Learning Objectives:

1. Define the basis of vector space
2. Describe how the Spanning Set Theorem generates a basis from a spanning set.
3. Find the bases of both null and column spaces

1 Deja vu

We previously discussed linear independence and linear dependence of vectors in $\mathbb{R}^n$. All of those concepts extend immediately to arbitrary vectors spaces.

Definition: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

has *only* the trivial solution $c_1 = c_2 = \cdots = c_p = 0$. If there is a non-trivial solution we call the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ **linearly dependent**.

Much of the intuition we built still holds:

1. A single vector $\mathbf{v}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.
2. A pair of vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent if and only if neither is a multiple of the other.
3. Any set containing the zero vector is linearly dependent.

Theorem: A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Idea of proof: We must show both directions of the "if and only if." We first assume the set is linearly dependent and show that $\mathbf{v}_j$ is a linear combination. We then assume that $\mathbf{v}_j$ is a linear combination and prove that the set is linearly dependent. In this way, we show that the statements are equivalent: both are simultaneously true or both are false.

Proof. If the set is linearly dependent then there exist $c_1, \ldots, c_p$ not all zero so that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

Let j be the largest index which is non-zero. That is, for all $k > j$ we have $c_k = 0$. So, we can write

$$c_1\mathbf{v}_1 + \cdots + c_j\mathbf{v}_j = \mathbf{0}.$$

(It may be the case that $j = p$). Then

$$\mathbf{v}_j = -\frac{c_1}{c_j}\mathbf{v}_1 - \cdots - \frac{c_{j-1}}{c_j}\mathbf{v}_{j-1},$$

so $\mathbf{v}_j$ is a linear combination of the previous vectors. This proves one direction.

Now, assume that some $\mathbf{v}_j$ is a linear combination of the previous vectors:

$$\mathbf{v}_j = c_1\mathbf{v}_1 + \cdots c_{j-1}\mathbf{v}_{j-1}.$$

This equation is equivalent to

$$c_1\mathbf{v}_1 + \cdots c_{j-1}\mathbf{v}_{j-1} - \mathbf{v}_j + 0\mathbf{v}_{j+1} + \cdots + 0\mathbf{v}_p = \mathbf{0}.$$

Since the coefficient of $\mathbf{v}_j$ is not zero, we conclude that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent. □

Example 1. *Consider the vector space of all polynomials* $\mathbb{P}$*. Show that the set* $\mathbf{p}_1(t) = 1$*,* $\mathbf{p}_2(t) = t^2$ *and* $\mathbf{p}_3(t) = 2t^2 - 4$ *is linearly dependent.*

Solution. Since $\mathbf{p}_3 = 2\mathbf{p}_2 - 4\mathbf{p}_1$ we conclude that the set is linearly dependent.

Example 2. *Let* $\mathbf{p}_1(t) = 1$*,* $\mathbf{p}_2(t) = t$ *and* $\mathbf{p}_3(t) = t^2$*. Explain why* $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ *is linearly independent.*

Solution. Suppose

$$c_1 + c_2t + c_3t^2 = 0.$$

This equation must be true for all t, and so it must be the case that $c_i = 0$ for all i and thus the set is linearly independent.

2 Bases

Definition: Let H be a subspace of a vector space V. Then $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

1. $\mathcal{B}$ is linearly independent and
2. the subspace spanned by $\mathcal{B}$ is H.

Intuition: A basis is a "Goldilocks" set: if a set is too small then it will not span the space. If it is too large then it will be linearly dependent. A basis is just the right size!

Example bases:

- The **standard basis** vectors for $\mathbb{R}^n$, are the vectors $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \ldots,\ \mathbf{e}_n\}$.
- The set $S = \{1, t, t^2, \ldots, t^n\}$ is called the **standard basis** for $\mathbb{P}_n$.

Example 3. *Do the vectors*

$$\begin{pmatrix} 3 \\ 0 \\ -6 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$$

form a basis of $\mathbb{R}^3$*?*

Solution. The invertible matrix theorem says that if the matrix whose columns are the given vectors is invertible, then the columns span $\mathbb{R}^3$ and are linearly independent. So we need only check the invertibility, e.g., by row reducing.

$$\begin{pmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{pmatrix} \sim \begin{pmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Since the matrix has 3 pivots it is invertible and thus the vectors do form a basis.

3 Spanning Set Theorem

The Spanning Set Theorem: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V and let $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

1. If some $\mathbf{v}_k$ is a linear combination of the remaining vectors in S then the set S' formed by removing $\mathbf{v}_k$ from S still spans H.
2. If $H \neq \{\mathbf{0}\}$ then some subset of S is a basis for H.

We can use the Spanning Set Theorem to find a basis for the column space of a matrix.

Finding a basis for the null space: Given a matrix A, solve

$$A\mathbf{x} = \mathbf{0}.$$

The vectors that make up the general solution span the null space and are linearly independent.

Example 4. *Knowing that*

$$A = \begin{pmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{pmatrix}$$

row reduces to

$$B = \begin{pmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

find a basis for $\operatorname{Col} A$.

Solution. Of course, $\operatorname{Col} A$ is the span of the columns, but there is no guarantee that the columns are linearly independent. So, we should eliminate any vectors which are linearly dependent. Since row reduces maintains linear dependence relationships, this means we should take the pivot columns! From B, the pivot columns are columns 1, 3, and 5. For matrix A these correspond to

$$\begin{pmatrix} 1 \\ 3 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 2 \\ 8 \end{pmatrix}.$$

In fact, we can see that $\mathbf{a}_2 = 4\mathbf{a}_1$ and $\mathbf{a}_4 = 2\mathbf{a}_1 - \mathbf{a}_3$.

Theorem: The pivot columns of a matrix A form a basis for the column space.

Example 5. *Let*

$$A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}.$$

Knowing that

$$\begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

find bases for $\operatorname{Nul} A$ *and* $\operatorname{Col} A$.

Solution. The first two columns are pivot columns, and so the column space is spanned by the first two vectors of A, so a basis for the column space is

$$\left\{ \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 4 \\ -6 \\ 8 \end{pmatrix} \right\}.$$

The general solution for $A\mathbf{x} = \mathbf{0}$ is

$$\begin{aligned} x_1 &= -6x_3 - 5x_4 \\ x_2 &= -5/2x_3 - 3/2x_4 \end{aligned}$$

so all vectors in the null space are of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -6x_3 - 5x_4 \\ -5/2x_3 - 3/2x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{pmatrix}.$$

Thus a basis for the null space is

$$\left\{ \begin{pmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Example 6. T/F: *If* $H = \operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *then a basis for* H *is* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Solution. False. The set may be linearly dependent and so one would have to use the Spanning Set Theorem to find a smaller set which is a basis.