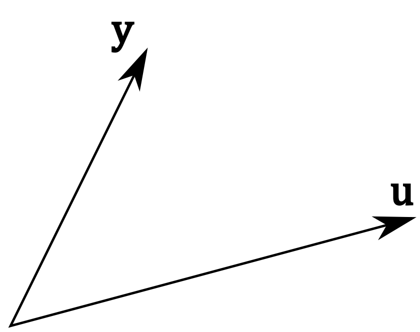
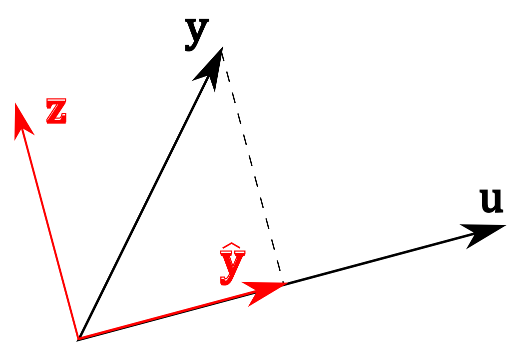
1 Orthogonal projections to lines

Example 1. *Suppose we are given two vectors* $\mathbf{u}$ *and* $\mathbf{y}$*. The* **orthogonal projection** *of* $\mathbf{y}$ *onto* $\mathbf{u}$ *is the vector* $\hat{\mathbf{y}}$ *that is the "shadow" of* $\mathbf{y}$ *cast down in the direction of* $\mathbf{u}$*.*

To make this more rigorous, the picture below shows the essential ideas: starting with $\mathbf{u}$ *and* $\mathbf{y}$*, if we drop a line segment from the tip of* $\mathbf{y}$ *perpendicular to* $\mathbf{u}$ *then we get the projection* $\hat{\mathbf{y}}$*. The vector* $\mathbf{z}$ *is the same as the perpendicular dotted line (just translated).*





So, we have

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ *is some scalar multiple of* $\mathbf{u}$*, and* $\mathbf{z}$ *and* $\hat{\mathbf{y}}$ *are orthogonal. Is there a nice formula for* $\hat{\mathbf{y}}$ *and* $\mathbf{z}$ *depending only on the starting vectors (*$\mathbf{y}$ *and* $\mathbf{u}$*)? Yes!*

Remark: The orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$ can be interpreted in the following equivalent way. Let $L = \text{span}\{\mathbf{u}\}$. That is, L is the line spanned by $\mathbf{u}$. Then $\hat{\mathbf{y}}$ is the point on the line L that is closest to $\hat{\mathbf{y}}$. Some people use the notation $\text{proj}_L(\mathbf{y})$ for this, so we have

$$\hat{\mathbf{y}} = \text{proj}_L(\mathbf{y}).$$

Also, notice that since $\mathbf{z}$ is perpendicular to L, it must be in the orthogonal complement, $L^\perp$.

Example 2. *Find a formula for* $\hat{\mathbf{y}}$ *from the example above.*

Solution. From our picture we want $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some value of α. To determine the value of α we need, take the dot product of both sides of the equation with $\mathbf{u}$. Then

$$\mathbf{y} \cdot \mathbf{u} = \alpha\mathbf{u} \cdot \mathbf{u} + \mathbf{z} \cdot \mathbf{u}.$$

From the above, we also want $\mathbf{z}$ to be perpendicular to $\mathbf{u}$ so we must have $\mathbf{z} \cdot \mathbf{u} = 0$! Thus we can solve our equation for α:

$$\alpha = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}.$$

That is, we have determined a formula for the projection. The **orthogonal projection** of $\mathbf{y}$ onto the vector $\mathbf{u}$ is given by

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}.$$

Once we know $\hat{\mathbf{y}}$ then the orthogonal component is easy to compute:

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}.$$

Take-away: Given vectors $\mathbf{y}$ and $\mathbf{u}$, the projection of $\mathbf{y}$ onto $\mathbf{u}$ is the point on the line spanned by $\mathbf{u}$, called L, that is closest to $\mathbf{y}$. We have

$$\hat{\mathbf{y}} = \mathrm{proj}_L(\mathbf{y}) = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u},$$

and the orthogonal component $\mathbf{z}$ is $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

Example 3. *Let* $\mathbf{y} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ *and* $\mathbf{u} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$. *Compute the orthogonal projection of* $\mathbf{y}$ *onto* $\mathbf{u}$. *In addition, compute the distance from* $\mathbf{y}$ *to the line spanned by* $\mathbf{u}$.

Solution. From the formulas above, the orthogonal projection is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{30}{100}\mathbf{u} = \frac{3}{10}\begin{pmatrix} 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 12/5 \\ 9/5 \end{pmatrix}.$$

The distance from $\mathbf{y}$ to the line spanned by $\mathbf{u}$ can be found by first finding the orthogonal vector

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 12/5 \\ 9/5 \end{pmatrix} = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}.$$

The distance between $\mathbf{y}$ and the line spanned by $\mathbf{u}$ is simply the length of this vector $\mathbf{z}$:

$$\|\mathbf{z}\| = \sqrt{(3/5)^2 + (-4/5)^2} = \sqrt{1} = 1.$$