Learning Objectives:

1. Compute inner products and distances between vectors from $\mathbb{R}^n$.
2. Determine whether two vectors are orthogonal and explain geometrically what it means.
3. Understand when a vector is in the orthogonal complement to a subspace.
4. Explain the orthogonality of the fundamental subspaces associated to a matrix A.

1 Inner Products

Definition: Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The product $\mathbf{u}^T\mathbf{v}$ is a single number. This product is called the **inner product** or **dot product**:

$$\mathbf{u}^T\mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$

We sometimes use the notation $\mathbf{u} \cdot \mathbf{v}$ or $\langle \mathbf{u}, \mathbf{v} \rangle$ to denote the inner product.

Example 1. *Compute* $\mathbf{u} \cdot \mathbf{v}$ *and* $\mathbf{v} \cdot \mathbf{u}$ *where*

$$\mathbf{u} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix}.$$

Solution. We have

$$\mathbf{u} \cdot \mathbf{v} = (2,\ -5,\ -1) \begin{pmatrix} 3 \\ 2 \\ -3 \end{pmatrix} = 6 - 10 + 3 = -1.$$

On the other hand

$$\mathbf{v} \cdot \mathbf{u} = (3,\ 2,\ -3) \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix} = -1.$$

Remark: The previous example makes it clear that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. Of course, matrix multiplication in general is not commutative, but in the case of dot products it is true!

Theorem: Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$.
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = 0$.

Example 2. *Let* $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. *Compute*

$$\sqrt{\mathbf{x} \cdot \mathbf{x}}.$$

If we draw a picture of $\mathbf{x}$ *on* $\mathbb{R}^2$*, then what does this quantity represent?*

Solution. We have

$$\sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x_1^2 + x_2^2}.$$

This quantity is the length of the vector (which can be seen by drawing a right triangle and applying the Pythagorean Theorem).

2 Length of a Vector

Definition: The **norm** or **length** of $\mathbf{v} \in \mathbb{R}^n$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

We have some nice properties of norms as well:

Theorem: Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

1. $\|u\| \geq 0$ and $\|u\| = 0$ if and only if $u = 0$.
2. $\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$.
3. $\|\mathbf{u} + \mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|$.

Note: The final inequality is called the **triangle inequality**. This is due to the fact from geometry that says that the length of any two side of a triangle are always at least as long as the third side!

Definition: A vector whose length is 1 is called a **unit vector**. In general, we can **normalize** a vector $\mathbf{v}$ by scaling it so that the resultant vector points in the same direction as $\mathbf{v}$ but has length 1.

Example 3. *Normalize the vector*

$$\mathbf{u} = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}.$$

Solution. First find the length of $\mathbf{u}$:

$$\|\mathbf{u}\| = \sqrt{4+16+9} = \sqrt{29}.$$

Then scale $\mathbf{u}$ by this to normalize it:

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \begin{pmatrix} -2/\sqrt{29} \\ 4/\sqrt{29} \\ -3/\sqrt{29} \end{pmatrix}.$$

3 Distance

Now that we can calculate lengths, we can calculate distances!

Definition: For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the **distance between** u**, and** v, written $\mathrm{dist}(\mathbf{u}, \mathbf{v})$ is the length of the vector $\mathbf{u} - \mathbf{v}$. That is

$$\mathrm{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Intuition: Many students wonder why we subtract and not add the vectors. If you draw a picture of vectors $\mathbf{u}$ and $\mathbf{v}$ then the vector $\mathbf{u} - \mathbf{v}$ is the vector going between the tips of $\mathbf{u}$ and $\mathbf{v}$! That is the quantity we care about to calculate distance between the vectors. Adding the vectors creates the 4th vertex of a parallelogram, which is not helpful when thinking about distance!

Example 4. *Compute the distance between the vectors* $\mathbf{u} = (7, 1)$ *and* $\mathbf{v} = (3, 2)$.

Solution. Since $\mathbf{u} - \mathbf{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ then $\mathrm{dist}(\mathbf{u}, \mathbf{v}) = \sqrt{16+1} = \sqrt{17}$.

