Learning Objectives:

1. Compute whether a given set of vectors is linearly independent or linearly dependent.

2. Describe linear independence and linear dependence geometrically.

1 Linear independence

Definition: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is **linearly independent** if the only solution to the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

is the trivial solution, $x_1 = \cdots = x_p = 0$.

On the other hand, if there exist coefficients $c_1, \ldots, c_p$ not all equal to 0 so that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is **linearly dependent**.

The columns of A are linearly independent if and only if A has no free variables.

Example 1. *Are the vectors* $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ *and* $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *linearly independent?*

Example 2.

- *Suppose* $\mathbf{b}$ *is in the span of the columns of* A. *What can be said about the solution set of* $A\mathbf{x} = \mathbf{b}$*?*
- *Suppose now in addition that the columns of* A *are linearly independent. What can be said about the solution set of* $A\mathbf{x} = \mathbf{b}$*?*
- *Suppose instead that the columns of* A *are linearly dependent. What can be said about the solution set of* $A\mathbf{x} = \mathbf{b}$*?*

Summary:

1. Span is related to *existence* of solutions whereas linear independence is related to *uniqueness* of solutions.
2. Given an $m \times n$ matrix A:
   - if there is a ____________________ of A then $A\mathbf{x} = \mathbf{b}$ is solvable for any $\mathbf{b} \in \mathbb{R}^m$.
   - if there is a _______________________ of A then $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} \in \mathbb{R}^n$ (whenever the equation is solvable).

2 Special cases

Example 3. *When is the set* $\{\mathbf{v}_1\}$ *linearly independent? What about two vectors* $\{\mathbf{v}_1, \mathbf{v}_2\}$*?*

Example 4. *What doe it mean for two vectors to be linearly dependent geometrically?*

3 General characterization of linear independent sets

Theorem: A set $S = \{\mathbf{v}_1, \dots \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the other vectors. That is, the set S is linearly dependent if there is some $\mathbf{v}_i$ which is contained in the span of the other vectors. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$ then some $\mathbf{v}_j$ is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

Example 5. *Suppose* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ *are linearly independent. What can you say about their span? What does it mean if the new vector* $\mathbf{w} \in \mathbb{R}^3$ *makes the set* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *linearly dependent? What does it mean if the set* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is linearly independent?*

We now may have intuition for the following Theorem:

Theorem: Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. (i). If __________ then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent. (ii). If one of the $\mathbf{v}_i = \mathbf{0}$ then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

Example 6. T/F: *The columns of any* 5×4 *matrix are linearly dependent.*