Topics: Maximum and minimum values, Extreme Value Theorem, local and absolute extrema, critical points, Closed Interval Method, First/Second derivative tests, concavity, inflection points, sketching functions

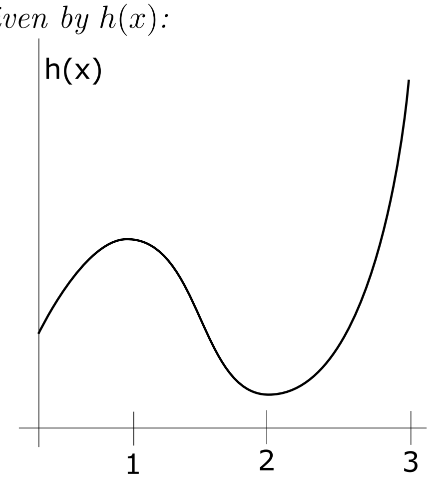
Maximum and minimum values.

Important applications of calculus are *optimization* problems: questions concerned with maximizing or minimizing some quantity. Optimization appears in all fields:

- (Physics/Engineering) Build a bridge that minimizes stresses.
- (Chemistry) Minimizing the energy in an arrangement of atoms in a molecule.
- (Biology/Medicine) Determine a treatment plan for patients that maximizes effectiveness while minimizing dosage.
- (Business) Build a consumer product while maximizing profits.

Although these problems all come from different areas, the mathematics underlying each is the same: find the x that maximizes/minimizes a function output $f(x)$.

Example 1. *Suppose you ride a bike from your apartment to TCNJ and your height after x miles is given by $h(x)$:*



Where do you achieve a maximum along the route?

If a number a is in a set A then we write $a \in A$ and say that "a is in A" or that "a is contained in A."

E.g., $x \in [0, 1]$ means that x is in the interval $[0, 1]$: $0 \leq x \leq 1$
$x \in (0, \infty)$ means that $0 < x$.

Definition: Let f be a function with domain D. Let $c \in D$ (c is a number in the domain of f). Then $f(c)$ is the

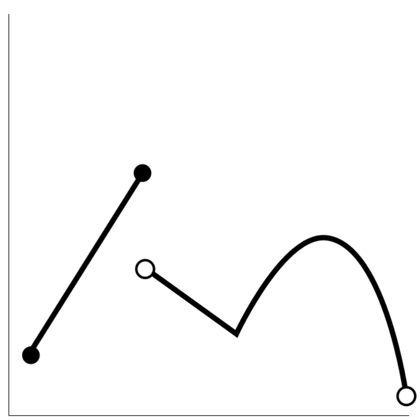
- *absolute/global maximum value* if
- *absolute/global minimum value* if

The number $f(c)$ is a

- *local maximum value* if
- *local minimum value* if

Remark: Local max/min cannot occur on endpoints of domains, since they require checking both left/right sides of c.

Example 2. *Label that local/global max and min in the following graph.*



How to find local extrema

Example 3. *Is finding local max/min the same as finding where $f'(x) = 0$?*

Critical numbers

Definition: A *critical number* of a function f is a number c in the domain of f such that

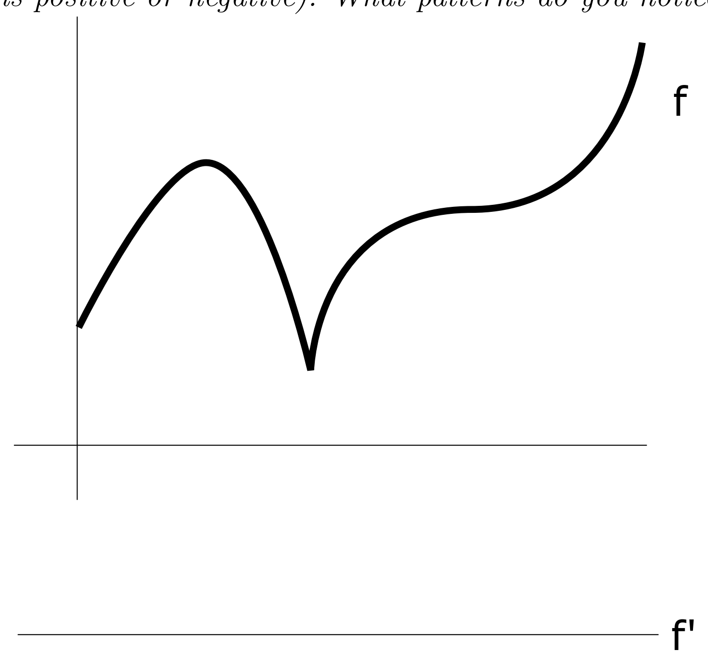
Example 4. *Find the critical numbers of* $f(x) = \frac{x^2}{x+1}$.

Example 5. *Find the critical number of the following function:*

$$h(t) = t^{3/4} - 6t^{1/4}.$$

> **Important:** Critical numbers are only *candidates* for local max/min! Must do more work to determine if they are max/min/neither.

