Learning Objectives:

1. Compute matrix vector products.
2. Relate matrix equations to vector equations/systems of linear equations.
3. Determine consistency of matrix equations to pivots, span of columns, and linear combinations.

1 Matrix equations

If A is an $m \times n$ matrix (m rows, n columns) with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and $\mathbf{x} \in \mathbb{R}^n$, then

$$Ax = (\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n.$$

Example 1. *An alternate way to multiply a matrix and a vector is using the "row-vector rule." Let's multiply* $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix}$ *and* $\mathbf{x} = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$.

Example 2. *If we have a* $1 \times n$ *vector* $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *and an* $n \times 1$ *vector* $\mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$, *then compute* $\mathbf{vw}$.

Theorem: If A is an $m \times n$ matrix, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and c is a scalar, then

1. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$.
2. $A(c\mathbf{u}) = c(A\mathbf{u})$.

2 Consistency of matrix equations

Example 3. *If* $A = \begin{pmatrix} 2 & 3 & 1 \\ -2 & -1 & 0 \\ 5 & 8 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, *and* $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ *then write* $A\mathbf{x} = \mathbf{b}$ *as a vector equation. What connections to previous concepts do you see?*

Theorem. Let A be an $m \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$. Then, the solution set of the vector equation

is the same as the solution set of the linear system with augmented matrix

which, in turn, is the same as the solution set of the matrix equation

In particular, we have

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if

Example 4. *Let* $A = \begin{pmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{pmatrix}$ *and* $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. *Is the equation* $A\mathbf{x} = \mathbf{b}$ *consistent for all possible* b_1, b_2, b_3*? That is, for any* $\mathbf{b}$*, does there exist an* $\mathbf{x} \in \mathbb{R}^n$ *solving the matrix equation?*

Hint: *Row reduce the associated augmented matrix and consider what happens when plugging different values of* b_1, b_2, b_3.

Example 5. *Thinking on the previous result, what does it mean if A has a pivot in every row of its coefficient matrix?*

A. The equation $A\mathbf{x} = \mathbf{b}$ is solvable for all $\mathbf{b} \in \mathbb{R}^m$.

B. Each $\mathbf{b} \in \mathbb{R}^m$ can be written as a linear combination of the columns of A.

C. The columns of A span $\mathbb{R}^m$.

D. All of the above.

Example 6. T/F: *A set of 3 vectors in $\mathbb{R}^4$ can span all of $\mathbb{R}^4$.*