Learning Objectives:

1. Combine and review the various notions seen thus far in the course.

2. Translate notions of invertible matrices to invertible linear transformations.

1 The Theorem

The Invertible Matrix Theorem. Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, they are all either true or all false.

a. A is an invertible matrix

b. A is row equivalent to

c. A has pivot positions

d. The equation $A\mathbf{x} = \mathbf{0}$ has

e. The columns of A form

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is

g. The equation $A\mathbf{x} = \mathbf{b}$ has

h. The columns of A span

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$

j. There is an $n \times n$ matrix C such that $CA =$

k. There is an $n \times n$ matrix D such that $AD =$

l. A^T is

Example 1. *Show that (j) implies (d).*

Example 2. **T/F:** *The matrix*

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{pmatrix}$$

is invertible.

Example 3. *Suppose A is 3×4 and that $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^3$. Does $A\mathbf{x} = \mathbf{0}$ have a unique solution?*

2 Invertible Linear Transformations

Definition: A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there exists a function $S\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = T(S(\mathbf{x})) = \mathbf{x}.$$

Example 4. *Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation that is one-to-one. Show that T is onto.*

Remark. For those who have seen discrete math, notice the distinction here: if a linear transformation T is one-to-one then it is invertible. This is not the case usually!

Example 5. *An $n \times n$* **upper triangular matrix** *is one whose entries below the diagonal are 0's. When is a square upper triangular matrix invertible?*