (ca_n)\mathbf{v}_n \in \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

Example 10. *Let H be the set of all vectors of the form $(a - 3b, b - a, a, b)$ where $a, b \in \mathbb{R}$. Is H a subspace of $\mathbb{R}^4$?*

Solution. Yes, we can write the vectors as

$$\begin{pmatrix} a-3b \\ b-a \\ a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

So, H is actually the span of two vectors:

$$H = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Thus, it is a subspace.

Example 11. *Show that* $\{f\colon f(0) = f(1)\}$ *is a subspace of* $C([0,1])$, *the set of all continuous functions* $f\colon [0,1] \to \mathbb{R}$.