Learning Objectives:

1. Use Cramer's Rule to solve systems of equations

2. Use Cramer's Rule to find a formula for inverse matrices

1 Cramer's Rule

Hot take: Cramer's rule itself is not particularly useful in the "real world" (there are a few uses of it, of course). That said, learning it can be useful, as it helps solidify and relate several ideas we have seen! Plus, it is somewhat satisfying in how "nicely" it works!
Goal: Cramer's rule is a way to solve the linear system

$$A\mathbf{x} = \mathbf{b}.$$

Let

$$A = (\mathbf{a}_1 \quad \mathbf{a}_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

As a linear transformation, we can think of A as the transformation which maps

$$A\mathbf{e}_1 = \mathbf{a}_1$$

and

$$A\mathbf{e}_2 = \mathbf{a}_2,$$

and given the target $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, our goal is to find the unknown $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ that maps to $\mathbf{b}$. This is represented visually in the figure below:

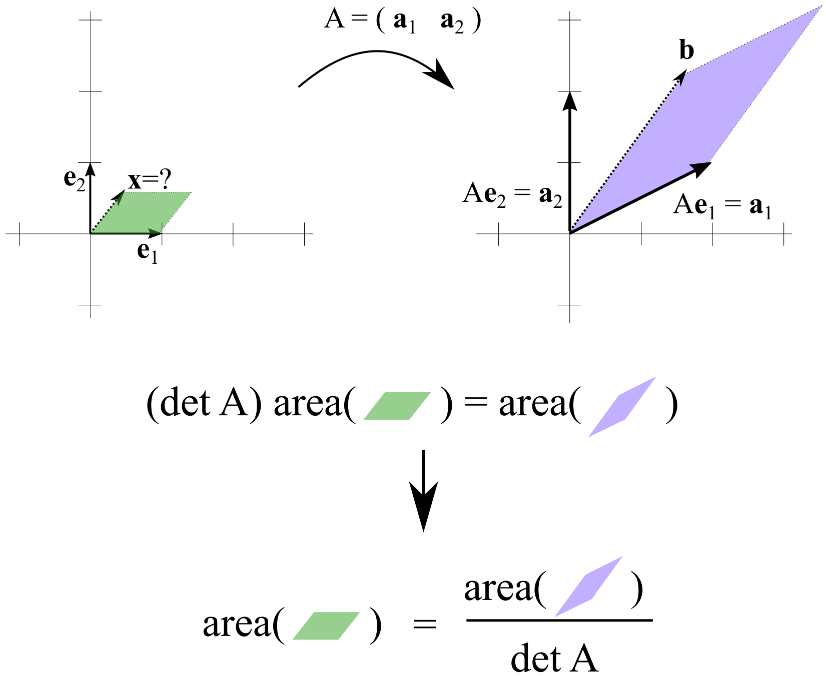
Now, consider drawing the (green) parallelogram between $\mathbf{e}_1$ and our unknown vector $\mathbf{x}$. When we apply the transformation A this parallelogram becomes the (purple) parallelogram formed between $\mathbf{a}_1$ and $\mathbf{b}$. How can we use this? Recall the following two facts:

1. The determinant of A measures the how much shapes scale after transformation.

2. The areas of parallelograms can be calculated using determinants.

The original (green) parallelogram has area

$$|\mathbf{e}_1 \quad \mathbf{x}| = \begin{vmatrix} 1 & x_1 \\ 0 & x_2 \end{vmatrix}.$$



The transformed (purple) parallelogram has area

$$|\mathbf{a}_1 \quad \mathbf{b}| = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}.$$

Finally, $\det A$ tells us how the areas scale:

$$(\det A) \cdot \begin{vmatrix} 1 & x_1 \\ 0 & x_2 \end{vmatrix} = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}.$$

After a bit of rearranging:

$$\begin{vmatrix} 1 & x_1 \\ 0 & x_2 \end{vmatrix} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\det A}.$$

Finally, notice that the left hand can be calculated to simplify

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\det A}.$$

The left hand side is the second component of our unknown vector. The right hand side is made up of all known quantities, so we can plug in numbers and solve!

