Learning Objectives:

1. Compute sums, scalar multiples, products, and transposes of matrices.

2. Recognize and apply properties of the above matrix operations.

1 Matrix operations

Definition: Given the matrix $A = (a_{ij})$, we call the entries $a_{11}, a_{22}, a_{33}, \ldots$ the **diagonal entries** of A. A **diagonal matrix** is an $n \times n$ matrix whose only non-zero entries are on the diagonal.

E.g. The identity matrix $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ is a diagonal matrix.

If A and B are matrices of the same size (same number of rows and columns), then their **sum** $A + B$ is the matrix obtained by summing the entries of A and B component-wise.

Given a matrix A and a scalar r, their **scalar multiplication** is the matrix rA obtained by multiplying each entry in A by r.

Theorem: Let A, B, C be matrices of the same size, and r, s be scalars. Then

1. $A + B = B + A$

2. $(A + B) + C = A + (B + C)$

3. $A + 0 = A$

4. $r(A + B) = rA + rB$

5. $(r + s) = rA + sA$

6. $r(sA) = (rs)A$

Here, 0 represents the zero matrix: the matrix whose entries are all zero.

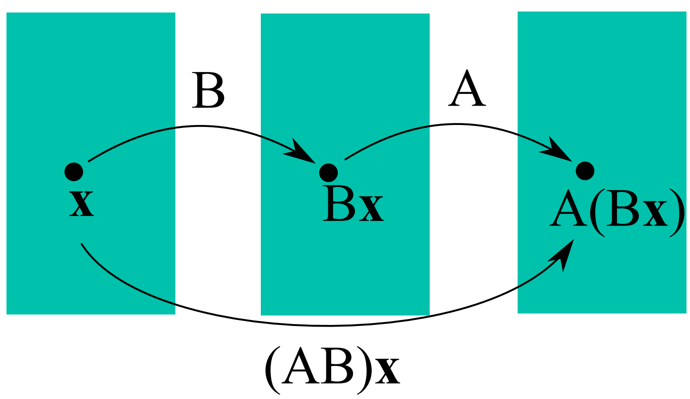
Example 1. *Let* $A = \begin{pmatrix} 1 & -2 \\ 4 & 4 \end{pmatrix}$. *Compute* $2A - 2I_2$.

2 Matrix Multiplication

Given A and B, two matrices, what should AB mean? We hope that the resultant matrix AB would have the property that

$$(AB)\mathbf{x} = A(B\mathbf{x})$$

for any vector $\mathbf{x}$. As a picture:



Example 2. *Suppose* A *is an* $m \times q$ *matrix and* B *is a* $p \times n$ *matrix.*

1. *What size must the vector* $\mathbf{x}$ *be in order for* $B\mathbf{x}$ *to be computed?*

2. *What size vector will* $B\mathbf{x}$ *be?*

3. *Are there any restrictions on the size of* A *so that* $A(B\mathbf{x})$ *makes sense?*

4. *What size vector will* $A(B\mathbf{x})$ *be?*

5. *What size matrix do you think* AB *will be?*

Definition: If A is size $m \times p$ and $B = (\mathbf{b}_1 \ \dots \ \mathbf{b}_n)$ is size $p \times n$, then the **matrix product** AB is the $m \times p$ matrix defined by

$$AB = (A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_n).$$

Practically: to find the i, jth entry of AB, compute the *dot product* of the ith row of A with the jth column of B.

Example 3. *If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?*

Example 4. *Let* $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 4 & 3 \\ 3 & 1 \\ 7 & -2 \end{pmatrix}$. *Compute AB and BA.*

Remark: In a better world, we would call AB the matrix *composition*! Thinking about A and B as transformations, AB is the transformation which first applies B and then applies A. An important observation here is that, in general, compositions are not commutative (perhaps you recall that $f \circ g \neq g \circ f$ in general). Thus, in general:

$$AB \neq BA.$$

However, other very nice properties of matrix multiplication still hold:

Theorem: Let A be $m \times n$, and B and C be appropriate sizes so that products/sums are defined:

1. $A(BC) =$
2. $A(B+C) =$
3. $(B+C)A =$
4. $r(AB) =$
5. $I_m A = A = AI_n$

where I_m, I_n are identity matrices.

If A is a square matrix, then we can multiply A by itself as many times as we would like:

$$A^k = A \cdots A.$$

If $k = 0$ then we interpret $A^0 = I_n$.

Example 5. *Suppose that $AB = 0$.* **T/F:** *It must be the case that either $A = 0$ or $B = 0$.*

3 Transposes

> **Definition:** Given a matrix A of size $m \times n$, the matrix A^T is called the **transpose** and is the $n \times m$ matrix whose columns are formed from the rows of A.

Example 6. *Suppose A is $m \times p$ and B is $p \times n$. Why can't $(AB)^T = A^T B^T$? Can you determine the correct formula?*

Theorem: Let A and B have appropriate sizes. Then:

1. $(A^T)^T =$
2. $(A+B)^T =$
3. For any scalar r, $(rA)^T =$
4. $(AB)^T =$

Example 7. *Suppose* $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ *and* $C = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$. *Check that* $AB = AC$. *What is surprising about the "algebra" of matrices that is different from algebra of numbers?*