Learning Objectives:

1. Know and check the axioms of vector spaces
2. Recognize several examples of vector spaces
3. Prove a subset of a vector space is a subspace
4. Prove and use the fact that spans of vectors are subspaces of vector spaces.

1 Vector spaces

Definition: A **vector space** is a nonempty set V of elements, called **vectors**, with two operations: *vector addition* and *scalar multiplication*. The vectors and scalars (real numbers) satisfy the following axioms: for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and for all scalars $c, d \in \mathbb{R}$,

i. The sum $\mathbf{u} + \mathbf{v} \in V$ (closure under addition)

ii. For each $\mathbf{u} \in V$ and scalar c, $c\mathbf{u} \in V$ (closure under scalar mult.)

iii. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associativity)

iv. There is a zero vector in V: $\mathbf{0} + \mathbf{u} = \mathbf{u}$. (additive identity)

v. For each $\mathbf{u} \in V$ there exists $\mathbf{v} \in V$ so that $\mathbf{u} + \mathbf{v} = \mathbf{0}$ (additive inverses, we often write $\mathbf{v} = -\mathbf{u}$.)

vi. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutativity)

vii. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (scalar distribution)

viii. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (vector distribution)

ix. $c(d\mathbf{u}) = (cd)\mathbf{u}$ (compatibility of multiplication)

x. $1\mathbf{u} = \mathbf{u}$ (multiplicative identity)

Example 1. *Let* $V = \mathbb{R}^+ = \{x : x > 0\}$*. Define vector addition by*

$$x \oplus y = xy$$

and scalar multiplication by

$$c \odot x = x^c.$$

Then V *forms a vector space! Check axioms (iii), (iv), (v), and (vii).*

Example 2. *Let H be the set of points within the unit circle of the xy plane: so*

$$H = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x^2 + y^2 \leqslant 1 \right\}.$$

Equip H with regular vector addition and scalar multiplication. Show that H is not a vector space.

2 Subspaces

Oftentimes a subset of a vector space is a vector space in its own right!

Definition: A **subspace** of a vector V is a subset H of V that has three properties

1.

2. if $\mathbf{u}, \mathbf{v} \in H$ then

3. if $\mathbf{u} \in H$ and c is a scalar then

If a subset H satisfies these three properties, then H automatically satisfies all of the axioms of a vector space (since all of the other axioms still hold automatically).

Example 3.

For any vector space V, the **zero subspace** *is the space consisting of exactly the zero vector $\{\mathbf{0}\}$.*

The space $\mathbb{P}$ consisting of all polynomials (of any degree) is a vector space. It has as subspaces $\mathbb{P}_n$ for any n.

Example 4. *Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?*

Example 5. *Show that* $H = \left\{ \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} : s, t \in \mathbb{R} \right\}$ *is a subspace of* $\mathbb{R}^3$.

3 Spanning set as a subspace

The definitions of **linear combination** and **span** are the same as we saw before.

Definition: The vector $\mathbf{w} \in V$ is said to be a **linear combination** of the vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ if there exist scalars $a_1, \ldots, a_n \in \mathbb{R}$ so that

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = \mathbf{w}.$$

The **span** of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ is the set of all their linear combinations, denoted $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Theorem: Given $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$, the set $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ forms a subspace of V.

Example 6. *Let H be the set of all vectors of the form $(a - 3b, b - a, a, b)$ where $a, b \in \mathbb{R}$. Is H a subspace of $\mathbb{R}^4$?*

Example 7. *Show that* $\{f \colon f(0) = f(1)\}$ *is a subspace of* $C([0,1])$, *the set of all continuous functions* $f \colon [0,1] \to \mathbb{R}$.