Topics: Rolle's Theorem, Mean Value Theorem, Indeterminate forms, L'Hospital's Rule

Mean Value Theorem

Example 1. *A racer is running back and forth along a straight path (no wide turns allowed!). She finishes the race at the place where she began.*
T/F: *There had to be at least one moment, other than the beginning and the end of the race, when she "stopped" (i.e., her speed was 0).*

Theorem. (Rolle's Theorem) Assume that a function f on a domain $[a, b]$

1. is continuous on
2. is differentiable on
3. and

Then,

Example 2. *Show that the function* $f(x) = x^3 - 2x^2 - 4x + 2$ *on* $[-2, 2]$ *satisfies the three hypotheses of Rolle's Theorem. Find the value of* c *satisfying the conclusion.*

Example 3. *I drove my car down I-95 (speed limit 65mph) and drove from Ewing to Washington, DC. The trip is approximately 171 miles. The tollways recorded my car leaving Ewing at 10am and arriving in Washington, DC at 12:15pm. I was promptly issued a speeding ticket in Washington, DC even though no police officer pulled me over on I-95. Was the ticket justified?*

Theorem. (Mean Value Theorem) Let f be a function that

1. is continuous on
2. is differentiable on

Then,

Take-away: The expression $\dfrac{f(b)-f(a)}{b-a}$ is exactly the slope of the secant line through points $(a, f(a))$ and $(b, f(b))$. So, the Mean Value Theorem states that there exists a point $c \in (a, b)$ so that the tangent line of f at c has the same slope as the secant line between (a, b).

Example 4. *Verify that* $f(x) = \ln(x)$ *on the interval* $[1, 4]$ *satisfies the hypotheses of the Mean Value Theorem, and find all* c *satisfying the conclusion.*

L'Hospital's Rule

Current known techniques to solve limits:

Definition: A limit $\lim_{x\to a} f(x)$ is said to be in **indeterminant form** if, upon plugging in $x = a$, we get

1. $\frac{0}{0}$,
2. $\frac{\infty}{\infty}$,
3. $0 \cdot \infty$,
4. $\infty - \infty$,
5. ∞^0
6. 0^0, or
7. 1^∞.

Theorem: (L'Hospital's Rule) Suppose that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, or that $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. Then

so long as the limit on the right exists (or is $\pm\infty$).

Example 5. *Compute*

$$\lim_{x\to 1} \frac{\ln x}{x-1}.$$

Example 6. *Find*

$$\lim_{x\to \pi^-} \frac{\sin x}{1-\cos x}.$$

Example 7. *(0/0) Compute*

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$$

Example 8. *(∞/∞) Compute*

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$$

Example 9. *($0 \cdot \infty$: rewrite as a fraction) Compute*

$$\lim_{x \to 0^+} x \ln x.$$

Example 10. *($\infty - \infty$: rewrite as single term) Compute*

$$\lim_{x \to 1} \left(\frac{3x}{x-1} - \frac{3}{\ln x} \right).$$