Learning Objectives:

1. Compute vector arithmetic
2. Relate systems of linear equations to vector equations
3. Describe the relationship between linear combinations, span, and consistency of linear systems
4. Describe the geometric interpretation of linear combinations, span, and consistency of systems

1 Vectors

Vectors are mathematical objects which arise naturally in all sciences.

Definition: A **vector** is an ordered list of numbers. We normally name vectors using lowercase, boldface letters, or letters with arrows over them:

$$\mathbf{v} = \vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

If the vector is arranged vertically, we call it a **column vector**. If it is arranged horizontally, we call it a **row vector**, e.g.,

$$\mathbf{v} = (2 \;\; 4).$$

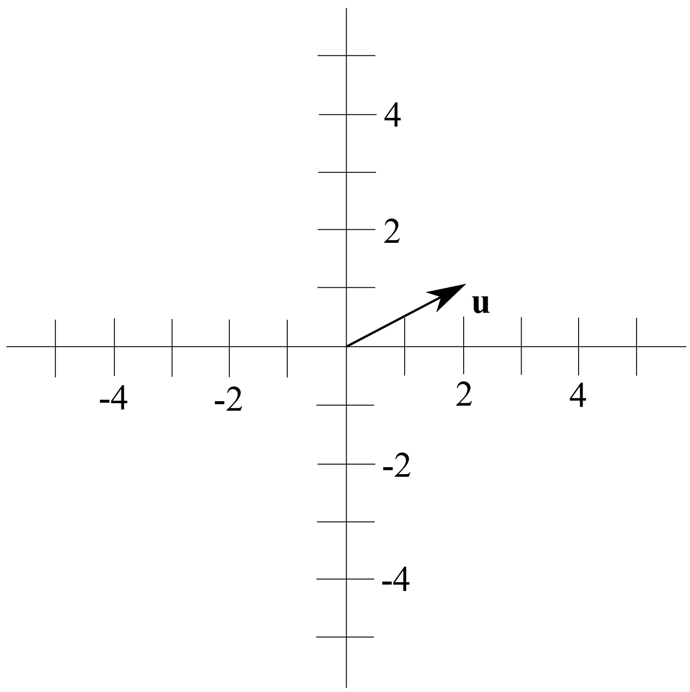
A vector we will use often this semester is the zero vector: $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The number of components in any particular zero vector will always be clear from context.

Uses: Vectors appear in any setting where a single object must be described using several independent quantities. For example:

1. In *physics* a vector can encode (x, y, z) coordinates of an object, or the direction and size of a force applied to an object.
2. In *computer science* a vector can encode colors by their red, green, and blue components.
3. In *finance* a vector can represent a portfolio of many stocks.

4. In *biology* a vector can represent different levels of protein expression in an organism.

Geometric interpretation: A vector with n components is from the space $\mathbb{R}^n$. For example, $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathbb{R}^2$. We can visualize this vector as the arrow pointing from the origin to the point $(2, 1)$.



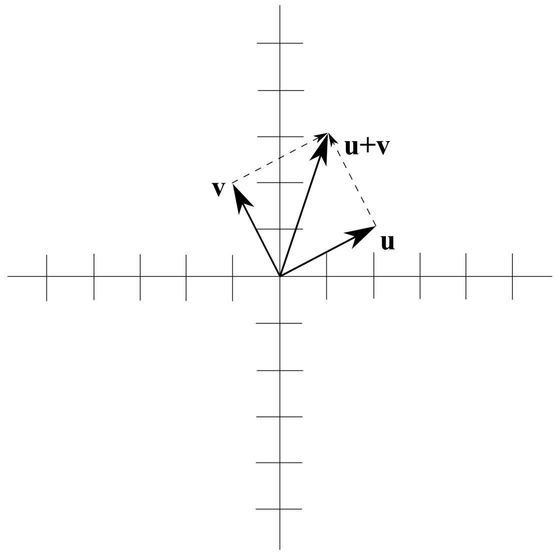
A vector from $\mathbb{R}^3$ would look like an arrow in 3D space.

1.1 Vector arithmetic

Vector addition: Two vectors from the same $\mathbb{R}^n$ can be added component-wise. If $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ then

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2-1 \\ 1+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

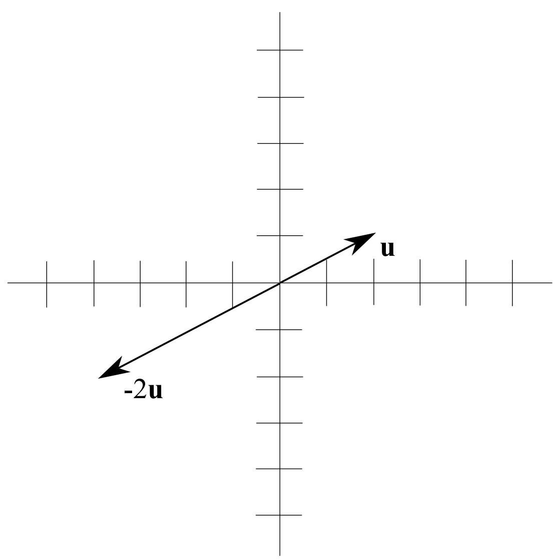
Geometric interpretation: The vector $\mathbf{u} + \mathbf{v}$ points to the 4th corner of a parallelogram with other corners at the origin, $\mathbf{u}$, and $\mathbf{v}$. You may have also seen this referred to as "head-to-tail" addition, since you can think of stacking vectors from head to tail to form their sum.



