**Learning Objectives:**

1. Given a non-singular square matrix, calculate its inverse.
2. Solve matrix equations using the inverse matrix.
3. Recognize and apply properties of invertible matrices.

# 1 Inverse of a matrix

**Motivation:** Suppose you had to solve

$$A\mathbf{x} = \mathbf{b}_1, \quad A\mathbf{x} = \mathbf{b}_2, \quad A\mathbf{x} = \mathbf{b}_3$$

for the same matrix $A$ but different vectors $\mathbf{b}_i$. Of course one could row reduce each of these one by one. However in algebra class we could solve

$$5x = 1, \quad 5x = 4 \;\; 5x = -3$$

each immediately

$$x = \frac{1}{5}, \quad x = \frac{4}{5} \;\; x = \frac{-3}{5}.$$

In some sense, the reason these are easy to solve is because no matter the right hand side, we can immediately solve $5x = b$ by writing $x = \frac{1}{5}b$. *In other words, once we know the inverse we can solve many equations quickly!*

So, one of our goals is to be able to solve matrix equations using algebra the same way we solve real number equations: we hope to find the *matrix inverse.*

One way of "seeing" that 5 and $\frac{1}{5}$ are inverses is to multiply them: $5 \cdot \frac{1}{5} = 1$. We know that whenever $x$ and $y$ are inverses then multiplying them will result in 1. Extending this idea lets us define the inverse of a matrix!

**Definition:** An $n \times n$ matrix $A$ is **invertible** or **nonsingular** if there exists an $n \times n$ matrix $C$ satisfying

$$CA = AC = I_n.$$

The matrix $C$ is called the **inverse** of $A$ and it is denoted $C = A^{-1}$.
A matrix that has no inverse is called **singular**.

**Example 1.** *If* $A = \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}$ *and* $C = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix}$ *then*

$$AC = \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix} \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

*and*

$$CA = \begin{pmatrix} -7 & -5 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

*So,* $C = A^{-1}$.

If we are able to determine the inverse, then we will be able to solve matrix equations easily.

> **Theorem:** If $A$ is an invertible $n \times n$ matrix, then for every $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

**Remark:** The power here is that no matter the choice of $\mathbf{b}$, we can quickly solve the linear equation $A\mathbf{x} = \mathbf{b}$ without having to row reduce each time. Instead we just perform a matrix-vector multiplication.

**Example 2.** *What can be said about the pivots of $A$ if it is invertible?*

**Solution.** Since $A\mathbf{x} = \mathbf{b}$ is always solvable, there must be a pivot in each row. Since the solution is unique, there cannot be any free variables, and so there is a pivot in each column. In fact, we see that $A$ must row reduce to the identity matrix!

For $2 \times 2$ matrices there is a very nice formula to determine the inverse:

> **Theorem:** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad - bc \neq 0$ then $A$ is invertible and
>
> $$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$
>
> If $ad - bc = 0$ then $A$ is singular.

**Remark:** The quantity $ad - bc$ is actually very important and will be generalized to larger square matrices in the next chapter. It is called the **determinant** of $A$, and is often written

$$\det A = ad - bc.$$

**Example 3.** *Solve the system*

$$3x_1 + 4x_2 = 3$$
$$5x_1 + 6x_2 = 7.$$

**Solution.** The system is equivalent to the matrix equation $A\mathbf{x} = \mathbf{b}$ where $A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$. Using the formula for inverses of $2 \times 2$ matrices, we calculate

$$A^{-1} = \frac{1}{18-20} \begin{pmatrix} 6 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 5/2 & -3/2 \end{pmatrix}.$$

Thus,

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{pmatrix} -3 & 2 \\ 5/2 & -3/2 \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}.$$

Thus $x_1 = 5$, $x_2 = -3$.

