Learning Objectives:

1. Compute sums, scalar multiples, products, and transposes of matrices.

2. Recognize and apply properties of the above matrix operations.

Motivation: We have so far discussed operations between two vectors, and operations between matrices and vectors. In order to more efficiently solve linear algebra problems, as well as solve more complex questions, we need to introduce new operations between two matrices.

1 Matrix operations

Setup: We assume A is an $m \times n$ matrix. We may think of A as n vectors from $\mathbb{R}^m$:

$$A = (\mathbf{a}_1 \ \dots \ \mathbf{a}_n).$$

We can denote the entry in the ith row and jth column of A by a_{ij} (typically, we use lower case letters for entries). We call this the (i, j)-entry of A (think, i-down, j-across).

Definition: Given the matrix $A = (a_{ij})$, We call the entries $a_{11}, a_{22}, a_{33}, \dots$ the **diagonal entries** of A. A **diagonal matrix** is an $n \times n$ matrix whose only non-zero entries are on the diagonal.

E.g. The identity matrix $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ is a diagonal matrix.

If A and B are matrices of the same size (same number of rows and columns), then their **sum** $A + B$ is the matrix obtained by summing the entries of A and B component-wise.

Example 1. *Let*

$$A = \begin{pmatrix} 1 & 3 & 10 \\ 2 & \pi & 3 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 0 & -3 \\ 0 & 3 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$A+B=\begin{pmatrix} 1+4 & 3+4 & 10-3 \\ 2+0 & \pi+3 & 3+1 \end{pmatrix}=\begin{pmatrix} 5 & 7 & 7 \\ 2 & \pi+3 & 4 \end{pmatrix}.$$

We see that $A+C$ is not defined because the sizes of A and C are different.

> Given a matrix A and a scalar r, their **scalar multiplication** is the matrix rA obtained by multiplying each entry in A by r.

Example 2. *Using the same matrices from Example 1:*

$$A-2B=\begin{pmatrix} 1 & 3 & 10 \\ 2 & \pi & 3 \end{pmatrix}-2\begin{pmatrix} 4 & 0 & -3 \\ 0 & 3 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 3 & 10 \\ 2 & \pi & 3 \end{pmatrix}+\begin{pmatrix} -8 & 0 & 6 \\ 0 & -6 & -2 \end{pmatrix}=\begin{pmatrix} -7 & 3 & 16 \\ 2 & \pi-6 & 1 \end{pmatrix}$$

> **Theorem:** Let A, B, C be matrices of the same size, and r, s be scalars. Then
>
> 1. $A+B=B+A$
> 2. $(A+B)+C=A+(B+C)$
> 3. $A+0=A$
> 4. $r(A+B)=rA+rB$
> 5. $(r+s)=rA+sA$
> 6. $r(sA)=(rs)A$
>
> Here, 0 represents the zero matrix: the matrix whose entries are all zero.

Example 3. *Let* $A=\begin{pmatrix} 1 & -2 \\ 4 & 4 \end{pmatrix}$. *Compute* $2A-2I_2$.

Solution. Can approach several ways. Using properties:

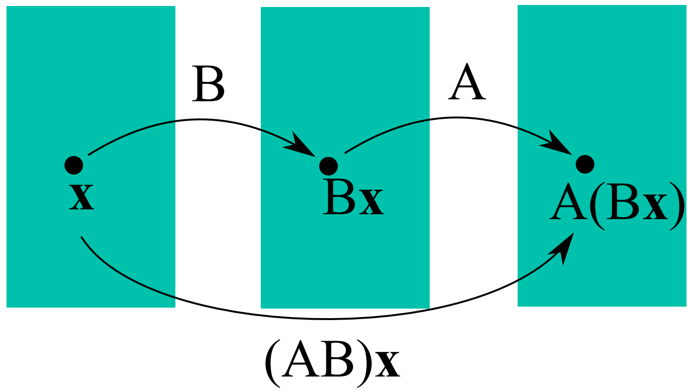
$$2A-2I_2=2(A-I_2)=2\begin{pmatrix} 0 & -2 \\ 4 & 3 \end{pmatrix}=\begin{pmatrix} 0 & -4 \\ 8 & 6 \end{pmatrix}.$$

2 Matrix Multiplication

Given A and B, two matrices, what should AB mean? We hope that the resultant matrix AB would have the property that

$$(AB)\mathbf{x}=A(B\mathbf{x})$$

for any $\mathbf{x}$. As a picture:



Example 4. *Suppose A is an $m \times q$ matrix and B is a $p \times n$ matrix.*

1. *What size must the vector $\mathbf{x}$ be in order for $B\mathbf{x}$ to be computed?*
2. *What size vector will $B\mathbf{x}$ be?*
3. *Are there any restrictions on the size of A so that $A(B\mathbf{x})$ makes sense?*
4. *What size vector will $A(B\mathbf{x})$ be?*
5. *What size matrix do you think AB will be?*

Solution. We require $\mathbf{x}$ to be from $\mathbb{R}^n$ so that $B\mathbf{x}$ makes sense. Then $B\mathbf{x}$ must be a vector from $\mathbb{R}^p$. Thus for $A(B\mathbf{x})$ to make sense, we need A to have p columns. So we must have $p = q$. The matrix AB is mapping a vector from $\mathbb{R}^n$ to $\mathbb{R}^m$, thus it must be a matrix of size $m \times n$.

