Learning Objectives:

1. Compute vector arithmetic
2. Relate systems of linear equations to vector equations
3. Describe the relationship between linear combinations, span, and consistency of linear systems
4. Describe the geometric interpretation of linear combinations, span, and consistency of systems

1 Vectors

Definition: A **vector** is an ordered list of numbers. We normally name vectors using lowercase, boldface letters, or letters with arrows over them:

$$\mathbf{v} = \vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

If the vector is arranged vertically, we call it a **column vector**. If it is arranged horizontally, we call it a **row vector**, e.g.,

$$\mathbf{v} = (2 \;\; 4).$$

Example 1. *Compute* $\mathbf{u} - 2\mathbf{v}$ *where*

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}.$$

The definitions of vector addition and scalar multiplication are the "right" ones because they result in the following familiar properties:

Algebraic properties of $\mathbb{R}^n$: For all $\mathbf{u},\ \mathbf{v},\ \mathbf{w} \in \mathbb{R}^n$ and all scalars c and d,

(i) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u}$-$\mathbf{u} = \mathbf{0}$

(v) $\mathrm{c}(\mathbf{u}+\mathbf{v}) = \mathrm{c}\mathbf{u}+ \mathrm{c}\mathbf{v}$

(vi) $(\mathrm{c}+\mathrm{d})\mathbf{u} = \mathrm{c}\mathbf{u}+ \mathrm{d}\mathbf{u}$

(vii) $\mathrm{c}(\mathrm{d}\ \mathbf{u}) = (\mathrm{cd})\mathbf{u}$.

2 Linear combinations

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ and scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} =$$

is called a ____________________ of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ with ________ $c_1, c_2, \ldots, c_p$. Note that some or all c_p may be zero.

Example 2. *Suppose we have* $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ *and* $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. *Are* $\mathbf{y} = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix}$ *and* $\mathbf{z} = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$ *linear combinations of* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*?*

In general, if we are given a set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$, and some target vector $\mathbf{y}$, how can we tell if $\mathbf{y}$ a linear combination of the vectors $\mathbf{v}_1, \dots \mathbf{v}_p$?

Example 3. *Let* $\mathbf{a}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$ *and* $\mathbf{b} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}$. *Do there exist weights* x_1 *and* x_2 *so that*

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}?$$

If so, give values of x_1 *and* x_2 *which solve this vector equation.*

We have discovered a fundamental result of vector equations:

Theorem: A vector equation

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solutions as __________________________

In particular, $\mathbf{b}$ is a linear combination if and only if

3 Span

Given $\mathbf{v}_1, \ldots, \mathbf{v}_p$, we can ask what are **all of the possible vectors b** for which we can solve

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}?$$

Definition: The **span** of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$ is ______________________________________.
That is, it is the set of all vectors $\mathbf{b} \in \mathbb{R}^n$ which can be written

for some scalar weights $x_1, \ldots, x_p$. We denote the span by $\text{Span}\{\mathbf{a_1}, \ldots, \mathbf{a_p}\}$. We also may call this set the *subset of* $\mathbb{R}^n$ *spanned (or generated) by the vectors* $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Remark: Sometimes it is helpful to build intuition about span using colors. Imagine $\mathbf{v}_1$ represents a can of yellow paint and $\mathbf{v}_2$ represents a can of blue paint. Then, a target color $\mathbf{b}$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ if there is a way of mixing yellow and blue paints in order to get the color $\mathbf{b}$. If $\mathbf{b}$ is the color green, then it is possible, so it is in the span. If $\mathbf{b}$ is the color red, then it is not possible, so it is not in the span!

Example 4. *What is span geometrically? How could we visualize the span of 1 vector? The span of 2 vectors? What does it mean geometrically that* $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$ *is solvable?*

Example 5. **T/F***: There is a unique value of* h *so that* $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ h \end{pmatrix}$ *is in the span of* $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.