1 Invertible Linear Transformations

We have seen:

1. $n \times n$ matrices are equivalent to linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$.

2. Non-singular/Invertible matrices are those whose actions can be "reversed."

Combining these two ideas, it is very natural to think about linear transformations which are invertible!

Definition: A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there exists a function $S\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = T(S(\mathbf{x})) = \mathbf{x}.$$

We write $S = T^{-1}$ and call S the *inverse* of T.

Theorem 1. The linear transformation T is invertible if and only if its standard matrix A is invertible. In that case, the standard matrix for T^{-1} is given by A^{-1}.

Note: If T is an invertible linear transformation then its inverse T^{-1} must also be a linear transformation (since it can be represented by a standard matrix).

Example 1. *Suppose $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation defined by*

$$T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2).$$

Show that T is invertible and find a formula for T^{-1}. (We wrote the column vectors here as row vectors for convenience, but the math does not change!)

Solution. The standard matrix for T is

$$A = \begin{pmatrix} -5 & 9 \\ 4 & -7 \end{pmatrix}.$$

The inverse matrix is

$$A^{-1} = \frac{1}{35-36}\begin{pmatrix} -7 & -9 \\ -4 & -5 \end{pmatrix} = \begin{pmatrix} 7 & 9 \\ 4 & 5 \end{pmatrix}.$$

So the inverse map is

$$T^{-1}(x_1, x_2) = (7x_1 + 9x_2, 4x_1 + 5x_2).$$