We previously saw how to use row operations to solve a system of equations. The process took place in two steps:

1. **Forward phase:** work down the matrix creating 0s below certain entries
2. **Backward phase:** work back up the matrix creating 0s above certain entries

This section will focus on exploring that process more deeply, including introducing some vocabulary to describe various parts of the algorithm.

Important: *You may need to read the following definition a few times! I recommend to try reading it once or twice, and then continue reading. Then come back to this definition after you have a few more examples in mind to see if it makes more sense.* This is very common in mathematics. Reading something may take several attempts and a few examples before it starts to "click."

Definition: A matrix is in **echelon form** if it has the following three properties:

1. Any rows of all zeros are at the bottom of the matrix.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it. (leading entry means first nonzero number in a row)
3. All entries in a column below a leading entry are zeros.

A matrix that is already in echelon form is in **reduced echelon form** (or **reduced row echelon form, rref**) if

1. The leading entry in each nonzero row is 1.
2. Each leading 1 is the only nonzero entry in its column.

Example 1. *The following matrices are in echelon form. The leading entries* ($\blacklozenge$) *can be any nonzero number. The numbers to the right of leading entries* ($*$) *can be any number (zero or nonzero).*

$$\begin{pmatrix} \blacklozenge & * & * & * \\ 0 & \blacklozenge & * & * \\ 0 & 0 & 0 & \blacklozenge \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \blacklozenge & * & * & * & * & * \\ 0 & 0 & \blacklozenge & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacklozenge & * \end{pmatrix}$$

The following matrices are in reduced echelon form. Now leading entries must be 1 and leading entries can be the only non-zero number in the entire column.

$$\begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}$$

Example 2. *The matrix*

$$\begin{pmatrix} 1 & -3/2 & 1 & 1/2 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is in echelon form, but not in reduced echelon form.

Given any matrix, we can always use row operations to reduce it to echelon form or reduced echelon form. Given a matrix, while there may be many different echelon forms we could create, the reduced echelon form is always unique. This could be surprising, considering the different ways we could row reduce in different orders. It is good that it always ends up creating the same final result!

Take-away: The forward phase of row reduction results in an echelon form of the matrix. The backward phase results in the reduced echelon form.

Definition: The entries corresponding to leading 1s in reduced row echelon form are called **pivot positions**. The columns with pivot positions are called **pivot columns**.

Which columns in the Example 1 matrices have pivots? How many pivots can an $m \times n$ matrix have?