1 Row space

Learning Objectives:

1. Define row space and find a basis for the row space.

2. Define rank and nullity, and be able to find both for a matrix.

3. Relate rank and nullity in the Rank-Nullity Theorem.

4. Relate column, row, and null spaces to rank and nullity.

Definition: Given an $m \times n$ matrix, the **row space** of A, denoted Row A is the set of all linear combinations of row vectors of A.

Theorem: If two matrices $A \sim B$, then Row A = Row B. If B is in echelon form, the nonzero rows of B form a basis not only for the row space of B, but also for the row space of A.

Remark: Remember that when we solve for the column space, row operations *do* change the span of the columns, so we need to return to the original matrix to find the basis! For row space, we do not need to return to the original rows to recover the basis.

Example 1. *Find bases for the row space, the column space, and the null space of*

$$A = \begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix}$$

knowing that

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 2. T/F: *The row space and column space always have the same dimension.*

Definition: The **rank** of A is ______________________________. The **nullity** of A is __________________________.

Example 3. *Let A be an $m \times n$ matrix. What do pivots and free variables tell you about the rank and nullity? Propose a rule for "rank + nullity."*

Rank-Nullity Theorem: If A is an $m \times n$ matrix, then

Example 4. *If the null space of a 5×6 matrix is 4-dimensional, what is the dimension of the row space of A?*

Example 5. *If A is a 5×7 matrix, then what is the largest possible rank of A?*

Example 6. *Why can't a 6×9 matrix have a 2 dimensional null space?*

These results additionally add a few statements to the invertible matrix theorem.

Theorem: Let A be $n \times n$. The following are equivalent to A being invertible:

1. The columns of A form
2. $\operatorname{Col} A =$
3. $\dim \operatorname{Col} A =$
4. $\operatorname{rank} A =$
5. $\operatorname{Nul} A =$
6. $\dim \operatorname{Nul} A =$

Example 7. *Let A be $m \times n$. Suppose that $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$. Show that $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.*