Learning Objectives:

1. Define the determinant for $n \times n$ matrices
2. Relate determinants to volume changes of linear transformations
3. Use cofactor expansions to determine the most efficient way to calculate the determinant of a matrix
4. Calculate the determinant of triangular matrices

Motivation: Given an $n \times n$ matrix, we want to extend our intuitions from the 2×2 case. So we want $\det A$ to satisfy

1. $\det A = 0$ if and only if A is not invertible.
2. $|\det A|$ is the volume of the image of the unit square/cube after applying the transformation A.

1 Determinants of 3×3 matrices

In the 2×2 case we row reduced to determine a formula for $\det A$. We do the same with an arbitrary 3×3 matrix A.

Example 1. *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Example 2. *What is the determinant of*

$$A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}?$$

Trick for 3×3 matrices. We can use the following diagonal trick to compute 3×3 matrices. Note however that this trick will **not** generalize to any other matrices.

Example 3. *Compute* $\det A$ *where*

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -2 \end{pmatrix}.$$

2 Determinants of $n \times n$ matrices

We defined the determinant of a 3×3 matrix in terms of 2×2 determinants. In general, we can *recursively* define determinants of $n \times n$ matrices using determinants of $(n-1) \times (n-1)$ submatrices.

Definition: For $n \geq 2$ the **determinant** of an $n \times n$ matrix $A = (a_{ij})$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$:

$$\det A =$$

Notation: We often will denote

$$\det(A) = |A|.$$

The **(i,j)-cofactor** of the matrix A is the number C_{ij}

$$C_{ij} =$$

Then, we can write the determinant is

$$\det A =$$

which is called the **cofactor expansion across the first row** of A.

Surprisingly, the determinant can be computed using the cofactor expansion along **any** row or column!

Theorem: The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column:

$$\det A =$$

or

$$\det A =$$

Note: The plus or minus sign of the cofactor C_{ij} depends on the following checkerboard of signs:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 4. *Compute*

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}.$$

Example 5. *Compute*

$$\begin{vmatrix} 2 & 2 & 4 & 5 \\ 0 & -2 & -1 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -5 \end{vmatrix}.$$

Definition: An **upper triangular matrix** is a matrix whose entries below the diagonal are zero. A **lower triangular matrix** is a matrix whose entries above the diagonal are zero. A **triangular matrix** is either upper or lower triangular.

Theorem: If A is a triangular matrix, then $\det A$ is

3 Volume

Theorem: If A is an $n \times n$ matrix, the area of the n-dimensional parallelogram determined by the columns of A is $|\det A|$.

Theorem: Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ have associated $n \times n$ matrix A. Given a set S with finite area/volume, then the area/volume of $T(S) = \{T(s) \mid s \in S\}$ is given by

$$(\text{area of } T(S)) =$$

Example 6. *What is the area enclosed by an ellipse given by*

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1?$$