Given a linear system we now know how to do a few things:

1. We can row reduce to determine whether a system is consistent or inconsistent.
2. If there are non-unique solutions, we can plug in particular values of free variables to find different solutions.

Something we do *not* yet know is how to clearly describe *all* solutions in some clear way which is easy to interpret. For example, one goal of this section is to visualize the solution set geometrically.

1 Homogeneous linear systems

Given the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

the simplest place to start is when $\mathbf{b} = \mathbf{0}$. We call such systems **homogeneous**. That is, a homogeneous system is of the form

$$A\mathbf{x} = \mathbf{0}.$$

Example 1. *When is the homogeneous system $A\mathbf{x} = \mathbf{0}$ solvable?*

Solution. Always! We always have the **trivial** solution $\mathbf{x} = \mathbf{0}$, no matter the choice of matrix A.

As such, we are truly interested in the existence of *non-trivial solutions*. We will use our previous knowledge about the existence/uniqueness theorem:

The homogeneous system $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions if and only if the system has free variables.

Example 2. *Determine the solution set of the system*

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0. \end{aligned}$$

Solution. We solve this system by row reducing.

$$\begin{pmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 6 & 1 & -8 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 3 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have solutions

$$\begin{cases} x_1 = \frac{4}{3}x_3 \\ x_2 = 0 \\ x_3 \text{ is free.} \end{cases}$$

Thus far, we have seen these steps already. At this point let's continue by writing the solution in vector form,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Since x_3 is a free variable, we will not be able to remove it, but we can write x_1 and x_2 in terms of constants and the free variable x_3:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}.$$

By choosing different values of x_3, we recover different solutions. Writing solutions in this way we can make the following observations:

- All solutions are points on the same line in $\mathbb{R}^3$ (each solution is some multiple of $\begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix}$).
- Another way of writing the set of all solutions is: Span $\left\{ \begin{pmatrix} \frac{4}{3} \\ 0 \\ 1 \end{pmatrix} \right\}$.