Learning Objectives:

1. Represent any linear transformation as a matrix multiplication.
2. Determine whether a given linear transformation is one-to-one and/or onto.

1 The matrix of a linear transformation

We previously saw that if A is a matrix then the transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ is linear. What about the other direction? If T is linear, can it always be represented by a matrix? The amazing fact is:

> **Every** linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ can be represented (uniquely) as a matrix transformation: $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix A.

The trick to understand this is to use the so-called *standard basis vectors*. In $\mathbb{R}^3$ they are

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In general, there are n standard basis vectors in $\mathbb{R}^n$, each one having 1 as one entry and 0 as all other entries.

The *identity matrix* is the matrix whose columns are the standard basis vectors: $I_n = (\mathbf{e}_1 \ \ldots \ \mathbf{e}_n)$.

Example 1. *Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ which maps*

$$T(\mathbf{e}_1) = \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix}, \ T(\mathbf{e}_2) = \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix}.$$

Determine a matrix A which represents T.

Solution. Given an arbitrary $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ we first realize that $\mathbf{x}$ can be written as a linear combination of the standard basis vectors: $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. Then, using linearity of T we get

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \\ &= x_1 T(\mathbf{e}_1) + x_2 T(\mathbf{e}_2) \\ &= x_1 \begin{pmatrix} 5 \\ -7 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 8 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= A\mathbf{x}, \end{aligned}$$

where $A = \begin{pmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{pmatrix}$. This works no matter the choice of $\mathbf{x}$.

Theorem: Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, there exists a unique matrix A of size $m \times n$ such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

The matrix A is given by

$$A = (T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)),$$

where $\mathbf{e}_i$ is the ith basis vector. We call A the *standard matrix of the linear transformation* T.

Example 2. *Write down the matrix representation of the rotation transformation* $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ *which rotates all vectors counterclockwise by* $45°$*.*

Solution. We see what happens to the standard basis vectors (it is easiest to draw a picture).

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ maps to } T(\mathbf{e}_1) = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ maps to } T(\mathbf{e}_2) = \begin{pmatrix} \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

So, the standard matrix is

$$A = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}.$$

In general, a rotation by φ radians will give a standard matrix

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

2 Transformation properties

We have seen a common theme of linear algebra is answering the question: "is $A\mathbf{x} = \mathbf{b}$ solvable? Are solutions unique?" We have also seen that the case $A\mathbf{x} = \mathbf{0}$ is particularly important.

We ask the same questions of transformations:

A mapping $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is **onto** or **surjective** if every $\mathbf{b} \in \mathbb{R}^m$ is the image of *at least one* $\mathbf{x} \in \mathbb{R}^n$.

That is, T is onto if given any $\mathbf{b}$ there is a solution to $T(\mathbf{x}) = \mathbf{b}$.

A mapping $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is **into** or **one-to-one** or **injective** if each $\mathbf{b} \in \mathbb{R}^m$ is the image of *at most one* $\mathbf{x} \in \mathbb{R}^n$.

That is, T is one-to-one if whenever $T(\mathbf{u}) = T(\mathbf{v})$ then it must be the case that $\mathbf{u} = \mathbf{v}$.

Example 3. *Let T be the linear transformation whose standard matrix row reduces to*

$$R = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix}.$$

Is $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ onto? into?

Solution. We first notice that R has a pivot in every row. Thus $R\mathbf{x} = \mathbf{b}$ is solvable for every $\mathbf{b}$ and so R is onto. Since $R\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{b}$ have the same solution set, then T is also onto. However, we see that there is a free variable (no pivot in column 3). Thus, there must be non-unique solutions to the equation $R\mathbf{x} = \mathbf{b}$, and likewise non-unique solutions to the equation $A\mathbf{x} = \mathbf{b}$. Thus, T is not into.

Theorem: Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

Proof. If T is one-to-one then $T(\mathbf{0}) = \mathbf{0}$ and no other $\mathbf{x}$ can map to $\mathbf{0}$, so $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. On the other hand, assume $T\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then suppose $T(\mathbf{u}) = T(\mathbf{v})$. We must show $\mathbf{u} = \mathbf{v}$. Notice that

$$T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0},$$

but then it must be that $\mathbf{u} - \mathbf{v} = \mathbf{0}$ since only the zero vector maps to $\mathbf{0}$. Thus $\mathbf{u} = \mathbf{v}$ and so T is one-to-one. □

Finally, we relate onto/into to span and linear independence. First, recall the relation between pivots, free variables, span, and linear independence.

Example 4. *Assume T is a linear transformation and A is its standard matrix. Fill in the following table with the following concepts:*

Columns of A linearly independent, columns of A span $\mathbb{R}^m$, pivot in each row of A, pivot in each column of A, row reducing never results in row $(0\ 0\ \ldots\ 0\ b)$, A has no free variables, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

T is one-to-one	*T is onto*

Example 5. *Let $T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$. Show that T is injective. Is it surjective?*

Solution. We first determine the standard matrix of T. One way to do this is to see that $T(\mathbf{e}_1) = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$ and $T(\mathbf{e}_2) = \begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}$. So,

$$A = \begin{pmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{pmatrix}.$$

T is injective if A has linearly independent columns. Since there are only two vectors, they are linearly independent if and only if they are not multiples of each other; this is clear from inspection. Thus A has linearly independent columns and T is injective.

T is surjective if and only if the columns of A span $\mathbb{R}^3$. Note that if we row reduce A then it is impossible to have a pivot in the last row. Thus, the columns of A do not span $\mathbb{R}^3$ and so T is not surjective.

Example 6. *Suppose that A is a 2×3 matrix.* **T/F***: The transformation $T(\mathbf{x}) = A\mathbf{x}$ can be one-to-one.*

Solution. False. A cannot have a pivot in each column, so there must be a free variable and so T is not one-to-one.