It turns out that most things we have been studying this semester are subspaces. We already know that the span of vectors forms a subspace. The first place we saw span of vectors was to describe the solution set of a system of equations, so perhaps solution sets form subspaces!

1 Null space of a matrix

Consider the solution to the homogeneous equation

$$A\mathbf{x} = \mathbf{0}$$

where

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

Remember that the *solution set* of this equation is the set of all $\mathbf{x}$ satisfying the equation. Sometimes we say that these $\mathbf{x}$ are *implicitly* defined by this equation. This simply means: they are well-defined but one cannot immediately tell what they are. Fortunately, we already know how to find an *explicit* description of the solutions: row reducing!

Example 1. *Find a spanning set of vectors for the solution set of*

$$A\mathbf{x} = \mathbf{0}$$

where

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

Solution. Row reducing the augmented matrix shows

$$(A \ \ \mathbf{0}) \sim \begin{pmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the general solution can be written

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Thus all solutions are given by the span of the vectors

$$\operatorname{Span}\left\{\begin{pmatrix}2\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\-2\\1\\0\end{pmatrix},\begin{pmatrix}-3\\0\\2\\0\\1\end{pmatrix}\right\}.$$

Since the solution set is a span of vectors it must be a subspace! To emphasize this fact, we often call the solution set of such homogeneous equations the *null space*.

Definition: The **null space** of an $m \times n$ matrix A, written $\operatorname{Nul} A$ is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation:

$$\operatorname{Nul} A = \{\mathbf{x} \colon A\mathbf{x} = \mathbf{0}\}.$$

Note: There is no difference between "solution set" and "null space" here! This is an example of the same concept in math being named several ways. Sometimes we will call it null space to really emphasize the fact that it is, in fact, a subspace.

Theorem: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Example 2. *Let A be an $m \times n$ matrix. Is the solution set of*

$$A\mathbf{x} = \mathbf{b}$$

always a subspace? If it is, what is it a subspace of?