MAT205 **Supplementary Notes**

Section 3.2 Properties of Determinants

Learning Objectives

1. Calculate determinants using row operations

2. Explain and apply properties of determinants

1 Calculating determinants using row operations

We previously saw that if a matrix is triangular, then the determinant is easy to calculate. This motivates the following idea: if we can determine how row operations affect the determinant, then we can row reduce as an alternative way to calculate determinants.

Example 1. *Given that* $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 4$, *calculate*

$$\begin{vmatrix} a & b \\ 4a+c & 4b+d \end{vmatrix}.$$

A. 16

B. -16

C. 4

D. -4

Solution. C. The determinant does not change.

Theorem: Let A be a square matrix.

a. If B is produced by adding a multiple of one row to another, then $\det A = \det B$.

b. If B is produced by interchanging two rows, then $\det B = -\det A$.

c. If B is produced by scaling one row by k then $\det B = k \cdot \det A$.

Example 2. *Compute*

$$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}.$$

We add the second row to the third row; so,

$$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ 0 & -9 & 6 & 8 \\ 1 & -4 & 0 & 6 \end{vmatrix}.$$

We factor out a 2 from the first row:

$$\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ 0 & -9 & 6 & 8 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ 0 & -9 & 6 & 8 \\ 1 & -4 & 0 & 6 \end{vmatrix},$$

and continue row reducing:

$$2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ 0 & -9 & 6 & 8 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

So, since the matrix is upper triangular, then

$$\det A = 2(1)(3)(-6)(1) = -36.$$

Notice that using this method, $\det A \neq 0$ if and only if it has n pivots, and so we confirm

Theorem: A square matrix A is invertible if and only if $\det A \neq 0$.

2 Properties of determinants

Example 3. *One of the following statements is false! Determine which one is false, and aim to prove the true ones using geometric and/or algebraic proofs!*

1. $\det(A^T) = \det(A)$.

2. $\det(AB) = \det(A)\det(B)$

3. $\det(A^{-1}) = \frac{1}{\det(A)}$

4. $\det(A + B) = \det(A) + \det(B)$

Example 4. *In light of Example 4 part 1, we can use* column operations *instead of row operations to manipulate matrices. These will have the same effects on the determinant as row operations. Compute*

$$\begin{vmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 0 & 2 & 2 \end{vmatrix}.$$

We use column operations:

$$\begin{vmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 0 & 2 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 2 & 6 \\ 1 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 & 6 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 0$$

Finally, we may also combine cofactor expansions with row operations to try and optimize computation of the determinants.

Example 5. *Compute*

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix}.$$

We get

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{vmatrix} = 2(1)(-3)(5) = -30.$$

3 Boot camp

Example 6. *What is*

$$\begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{vmatrix}?$$

Solution. Since

$$\begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{vmatrix},$$

we see that two rows are the same, so the matrix is not invertible. Thus $\det A = 0$.

Example 7. *Compute*

$$\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 11 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix}.$$

Solution. B. One method may be:

$$\begin{aligned}
\begin{vmatrix} -1 & 2 & 3 & 0 \\ 3 & 4 & 3 & 0 \\ 11 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix} &= -\begin{vmatrix} 1 & -2 & -3 & 0 \\ 3 & 4 & 3 & 0 \\ 11 & 4 & 6 & 6 \\ 4 & 2 & 4 & 3 \end{vmatrix} \\
&= -\begin{vmatrix} 1 & -2 & -3 & 0 \\ 0 & 10 & 12 & 0 \\ 0 & 26 & 39 & 6 \\ 0 & 10 & 16 & 3 \end{vmatrix} \\
&= -\begin{vmatrix} 10 & 12 & 0 \\ 26 & 39 & 6 \\ 10 & 16 & 3 \end{vmatrix} \\
&= -2\begin{vmatrix} 5 & 6 & 0 \\ 26 & 39 & 6 \\ 10 & 16 & 3 \end{vmatrix} \\
&= -2(5 \cdot 39 \cdot 3 + 6 \cdot 6 \cdot 10 - 16 \cdot 6 \cdot 5 - 6 \cdot 26 \cdot 3) = 6
\end{aligned}$$

Another method:

$$\det A = -6\begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ 4 & 2 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 & 3 \\ 3 & 4 & 3 \\ 11 & 4 & 6 \end{vmatrix}$$

$$= 12\begin{vmatrix} 1 & -1 & -3 \\ 3 & 2 & 3 \\ 4 & 1 & 4 \end{vmatrix} - 6\begin{vmatrix} 1 & -1 & -3 \\ 3 & 2 & 3 \\ 11 & 2 & 6 \end{vmatrix}$$

$$= 12\begin{vmatrix} 1 & -1 & -3 \\ 0 & 5 & 12 \\ 0 & 5 & 16 \end{vmatrix} - 6\begin{vmatrix} 1 & -1 & -3 \\ 0 & 5 & 12 \\ 0 & 13 & 39 \end{vmatrix}$$

$$= 12\begin{vmatrix} 1 & -1 & -3 \\ 0 & 5 & 12 \\ 0 & 0 & 4 \end{vmatrix} + 6 \cdot 13\begin{vmatrix} 1 & -1 & -3 \\ 0 & 1 & 3 \\ 0 & 5 & 12 \end{vmatrix}$$

$$= 12\begin{vmatrix} 1 & -1 & -3 \\ 0 & 5 & 12 \\ 0 & 0 & 4 \end{vmatrix} + 6 \cdot 13\begin{vmatrix} 1 & -1 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{vmatrix}$$

$$= 12(20) + 6(13)(-3) = 6$$

Example 8. *Suppose that A is a square matrix satisfying $A^2 = I$. Prove that* $\det A = \pm 1$.