Learning Objectives:

1. Compute inner products and distances between vectors from $\mathbb{R}^n$.
2. Determine whether two vectors are orthogonal and explain geometrically what it means.
3. Understand when a vector is in the orthogonal complement to a subspace.
4. Explain the orthogonality of the fundamental subspaces associated to a matrix A.

1 Inner Products

Definition: Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The product $\mathbf{u}^T\mathbf{v}$ is a single number. This product is called the **inner product** or **dot product**:

$$\mathbf{u}^T\mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$

We sometimes use the notation $\mathbf{u} \cdot \mathbf{v}$ or $\langle \mathbf{u}, \mathbf{v} \rangle$ to denote the inner product.

Theorem: Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$.
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = 0$.

2 Length of a Vector

Definition: The **norm** or **length** of $\mathbf{v} \in \mathbb{R}^n$ is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Theorem: Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

1. $\|u\| \geq 0$ and $\|u\| = 0$ if and only if $u = 0$.
2. $\|c\mathbf{u}\| = |c|\|\mathbf{u}\|$.
3. $\|\mathbf{u} + \mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|$.

Definition: A vector whose length is 1 is called a **unit vector**. In general, we can **normalize** a vector $\mathbf{v}$ by scaling it so that the resultant vector points in the same direction as $\mathbf{v}$ but has length 1.

Example 1. *Normalize the vector*

$$\mathbf{u} = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}.$$

3 Distance

Now that we can calculate lengths, we can also calculate distances!

Definition: For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the **distance between u, and v** is

$$\text{dist}(\mathbf{u}, \mathbf{v}) =$$

Intuition: Many students wonder why we subtract and not add the vectors. If you draw a picture of vectors $\mathbf{u}$ and $\mathbf{v}$ then the vector $\mathbf{u} - \mathbf{v}$ is the vector going between the tips of $\mathbf{u}$ and $\mathbf{v}$! That is the quantity we care about to calculate distance between the vectors. Adding the vectors creates the 4th vertex of a parallelogram, which is not helpful when thinking about distance!

Example 2. *Compute the distance between the vectors* $\mathbf{u} = (7, 1)$ *and* $\mathbf{v} = (3, 2)$.

Theorem: For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ we have

1. $\operatorname{dist}(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$.
2. $\operatorname{dist}(\mathbf{u}, \mathbf{v}) = \operatorname{dist}(\mathbf{v}, \mathbf{u})$.
3. $\operatorname{dist}(\mathbf{u}, \mathbf{w}) \leqslant \operatorname{dist}(\mathbf{u}, \mathbf{v}) + \operatorname{dist}(\mathbf{v}, \mathbf{w})$.

4 Orthogonality

Example 3. *Let* $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. *Find a vector that is perpendicular to* $\mathbf{u}$. *It may be helpful to think about the* slope *of the vector* $\mathbf{u}$.

> **Definition:** Two vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** (perpendicular) if

Example 4. *Use the properties of inner products to expand and simplify*

$$\|\mathbf{u} + \mathbf{v}\|^2.$$

Pythagorean Theorem: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

4.1 Orthogonal complements

Definition: Given a subspace W of $\mathbb{R}^n$ we define

$$W^{\perp} =$$

That is, $W^{\perp}$ (W perp) is the set of vectors which are perpendicular to all vectors of W.

Example 5. *If $W \subseteq \mathbb{R}^3$ is the plane spanned by* $\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ *then* $W^{\perp} = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

Theorem. (i). A vector $\mathbf{x}$ is in $W^{\perp}$ if and only if $\mathbf{x}$ is orthogonal to every vector in a spanning set of W.
(ii). $W^{\perp}$ is always a subspace of $\mathbb{R}^n$.

Example 6. *If A is an $m \times n$ matrix then suppose that*

$$\mathbf{x} \in (\text{Row}\, A)^{\perp}.$$

Compute $A\mathbf{x}$.

Theorem: Let A be an $m \times n$ matrix. Then

$$(\text{Row}\, A)^{\perp} = \qquad\qquad\qquad , \quad (\text{Col}\, A)^{\perp} =$$

Example 7. *Let W be a subspace of $\mathbb{R}^n$. Prove that $W^{\perp}$ is a subspace of $\mathbb{R}^n$.*