1 Orthogonal sets

Definition: A set $\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ is said to be an **orthogonal set** if each pair of distinct vectors is orthogonal: $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$.

Definition: We say that a set $\{u_1, \ldots u_p\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors (all have norm 1).

Example 1. *Show that the set* $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *is orthogonal, where*

$$\mathbf{u}_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \ \mathbf{u}_3 = \begin{pmatrix} -1/2 \\ -2 \\ 7/2 \end{pmatrix}.$$

Is the set orthonormal?

Solution. We can compute

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = (-3) + (2) + (1) = 0,$$

$$\mathbf{u}_2 \cdot \mathbf{u}_3 = (1/2) + (-4) + (7/2) = 0,$$

and

$$\mathbf{u}_1 \cdot \mathbf{u}_3 = (-3/2) + (-2) + (7/2) = 0.$$

Thus all the vectors are pairwise orthogonal and thus this forms an orthogonal set. This set is *not* orthonormal because the vectors are not all length 1. For example we see

$$\|\mathbf{u}_1\| = \sqrt{3^2 + 1^2 + 1^2} = \sqrt{11}.$$

We could make this set orthonormal by normalizing all of the vectors! Computing

$$\|\mathbf{u}_2\| = \sqrt{6}$$

and

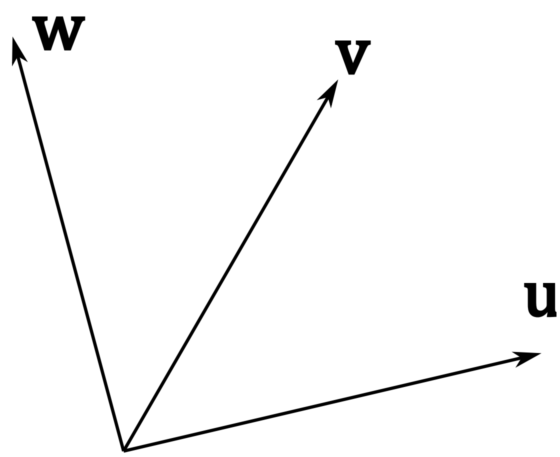
$$\|\mathbf{u}_3\| = \sqrt{33/2},$$

we thus see that the new set

$$\left\{ \begin{pmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \begin{pmatrix} -1/(2\sqrt{33/2}) \\ -2/\sqrt{33/2} \\ 7/(2\sqrt{33/2}) \end{pmatrix} \right\}$$

is orthonormal.

Intuition: Linearly independent sets are sets where no vectors are in the span of the remaining vectors. Intuitively, this means that all vectors "point in different directions." A set of orthogonal vectors is the **extreme** version of this: we not only want vectors to point in different directions, we want them to all be perpendicular. In some sense, they are pointing in *the most different directions* possible. For example: in the picture below we see that the set $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, but it is not orthogonal. On the other hand, the set $\{\mathbf{u}, \mathbf{w}\}$ is not just linearly independent, it is orthogonal! Thus, *orthogonality is an even stronger requirement!*



Theorem: If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and thus is a basis for the subspace spanned by S.