1 The Definite Integral

Definition: If f is a function on the interval $[a, b]$, then we divide it into n subintervals of width $\Delta x = (b-a)/n$. Call the intervals $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$. Then, the **definite integral of f from a to b** is

where $x_i^* \in [x_{i-1}, x_i]$. If this limit exists then we say that f is **integrable**.

Theorem: If f is continuous on $[a, b]$, or has at most a finite number of discontinuities, then f is integrable on $[a, b]$.

Example 1. *Compute*

$$\int_0^3 (x-1)dx.$$

Take-away:

We have the following properties of integrals:

1. $\displaystyle\int_a^b f(x)dx =$
2. $\displaystyle\int_a^a f(x)dx =$
3. $\displaystyle\int_a^b c dx =$
4. $\displaystyle\int_a^b (f(x) + g(x))dx =$
5. $\displaystyle\int_a^b cf(x)dx =$
6. $\displaystyle\int_a^b f(x)dx + \int_b^c f(x)dx =$
7. If $f(x) \geq g(x)$ then .

Example 2. *Assuming*

$$\int_0^1 f(x)dx = 1$$

and

$$\int_2^1 2f(x)dx = 3$$

compute $\displaystyle\int_0^2 f(x)dx$.

Example 3. *Given* $\int_0^2 f(x)dx = e^2 - 1$ *compute*

$$\int_0^2 (-2f(x) - 3)dx.$$

Example 4. *Confirm that*

$$\int_{-1}^{1} \sqrt{1 + x^2}dx \leqslant 2\sqrt{2}.$$

2 The Fundamental Theorem of Calculus

Example 5. *The function*

$$g(x) = \int_0^x f(t)dt$$

measures the area under f from 0 to x.

Use geometry to simplify:

$$g(x) = \int_0^x 1dt,$$

and

$$h(x) = \int_0^x tdt.$$

Example 6. *What is $g'(x)$? What is $h'(x)$?*

Fundamental Theorem of Calculus, Part 1(FTC 1): If f is continuous on $[a, b]$, then the function g defined by

is continuous on $[a, b]$, differentiable on (a, b), and $g'(x) = f(x)$. That is:

Take-away: integration and differentiation are inverse operations.

Note: It doesn't matter what value a is!

Understanding why: We want to show that $g'(x) = f(x)$. Notice that:

Example 7. *Compute the derivative of*

$$g(x) = \int_1^x \sqrt{1+t^2}dt.$$

The second part of the fundamental theorem gives us a way to compute definite integrals:

FTC2: If f is continuous on $[a, b]$, then

where F is any antiderivative of $f(x)$.

Example 8. *Compute* $\int_1^9 \sqrt{x}dx$.

Example 9. Why do we need f to be continuous? *Compute*

$$\int_{-2}^{1} \frac{1}{x^2} dx.$$

Example 10. *Compute*

$$\frac{d}{dx}\int_{-1}^{x^4} \sec t dt.$$

Example 11. *Compute*

$$\frac{d}{dx}\int_{x}^{x^2} \sin^3(t) dt.$$

Example 12. *Compute*

$$\int_2^4 t dt.$$

Example 13. *Evaluate*

$$\int_1^4 \frac{x-3}{\sqrt{x}} dx.$$