1 Cramer's Rule

Goal: Cramer's rule is a way to solve the linear system

$$A\mathbf{x} = \mathbf{b}.$$

For us, the main goal of learning Cramer's rule is to link and review several concepts we have seen!

Let

$$A = (\mathbf{a}_1 \quad \mathbf{a}_2) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

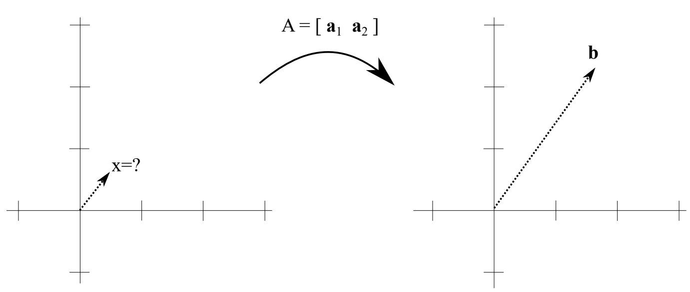
As a linear transformation, we can think of A as the transformation which maps

$$A\mathbf{e}_1 = \mathbf{a}_1$$

and

$$A\mathbf{e}_2 = \mathbf{a}_2.$$

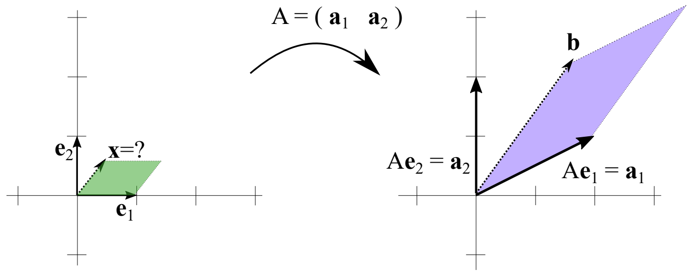
Given the target $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, our goal is to find the unknown $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ that maps to $\mathbf{b}$. This is represented visually in the figure below.

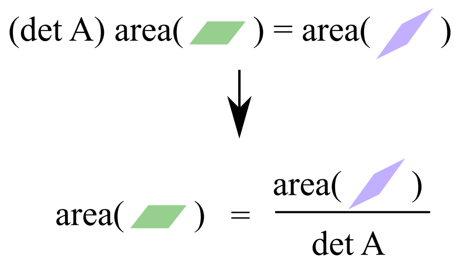


Now, consider drawing a (green) parallelogram between $\mathbf{e}_1$ and our unknown vector $\mathbf{x}$. When we apply the transformation A this parallelogram becomes a (purple) parallelogram formed between $\mathbf{a}_1$ and $\mathbf{b}$. This is shown in the next figure below. How can we use this? Recall the following two facts:

1. The determinant of A measures the how much shapes scale after transformation.

2. The areas of parallelograms can be calculated also using determinants.





The original (green) parallelogram has area

$$|\mathbf{e}_1 \quad \mathbf{x}| = \begin{vmatrix} 1 & x_1 \\ 0 & x_2 \end{vmatrix}.$$

The transformed (purple) parallelogram has area

$$|\mathbf{a}_1 \quad \mathbf{b}| = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}.$$

Finally, $\det A$ tells us how the first area scales to the second one:

$$(\det A) \cdot \begin{vmatrix} 1 & x_1 \\ 0 & x_2 \end{vmatrix} = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}.$$

After a bit of rearranging:

$$\begin{vmatrix} 1 & x_1 \\ 0 & x_2 \end{vmatrix} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\det A}.$$

Finally, notice that the left hand can be calculated to simplify

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\det A}.$$

Pause a moment: the left hand side is the second component of our unknown vector. The right hand side is made up of all known quantities, so we can plug in numbers and solve!

Can you see how to modify the above argument in order to get the other equation

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\det A}?$$

We have just discovered *Cramer's Rule*. By dividing two determinants we can calculate each of the unknown values of $\mathbf{x}$. Perhaps you noticed the following pattern for the matrices on the right hand side: if we are finding the x_i component, we need to replace the ith column of A by the vector $\mathbf{b}$.

Cramer's Rule: Let A be an invertible $n \times n$ matrix. Let

$$A_i(\mathbf{b}) = (\mathbf{a}_1 \cdots \mathbf{b} \cdots \mathbf{a}_n),$$

be the matrix A with the ith column replaced by $\mathbf{b}$. Then the unique solution $\mathbf{x}$ solving $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det A}, \quad \text{for all } i = 1, \ldots, n.$$