1 Inverse of a matrix

Motivation: Suppose you had to solve

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad A\mathbf{x} = \mathbf{b}_3$$

for the same matrix A but different vectors $\mathbf{b}_i$. Of course you could row reduce each of these one by one. This could take a long time.
If we think back to algebra class, we could solve

$$5x = 1, \quad 5x = 4 \quad 5x = -3$$

each immediately

$$x = \frac{1}{5}, \quad x = \frac{4}{5} \quad x = \frac{-3}{5}.$$

In some sense, the reason these are easy to solve is because no matter the right hand side, we can immediately solve $5x = b$ by writing $x = \frac{1}{5}b$. *In other words, once we know the inverse of* 5 *(namely,* $1/5$*) we can solve many equations quickly!*

So this may inspire us to try to solve matrix equations using algebra the same way we solve real number equations... we can hope to find the *matrix inverse*.

How to define the inverse? How will we define the inverse matrix, A? We cannot write $1/A$, since that doesn't make sense. Going back to regular numbers, one way of seeing that 5 and $\frac{1}{5}$ are inverses is to multiply them: $5 \cdot \frac{1}{5} = 1$. We know that if x and y are inverses then multiplying them will result in 1. Extending this idea lets us define the inverse of a matrix!

Definition: An $n \times n$ matrix A is **invertible** or **nonsingular** if there exists an $n \times n$ matrix C satisfying

$$CA = AC = I_n.$$

The matrix C is called the **inverse** of A and it is denoted $C = A^{-1}$.
A matrix that has no inverse is called **singular**.

Remark: Notice that we only talk about square matrices when considering invertible matrices. While there are more advanced techniques to talk about inverses when dealing with non-square matrices, our course will not talk about those.

Example 1. *If* $A = \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}$ *and* $C = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}$ *then*

$$AC = \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix} \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$CA = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So, $C = A^{-1}$.

If we are able to determine the inverse, then we will be able to solve matrix equations easily.

Theorem: If A is an invertible $n \times n$ matrix, then for every $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example 2. *Using the same matrix* A *as above, solve*

$$A\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Solution. While we could solve this using row reduction, let's instead use the fact that we know the inverse matrix already from the last exercise:

$$A^{-1} = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}.$$

So,

$$\mathbf{x} = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -19 \\ 8 \end{pmatrix}.$$

This was much easier!

Remark: The power here is that even if we have several different $\mathbf{b}$, we can quickly solve the linear equation $A\mathbf{x} = \mathbf{b}$ without having to row reduce each time. Instead we just perform a matrix-vector multiplication, which is quite easy. Of course a big question that remains is: if we have a matrix A, *how* do we find its inverse?