Learning Objectives:

1. Given a non-singular square matrix, calculate its inverse.

2. Solve matrix equations using the inverse matrix.

3. Recognize and apply properties of invertible matrices.

1 Inverse of a matrix

Definition: An $n \times n$ matrix A is **invertible** or **nonsingular** if there exists an $n \times n$ matrix C satisfying

$$CA = AC = I_n.$$

The matrix C is called the **inverse** of A and it is denoted $C = A^{-1}$.
A matrix that has no inverse is called **singular**.

Theorem: If A is an invertible $n \times n$ matrix, then for every $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example 1. *What can be said about the pivots of A if it is invertible?*

For 2×2 matrices there is a very nice formula to determine the inverse:

Theorem: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $ad - bc \neq 0$ then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If $ad - bc = 0$ then A is singular.

Remark: The quantity $ad - bc$ is actually very important and will be generalized to larger square matrices in the next chapter. It is called the **determinant** of A, and is often written

$$\det A = ad - bc.$$

Example 2. *Solve the system*

$$\begin{aligned} 3x_1 + 4x_2 &= 3 \\ 5x_1 + 6x_2 &= 7. \end{aligned}$$

Example 3. *If A and B are invertible matrices of size $n \times n$, then what is the inverse of AB?*

Theorem: Let A and B be $n \times n$ invertible matrices. Then

1. A^{-1} is invertible and $(A^{-1})^{-1} =$.
2. A^T is invertible and $(A^T)^{-1} =$.
3. $(AB)^{-1} =$.

2 Elementary Matrices

Recall the elementary operations for row reducing: *replacement, scaling, swapping.* It turns out that each of these operations correspond to matrix multiplication:

Example 4. *Let* $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$*, and* $E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$*. Given an arbitrary matrix* $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$*, compute the matrices E_1A and E_2A.*

Example 5. *What is the matrix for scaling the third row by* 2*?*

Example 6. *How would we find the matrix for switching rows 1 and 2 and then scaling the third row by 2?*

Example 7. *Are all elementary matrices invertible? If so, find the inverse of* E_1*.*

Theorem: An $n \times n$ matrix A is invertible if and only if

3 Algorithm to find A^{-1}

Algorithm to find A^{-1}: Row reduce the augmented matrix $[A \;\; I]$. If A is row equivalent to I, then $[A \;\; I] \sim [I \;\; A^{-1}]$. Otherwise, A does not have an inverse.

Example 8. *Find the inverse of the matrix*

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{pmatrix}.$$

Example 9. *Find the inverse of the matrix*

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{pmatrix}.$$