Scalar multiplication: A vector can also be scaled by a real number by multiplying each component. If $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $c = -2$ then

$$c\mathbf{u} = -2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}.$$

Geometric interpretation: The number c is often called a *scalar* because it scales the original vector by a factor of c (where negative values additionally flip the vector to point in the opposite direction).



Question 1. *Using the above definitions, compute* $\mathbf{u} - 2\mathbf{v}$ *where*

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}.$$

Solution. We compute

$$\mathbf{u} - 2\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}.$$

These definitions of vector addition and scalar multiplication are the "right" ones because they result in the following familiar properties:

Algebraic properties of $\mathbb{R}^n$: For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^n$ and all scalars c and d,

(i) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\mathbf{u}+\mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(iv) $\mathbf{u}\text{-}\mathbf{u} = \mathbf{0}$

(v) $\mathrm{c}(\mathbf{u}+\mathbf{v}) = \mathrm{c}\mathbf{u}+ \mathrm{c}\mathbf{v}$

(vi) $(\mathrm{c}+\mathrm{d})\mathbf{u} = \mathrm{c}\mathbf{u}+ \mathrm{d}\mathbf{u}$

(vii) $\mathrm{c}(\mathrm{d}\,\mathbf{u}) = (\mathrm{cd})\mathbf{u}$.

2 Linear combinations

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ and scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ with **weights** $c_1, c_2, \ldots, c_p$. Note that some or all c_p may be zero.

Example 1. *Suppose we have* $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ *and* $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. *Then* $\mathbf{y} = \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix}$ *is a linear combination of* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *since*

$$\mathbf{y} = 3\mathbf{v}_1 - 5\mathbf{v}_2.$$

However, it is pretty clear that $\mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ 9 \end{pmatrix}$ *is* **not** *a linear combination, since no matter how I choose weights* c_1 *and* c_2*, I will never create a non-zero third component.*

Remark: In general, if we are given a set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$, and some target vector $\mathbf{y}$, how can we tell if $\mathbf{y}$ a linear combination of the vectors $\mathbf{v}_1, \ldots \mathbf{v}_p$?

Example 2. *Let* $\mathbf{a}_1 = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$ *and* $\mathbf{b} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}$. *Do there exist weights* x_1 *and* x_2 *so that*

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}?$$

If so, give values of x_1 *and* x_2 *which solve this vector equation.*

Solution. We can write the equation as

$$\begin{pmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ -3 \end{pmatrix}.$$

Two vectors are equal if and only if their individual components are equal, so we have the following system of equations:

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3. \end{aligned}$$

If this system is solvable (consistent) then $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. If it is not solvable, then no values of x_1, x_2 will give the vector $\mathbf{b}$ and so it would not be a linear combination.

We write the system in matrix form and row reduce:

$$\begin{pmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, we get that $x_1 = 3$ and $x_2 = 2$ is the unique solution.

We have discovered a fundamental result of vector equations:

Theorem: A vector equation

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solutions as the linear system with matrix

$$\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{pmatrix}.$$

In particular, $\mathbf{b}$ is a linear combination if and only if the system associated to this matrix is consistent.

3 Span

In general, given $\mathbf{v}_1$, ..., $\mathbf{v}_p$, we can ask what are **all of the possible vectors b** for which we can solve

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{b}?$$

Definition: The **span** of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$ is the set of all their linear combinations. That is, it is the set of all vectors $\mathbf{b} \in \mathbb{R}^n$ which can be written

$$\mathbf{b} = x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p$$

for some scalar weights $x_1, \ldots, x_p$. We denote the span by $\text{Span}\{\mathbf{a_1}, \ldots, \mathbf{a_p}\}$. We also may call this set the *subset of* $\mathbb{R}^n$ *spanned (or generated) by the vectors* $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Remark: Sometimes it is helpful to build intuition about span using colors. Imagine $\mathbf{v}_1$ represents a can of yellow paint and $\mathbf{v}_2$ represents a can of blue paint. Then, a target color $\mathbf{b}$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ if there is a way of mixing yellow and blue paints in order to get the color $\mathbf{b}$. If $\mathbf{b}$ is the color green, then it is possible, so it is in the span. If $\mathbf{b}$ is the color red, then it is not possible, so it is not in the span!

Example 3. *What is span geometrically? How could we visualize the span of 1 vector? The span of 2 vectors? What does it mean geometrically that* $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$ *is solvable?*

Solution. The span of the vector $\mathbf{a}_1$ consists of all linear combinations of just the vector $\mathbf{a}_1$. That is, it is all scalar multiples: $c\mathbf{a}_1$. Graphically, this is a line through the origin. The span of two vectors may be either a line or a plane. Geometrically, if $\mathbf{b}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ then it lies in the line or plane spanned by the vectors. If it is not then it lies somewhere off the plane.

Example 4. **T/F**: *There is a unique value of h so that $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ h \end{pmatrix}$ is in the span of $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.*

Solution. We row reduce the augmented matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & h \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & h \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & h+2 \end{pmatrix}.$$

This system is consistent if and only if $h + 2 = 0$ so $h = -2$ is the only value. So this is true.