Learning Objective:

1. Translate concepts of matrix multiplication to those of linear transformations.
2. Determine domain, codomain, range, and whether a vector is in the image of a transformation.
3. Introduce geometric interpretation of transformations such as projections, shears, and dilations.

1 Transformations

Motivation: Why have we spent so much time reformulating systems of equations into matrix equations: $A\mathbf{x} = \mathbf{b}$?

Answer: A benefit of the matrix formulation is that we can think of the matrix A as a function: it inputs a vector $\mathbf{x}$ and outputs another vector $\mathbf{b}$. Thinking this way, we can develop a lot more intuition and build stronger results.

Example 1. *Given the matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 2 & -1 \end{pmatrix},$$

let T be the function

$$T(\mathbf{x}) = A\mathbf{x}.$$

Letting $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$, *compute $T(\mathbf{u})$.*

Solution. We have

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}.$$

In fact, we see that T will map every vector in $\mathbb{R}^4$ to some vector in $\mathbb{R}^2$.

We have generalized the idea of *function*:

Definition: A **transformation** T (or **function** or **mapping** or **operator**) from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each input vector $\mathbf{x} \in \mathbb{R}^n$ an output vector $T(\mathbf{x}) \in \mathbb{R}^m$. We write $T\colon \mathbb{R}^n \to \mathbb{R}^m$.

The set $\mathbb{R}^n$ is the **domain** of T.

The set $\mathbb{R}^m$ is the **codomain** of T.

A vector $\mathbf{v} \in \mathbb{R}^m$ is in the **image** of T if there exists $\mathbf{x} \in \mathbb{R}^n$ so that $T\mathbf{x} = \mathbf{v}$. The set of all images of T is the **range** of T.

Example 2. *Let* $A = \begin{pmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \end{pmatrix}$. *Then, define the transformation*

$$T(\mathbf{x}) = A\mathbf{x}.$$

What are the domain/codomain of T*? Is* $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *in the range of* T*?*

Solution. We see that for $A\mathbf{x}$ to make sense, we require that $\mathbf{x} \in \mathbb{R}^3$. So, the domain of T is $\mathbb{R}^3$. Similarly, since $A\mathbf{x} \in \mathbb{R}^2$, the codomain of T is $\mathbb{R}^2$. To determine if $\mathbf{u}$ is in the range we must see if we can solve

$$A\mathbf{x} = \mathbf{u}.$$

We row reduce the associated augmented matrix

$$\begin{pmatrix} 1 & 0 & -2 & 1 \\ -2 & 1 & 6 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

We see that in general the solution is given by

$$\mathbf{x} = \begin{pmatrix} 2x_3 + 1 \\ -2x_3 + 3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

Any of these $\mathbf{x}$ will map to $\mathbf{u}$, so indeed $\mathbf{u}$ is in the range, and in fact there are infinitely many vectors $\mathbf{x}$ which have $\mathbf{u}$ as their image.

Example 3. *Let* $A = \begin{pmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$. *Define* T *by* $T(\mathbf{x}) = A\mathbf{x}$.

1. *What are the domain and codomain of T?*
2. *Find $T(\mathbf{u})$*
3. *Find an $\mathbf{x} \in \mathbb{R}^2$ whose image under T is $\mathbf{b}$.*
4. *Is there more than one choice of $\mathbf{x}$ in the previous question?*
5. *Determine if $\mathbf{c}$ is in the range of T.*

Solution. 1. $T\colon \mathbb{R}^2 \to \mathbb{R}^3$.

2. $T(\mathbf{u}) = \begin{pmatrix} 5 \\ 1 \\ -9 \end{pmatrix}$.

3. Row reduce:

$$\begin{pmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{pmatrix}.$$

So $\mathbf{x} = \begin{pmatrix} 1.5 \\ -.5 \end{pmatrix}$ works.

4. There are no free variables in the row reduced matrix, so this choice is unique.

5. Row reducing shows:

$$\begin{pmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{pmatrix}.$$

This augmented matrix is not consistent; so $\mathbf{c}$ is not in the range of T.

2 Geometric transformations

Example 4. *The matrix*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

defines a transformation T via $T(\mathbf{x}) = A\mathbf{x}$ which acts $T\colon \mathbb{R}^3 \to \mathbb{R}^3$. This matrix is an example of a projection*; why?*

Solution. Taking an arbitrary vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, we see that $A\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}$. So, every vector is projected onto the x_1-x_3 plane, since automatically the x_2 component becomes 0. Graphically, this can be thought of as all points in $\mathbb{R}^3$ being pushed onto a single plane.

Example 5. *Consider the matrix*

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

Then A gives rise to a shear transformation. *Consider what happens to a few vectors:*

$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ is mapped to } A\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$
$$\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ is mapped to } A\mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$
$$\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is mapped to } A\mathbf{w} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

Overall, we see that this transformation "pulls" the top of these vectors to the right. Such transformations appear frequently in geology and chemistry in applications of crystallography.

3 Linear transformations

What if we define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\mathbf{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This transformation is *constant* and maps everything to the same output. Unlike the previous examples, this doesn't come from matrix multiplication. This gives us the question:

Question: How can we tell if a transformation T can be represented as a matrix multiplication?

Idea: Recall that matrices satisfy:

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}, \quad A(c\mathbf{u}) = c(A\mathbf{u}).$$

Thus, if we have a transformation T, for there to be a hope of representing it using matrices, it must have these same properties.

Definition: A transformation T is **linear** if the following equalities hold for all vectors $\mathbf{u}, \mathbf{v}$ and scalars c:

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad T(c\mathbf{u}) = cT(\mathbf{u}).$$

Theorem: If T is a linear transformation then

(i) $T(\mathbf{0}) = \mathbf{0}$,

(ii) $T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$.

Proof. We can write

$$T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}),$$

so

$$T(\mathbf{0}) = 2T(\mathbf{0}),$$

and subtracting $T(\mathbf{0})$ from both sides yields

$$\mathbf{0} = T(\mathbf{0}).$$

$\square$

Remark: The second property is often called the *superposition principle*, which comes up often in physics and engineering. One way to think about its importance is the following: if I already know the action of a linear transformation on vectors $\mathbf{u}$ and $\mathbf{v}$, then I know the action of the linear transformation on all linear combinations of $\mathbf{u}$ and $\mathbf{v}$ making both calculations and storage very efficient. This is not true for transformations which are not linear!

Question 1. *Show that the mapping $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = 3\mathbf{x}$ is a linear transformation. This transformation is called a* dilation *(why?)*

Solution. We will use the theorem. Let c, d be arbitrary scalars and $\mathbf{u}, \mathbf{v}$ be arbitrary vectors. Then,

$$\begin{aligned} T(c\mathbf{u} + d\mathbf{v}) &= 3(c\mathbf{u} + d\mathbf{v}) \\ &= 3c\mathbf{u} + 3d\mathbf{v} \\ &= c(3\mathbf{u}) + d(3\mathbf{v}) \\ &= cT(\mathbf{u}) + dT(\mathbf{v}). \end{aligned}$$

We see that property (ii) also holds, so indeed T is a linear transformation.

Question: We see that dilations are linear transformations, but how do we know if they can be written as $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A?