MAT205 **Supplementary Notes**

Section 4.3 Linearly independent sets; bases

Learning Objectives:

1. Define the basis of vector space
2. Describe how the Spanning Set Theorem generates a basis from a spanning set.
3. Find the bases of both null and column spaces

1 Deja vu

Definition: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

has *only* the trivial solution $c_1 = c_2 = \cdots = c_p = 0$. If there is a non-trivial solution we call the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ **linearly dependent**.

Much of the intuition we built still holds:

1. A single vector $\mathbf{v}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.
2. A pair of vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent if and only if neither is a multiple of the other.
3. Any set containing the zero vector is linearly dependent.

Theorem: A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

2 Bases

Definition: Let H be a subspace of a vector space V. Then $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

1. $\mathcal{B}$ is linearly independent and
2. the subspace spanned by $\mathcal{B}$ is H.

Intuition: A basis is a "Goldilocks" set: if a set is too small then it will not span the space. If it is too large then it will be linearly dependent. A basis is just the right size!

Example bases:

- The **standard basis** vectors for $\mathbb{R}^n$, are the vectors $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \ldots,\ \mathbf{e}_n\}$.
- The set $S = \{1, t, t^2, \ldots, t^n\}$ is called the **standard basis** for $\mathbb{P}_n$.

Example 1. *Do the vectors*

$$\begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$$

form a basis of $\mathbb{R}^3$*?*

3 Spanning Set Theorem; finding bases for $\operatorname{Col} A$ and $\operatorname{Nul} A$

The Spanning Set Theorem: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V and let $H = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

1. If some $\mathbf{v}_k$ is a linear combination of the remaining vectors in S then

2. If $H \neq \{\mathbf{0}\}$ then some subset of S is

Finding a basis for the null space: Given a matrix A, solve

Example 2. *Knowing that*

$$A = \begin{pmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{pmatrix}$$

row reduces to

$$B = \begin{pmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

find a basis for $\mathrm{Col}\, A$.

Theorem:	form a basis for the column space.

Example 3. *Let*

$$A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}.$$

Knowing that

$$\begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

find bases for $\operatorname{Nul} A$ *and* $\operatorname{Col} A$.

Example 4. T/F: *If* $H = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *then a basis for* H *is* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.