Learning Objectives:

1. Describe solution sets of homogeneous and non-homogeneous systems in parametric forms.
2. Relate solution sets of homogeneous and non-homogeneous systems as geometric translations of each other.

1 Homogeneous linear systems

Definition: A **homogeneous system** is of the form

$$A\mathbf{x} = \mathbf{0}.$$

Recall: We always have the **trivial** solution $\mathbf{x} = \mathbf{0}$.

The homogeneous system $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions if and only if the system has free variables.

Example 1. *Determine the solution set of* $A\mathbf{x} = \mathbf{0}$ *where*

$$A = \begin{pmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Take-away: We have seen that the solution set of a homogeneous system can always be written as the span of a collection of vectors:

$$\mathbf{x} =$$

This is called a **parametric vector equation** or **parametric vector form** of the solution since we can get all the solutions by simply plugging in different scalars c_i.

2 Non-homogeneous linear systems

Definition: A *non-homogeneous* system is of the form

$$A\mathbf{x} = \mathbf{b}$$

for $\mathbf{b} \neq \mathbf{0}$.

Luckily, the same approach we just used still works! There will just be one small difference.

Example 2. *Find all solutions of* $A\mathbf{x} = \mathbf{b}$ *where*

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}.$$

Take-away: If the non-homogeneous system $A\mathbf{x} = \mathbf{b}$ is solvable, then its solution set is of the form

$$\mathbf{x} =$$

where c_i are scalars.

Example 3. *Suppose that* $\mathbf{p}$ *is a* particular solution *of* $A\mathbf{x} = \mathbf{b}$*, so that* $A\mathbf{p} = \mathbf{b}$*. If* $\mathbf{v}_h$ *is a solution of the homogeneous equation* $A\mathbf{x} = \mathbf{0}$*, then what equation does* $\mathbf{p} + \mathbf{v}_h$ *solve?*

Theorem: If $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and $\mathbf{p}$ is some vector satisfying $A\mathbf{p} = \mathbf{b}$, then the solution set of $A\mathbf{x} = \mathbf{b}$ consists of all vectors of the form $\mathbf{x} = \mathbf{p} + \mathbf{v}$ where $\mathbf{v}$ is a solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Geometrically: We can think of solutions of homogeneous equations as say a line or plane through the origin (since we know $\mathbf{0}$ must be a solution). Then, solutions of the non-homogeneous system are the same line or plane but translated or shifted by the vector $\mathbf{p}$.

Algorithm to determine the solution set of a matrix equation:

1. Row reduce the associated augmented matrix to RREF.
2. Express the basic variables in terms of free variables.
3. Write the solutions $\mathbf{x}$ as a vector whose entries depend on the free variables.
4. Decompose and factor $\mathbf{x}$ into a linear combination of vectors with weights comprised of free variables.

Example 4. *Suppose A is a 3×3 matrix and $\mathbf{y} \in \mathbb{R}^3$ is such that $A\mathbf{x} = \mathbf{y}$ does not have a solution. Does there exist a vector $\mathbf{z} \in \mathbb{R}^3$ so that $A\mathbf{x} = \mathbf{z}$ has a unique solution?*

Example 5. *The following equations describe planes in $\mathbb{R}^3$. Describe their intersection.*

$$\begin{aligned} x_1 + 4x_2 - 5x_3 &= 0 \\ 2x_1 - x_2 + 8x_3 &= 9. \end{aligned}$$