Learning Objectives:

1. Determine whether a matrix is diagonalizable or not.

2. If it is, diagonalize the matrix.

3. Describe why diagonalization is a useful technique for simplifying calculations in applications.

1 Diagonalization

Recall: Remember that two matrices A and B are similar if there exists an invertible P so that $A = PBP^{-1}$.

Definition: We say that a square matrix A is **diagonalizable** if it is similar to a diagonal matrix.

Motivation: We saw in the reading that if a matrix A is diagonalizable, then calculating A^k is very easy, since we can write $A = PDP^{-1}$ where D is diagonal, and thus $A^k = PD^kP^{-1}$.

Problems:

- How do we know when a matrix is diagonalizable?

- Even if we know it is diagonalizable, how do we find the matrix P that transforms A to the diagonal matrix?

Example 1. *In the reading you used the matrices*

$$A = \begin{pmatrix} 7 & 4 \\ -2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}.$$

You saw that by taking

$$P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

and that

$$A = PDP^{-1}.$$

So A is diagonalizable.

Example 2. *Calculate the eigenvalues and eigenvectors of A from above. What does this show you?*

Example 3. *(a). The eigenvectors of* $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ *are*

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \textit{and } \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

corresponding to the eigenvalues 2 *and* 1 *respectively. What does this show?*

(b). The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *has its* 1*-eigenspace spanned by*

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and its 2*-eigenspace spanned by*

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In this case A *is not diagonalizable. Why do you think not?*

Theorem: An $n \times n$ matrix A is diagonalizable if and only if

If A is diagonalizable with $A = PDP^{-1}$, then P is formed by

and the entries of D are

Remark: Another ways of interpreting this Theorem is to say: A is diagonalizable if and only if the n eigenvectors for A form a basis of $\mathbb{R}^n$ (since they are all linearly independent).

Theorem: Let A be $n \times n$ with eigenvalues $\lambda_1, \ldots, \lambda_p$. Then, the matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n. This happens if and only if the characteristic polynomial factors into linear factors and the dimension of the eigenspace for each λ_k is the multiplicity of λ_k.

Example 4. *Let*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\lambda = 1$ is the only eigenvalue of A, and $(A - I)$ has null space spanned only by the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Thus A is not diagonalizable.

Example 5. *Diagonalize*

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}.$$

Remember that eigenvectors corresponding to distinct eigenvalues are linearly independent. So,

Theorem: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Example 6. T/F: *The following matrix is diagonalizable:*

$$A = \begin{pmatrix} 5 & 8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 7. T/F: *A 5×5 matrix has only two eigenvalues and is diagonalizable. One of the eigenspaces has odd dimension.*