Learning Objectives:

1. Transform a matrix into echelon or reduced echelon form
2. Use pivots and free variables to deduce information about existence and uniqueness of solutions to the linear system.

1 Row reduction revisited

We previously saw how to use row operations to solve a system of equations. The process took place in two steps:

1. **Forward phase:** work down the matrix creating 0s below certain entries
2. **Backward phase:** work back up the matrix creating 0s above certain entries

Today we will explore this algorithm more carefully, working to understand *which entries we use to create those 0s* and *what happens if not all columns in the matrix let us create these 0s?*

Definition: A matrix is in **echelon form** if it has the following three properties:

1. Any rows of all zeros are at the bottom of the matrix.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it. (leading entry means first nonzero number in a row)
3. All entries in a column below a leading entry are zeros.

A matrix that is already in echelon form is in **reduced echelon form** (or **reduced row echelon form, rref**) if

1. The leading entry in each nonzero row is 1.
2. Each leading 1 is the only nonzero entry in its column.

Example 1. *Put the matrix*

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{pmatrix}$$

into echelon form and reduced echelon form.

Solution. We have

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 5 & 7 & 9 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{pmatrix}$$

At this point we are technically in echelon form. Notice that it is fine to not have all leading entries 1. Sometimes it will be beneficial but sometimes it just creates more work for us!

If we wanted to get to reduced row echelon form, then we can continue the backward phase:

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -10 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now our matrix is in reduced row echelon form.

2 Pivots and free variables

Definition: The entries corresponding to leading 1s in reduced row echelon form are called **pivot positions**. The columns with pivot positions are called **pivot columns**.

Example 2. *Reduce the following augmented matrix to rref:*

$$\begin{pmatrix} 2 & 2 & -8 & 10 \\ 0 & 1 & 1 & 4 \\ 0 & -1 & -1 & -4 \end{pmatrix}.$$

Try to understand the solution set of the linear system: is the system consistent? If so how many solutions are there? What can the pivots tell us?

Solution. We have

$$\begin{pmatrix} 2 & 2 & -8 & 10 \\ 0 & 1 & 1 & 4 \\ 0 & -1 & -1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So we have

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0. \end{aligned}$$

The last equation doesn't tell us anything! In fact, we can let x_3 be any number, and it will give us different solutions... if $x_3 = 0$ then we get $(x_1, x_2, x_3) = (1, 4, 0)$ which is a solution. If we let $x_3 = 1$ then we get $(x_1, x_2, x_3) = (6, 3, 1)$ which is another. We have infinitely many solutions. We notice that the pivots correspond to variables which are "fixed" by equations while columns without pivots gave us "free" variables we could make whatever we wanted! We could write all solutions as

$$\begin{cases} x_1 & = 1 + 5x_3 \\ x_2 & = 4 - x_3 \\ x_3 & \text{is free} \end{cases}$$

Definition: Variables corresponding to pivot columns are called **basic variables**. Variables corresponding to non-pivot columns are called **free variables**.

Example 3. *Suppose a system row reduced to*

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

Describe the solution set.

Solution. We have pivots in columns 1, 2, and 5, and so x_1, x_2, and x_5 will be basic variables, and x_3, x_4 will be free variables. The solutions are

$$\begin{cases} x_1 & = 2x_3 - 3x_4 - 24 \\ x_2 & = 2x_3 - 2x_4 - 7 \\ x_3 & \text{is free} \\ x_4 & \text{is free} \\ x_5 & = 4 \end{cases}$$

3 Revisiting existence and Uniqueness

One of the fundamental questions we have seen so far is

> Given a linear system of equations:
>
> - Is the system consistent? That is, does there exist at least one solution?
> - If a solution exists, is it unique?

> **Theorem:** A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column. That is, there is no row of the form
>
> $$[0, \cdots, 0, b] \text{ with } b \text{ nonzero.}$$
>
> If a linear system is consistent, then:
>
> 1. there is a *unique* solution if there are no free variables
> 2. there are *infinitely many* solutions if there is at least one free variable.

Question 1. *Is the following system consistent?*

$$\begin{pmatrix} 1 & 2 & -1 & 10 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{pmatrix}$$

A. Yes, and the solution is unique

B. Yes, and the solution is not unique

C. No

D. There is not enough information without further calculation

Solution. A.

Question 2. *Is it possible to have a system with free variables which is inconsistent?*

A. Yes, and I'm very confident.

B. Yes, but I'm not confident.

C. No, but I'm not confident.

D. No, and I'm very confident.

Solution. Yes! Take

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As an overall review of using row reduction to solve a linear system:

> **Using row reduction to solve a linear system**
>
> 1. Write the augmented matrix of the system
> 2. Use elementary row operations to obtain the equivalent echelon form (forward phase). Determine whether the system is consistent.
> 3. If it is consistent, continue row reducing to obtain the RREF (backward phase).
> 4. Write the final system of equations obtained from Step 3 so that all basic variables are expressed in terms of free variables (explicit description of solution set).

Question 3. *Suppose a* 4×7 *coefficient matrix has* 4 *pivots. Is the system consistent? If so, how many solutions are there?*

Solution. Since each row has a pivot, then in echelon form, no row has all zeros. Thus, the augmented matrix does not have a row of the form $[0, 0, \cdots, 0, b]$. Thus the matrix is consistent. Since there are columns without pivots, there are three free variables, and so there are infinitely many solutions.

Question 4. *Find the general solution of the linear system whose augmented matrix is*

$$\begin{pmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}.$$

Solution. We find the RREF:

$$\begin{pmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}.$$

So, we have that

$$\begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 4x_4 + 5 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

Question 5. *A system with fewer equations than unknowns is called* underdetermined. *Show that if an underdetermined system is consistent, then it must have infinitely many solutions.*

Solution. Suppose there are m equations for n unknowns. Then, the coefficient matrix for the system is an $m \times n$ system. If the system is underdetermined, then $m < n$. We see that there must be free variables, since at least one column cannot have a pivot. So, if the system is consistent, then it must have infinitely many solutions.