Learning Objectives:

1. Define the determinant for $n \times n$ matrices
2. Relate determinants to volume changes of linear transformations
3. Use cofactor expansions to determine the most efficient way to calculate the determinant of a matrix
4. Calculate the determinant of triangular matrices

Recall the determinant for a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is $\det A = ad - bc$. Sometimes we use the notation $\Delta = ad - bc$. We previously saw that A is invertible if and only if $\det A \neq 0$. However this was not at all motivated or explained, so let's pause and try to understand

1. Why $\det A$ is calculated the way it is, and
2. How to interpret $\det A$ in a geometric way.

Together, these will help us understand how to define $\det A$ for larger matrices.

1 Deriving the formula

So far we have seen:

$$\boxed{\det A = ad - bc \neq 0 \text{ if and only if } A \text{ is invertible.}}$$

Where does the formula for the determinant come from? In the 2×2 case notice that given an arbitrary 2×2 matrix, we can row reduce (and we will assume $a \neq 0$ and $c \neq 0$ for simplicity)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} ac & bc \\ ac & ad \end{pmatrix} \sim \begin{pmatrix} ac & bc \\ 0 & ad - bc \end{pmatrix}.$$

For this to be invertible we need 2 pivots, so we need $ad - bc \neq 0$. This idea shows us how we can derive the formula $\Delta = ad - bc$ using the connection to *invertibility*.

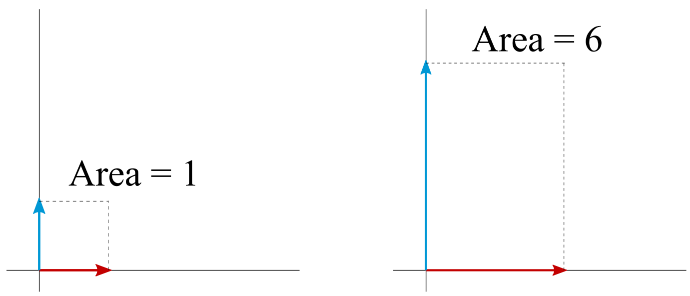
2 Geometric intuition: volume change

Now, we interpret the determinant geometrically: for simplicity, consider the diagonal matrix $A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. This corresponds to the linear transformation

$$T(x_1, x_2) = (ax_1, dx_2),$$

which scales all vectors by a in the $\mathbf{e}_1$ direction and by d in the $\mathbf{e}_2$ direction.

We will consider what happens to the unit square with vertices at $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2$ before and after applying the transformation T. To have specific numbers, take $a = 2$ and $d = 3$. We start from a square of area 1, and after applying T we get a rectangle of area 6. It is no coincidence that $\det A = ad - bc = (2)(3) - 0 = 6$.



In fact, even if b and c are non-zero, this interpretation still holds (you are welcome to explore what happens with a few examples!) This leads to the following important geometric intuition for the determinant.

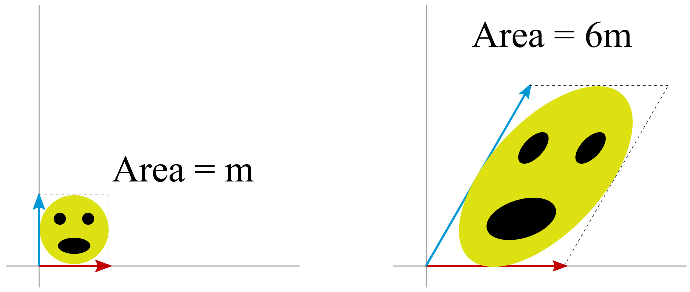
> The value of $\det A$ is the area of the unit square after applying the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Remarks:

- A negative determinant corresponds to "flipping" the plane (for example, leaving $\mathbf{e}_2$ in place but mapping $\mathbf{e}_1$ to $-\mathbf{e}_1$). In this case, the absolute value of the determinant still tells us the resultant area of the unit square.

- It turns out that this "area scaling" property is true *for any starting shape*. So if you start with any shape with area m then after the transformation A the image of the shape will have area $|\det A| \cdot m$. The figure below shows the effect of a linear transformation (that both scales and shears). The areas of both the unit square and an emoji face are scaled by the same factor.

  Another way of interpreting the idea above is



Theorem: If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.

