Learning Objectives:

1. Compute whether a given set of vectors is linearly independent or linearly dependent.
2. Describe linear independence and linear dependence geometrically.

1 Linear independence

Intuition: We have already asked whether a given vector is in the span of a set. On the other hand, we may ask if a given vector in the span can be described uniquely! This is to say, are some of the vectors in our set redundant?

Returning to paint as an intuition: suppose one paint can (vector $\mathbf{v}_1$) is yellow, one paint can (vector $\mathbf{v}_2$) is blue, and one paint can (vector $\mathbf{v}_3$) is green. The span of these vectors is the collection of all possible colors we can get by mixing them in various ratios. However, since green is already just a mix of yellow and blue, we could remove it and still have the same range of colors possible. This vector was redundant!

Definition: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is **linearly independent** if the only solution to the vector equation

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

is the trivial solution, $x_1 = \cdots = x_p = 0$.

On the other hand, if there exist coefficients $c_1, \ldots, c_p$ not all equal to 0 so that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is **linearly dependent**.

As we have done before, we may translate these definitions into a statement about a matrix equation: let $A = (\mathbf{v}_1 \ \ldots \ \mathbf{v}_p)$. Then, the set $\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ is linearly independent if and only if

$$A\mathbf{x} = \mathbf{0}$$

has only the trivial solution $\mathbf{x} = \mathbf{0}$.

The columns of A are linearly independent if and only if A has no free variables.

Example 1. *Is the set* $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}\right\}$ *linearly independent?*

Solution. We solve $A\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix:

$$\begin{pmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see that x_3 is a free variable. So, there exist infinitely many solutions to $A\mathbf{x} = \mathbf{0}$ and thus the set is linearly dependent.

Example 2.

- *Suppose* $\mathbf{b}$ *is in the span of the columns of* A*. What can be said about the solution set of* $A\mathbf{x} = \mathbf{b}$*?*
- *Suppose now in addition that the columns of* A *are linearly independent. What can be said about the solution set of* $A\mathbf{x} = \mathbf{b}$*?*
- *Suppose instead that the columns of* A *are linearly dependent. What can be said about the solution set of* $A\mathbf{x} = \mathbf{b}$*?*

Solution. First, if $\mathbf{b}$ is in the span, then the equation $A\mathbf{x} = \mathbf{b}$ is solvable. If the columns of A are linearly independent, then row reducing results in no free variables, so the solution is unique. If the columns of A are linearly dependent, then row reducing results in free variables, so the solution is non-unique.

Summary:

1. Span is related to *existence* of solutions whereas linear independence is related to *uniqueness* of solutions.
2. Given an $m \times n$ matrix A:
   - if there is a pivot in every row of A then $A\mathbf{x} = \mathbf{b}$ is solvable for any $\mathbf{b} \in \mathbb{R}^m$.
   - if there is a pivot in every column of A then $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x} \in \mathbb{R}^n$ (whenever the equation is solvable).

2 Special cases

Example 3. *When is the set* $\{\mathbf{v}_1\}$ *linearly independent? What about two vectors* $\{\mathbf{v}_1, \mathbf{v}_2\}$*?*

Solution. A set with one vector is linearly independent if $\mathbf{v}_1 \neq \mathbf{0}$. This is true because if we consider the equation $x_1\mathbf{v}_1 = \mathbf{0}$, then if $\mathbf{v}_1 \neq \mathbf{0}$ then $x_1 = 0$ is the only solution and so the set $\{\mathbf{v}_1\}$ is linearly independent. On the other hand, if $\mathbf{v}_1 = \mathbf{0}$ then taking $x_1 = 1$ gives a non-trivial solution to $x_1\mathbf{v}_1 = \mathbf{0}$. A set with two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent if and only if neither vector is a scalar multiple of the other. We see that this follows from the simple equation:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0}.$$

If $x_1 \neq 0$ then we can solve for $\mathbf{v}_1$:

$$\mathbf{v}_1 = -(x_2/x_1)\mathbf{v}_2.$$

So, $\mathbf{v}_1$ is a scalar multiple of $\mathbf{v}_2$. We see something similar happens if $x_2 \neq 0$.

Example 4. *What doe it mean for two vectors to be linearly dependent geometrically?*

Solution. Two vectors are linearly dependent if they lie on the same line.

General characterization of linear independent sets

Theorem: A set $S = \{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the other vectors. That is, the set S is linearly dependent if there is some $\mathbf{v}_i$ which is contained in the span of the other vectors. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$ then some $\mathbf{v}_j$ is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Example 5. *Suppose* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ *are linearly independent. What can you say about their span? What does it mean if the new vector* $\mathbf{w} \in \mathbb{R}^3$ *makes the set* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *linearly dependent? What does it mean if the set* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is linearly independent?*

Solution. If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent they do not point in the same direction, so their span is a plane in $\mathbb{R}^3$. If $\mathbf{w}$ makes the set linearly dependent, then it must also lie in the same plane already described by $\mathbf{u}$ and $\mathbf{v}$. If the set with all vectors is still linearly independent, then $\mathbf{w}$ lies somewhere off the plane, and in fact the span of all vectors is now all of $\mathbb{R}^3$.

We now may have intuition for the following Theorem:

Theorem: Let $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n$. (i). If $p > n$ then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent. (ii). If one of the $\mathbf{v}_i = \mathbf{0}$ then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

We see in (i) that if there are more vectors than the dimension of the space, then there are "no more directions" for extra vectors so at least one of them must be a linear combination of the others.

Example 6. T/F: *The columns of any* 5×4 *matrix are linearly dependent.*

Solution. False. We can take $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. There is a pivot in each column so the columns are linearly independent.