1 Diagonalization

Motivation: In the last section we said that similar matrices can help us simplify calculations. In this section we see how to this works using **diagonalization.**

Example 1. *Consider the matrices*

$$A = \begin{pmatrix} 7 & 4 \\ -2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}.$$

Which of these is easier to compute powers of?

Solution. Just by looking, we guess that computing powers of D should be easier. Let's see why:

$$A^2 = \begin{pmatrix} 7 & 4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 41 & 32 \\ -16 & -7 \end{pmatrix}.$$

On the other hand

$$D^2 = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}.$$

In fact, computing A^3 already makes me want to use MATLAB since the calculations are becoming unwieldy, but computing higher powers of D is easy:

$$D^3 = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 125 & 0 \\ 0 & 81 \end{pmatrix}$$

and in general

$$D^k = \begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix}.$$

Recall: Remember that two matrices A and B are similar if there exists an invertible P so that $A = PBP^{-1}$.

Example 2. The plot twist: *It turns out that A and D from the last example are* **similar matrices**. *Indeed, take*

$$P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

and one can check that

$$A = PDP^{-1}.$$

Is this observation at all useful?

Solution. It is! Notice that another way of calculating A^2 is

$$A^2 = AA = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}. \tag{$*$}$$

Let's first check that this works.

$$\begin{aligned} PD^2P^{-1} &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 25 & 25 \\ 9 & 18 \end{pmatrix} \\ &= \begin{pmatrix} 41 & 32 \\ -16 & -7 \end{pmatrix}. \end{aligned}$$

This matches our calculation of A^2 from above! OK, this was actually more work than just computing A^2 normally, *but* what if we wanted to compute A^{100}?

Notice that in the line labeled $(*)$ there is cancellation resulting in the "inner" P^{-1} and P matrices to disappear. If we kept multiplying more (PDP^{-1}) terms, then this cancellation would always happen, resulting in just higher powers of the diagonal matrix D (perhaps you should check yourself what happens for A^3 to really confirm what happens). So we have found a nice formula for A^k:

$$A^k = PD^kP^{-1}.$$

Take a second to appreciate how much easier this would be to calculate: D^k is easy to compute no matter how big k is (from the previous example), and once we know that we just need to multiply the three matrices together, rather than multiply A together k times!

Definition: We say that a square matrix A is **diagonalizable** if it is similar to a diagonal matrix.

Problem: The examples above should convince us that if a matrix A is diagonalizable, then it is easier to use it for calculations. *However*, we have two problems:

- How do we know when a matrix is diagonalizable?
- Even if we know it is diagonalizable, how do we find the matrix P that transforms A to the diagonal matrix?