Motivation: We already know how to determine whether a given vector $\mathbf{b}$ is in the span of a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. This is the question of *existence:* does there *exist* a linear combination so that

$$\mathbf{b} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n?$$

On the other hand, we may ask if any such vector in the span of $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ can be described *uniquely* as a linear combination. This is the other major question we continually revisit this semester.

Another way to think about the uniqueness question is: are some of the vectors in our set redundant? Could we remove one or more vectors from $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and still have the same span as before?

1 Linear independence

Definition: A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is **linearly independent** if the only solution to the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

is the trivial solution, $x_1 = \cdots = x_p = 0$.

On the other hand, if there exist coefficients $c_1, \dots, c_p$ not all equal to 0 so that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is **linearly dependent**.

As we have done before, we may translate these definitions into a statement about a matrix equation: let $A = (\mathbf{v}_1 \ \dots \ \mathbf{v}_p)$. Then, the set $\{\mathbf{v}_1, \dots \mathbf{v}_p\}$ is linearly independent if and only if

$$A\mathbf{x} = \mathbf{0}$$

has only the trivial solution $\mathbf{x} = \mathbf{0}$. Recall that this homogeneous system has a unique solution if and only if it has no free variables.

The columns of A are linearly independent if and only if A has no free variables.

Example 1. *Is the set* $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$ *linearly independent?*

Solution. We solve $A\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix. You can check:

$$\begin{pmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that x_3 is a free variable. Take, for example, $x_3 = 1$. Then $x_1 = 2x_3 = 2$ and $x_2 = -x_3 = -1$. Writing $A\mathbf{x} = \mathbf{0}$ as a vector equation we thus have

$$2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This shows that there is a non-trivial solution to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}.$$

As such the set of vectors is linearly dependent. In terms of uniqueness, this shows us that there must be infinitely many ways to represent the vector $\mathbf{0}$ as a linear combination! We of course could take

$$0\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 0\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

but we also have infinitely many other choices, which we can obtain by simply choosing different values of x_3 above.

Take-away: A set of vectors is *linearly independent* if there is a *unique* way to represent $\mathbf{0}$ as a linear combination (namely, only the trivial solution). Linear independence will tell us about *redundancy* of vectors in the set. We will later see that if a set is linearly dependent, then we could remove one of the vectors and not change the span at all.

It may also be helpful to remember paint cans for intuition: suppose one paint can (vector $\mathbf{v}_1$) is yellow, one paint can (vector $\mathbf{v}_2$) is blue, and one paint can (vector $\mathbf{v}_3$) is green. The span of these vectors is the collection of all possible colors we can get by mixing them in various ratios. However, since green is already just a mix of yellow and blue, we could remove it and still have the same range of colors possible. This vector was redundant!