Definition: If A is size $m \times p$ and $B = (\mathbf{b}_1 \ \dots \ \mathbf{b}_n)$ is size $p \times n$, then the **matrix product** AB is the $m \times p$ matrix defined by

$$AB = (A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_n).$$

Practically: to find the i, jth entry of AB, compute the *dot product* of the ith row of A with the jth column of B.

Example 5. *If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?*

Remarks:

- In a better world, we would call AB the matrix *composition*! Thinking about A and B as transformations, AB is the transformation which first applies B and then applies A.

To compute:

$$\begin{pmatrix}1 & 2 & -1\\ 0 & -5 & 3\end{pmatrix}\begin{pmatrix}4 & 3\\ 3 & 1\\ 7 & -2\end{pmatrix}$$

Step 1:

$$\begin{pmatrix}4 & 3\\ 3 & 1\\ 7 & -2\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & -1\\ 0 & -5 & 3\end{pmatrix}\begin{pmatrix}\bullet & \\ & \end{pmatrix} 1\cdot 4 + 2\cdot 3 + -1\cdot 7 = 3 = \bullet$$

Step 2:

$$\begin{pmatrix}4 & 3\\ 3 & 1\\ 7 & -2\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & -1\\ 0 & -5 & 3\end{pmatrix}\begin{pmatrix}3 & \\ \bullet & \end{pmatrix} 0\cdot 4 + -5\cdot 3 + 3\cdot 7 = 6 = \bullet$$

Step 3:

$$\begin{pmatrix}4 & 3\\ 3 & 1\\ 7 & -2\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & -1\\ 0 & -5 & 3\end{pmatrix}\begin{pmatrix}3 & \bullet\\ 6 & \end{pmatrix} 1\cdot 3 + 2\cdot 1 + -1\cdot -2 = 7 = \bullet$$

Step 4:

$$\begin{pmatrix}4 & 3\\ 3 & 1\\ 7 & -2\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & -1\\ 0 & -5 & 3\end{pmatrix}\begin{pmatrix}3 & 7\\ 6 & \bullet\end{pmatrix} 0\cdot 3 + -5\cdot 1 + 3 \cdot -2 = -11 = \bullet$$

So,

$$\begin{pmatrix}1 & 2 & -1\\ 0 & -5 & 3\end{pmatrix}\begin{pmatrix}4 & 3\\ 3 & 1\\ 7 & -2\end{pmatrix} = \begin{pmatrix}3 & 7\\ 6 & -11\end{pmatrix}$$

Example 6. *Let* $A = \begin{pmatrix}1 & 2 & -1\\ 0 & -5 & 3\end{pmatrix}$ *and* $B = \begin{pmatrix}4 & 3\\ 3 & 1\\ 7 & -2\end{pmatrix}$. *Compute* AB *and* BA.

Solution. The product BA is not possible to compute!

An important observation here is that, in general, compositions are not commutative (perhaps you recall that $f \circ g \neq g \circ f$ in general). Thus, in general:

$$AB \neq BA.$$

However, other very nice properties of matrix multiplication still hold:

Theorem: Let A be $m \times n$, and B and C be appropriate sizes so that products/sums are defined:

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $I_m A = A = AI_n$

where I_m, I_n are identity matrices.

Notice that if A is a square matrix, then we can multiply A by itself as many times as we would like:

$$A^k = A \cdots A.$$

If $k = 0$ then we interpret $A^0 = I_n$.

Example 7. *Suppose that $AB = 0$.* **T/F:** *It must be the case that either $A = 0$ or $B = 0$.*

Solution. No, let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $AB = 0$.

3 Transposes

Definition: Given a matrix A of size $m \times n$, the matrix A^T is called the **transpose** and is the $n \times m$ matrix whose columns are formed from the rows of A.

Example 8. *Given $B = \begin{pmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}$, what is B^T?*

Given

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{pmatrix},$$

what is C^T?

Solution. $B^T = \begin{pmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{pmatrix}$ and $C^T = \begin{pmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{pmatrix}$.

Example 9. *Suppose A is $m \times p$ and B is $p \times n$. Why can't $(AB)^T = A^T B^T$? Can you determine the correct formula?*

Solution. We know AB is size $m \times n$ and so $(AB)^T$ must be $n \times m$. We notice that A^T is $p \times m$ and B^T is $n \times p$ and so $A^T B^T$ is not defined unless $m = n$! However we do see that $B^T A^T$ is defined and has size $n \times m$.

Indeed, the ij entry of $(AB)^T$ is the same as the ji entry of AB. This entry is obtained from the dot product of the jth row of A with the ith column of B. Taking transposes this means the jth column of A^T dot product with the ith row of B^T, which is $B^T A^T$.

The transpose satisfies the following properties:

Theorem: Let A and B have appropriate sizes. Then:

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. For any scalar r, $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

Example 10. *Suppose* $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ *and* $C = \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}$. *Check that $AB = AC$. What is surprising about the "algebra" of matrices that is different from algebra of numbers?*