Learning Objectives:

1. Use the characteristic equation to find the eigenvalues of a matrix.

2. Define when two matrices are similar.

1 Characteristic polynomial

Recall: So far we have seen how to find eigenvectors if we are given eigenvalues. So far we do not know how to find the eigenvalues themselves.

Example 1. *How do we find the eigenvalues of a matrix A?*

Solution. In order to determine the eigenvalues of a matrix A, we must determine for which values λ

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

has non-trivial solutions. Note that $A - \lambda I$ has non-trivial solutions if and only if it is non-invertible (by the IMT). Thus, an equivalent criteria for $A - \lambda I$ to have non-trivial null space is

$$\det(A - \lambda I) = 0.$$

The determinant of $A - \lambda I$ is simply a polynomial in the variable λ (why?) So, solving

$$\det(A - \lambda I) = 0$$

is equivalent to solving for the roots of some polynomial equation. We call this equation the *characteristic equation*. We often write the polynomial $p_A(\lambda) = \det(A - \lambda I)$ to signify the fact that the determinant is a polynomial of the variable λ. We call $p_A(\lambda)$ the **characteristic polynomial**.

Example 2. *Find the eigenvalues of and eigenvectors of*

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

Solution. Computing

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -4 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 4 = 4 - 4\lambda + \lambda^2 - 4 = \lambda^2 - 4\lambda.$$

Thus setting $\det(A - \lambda I) = 0$ yields $\lambda(\lambda - 4) = 0$ so $\lambda = 0, 4$ are the eigenvalues. To find the 0 eigenspace we solve

$$(A - 0I)\mathbf{x} = \mathbf{0},$$

so

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 \\ 0 & \end{pmatrix},$$

so $x_1 = 2x_2$ and thus $\mathbf{x} = \begin{pmatrix} 2x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ so one eigenvector is $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

The 4 eigenspace is found by reducing

$$A - 4I = \begin{pmatrix} -2 & -4 \\ -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

so $x_1 = -2x_2$ and thus $\mathbf{x} = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and an eigenvector is $\mathbf{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Example 3. *Find the eigenvalues of*

$$A = \begin{pmatrix} 6 & -2 & 0 \\ -2 & 9 & 0 \\ 5 & 8 & 3 \end{pmatrix}.$$

Solution. Computing

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & 9-\lambda & 0 \\ 5 & 8 & 3-\lambda \end{vmatrix} \\
&= (3-\lambda)\begin{vmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{vmatrix} = (3-\lambda)[(6-\lambda)(9-\lambda) - 4] \\
&= (3-\lambda)(54 - 15\lambda + \lambda^2 - 4) = (3-\lambda)(\lambda^2 - 15\lambda + 50) \\
&= (3-\lambda)(\lambda - 5)(\lambda - 10).
\end{aligned}$$

Setting the characteristic polynomial to zero we see that $\lambda = 3, 5, 10$ are the roots.

2 Similar matrices

In practice, it is often hard to compute eigenvalues for large matrices. Instead we utilize **similar matrices** to simplify the calculation.

Definition: Two matrices A and B are **similar** if there exists an invertible matrix P so that $A = P^{-1}BP$. Of course, this equation also means that $B = PAP^{-1}$, so A is similar to B if and only if B is similar to A.

Theorem: If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomials and eigenvalues.

Proof. Since A and B are similar, there exists an invertible matrix P so that $B = PAP^{-1}$. Thus,

$$\begin{aligned}\det(A - \lambda I) &= \det(PP^{-1})\det(A - \lambda I) = \det(P)\det(A - \lambda I)\det(P^{-1}) \\ &= \det(P(A - \lambda I)P^{-1}) = \det(PAP^{-1}\lambda PP^{-1}) = \det(B - \lambda I).\end{aligned}$$

Example 4. T/F: *If A and B are similar then* $\det A = \det B$.

Solution. True. Notice that $A = P^{-1}BP$ for some invertible matrix P. Then

$$\det(A) = \det(P^{-1}BP) = \det(P^{-1})\det(B)\det(P) = \frac{1}{\det(P)}\det(B)\det(P) = \det(B).$$

Example 5. T/F: *The matrices*

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

are similar.

Solution. False, they have different characteristic polynomials and so cannot be similar.