Learning Objectives:

1. Calculate the orthogonal projection of a vector onto a subspace.
2. Interpret the orthogonal projection geometrically.

1 Orthogonal projections to lines

Take-away: Given vectors $\mathbf{y}$ and $\mathbf{u}$, the projection of $\mathbf{y}$ onto $\mathbf{u}$ is the point on the line spanned by $\mathbf{u}$, called L, that is closest to $\mathbf{y}$. We have

$$\hat{\mathbf{y}} = \mathrm{proj}_L(\mathbf{y}) = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u},$$

and the orthogonal component $\mathbf{z}$ is $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

Example 1. *Let* $\mathbf{y} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$ *and* $\mathbf{u} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. *Find the projection of* $\mathbf{y}$ *onto* $\mathbf{u}$ *and the orthogonal component* $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

2 Orthogonal projection to more general subspaces

Our previous discussion can generalize quite nicely to other subspaces!

Example 2. *Suppose that W is a plane in $\mathbb{R}^3$ spanned by the orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$. If $\mathbf{y} \in \mathbb{R}^3$, then write the orthogonal projection of $\mathbf{y}$ onto the space W.*

Theorem: (The Orthogonal Decomposition Theorem) Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y} \in \mathbb{R}^n$ can be written

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$. In fact, letting $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be any orthogonal basis of W then

$$\hat{\mathbf{y}} =$$

and $\mathbf{z} =$

Example 3. *Let* $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ *and* $\mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. *Then,* $\{\mathbf{u}_1, \mathbf{u}_2\}$ *is an orthogonal set. Letting* $W = \mathrm{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, *find the distance from* $\mathbf{y}$ *to* W.

Example 4. T/F: *If* W *has orthogonal basis* $\{\mathbf{u}_1, \mathbf{u}_2\}$ *and* $\mathbf{z}$ *is orthogonal to both* $\mathbf{u}_1$ *and* $\mathbf{u}_2$, *then* $\mathbf{z} \in W^{\perp}$.

Example 5. *Confirm that if* $W \subseteq \mathbb{R}^n$ *has orthogonal basis* $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ *and* $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ *with* $\hat{\mathbf{y}} \in W$ *the orthogonal projection, then* $\mathbf{z} \in W^{\perp}$*.*