Learning Objectives:

1. Transform a matrix into echelon or reduced echelon form
2. Use pivots and free variables to deduce information about existence and uniqueness of solutions to the linear system.

1 Row reduction revisited

We previously saw how to use row operations to solve a system of equations. The process took place in two steps:

1. **Forward phase:** work down the matrix creating 0s below certain entries
2. **Backward phase:** work back up the matrix creating 0s above certain entries

Today we will explore this algorithm more carefully, working to understand *which entries we use to create those 0s* and *what happens if not all columns in the matrix let us create these 0s?*

Definition: A matrix is in **echelon form** if it has the following three properties:

1. Any rows of all zeros are at the bottom of the matrix.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it. (leading entry means first nonzero number in a row)
3. All entries in a column below a leading entry are zeros.

A matrix that is already in echelon form is in **reduced echelon form** (or **reduced row echelon form, rref**) if

1. The leading entry in each nonzero row is 1.
2. Each leading 1 is the only nonzero entry in its column.

Example 1. *Put the matrix*

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix}$$

into echelon form and reduced echelon form.

2 Pivots and free variables

Definition: The entries corresponding to leading 1s in reduced row echelon form are called **pivot positions**. The columns with pivot positions are called **pivot columns**.

Example 2. *Reduce the following augmented matrix to rref:*

$$\begin{pmatrix} 2 & 2 & -8 & 10 \\ 0 & 1 & 1 & 4 \\ 0 & -1 & -1 & -4 \end{pmatrix}.$$

Try to understand the solution set of the linear system: is the system consistent? If so how many solutions are there? What can the pivots tell us?

Definition: Variables corresponding to pivot columns are called **basic variables**. Variables corresponding to non-pivot columns are called **free variables**.

Example 3. *Suppose a system row reduced to*

$$\begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}.$$

Describe the solution set.

3 Revisiting existence and uniqueness

Given a linear system of equations:

- Is the system consistent? That is, does there exist at least one solution?
- If a solution exists, is it unique?

Theorem: A linear system is consistent if and only if

If a linear system is consistent, then:

1. there is a *unique* solution if ____
2. there are *infinitely many* solutions if ____

As an overall review of using row reduction to solve a linear system:

Using row reduction to solve a linear system

1. Write the augmented matrix of the system
2. Use elementary row operations to obtain the equivalent echelon form (forward phase). Determine whether the system is consistent.
3. If it is consistent, continue row reducing to obtain the RREF (backward phase).
4. Write the final system of equations obtained from Step 3 so that all basic variables are expressed in terms of free variables (explicit description of solution set).

Example 4. *Find the general solution of the linear system whose augmented matrix is*

$$\begin{pmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}.$$

Example 5. *Suppose a given linear system has a* 4×7 **coefficient matrix** *with* 4 *pivots. Is the system necessarily consistent? If it is, how many solutions are there?*