Example 6. *Find all critical numbers of the following function. Then determine the* sign *of f' on each interval between critical numbers (you don't have to sketch the entire graph of f', just determine if it is positive or negative). What patterns do you notice?*



First derivative test. Suppose that c is a critical number of a continuous function f.

1. If f' goes from ______________ to ______________ at c, then $f(c)$ is a local __________.
2. If f' goes from ______________ to ______________ at c, then $f(c)$ is a local __________.
3. If f' __________________________________, then $f(c)$ is ______________.

Example 7. *Let* $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 + 4$. *Determine the intervals where* f *is increasing and decreasing.*

Example 8. *Using the same* $f(x)$*, identify the local max and min.*

Global max/min

Techniques from the last section let us find all of the local max/min. How do we find the global ones? First, we must determine *when* global ones exist!

Example 9. *Is there a function that is...*

			Y/N
continuous	*on domain* $(0, 1)$	*without a global max and min?*	
discontinuous	*on domain* $[0, 1]$	*without a global max and min?*	
continuous	*on domain* $[0, 1]$	*without a global max and min?*	

The Extreme Value Theorem.

Theorem. **(The Extreme Value Theorem)** If f is ____________________ on the ____________________, then f attains both an ____ at numbers $c, d \in [a, b]$.

Closed interval method

Given a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at its critical numbers in (a, b).
2. Compute the values $f(a)$ and $f(b)$ (endpoints).
3. Compare all values from steps 1 and 2. The largest value is the absolute maximum and the smallest value is the absolute minimum of f on $[a, b]$.

Example 10. *Find the absolute maximum and minimum of*

$$f(x) = x^3 - 3x^2 + 1$$

on the interval $[-1/2, 4]$.

Example 11. *Use the closed interval method on*

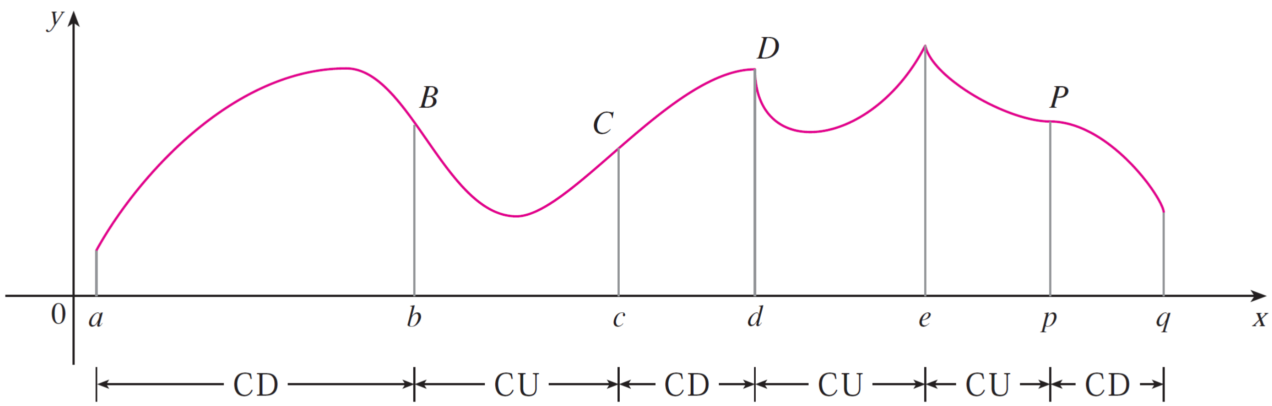
$$\frac{x^3}{x^2 - 1},$$

on the interval $(-1, 1)$. *What do you notice?*

The Second Derivative.

1. If $f''(x) > 0$ for all $x \in I$ then f is ________________________ on I
2. If $f''(x) < 0$ for all $x \in I$ then f is ________________________ on I.

Example 12. *Concavity visual*



A point P on the curve $y = f(x)$ is called an ________________________ if f is continuous at P and the curve changes from concave upward to downward (or vice versa).

Example 13. *Find intervals of concavity and inflection points of*

$$x^3 - 12x + 2.$$

Example 14. *Sketch a graph of a function f satisfying all of the following conditions:*

1. $f'(x) > 0$ *on* $(-\infty, 1)$ *and* $f'(x) < 0$ *on* $(1, \infty)$.

2. $f''(x) > 0$ *on* $(-\infty, -2) \cup (2, \infty)$, $f''(x) < 0$ *on* $(-2, 2)$

3. $\lim_{x \to -\infty} f(x) = -2$ *and* $\lim_{x \to \infty} f(x) = 0$

Difference between inflection points and critical points: Critical numbers are places where $f'(x) = 0$ or f' does not exist. They may or may not correspond to max/min. Inflection points are places where f'' changes sign. It is necessary that $f''(x) = 0$ or $f''(x)$ does not exist, but it is not sufficient.

Example 15. *Suppose a function f has second derivative*

$$f''(x) = (x-1)^2(x+2)(x-3).$$

Find the inflection points and the intervals of concavity.

Example 16. *Sketch*

$$y = \frac{2x^2}{x^2 - 1}.$$