Motivation: In this section we relate the "fundamental" subspaces: the column space, null space, and a new space called the *row space*. It turns out that all of these spaces are intertwined in some nice ways!

1 Row space

Definition: Given an $m \times n$ matrix, the **row space** of A, denoted $\operatorname{Row} A$ is the set of all linear combinations of row vectors of A.

Example 1. *How are* $\operatorname{Col} A$, $\operatorname{Row} A$, $\operatorname{Col} A^T$ *and* $\operatorname{Row} A^T$ *related?*

Solution. Notice that if A is an $m \times n$ matrix, then $\operatorname{Col} A$ is comprised of vectors from $\mathbb{R}^m$ whereas $\operatorname{Row} A$ is comprised of vectors from $\mathbb{R}^n$. So we do not expect these to be related. However by transposing A the roles of rows and columns swap, so we discover that

$$\operatorname{Col} A = \operatorname{Row} A^T$$

and

$$\operatorname{Col} A^T = \operatorname{Row} A.$$

1.1 Finding a basis for the row space

Just as in the column space, we can quickly determine the row space by simply writing down all rows of a matrix. However, this does not necessarily give a basis as some rows may be redundant.

Example 2. *Find a basis for the row space of*

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & 7 \\ 0 & 3 & 4 \\ 2 & 7 & 10 \end{pmatrix}.$$

Solution. The row space is certainly spanned by the set of all row vectors:

$$\operatorname{Row}(A) = \operatorname{Span}\{(1,2,3),(0,3,4),(1,5,7),(2,7,10)\}.$$

Since we have row vectors in $\mathbb{R}^3$ and there are 4 rows total, certainly this set must be linearly dependent. How do we determine which vectors to remove? Notice that if we were to complete a replacement operation on row j, it would be replaced by something of the form:

$$c\text{Row}_i + \text{Row}_j,$$

which is simply a linear combination of row i and row j. In other words: row operations maintain the span of the rows! Let's see why that is useful. If we row reduce the given matrix we get

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & 7 \\ 0 & 3 & 4 \\ 2 & 7 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \\ 2 & 7 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rows with zeros are certainly not needed and are redundant. The first two rows are linearly independent since they have pivots in different locations. So if we take the top two rows of the reduced matrix, our previous observation tells us that they span *the original row space*! So a basis for the row space is

$$\text{Row}(A) = \text{Span}\{(1,2,3),\ (0,3,4)\}.$$

Theorem: If two matrices $A \sim B$, then $\text{Row}\, A = \text{Row}\, B$. If B is in echelon form, the nonzero rows of B form a basis not only for the row space of B, but also for the row space of A.

Remark: Remember that when we solve for the column space, row operations *do* change the span of the columns, so we need to return to the original matrix to find the basis! For row space, we do not need to return to the original rows to recover the basis.

What can you deduce about the dimensions of $\text{Col}\, A, \text{Row}\, A, \text{Col}\, A^T, \text{Row}\, A^T$?