Learning Objectives:

1. Use Cramer's Rule to solve systems of equations
2. Use Cramer's Rule to find a formula for inverse matrices

1 Cramer's Rule

Cramer's Rule: Let A be an invertible $n \times n$ matrix. Let

$$A_i(\mathbf{b}) = (\mathbf{a}_1 \cdots \mathbf{b} \cdots \mathbf{a}_n),$$

be the matrix A with the ith column replaced by $\mathbf{b}$. Then the unique solution $\mathbf{x}$ solving $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det A}, \text{ for all } i = 1, \ldots, n.$$

Example 1. *Solve the system*

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8,$$

using Cramer's Rule.

2 Formula for A^{-1}

We can exploit Cramer's Rule to find a formula for the inverse of A. Consider

$$A\mathbf{x} = \mathbf{e}_j.$$

By Cramer's rule,

On the other hand, multiplying both sides by A^{-1} we have

Example 2. *Find the inverse of the matrix* $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix}$.

Example 3. *Find the inverse of*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

using the adjugate.

Example 4. *Solve*

$$\begin{aligned} -5x_1 + 2x_2 &= 9 \\ 3x_1 - x_2 &= -4. \end{aligned}$$