Learning Objectives:

1. Calculate null and column spaces of a given matrix.
2. Find a matrix whose column space matches a given subspace.
3. Compare and contrast null and column spaces for a matrix.
4. Extend these concepts to more general linear transformations on vector spaces.

1 Null space of a matrix

It turns out that most things we have been studying this semester are subspaces! We already know that the span of vectors forms a subspace. The first place we saw span of vectors was to describe the solution set of a system of equations!

Consider the system of linear equations

$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -5x_1 + 9x_2 + x_3 &= 0. \end{aligned}$$

In matrix form, this system is written $A\mathbf{x} = \mathbf{0}$ with

$$A = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}.$$

We know the *solution set* of this equation is the set of all $\mathbf{x}$ solving it.

Definition: The **null space** of an $m \times n$ matrix A, written $\operatorname{Nul} A$ is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation:

$$\operatorname{Nul} A = \{\mathbf{x} \colon A\mathbf{x} = \mathbf{0}\}.$$

Note: There is no difference between "solution set" and "null space" here! This is an example of the same concept in math being named several ways. Sometimes we will call it null space to really emphasize the fact that it is, in fact, a subspace.

Theorem: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Proof. Note that $\mathbf{0} \in \mathbb{R}^n$ satisfies $A\mathbf{0} = \mathbf{0}$ so $\operatorname{Nul} A$ is always non-empty. Suppose that $\mathbf{u}, \mathbf{v} \in \operatorname{Nul} A$. Then

$$A\mathbf{u} = A\mathbf{v} = \mathbf{0}.$$

Then,

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0},$$

so $\mathbf{u} + \mathbf{v} \in \operatorname{Nul} A$. Here, we used the linearity property of the matrix A. Finally, if $c \in \mathbb{R}$ then

$$A(c\mathbf{u}) = cA\mathbf{u} = c\mathbf{0} = \mathbf{0}.$$

Thus $c\mathbf{u} \in \operatorname{Nul} A$, so $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^n$. $\square$

In the above, we see that we used both linearity properties of the matrix A to prove that $\operatorname{Nul} A$ is a subspace. Thus, you may already expect that a similar result should hold in general for linear transformations.

Example 1. T/F: *Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates (a, b, c, d) satisfy*

$$\begin{aligned} a - 2b + 5c &= d \\ c - a &= b. \end{aligned}$$

Then H a subspace of $\mathbb{R}^4$.

Solution. True. The equations can be written as a homogeneous linear system so by the previous theorem it must be a subspace.

Example 2. T/F: *Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates (a, b, c, d) satisfy*

$$\begin{aligned} a - 2b + 5c - d &= 4 \\ c - a - b &= 1. \end{aligned}$$

Then H a subspace of $\mathbb{R}^4$.

Solution. This is false. This is not a homogeneous equation and so the previous theorem does not apply. In fact, notice that the zero vector is not in H so H cannot be a subspace.

2 Calculating $\operatorname{Nul} A$

So far, we have only described the null space implicitly. However, we already know how to explicitly describe the null space! We just need to row reduce to find the solution set.

Example 3. *Find a spanning set of the null space of the matrix:*

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

Row reducing shows

$$(A\ \mathbf{0}) \sim \begin{pmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the general solution can be written

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

So, we can write a spanning set for $\operatorname{Nul} A$.

Example 4. *What is the relationship between pivots and the number of vectors in the spanning set of* $\operatorname{Nul} A$*?*

Solution. The number of vectors in the spanning set is the same as the number of free variables. So if A is $m \times n$ then the number of vectors will be n minus the number of pivots.

3 Column space of a matrix

Definition: The **column space** of an $m \times n$ matrix A, written $\operatorname{Col} A$, is the set of all linear combinations of the columns of A. That is, if $A = (\mathbf{a}_1 \ \cdots \mathbf{a}_n)$ then

$$\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

Theorem. The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$.

