Learning Objectives:

1. Define dimension of a vector space
2. Describe the relationship between dimensions of a vector space and its subspaces, including geometric intuition from $\mathbb{R}^n$
3. Calculate the dimension of subspaces

1 Dimension

Intuition: We often think of each new dimension as a new spatial "direction", but this intuition fails past 3D. It also may not be useful when thinking about more abstract vector spaces. Instead, we should think of dimension as "degrees of freedom:" *how many coordinates do we need to uniquely describe any vector in the vector space?* To describe any vector sounds roughly like spanning the vector space. To uniquely do so sounds roughly like linear independence. So, dimension of a space must have something to do with basis sets!

Theorem: If a vector space V has a basis of n vectors, then every basis of V has n vectors.

Remark: This theorem tells us that any set which has more than n vectors must be linearly dependent and any set which has fewer than n vectors cannot span the vector space.

Definition: If V is spanned by a finite set, then V is **finite-dimensional** and the dimension of V, written $\dim V$ is the number of vectors in any basis for V. The dimension of $\{\mathbf{0}\}$ is 0. Any vector space not spanned by a finite set is infinite-dimensional.

Example 1.

- *The vector space $\mathbb{R}^n$ has dimension n.*
- *The vector space $\mathbb{P}_n$ has dimension $n+1$.*

Example 2. *Find the dimension of the subspace*

$$\left\{ \begin{pmatrix} s-2t \\ s+t \\ 3t \end{pmatrix} : s,t \in \mathbb{R} \right\}.$$

Solution. We need to find a basis for this subspace of $\mathbb{R}^3$. Writing

$$\begin{pmatrix} s-2t \\ s+t \\ 3t \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix},$$

we see that the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$$

are linearly independent (as they are not multiples of each other). As such they are linearly independent and thus must form a basis. Thus the dimension of the space is 2.

Example 3. *In general, the subspaces of* $\mathbb{R}^3$ *are classified by their dimension:*

- 1 *dimensional subspaces are the lines through the origin*
- *2-dimensional subspaces are the planes through the origin*
- *The only 3-dimensional subspace is all of* $\mathbb{R}^3$.

Theorem: If H is a subspace of a finite dimensional space V then $\dim H \leqslant \dim V$. Moreover any basis of H can be extended to a basis of V.

As a quick corollary of this theorem we immediately have:

Theorem: If $\dim V = p$ then any set of p linearly independent vectors is a basis of V, and any set of p vectors that spans V is a basis of V.

Example 4. *Given a matrix A, what are the dimensions of* $\operatorname{Nul} A$ *and* $\operatorname{Col} A$*?*

Solution. A basis for the null space is found by row reducing $A\mathbf{x} = \mathbf{0}$, and the dimension is the number of free variables.

A basis for the column space is given by the pivot columns of A, so the number of pivots is the dimension.

Example 5. T/F: *If* $\dim V = p$ *and S is a linearly dependent set in V, then S has more than p vectors.*

Solution. False. $S = \{0\}$ is a counter example. The converse statement is true.