# 1 Matrix multiplication

We saw in the last section that we can multiply a scalar and a vector; in this section we learn how to multiply matrices and vectors.

If $A$ is an $m \times n$ matrix ($m$ rows, $n$ columns) with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, then

$$A\mathbf{x} = (\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n.$$

Written out:

$$A\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

**Notation:** We sometimes write $A = (\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n)$ to say that $A$ is a matrix whose columns are the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

**Example 1.** *Multiply* $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix}$ *and* $\mathbf{x} = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix}$.

**Solution.** We have

$$\begin{aligned} Ax = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix} &= 4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ -5 \end{pmatrix} + 7 \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ -15 \end{pmatrix} + \begin{pmatrix} -7 \\ 21 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}. \end{aligned}$$

**Remark:** Notice that the result of $A\mathbf{x}$ was in $\mathbb{R}^2$. In general, if $A$ is $m \times n$ then $A\mathbf{x}$ must be a vector in $\mathbb{R}^m$.

**Example 2.** *Can we multiply* $A = \begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 5 & 0 \end{pmatrix}$ *and* $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$*?*

**Solution.** No. In order for us to multiply a matrix and vector, we need the number of columns in $A$ to match the number of rows in $\mathbf{x}$.

**Example 3.** *Finally, let's work backward. Suppose* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^m$*. Write the linear combination* $3\mathbf{v}_1 - \mathbf{v}_2 + 8\mathbf{v}_3$ *as a matrix times a vector.*

**Solution.** We can write

$$3\mathbf{v}_1 - \mathbf{v}_2 + 8\mathbf{v}_3 = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3)\begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix},$$

so the matrix is the $m \times 3$ matrix $A = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3)$ and the vector is $\mathbf{x} = \begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix}$.