Motivation: Why have we spent so much time reformulating systems of equations into matrix equations: $A\mathbf{x} = \mathbf{b}$?

Answer: A benefit of the matrix formulation is that we can think of the matrix A as a **function**: it inputs a vector $\mathbf{x}$ and outputs another vector $\mathbf{b}$. Thinking this way, we can develop a lot more intuition and build stronger results.

1 Transformations

Given an $m \times n$ matrix A, we can think about it as a function: given an "input" vector $\mathbf{x} \in \mathbb{R}^n$, we can compute an "output" vector $\mathbf{b} \in \mathbb{R}^m$ by simply computing the matrix vector product $A\mathbf{x}$.

Example 1. *Given the matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 2 & -1 \end{pmatrix},$$

let T be the function

$$T(\mathbf{x}) = A\mathbf{x}.$$

Letting $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$, *compute $T(\mathbf{u})$.*

Solution. We have

$$T(\mathbf{u}) = A\mathbf{u} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}.$$

In fact, we see that T will map every vector in $\mathbb{R}^4$ (in this case, $\mathbf{u}$) to some vector in $\mathbb{R}^2$ (in this case, $T(\mathbf{u})$).

So, T really is a function like you saw in calculus or other classes. The only difference is that the inputs and outputs are vectors as opposed to numbers.

Definition: A **transformation** T (or **function** or **mapping** or **operator**) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each input vector $\mathbf{x} \in \mathbb{R}^n$ an output vector $T(\mathbf{x}) \in \mathbb{R}^m$. We write $T\colon \mathbb{R}^n \to \mathbb{R}^m$.

The set $\mathbb{R}^n$ is the **domain** of T.

The set $\mathbb{R}^m$ is the **codomain** of T.

A vector $\mathbf{v} \in \mathbb{R}^m$ is in the **image** of T if there exists $\mathbf{x} \in \mathbb{R}^n$ so that $T\mathbf{x} = \mathbf{v}$. The set of all images of T is the **range** of T.

So many names! Why do we use the word *transformation* instead of just using the familiar word *function*? We will later see that using *transformation* emphasizes the actual geometric movement of vectors. For example, we will see how transformations are used to rotate, stretch, or project vectors.

Example 2. *Let* $A = \begin{pmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \end{pmatrix}$*. Then, define the transformation*

$$T(\mathbf{x}) = A\mathbf{x}.$$

What are the domain/codomain of T*? What would you do to figure out whether* $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *is in the range of* T*?*

Our work will primarily focus on special transformations which additionally satisfy the following properties.

Definition: A transformation T is **linear** if the following equalities hold for all vectors $\mathbf{u}, \mathbf{v}$ and scalars c:

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad T(c\mathbf{u}) = cT(\mathbf{u}).$$

Do these equations look familiar?