1 Row space

Definition: Given an $m \times n$ matrix, the **row space** of A, denoted $\operatorname{Row} A$ is the set of all linear combinations of row vectors of A.

Example 1. *How are* $\operatorname{Col} A$, $\operatorname{Row} A$, $\operatorname{Col} A^T$ *and* $\operatorname{Row} A^T$ *related?*

Solution. We see $\operatorname{Col} A = \operatorname{Row} A^T$ and $\operatorname{Col} A^T = \operatorname{Row} A$.

1.1 Finding a basis for the row space

Just as in the column space, we can quickly determine the row space by simply writing down all rows of a matrix. However, this does not necessarily give a basis as some rows may be redundant.

Example 2. *Find a basis for the row space of*

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & 7 \\ 0 & 3 & 4 \\ 2 & 7 & 10 \end{pmatrix}.$$

Solution. The row space is certainly spanned by the set of all row vectors:

$$\{(1,2,3),(0,3,4),(1,5,7),(2,7,10)\}.$$

Since we have row vectors in $\mathbb{R}^3$ and there are 4 rows total, certainly this set must be linearly dependent. How do we determine which vectors to remove? First notice that if we were to complete a replacement operation on row j, it would be replaced by something of the form:

$$c\text{Row}_i + \text{Row}_j,$$

which is simply a linear combination of row i and row j. In other words: row operations maintain the span of the rows! Let's see why that is useful. If we row reduce the given matrix we get

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & 7 \\ 0 & 3 & 4 \\ 2 & 7 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \\ 2 & 7 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rows with zeros are certainly not needed, and the first two rows are definitely linearly independent since they have pivots in different locations. So if we take the top two rows of the reduced matrix, our previous observation tells us that they span *the original row space*! So a basis for the row space is

$$\{(1,2,3),\ (0,3,4)\}.$$

Theorem: If two matrices $A \sim B$, then Row $A =$ Row B. If B is in echelon form, the nonzero rows of B form a basis not only for the row space of B, but also for the row space of A.

Remark: Remember that when we solve for the column space, row operations *do* change the span of the columns, so we need to return to the original matrix to find the basis!

Example 3. *Find bases for the row space, the column space, and the null space of*

$$A = \begin{pmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{pmatrix}$$

knowing that

$$A \sim \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution. So, the row space has a basis

$$\{(1,0,1,0,1),\ (0,1,-2,0,3),\ (0,0,0,1,-5)\}.$$

The column space has a basis

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \\ 11 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 7 \\ 5 \end{pmatrix} \right\}.$$

Finally, the null space is determined by finding the general solution to $A\mathbf{x} = \mathbf{0}$, so

$$\begin{aligned} x_1 &= -x_3 - x_5 \\ x_2 &= 2x_3 - 3x_5 \\ x_4 &= 5x_5 \end{aligned}$$

with x_3 and x_5 free. An element of the null space is thus

$$\mathbf{x} = \begin{pmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{pmatrix}.$$

The basis is thus

$$\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{pmatrix} \right\}$$

Example 4. T/F: *The row space and column space always have the same dimension.*

Solution. True! The row space is the number of non-zero rows in the matrix, and the column space is the number of pivots. Each pivot occurs in a new row, so there are the same number of each.

Definition: The **rank** of A is the dimension of the column space of A. The **nullity** of A is the dimension of the null space.

Example 5. *Let A be an $m \times n$ matrix. What do pivots and free variables tell you about the rank and nullity? Propose a rule for "rank + nullity."*

Solution. The number of pivots is the rank. The number of free variables is the nullity. Since those are all of the columns, and there are n columns, it must be the case that rank + nullity $= n$.

Rank-Nullity Theorem: If A is an $m \times n$ matrix, then

$$\operatorname{rank} A + \operatorname{nullity} A = n.$$

Example 6. *If the null space of a 5×6 matrix is 4-dimensional, what is the dimension of the row space of A?*

Solution. The rank-nullity theorem implies that the rank is $6 - 4 = 2$. Since the dimension of the row space is the same as the rank, it is also 2 dimensional.

Example 7. *If A is a 5×7 matrix, then what is the largest possible rank of A?*

Solution. The matrix A can have rank at most 5, since it can have at most 5 pivots.

Example 8. *Why can't a 6×9 matrix have a 2 dimensional null space?*

Solution. If such a matrix had 2 dimensional null space then the rank is 7, but the matrix cannot have 7 pivots since it only has 6 rows.

These results additionally add a few statements to the invertible matrix theorem (this brings us up to r statements).

Theorem: Let A be $n \times n$. The following are equivalent to A being invertible:

1. The columns of A form a basis of $\mathbb{R}^n$.
2. $\operatorname{Col} A = \mathbb{R}^n$
3. $\dim \operatorname{Col} A = n$.
4. $\operatorname{rank} A = n$
5. $\operatorname{Nul} A = \{\mathbf{0}\}$
6. $\dim \operatorname{Nul} A = 0$.

Example 9. *Let A be $m \times n$. Suppose that $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$. Show that $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

Solution. We can use the rank-nullity theorem here. If $A\mathbf{x} = \mathbf{b}$ is always solvable, then the rank of the matrix is m. But, then the row space is also dimension m. Of course, this means that the dimension of the column space of A^T is m. Applying the rank-nullity theorem to A^T we have that the nullity of A^T is 0. Thus $A^T\mathbf{x} = \mathbf{0}$ is uniquely solvable.