Example 5. *Find a matrix A so that $W = \operatorname{Col} A$ where*

$$W = \left\{ \begin{pmatrix} 6a - b \\ a + b \\ -7a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Solution. Notice that $W = \operatorname{Span} \left\{ \begin{pmatrix} 6 \\ 1 \\ -7 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. So A has these vectors as columns.

4 Contrasting the null and column spaces

Example 6. *Let*

$$A = \begin{pmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{pmatrix}.$$

In which $\mathbb{R}^k$ do $\operatorname{Col} A$ and $\operatorname{Nul} A$ live?

A. Column space: $k = 4$, Null space: $k = 3$

B. Column space: $k = 3$, Null space: $k = 4$

C. Column space: $k = 3$, Null space: $k = 3$

D. Column space: $k = 4$, Null space: $k = 4$

Solution. B.

So, if A is not square, then the null and column spaces live in different spaces completely.

Table in book. The table in the book shows several comparisons between the null and column spaces so that we do not confuse them.

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

5 Linear transformations

We very slightly generalize the notion of linear transformation:

Definition: A **linear transformation** T from vectors space V to W is a rule assigning to each $\mathbf{x} \in V$ a unique vector $T(\mathbf{x}) \in W$ so that

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad T(c\mathbf{u}) = cT(\mathbf{u}).$$

Definition: The **kernel** of T is the set of all $\mathbf{u} \in V$ so that $T(\mathbf{u}) = \mathbf{0}$. The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x} \in V$.

If V, W are $\mathbb{R}^n$ and $\mathbb{R}^m$, then the kernel and range are precise the null and column spaces respectively.

Theorem: The kernel and range of a linear transformation constitute subspaces of V and W, respectively.

Example 7. *Define* $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ *by* $T(\mathbf{p}) = \begin{pmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{pmatrix}$. *Show that* T *is a linear transformation. Moreover, find a polynomial* $\mathbf{p}$ *that spans the kernel of* T*, and find the range of* T.

Solution. Let $\mathbf{p}, \mathbf{q} \in \mathbb{P}_2$. Then

$$T(\mathbf{p}+\mathbf{q}) = \begin{pmatrix} \mathbf{p}(0)+\mathbf{q}(0) \\ \mathbf{p}(1)+\mathbf{q}(1) \end{pmatrix} = \begin{pmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{pmatrix} + \begin{pmatrix} \mathbf{q}(0) \\ \mathbf{q}(1) \end{pmatrix} = T\mathbf{p} + T\mathbf{q}.$$

Let $c \in \mathbb{R}$. Then

$$T(c\mathbf{p}) = \begin{pmatrix} c\mathbf{p}(0) \\ c\mathbf{p}(1) \end{pmatrix} = c\begin{pmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{pmatrix} = cT(\mathbf{p}).$$

Thus T is a linear transformation.

To find the kernel of T, we seek all $\mathbf{p}$ so that $T(\mathbf{p}) = \mathbf{0}$, so we need

$$\begin{pmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\mathbb{P}_2$ are all polynomials of degree at most 2, we know $\mathbf{p}(x) = ax^2 + bx + c$, so $\mathbf{p}(0) = c$ and $\mathbf{p}(1) = a + b + c$. That is, $c = 0$ and thus $a + b = 0$. So, the kernel must be polynomials of the form $\mathbf{p}(x) = ax^2 - ax = a(x^2 - x)$. Thus, taking $\mathbf{p}(x) = x^2 - x$ we see that the kernel of T is the span of $\mathbf{p}$.

The range of T is all of $\mathbb{R}^2$. To see that this is the case, let $\begin{pmatrix} y \\ z \end{pmatrix} \in \mathbb{R}^2$. We need to find a polynomial so that $\mathbf{p}(0) = y$ and $\mathbf{p}(1) = z$. Take for example $\mathbf{p}(x) = (z - y)x + y$.