# 1 Inner Products

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Notice then that $\mathbf{u}^T$ is $1 \times n$ and $\mathbf{v}$ is $n \times 1$, and so the product $\mathbf{u}^T\mathbf{v}$ is well defined, and in fact is a single number!

**Definition:** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The product $\mathbf{u}^T\mathbf{v}$ is called the **inner product** or **dot product**:

$$\mathbf{u}^T\mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$

We sometimes use the notation $\mathbf{u} \cdot \mathbf{v}$ or $\langle \mathbf{u}, \mathbf{v} \rangle$ to denote the inner product.

**Example 1.** *Compute* $\mathbf{u} \cdot \mathbf{v}$ *and* $\mathbf{v} \cdot \mathbf{u}$ *where*

$$\mathbf{u} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}.$$

**Solution.** We have

$$\mathbf{u} \cdot \mathbf{v} = (2,\ -5,\ -1) \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} = 6 - 10 + 3 = -1.$$

On the other hand

$$\mathbf{v} \cdot \mathbf{u} = (3,\ 2,\ -3) \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix} = -1.$$

**Remark:** The previous example makes it clear that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. Of course, matrix multiplication in general is not commutative, but in the case of dot products it is true!

**Theorem:** Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.

3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$.

4. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = 0$.

## 2 Length of a Vector

**Example 2.** *Let* $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. *Compute*

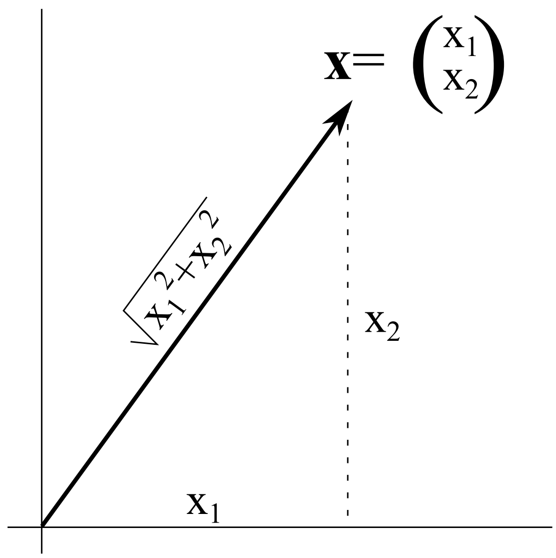
$$\sqrt{\mathbf{x} \cdot \mathbf{x}}.$$

*If we draw a picture of* $\mathbf{x}$ *on* $\mathbb{R}^2$*, then what does this quantity represent?*

**Solution.** We have

$$\sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1 x_1 + x_2 x_2} = \sqrt{x_1^2 + x_2^2}.$$

This quantity is the length of the vector which can be seen by drawing a right triangle and applying the Pythagorean Theorem:




**Definition:** The **norm** or **length** of $\mathbf{v} \in \mathbb{R}^n$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

We have some nice properties of norms as well:

**Theorem:** Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

1. $\|\mathbf{u}\| \geq 0$ and $\|\mathbf{u}\| = 0$ if and only if $\mathbf{u} = 0$.
2. $\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$.
3. $\|\mathbf{u} + \mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|$.

**Note:** The final inequality is called the **triangle inequality**. This is due to the fact from geometry that says that the length of any two side of a triangle are always at least as long as the third side!