Can you see how to modify the above argument in order to get the other equation

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\det A}?$$

2 Cramer's Rule

We have just discovered *Cramer's Rule*. By dividing two determinants we can calculate each of the unknown values of $\mathbf{x}$. Perhaps you noticed the following pattern for the right hand side matrix: to find the x_i component, we start by replacing the ith column of A by the vector $\mathbf{b}$.

Cramer's Rule: Let A be an invertible $n \times n$ matrix. Let

$$A_i(\mathbf{b}) = (\mathbf{a}_1 \cdots \mathbf{b} \cdots \mathbf{a}_n),$$

be the matrix A with the ith column replaced by $\mathbf{b}$. Then the unique solution $\mathbf{x}$ solving $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det A}, \text{ for all } i = 1, \ldots, n.$$

Example 1. *We can solve the system*

$$\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8, \end{aligned}$$

using Cramer's Rule. We have

$$A = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}, \; A_1(\mathbf{b}) = \begin{pmatrix} 6 & -2 \\ 8 & 4 \end{pmatrix}, \; A_2(\mathbf{b}) = \begin{pmatrix} 3 & 6 \\ -5 & 8 \end{pmatrix}.$$

Then, $\det A = 2$, $\det A_1(\mathbf{b}) = 40$, *and* $\det A_2(\mathbf{b}) = 54$. *So,*

$$x_1 = \frac{40}{2} = 20, \quad x_2 = \frac{54}{2} = 27.$$

3 Formula for A^{-1}

We can exploit this idea to find a formula for the inverse of A. Consider

$$A\mathbf{x} = \mathbf{e}_j.$$

By Cramer's rule,

$$x_i = \frac{\det A_i(\mathbf{e}_j)}{\det A}.$$

On the other hand, multiplying both sides by A^{-1} we have

$$\mathbf{x} = A^{-1}\mathbf{e}_j = \; j\text{th column of } A^{-1}.$$

So x_i is the (i,j) entry of A^{-1}. Now, the trick is to notice that we can easily solve $\det A_i(\mathbf{e}_j)$ via a cofactor expansion down the jth column (since it is mostly zeros).
Recall that the notation A_{ji} represents the matrix A with the jth row and ith column deleted. Then,

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}.$$

So,

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}.$$

This matrix of cofactor expansions is called the **adjugate** of A, denoted $\operatorname{adj} A$. Thus

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

Example 2. *Find the inverse of the matrix* $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}$.

Solution. We compute the cofactors:

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, \qquad C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, C_{13} = \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, \qquad C_{22} = \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = -7, C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -7$$

$$C_{31} = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, \qquad C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, C_{33} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3.$$

Then,

$$\operatorname{adj} A = \begin{pmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{pmatrix}.$$

Moreover, $\det A = 14$. So,

$$A^{-1} = \begin{pmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{pmatrix}.$$

Example 3. *Find the inverse of*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

using the adjugate.

Solution. The cofactors are

$$C_{11} = d, \quad C_{12} = -c$$

$$C_{21} = -b, \quad C_{22} = a.$$

Thus the adjugate is

$$\operatorname{adj} A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since $\det A = ad - bc$ we have

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example 4. *Solve*

$$\begin{aligned} -5x_1 + 2x_2 &= 9 \\ 3x_1 - x_2 &= -4. \end{aligned}$$

Solution. By Cramer's rule:

$$x_1 = \frac{\begin{vmatrix} 9 & 2 \\ -4 & -1 \end{vmatrix}}{\det A} = \frac{-1}{-1} = 1,$$

and

$$x_2 = \frac{\begin{vmatrix} -5 & 9 \\ 3 & -4 \end{vmatrix}}{\det A} = \frac{-7}{-1} = 7.$$

So $\mathbf{x} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$.