Learning Objective:

1. Translate concepts of matrix multiplication to those of linear transformations.
2. Determine domain, codomain, range, and whether a vector is in the image of a transformation.
3. Introduce geometric interpretation of transformations such as projections, shears, and dilations.

1 Transformations

Definition: A **transformation** T (or **function** or **mapping** or **operator**) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each input vector $\mathbf{x} \in \mathbb{R}^n$ an output vector $T(\mathbf{x}) \in \mathbb{R}^m$. We write $T\colon \mathbb{R}^n \to \mathbb{R}^m$.

The set $\mathbb{R}^n$ is the **domain** of T.

The set $\mathbb{R}^m$ is the **codomain** of T.

A vector $\mathbf{v} \in \mathbb{R}^m$ is in the **image** of T if there exists $\mathbf{x} \in \mathbb{R}^n$ so that $T\mathbf{x} = \mathbf{v}$. The set of all images of T is the **range** of T.

Example 1. *Let* $A = \begin{pmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \end{pmatrix}$. *Then, define the transformation*

$$T(\mathbf{x}) = A\mathbf{x}.$$

What are the domain/codomain of T*? Is* $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *in the range of* T*?*

Example 2. *Let* $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$. *Define* T *by* $T(\mathbf{x}) = A\mathbf{x}$.

1. *What are the domain and codomain of* T*?*
2. *Find* $T(\mathbf{u})$
3. *Find an* $\mathbf{x} \in \mathbb{R}^2$ *whose image under* T *is* $\mathbf{b}$.
4. *Is there more than one choice of* $\mathbf{x}$ *in the previous question?*
5. *Determine if* $\mathbf{c}$ *is in the range of* T.

Example 3. T/F: *If A is an $m \times n$ matrix, then the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.*

2 Geometric transformations

Example 4. *The matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

defines a transformation T *via* $T(\mathbf{x}) = A\mathbf{x}$ *which acts* $T\colon \mathbb{R}^3 \to \mathbb{R}^3$*. This matrix is an example of a* projection*; why?*

Example 5. *Consider the matrix*

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

Then A gives rise to a shear transformation. *Consider what happens to a few vectors:*

$$\begin{aligned} \mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \quad \textit{is mapped to } A\mathbf{u} = \\ \mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \quad \textit{is mapped to } A\mathbf{v} = \\ \mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \quad \textit{is mapped to } A\mathbf{w} = \end{aligned}$$

3 Linear transformations

What if we define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\mathbf{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This transformation is *constant* and maps everything to the same output. Unlike the previous examples, this doesn't come from matrix multiplication. This gives us the question:

Question: How can we tell if a transformation T can be represented as a matrix multiplication?

Idea: Recall that matrices satisfy:

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}, \quad A(c\mathbf{u}) = c(A\mathbf{u}).$$

Thus, if we have a transformation T, for there to be a hope of representing it using matrices, it must have these same properties.

Definition: A transformation T is **linear** if the following equalities hold for all vectors $\mathbf{u}, \mathbf{v}$ and scalars c:

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad T(c\mathbf{u}) = cT(\mathbf{u}).$$

Theorem: T is a linear transformation if and only if

(i) $T(\mathbf{0}) = \mathbf{0}$,

(ii) $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.

Remark: The second property is often called the *superposition principle*, which comes up often in physics and engineering. One way to think about its importance is the following: if I already know the action of a linear transformation on vectors $\mathbf{u}$ and $\mathbf{v}$, then I know the action of the linear transformation on all linear combinations of $\mathbf{u}$ and $\mathbf{v}$ making both calculations and storage very efficient. This is not true for transformations which are not linear!

Example 6. *Show that the mapping $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = 3\mathbf{x}$ is a linear transformation. This transformation is called a* dilation *(why?)*

Question: We see that dilations are linear transformations, but how do we know if they can be written as $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A?