1 Deja vu

We previously discussed linear independence and linear dependence of vectors in $\mathbb{R}^n$. All of those concepts extend immediately to arbitrary vectors spaces.

Definition: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

has *only* the trivial solution $c_1 = c_2 = \cdots = c_p = 0$. If there is a non-trivial solution we call the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ **linearly dependent**.

Much of the intuition we built in Chapter 1 still holds:

1. A single vector $\{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.
2. A pair of vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ are linearly independent if and only if neither is a multiple of the other.
3. Any set containing the zero vector is linearly dependent.

Theorem: A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Example 1. *Consider the vector space of polynomials $\mathbb{P}$ (remember: each polynomial function is a vector. Addition and scalar multiplication are regular function addition and multiplication). Show that the set consisting of $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t^2$ and $\mathbf{p}_3(t) = 2t^2 - 4$ is linearly dependent.*

Solution. Method 1. (Using the definition.) Notice that $\mathbf{p}_3 = 2\mathbf{p}_2 - 4\mathbf{p}_1$. Rearranging we have

$$4\mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3 = 0$$

and so we conclude that the set is linearly dependent.

Method 2. (Using the theorem.) Once we see that $\mathbf{p}_3 = 2\mathbf{p}_2 - 4\mathbf{p}_1$, we know that $\mathbf{p}_3$ is a linear combination of the preceding vectors, so the set is linearly dependent.

Example 2. *Let* $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$ *and* $\mathbf{p}_3(t) = t^2$. *Explain why* $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ *is linearly independent.*

Solution. Suppose

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = 0.$$

Plugging in the definitions of $\mathbf{p}_i$ this is

$$c_1 + c_2 t + c_3 t^2 = 0.$$

However, remember that the right hand side is really the 0 polynomial! So we can re-write the right hand side as

$$c_1 + c_2 t + c_3 t^2 = 0 + 0t + 0t^2.$$

By matching coefficients, it must be the case that $c_i = 0$ for all i and thus the set is linearly independent.

2 Bases

Definition: Let H be a subspace of a vector space V. Then $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

1. $\mathcal{B}$ is linearly independent and
2. the subspace spanned by $\mathcal{B}$ is H.

Intuition: A basis is a "Goldilocks" set: if a set is too small then it will not span the space. If it is too large then it will be linearly dependent. A basis is just the right size!
Remark: Remember that H can be any subspace, including the zero subspace, or the entire space V!

Example bases:

- In $\mathbb{R}^n$, the vectors $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \ldots,\ \mathbf{e}_n\}$ form a basis which is called **the standard basis**.
- In $\mathbb{R}^2$, the vectors $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ also form a basis! So there is no such thing as a "unique" basis.
- The set $S = \{1, t, t^2, \ldots, t^n\}$ form a basis for $\mathbb{P}_n$, also called **the standard basis**.