**Example 4.** *If $A$ and $B$ are invertible matrices of size $n \times n$, then which of the following is the inverse of $AB$?*

*A.* $A^{-1}B^{-1}$

*B.* $B^{-1}A^{-1}$

*C.* $A^{-1}B + AB^{-1}$

*D.* $AB$ *is singular.*

**Solution.** The answer is $B$. Indeed, by calculation we see that

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I_n$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I_n.$$

Thus,

$$(AB)^{-1} = B^{-1}A^{-1}.$$

The following Theorem lists a few facts about inverses:

**Theorem:** Let $A$ and $B$ be $n \times n$ invertible matrices. Then

1. $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$.
2. $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$.
3. $(AB)^{-1} = B^{-1}A^{-1}$.

## 2 Elementary Matrices

Recall the elementary operations for row reducing: *replacement, scaling, swapping.* It turns out that each of these operations correspond to matrix multiplication:

**Example 5.** *Let* $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *and* $E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *Given an arbitrary matrix* $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, *compute the matrices* $E_1A$ *and* $E_2A$.

**Solution.** We compute

$$E_1A = \begin{pmatrix} a & b & c \\ 2a+d & 2b+e & 2c+f \\ g & h & i \end{pmatrix}, \quad E_2A = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}.$$

Thus, we see that $E_1$ corresponds to row replacement, and $E_2$ corresponds to switching.

**Example 6.** *What is the matrix for scaling the third row by* $2$*?*

**Solution.** The corresponding matrix is $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

**Example 7.** *How would we find the matrix for switching rows 1 and 2 and then scaling the third row by 2?*

**Solution.** We will simply multiply the previous matrices, since multiplying matrices lets us perform one operation after the other:

$$E_3E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

**Example 8.** *Are all elementary matrices invertible? If so, find the inverse of $E_1$.*

**Solution.** If you remember that all elementary row operations are reversible, it must be the case that all elementary matrices are invertible! For example, the inverse of $E_1$ is

$$E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We may use this fact to help us compute inverse matrices:

> **Theorem:** An $n \times n$ matrix $A$ is invertible if and only if $A$ row reduces to $I_n$. In particular, the series of elementary row operations which transforms $A$ to $I_n$ also transforms $I_n$ to $A^{-1}$.

**Proof idea:** Suppose that the sequence of elementary row operations $E_1, E_2, \ldots, E_p$ transforms $A$ into $I_n$. That is: $E_p E_{p-1} \cdots E_2 E_1 A = I_n$ (note the order of operations; first we multiply by $E_1$, then $E_2$, and so on). Since each of the $E_i$ is invertible, the product $E_p \cdots E_1$ is invertible. Multiply both sides of the equation by the inverse of the product:

$$(E_p E_{p-1} \cdots E_2 E_1)^{-1}(E_p E_{p-1} \cdots E_2 E_1)A = (E_p E_{p-1} \cdots E_2 E_1)^{-1} I_n = (E_p E_{p-1} \cdots E_2 E_1)^{-1}.$$

So,

$$A = (E_p E_{p-1} \cdots E_2 E_1)^{-1},$$

or, taking the inverse of both sides,

$$A^{-1} = E_p E_{p-1} \cdots E_2 E_1.$$

## Algorithm to find $A^{-1}$

The previous Theorem motivates the following algorithm which allows us to find $A^{-1}$:

> **Algorithm to find $A^{-1}$:** Row reduce the augmented matrix $[A \;\; I]$. If $A$ is row equivalent to $I$, then $[A \;\; I] \sim [I \;\; A^{-1}]$. Otherwise, $A$ does not have an inverse.

**Example 9.** *Find the inverse of the matrix*

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{pmatrix}.$$

**Solution.** We first write the matrix $[A \ \ I]$:

$$\begin{pmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{pmatrix}$$

After row reducing, we find

$$\begin{pmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{pmatrix}.$$

We see that $A$ only has two pivots and so it cannot be row equivalent to the identity matrix; we conclude that $A$ is not invertible.

**Example 10.** *Find the inverse of the matrix*

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}.$$

**Solution.** We row reduce $[A \ \ I]$.

$$\begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{pmatrix}.$$

Thus, since $A \sim I$ we have that $A$ is invertible and

$$A^{-1} = \begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{pmatrix}.$$