Example 1. *Calculate the area of the parallelogram whose vertices are* $(-2,0),(0,3),(1,3),(-1,0)$. *We shift the points so that one of them is the origin, for example, adding* $(2,0)$ *to every point:* $(0,0),(2,3),(3,3),(1,0)$. *Then, note that* $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ *and* $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *correspond to two sides of the parallelogram. Then the area is given by*

$$|\mathbf{u} \ \ \mathbf{v}| = |-3| = 3.$$

What do you think it means that $\det A = 0$ in terms of linear transformations? Why does this mean that the matrix (and linear transformation) is not invertible?

3 3×3 matrices

Given an $n \times n$ matrix, we want to define $\det A$ in such a way that

1. $\det A = 0$ if and only if A is not invertible.
2. $|\det A|$ is the volume of the image of the unit square/cube/? after applying the transformation A.

In the 2×2 case we row reduced to determine a formula for $\det A$. We do the same with an arbitrary 3×3 matrix A.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

After some calculations one can confirm that

$$A \sim \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{pmatrix},$$

where $\Delta = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$.
Note that after some regrouping we can write Δ as

$$\begin{aligned} \Delta &= a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \cdot \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \cdot \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \\ &= a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + a_{13} \cdot \det A_{13}, \end{aligned}$$

where A_{ij} is the matrix A with the ith row and jth column deleted. This is the determinant!

Example 2. *What is the determinant of*

$$A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}?$$

Solution.

$$\det A = 1 \cdot (4 \cdot 0 - 2) - 5 \cdot (0 - 0) + 0 \cdot (-4 - 0) = -2.$$

Trick for 3×3 matrices. We can use the following diagonal trick to compute 3×3 matrices. Note however that this trick will **not** generalize to any other matrices.

Example 3. *Compute* $\det A$ *where*

$$A = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 3 & 2 \\ 0 & 5 & -2 \end{pmatrix}.$$

Solution. We have

$$\det A = (1)(3)(-2) + (0) + (4)(2)(5) - (0) - (1)(2)(5) - (0) = -6 + 40 - 10 = 24$$

4 General definition of determinant

We defined the determinant of a 3×3 matrix in terms of 2×2 determinants. In general, it turns uot that we can *recursively* define determinants of $n \times n$ matrices using determinants of $(n-1) \times (n-1)$ submatrices.

Definition: For $n \geq 2$ the **determinant** of an $n \times n$ matrix $A = (a_{ij})$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$:

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}.$$

Notation: We often will denote

$$\det(*) = |*|.$$

The **(i,j)-cofactor** of the matrix A is the number C_{ij}

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Then, we can write the determinant is

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n},$$

which is called the **cofactor expansion across the first row** of A.

It turns out that the determinant can be computed using the cofactor expansion along **any** row or column.

Theorem: The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

or

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Note: The plus or minus sign of the cofactor C_{ij} depends on the following checkerboard of signs:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This Theorem is very useful to compute determinants whenever there are many zeros in a row or column.

Example 4. *Compute*

$$\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}.$$

We use the cofactor expansion down the third column. We see that

$$\begin{aligned} \det A &= +2\begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix} - 0 + 0 - 0 \\ &= 2\left(0 - (-5)\begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} + 0\right) \\ &= 2(5(-18+20)) = 20. \end{aligned}$$

Example 5. *Compute*

$$\begin{vmatrix} 2 & 2 & 4 & 5 \\ 0 & -2 & -1 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -5 \end{vmatrix}.$$

Solution. We have $\det A = (2)(-2)(1)(-5) = 20$.

Theorem: If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

5 Volume

Theorem: If A is an $n \times n$ matrix, the area of the n-dimensional parallelogram determined by the columns of A is $|\det A|$.

This lets us say:

Theorem: Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ have associated $n \times n$ matrix A. Given a set S with finite area/volume, then the area/volume of $T(S) = \{T(s) \mid s \in S\}$ is given by

$$(\text{area of } T(S)) = |\det A| \cdot (\text{area of } S).$$

Example 6. *What is the area enclosed by an ellipse given by*

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1?$$

In fact, an ellipse is the image of the unit circle under the linear transformation T determined by the matrix $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. So, by the Theorem we have

$$(\textit{area of ellipse}) = |ab| \cdot (\textit{area of unit circle}) = |ab| \cdot \pi(1)^2 = \pi ab.$$