Learning Objectives:

1. Define orthogonal set, orthogonal projection, orthonormal basis, and orthogonal matrix.
2. Represent vectors as a linear combination of orthogonal basis vectors.
3. Explain properties of matrices with orthonormal columns.

1 Orthogonal sets

Definition: A set $\{\mathbf{v}_1, \dots \mathbf{v}_p\}$ is said to be an **orthogonal set** if each pair of distinct vectors is orthogonal: $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$.

Definition: We say that a set $\{u_1, \dots u_p\}$ is an **orthonormal set** if it is an orthogonal set of unit vectors (all have norm 1).

Theorem: If $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and thus is a basis for the subspace spanned by S.

Example 1. *Why are orthogonal bases easier to work with? Consider the following bases of* $\mathbb{R}^2$*:*

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\},$$

and

$$\left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Write $\mathbf{y} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$ *as a linear combination of the basis vectors in each case.*

Take-away: In general, if a set is orthogonal, then it is much easier to represent other vectors as linear combinations!

Theorem: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for W. For each $y \in W$, we have

$$y = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p,$$

where

$$c_i =$$

Example 2. *Take* $\mathbf{u}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ *and* $\mathbf{u}_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$. *Write* $\mathbf{x} = \begin{pmatrix} 9 \\ -7 \end{pmatrix}$ *as a linear combination of the* $\mathbf{u}_i$.

2 Orthogonal matrices

Example 3. *Suppose* $\{\mathbf{v}_1, \cdots, \mathbf{v}_3\}$ *is an orthonormal set of vectors from* $\mathbb{R}^3$. *Let* U *have columns formed by these vectors. What is*

$$U^T U?$$

Definition: An $m \times n$ matrix U has orthonormal columns if and only if ____________. If U is square then we say U is **orthogonal**.

Theorem: Let U be an $m \times n$ matrix with orthonormal columns and $x, y \in \mathbb{R}^n$. Then

1. $\|Ux\| =$
2. $(Ux) \cdot (Uy) =$
3. $(Ux) \cdot (Uy) = 0$ if and only if

Example 4. **T/F:** *An orthogonal matrix is necessarily invertible.*

Example 5. *Let* $\theta = \pi/13$. *Is*

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

orthogonal?