Theorem: For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ we have

1. $\operatorname{dist}(u, v) = 0$ if and only if $u = v$.
2. $\operatorname{dist}(u, v) = \operatorname{dist}(v, u)$.
3. $\operatorname{dist}(u, w) \leqslant \operatorname{dist}(u, v) + \operatorname{dist}(v, w)$.

4 Orthogonality

Example 5. *Let* $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. *Find a vector that is perpendicular to* $\mathbf{u}$. *It may be helpful to think about the* slope *of the vector* $\mathbf{u}$.

Solution. If we think about the vector $\mathbf{u}$ as a line segment with slope $\frac{\Delta y}{\Delta x} = \frac{2}{1}$ then we remember that a perpendicular slope is given by negative reciprocal, so $\frac{-1}{2}$. So we propose vector $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Notice that

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

Definition: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** (perpendicular) if $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{0}$.

Example 6. *Use the properties of inner products to expand and simplify*

$$\|\mathbf{u} + \mathbf{v}\|^2.$$

Solution. We have

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 + \|v\|^2 + 2\langle u, v \rangle. \end{aligned}$$

Notice that if the vectors are orthogonal, then we get a nice formula!

Pythagorean Theorem: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

4.1 Orthogonal complements

Definition: Given a subspace W of $\mathbb{R}^n$ we define

$$W^\perp = \{x \in \mathbb{R}^n : x \cdot w = 0 \text{ for all } w \in W\}.$$

That is, $W^\perp$ (W perp) is the set of vectors which are perpendicular to all vectors of W.

Example 7. *If $W \subseteq \mathbb{R}^3$ is the plane spanned by* $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ *then* $W^\perp = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

Theorem. (i). A vector $\mathbf{x}$ is in $W^\perp$ if and only if $\mathbf{x}$ is orthogonal to every vector in a spanning set of W.
(ii). $W^\perp$ is always a subspace of $\mathbb{R}^n$.

Example 8. *If A is an $m \times n$ matrix then suppose that*

$$\mathbf{x} \in (\operatorname{Row} A)^\perp.$$

Compute $A\mathbf{x}$.

Solution. Assuming $A = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}$ (writing the row vectors), then $\mathbf{x}$ being perpendicular to each row of A means $\mathbf{x} \cdot \mathbf{a}_i = 0$ for each i. Thus, each component of $A\mathbf{x}$ is 0 and so $A\mathbf{x} = \mathbf{0}$.

Theorem: Let A be an $m \times n$ matrix. Then

$$(\operatorname{Row} A)^\perp = \operatorname{Nul} A, \quad (\operatorname{Col} A)^\perp = \operatorname{Nul} A^T.$$

Example 9. *Show that if $\mathbf{x} \in W$ and $\mathbf{x} \in W^\perp$ then $\mathbf{x} = 0$.*

Solution. If $\mathbf{x}$ is in both then $\mathbf{x} \cdot \mathbf{x} = 0$.

Question 1. *Let W be a subspace of $\mathbb{R}^n$. Prove that $W^\perp$ is a subspace of $\mathbb{R}^n$.*

Solution. Since $\mathbf{0} \cdot \mathbf{w} = 0$ for any vector $\mathbf{w}$, it is clear that $\mathbf{0} \in W^\perp$. Suppose that $w_1, w_2 \in W^\perp$. Then for any $w \in W$ we have $w_1 \cdot w = 0$ and $w_2 \cdot w = 0$. Now $(w_1+w_2)\cdot w = w_1 \cdot w + w_2 \cdot w = 0+0 = 0$ so $w_1 + w_2 \in W^\perp$.

Finally suppose $c \in \mathbb{R}$ and $v \in W^\perp$. Then for any $w \in W$ we have $w \cdot v = 0$ and so $(cv) \cdot w = c(v \cdot w) = c0 = 0$. Thus $cv \in W^\perp$ so $W \perp$ is a subspace.