Topics: Continuity, Intermediate Value Theorem

Continuity

Definition: A function f is *continuous* at a point a if

A function is *continuous on an interval* if it is continuous at every point in the interval.

To check continuity, we need to confirm three things:

Checking continuity: A function f is continuous at the point a if

1.
2.
3.

If a function fails to be continuous, then it is **discontinuous**.

Example 1. *Draw three functions that are discontinuous at $x = a$. Make each one fail a different rule of continuity.*

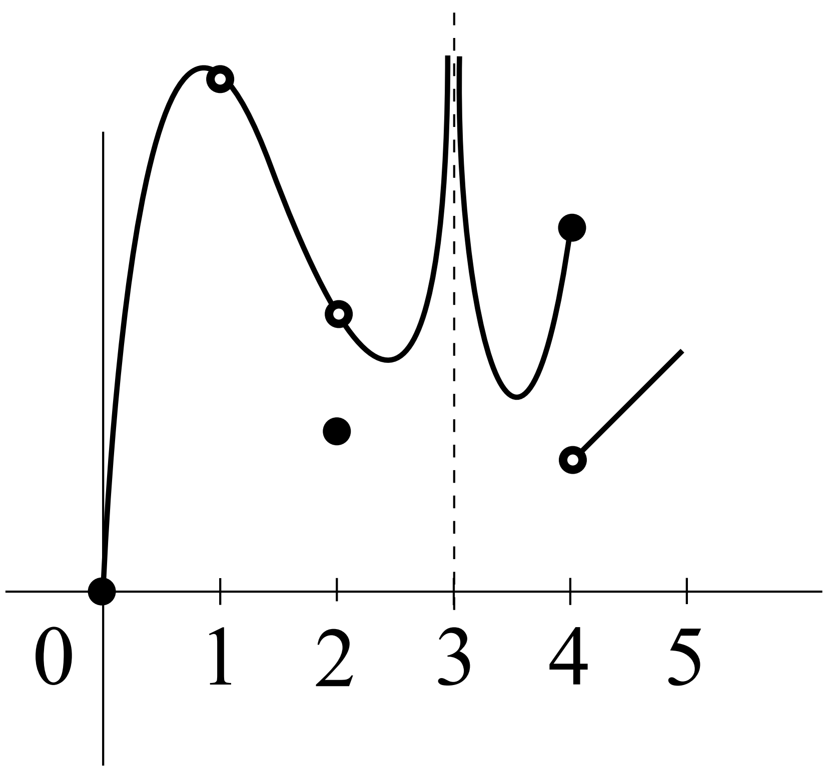
One-sided continuity

Using one-sided limits, we can consider *one-sided continuity:*

Definition: A function f is *continuous from the right at a* if

and f is *continuous from the left at a* if

Example 2. *Classify the discontinuities and identify the one-sided continuity of the following function:*



Examples of continuous functions

Theorem. *The following types of functions are continuous at all points of their domains.*

polynomials
rational functions
root functions
trigonometric functions
inverse trig. functions
exponential functions
logarithmic functions

We also may combine continuous functions in natural ways to build more complicated functions which are still continuous:

Theorem. *If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:*

1. $f+g$
2. $f-g$
3. cf
4. fg
5. $\dfrac{f}{g}$ *if* $g(a) \neq 0$.

Theorem. *If g is continuous at $x=a$ and f is continuous at $g(a)$ then*

That is, $f \circ g$ is continuous at $x=a$.

Take-away: If a function is made up of combinations of polynomials, roots, trig, exponential, and log functions, then it is continuous wherever it is defined!

Example 3. *Evaluate*

$$\lim_{x\to 1} \frac{\sin(1+e^{-x^2})}{\ln(x+1)}.$$

Example 4. *Where is the function* $f(x) = \frac{\ln x + \tan^{-1} x}{x^2-1}$ *continuous?*

Example 5. *What value of* c *makes the following function continuous at all real numbers?*

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \textit{if } x < 2 \\ x^2 - cx + 3 & \textit{if } x \geq 2. \end{cases}$$

Intermediate Value Theorem

Example 6. Challenge: *Draw a continuous function with* $f(0) = -1$ *and* $f(2) = 1$ *in such a way that* $f(x)$ *never passes through* $y = 0$*. Is it possible or impossible?*

Intermediate Value Theorem. Let f be continuous on $[a, b]$. Let N be any number between $f(a)$ and $f(b)$. Then,

Example 7. *Show that there is a solution of the equation*

$$x^4 - 5x^3 + 2x + 1 = 0$$

on the interval $[0, 1]$.