The determinant for a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is $\det A = ad - bc$. Sometimes we use the notation $|A| = ad - bc$ or $\Delta = ad - bc$. We previously saw that A is invertible if and only if $\det A \neq 0$. However this was not motivated or explained, so let's pause and try to understand

1. why $\det A$ is calculated the way it is, and
2. how to interpret $\det A$ in a geometric way.

Together, these will help us understand how to define $\det A$ for larger matrices, which is our first goal for Chapter 3.

1 Deriving the formula from row reduction

A 2×2 matrix A is invertible if and only if $\det A = ad - bc \neq 0$.

Where does the formula for the determinant come from? In the 2×2 case notice that given an arbitrary 2×2 matrix, we can row reduce (and we will assume $a \neq 0$ and $c \neq 0$ for simplicity)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} ac & bc \\ ac & ad \end{pmatrix} \sim \begin{pmatrix} ac & bc \\ 0 & ad - bc \end{pmatrix}.$$

(I didn't explain the row operations here, but you should think about what operations were applied to get these row equivalencies!) Now, for A to be invertible we need 2 pivots, so we need $ad - bc \neq 0$, otherwise we would have a row of all zeros! This idea shows us how we can derive the formula $\Delta = ad - bc$ using the connection to row reduction and pivots.

2 Geometric intuition: volume change

Now, we interpret the determinant geometrically: for simplicity, consider the diagonal matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

This corresponds to the linear transformation

$$T(x_1, x_2) = (2x_1, 3x_2),$$

which scales all vectors by 2 in the $\mathbf{e}_1$ direction and by 3 in the $\mathbf{e}_2$ direction.

We will consider what happens to the unit square whose vertices are determined by the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ after transformation. That is, start from the square whose corners are at $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_1 + \mathbf{e}_2$. Looking at the figure below, we start from a square of area 1, and after applying T to each vector, we end up with an new rectangle with area 6.

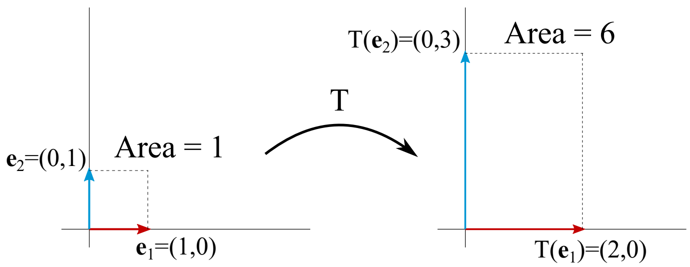


Figure 1: Since $\det A = 6$ then the unit square maps to a rectangle whose area is 6.

It is also no coincidence that $\det A = ad - bc = (2)(3) - 0 = 6$.

Take-away: The value of $|\det A|$ is the area of the unit square after applying the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

Remarks:

- We take the absolute value of the determinant, because $\det A$ can be negative. A negative determinant means we performed a reflection of the plane!

- It turns out that this "area scaling" property is true *for any starting shape.* So if you start with any shape with area m then after the transformation A the image of the shape will have area $|\det A| \cdot m$. The figure below shows the effect of a linear transformation (that both scales and shears). The areas of both the unit square and an emoji face are scaled by the same factor.

Notice that when the transformation also shears, the unit square in general becomes a parallelogram (and not just a rectangle). Also, remember that the columns of A are the same as $T(\mathbf{e}_i)$, and so another way of interpreting the above discussion is to say:

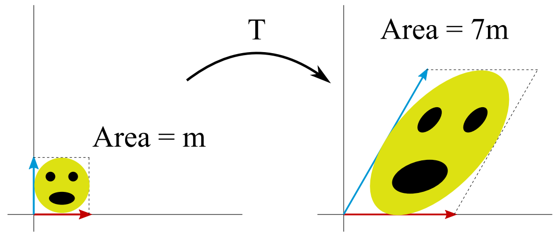
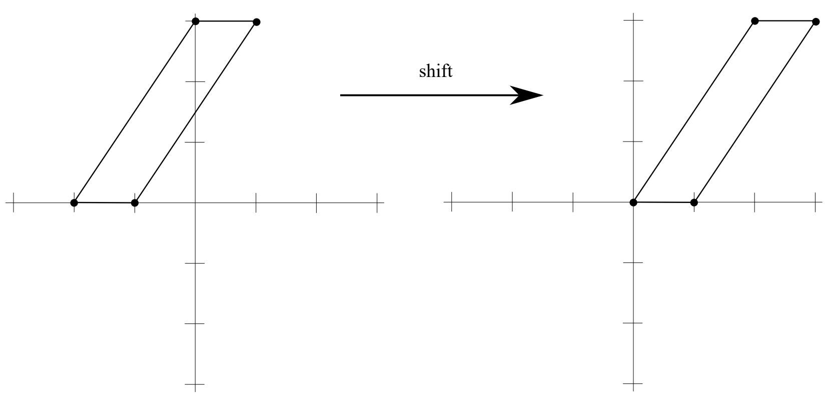


Figure 2: A different linear transformation with shear and scaling (notice the unit square becomes a parallelogram). Since all areas scale by 7 we have $\det A = 7$.

Theorem: If A is a 2×2 matrix, then the area of the parallelogram determined by the columns of A is $|\det A|$.

Example 1. *Calculate the area of the parallelogram whose vertices are* $\begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

Solution. To save space, I will write column vectors as row vectors. If we draw a picture, notice that no vertex is at the origin.



By adding $(2, 0)$ to every point, the parallelogram has the same area, but the points are now

$$(0, 0), (2, 3), (3, 3), (1, 0).$$

The parallelogram can now be described by the two vectors coming from the origin: $(2, 3)$ and $(1, 0)$. Put them as columns in a matrix A and calculate the area of the parallelogram using the determinant:

$$\left| \det \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \right| = |-3| = 3.$$

What do you think it means that $\det A = 0$ in terms of linear transformations? Why does this mean that the matrix (and linear transformation) is not invertible?