Learning Objectives:

1. Determine whether a matrix is diagonalizable or not.
2. If it is, diagonalize the matrix.
3. Describe why diagonalization is a useful technique for simplifying calculations in applications.

1 Diagonalization

Recall: Remember that two matrices A and B are similar if there exists an invertible P so that $A = PBP^{-1}$.
Motivation: In the last section we said that similar matrices can help us simplify calculations. In this section we see how to this works using **diagonalization.**

Example 1. *Consider the matrices*

$$A = \begin{pmatrix} 7 & 4 \\ -2 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}.$$

Which of these is easier to compute powers of?

Solution. Just by looking, we guess that computing powers of D should be easier. Let's see why:

$$A^2 = \begin{pmatrix} 7 & 4 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 41 & 32 \\ -16 & -7 \end{pmatrix}.$$

On the other hand

$$D^2 = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix}.$$

In fact, computing A^3 already makes me want to use MATLAB since the numbers are becoming unwieldy, but computing higher powers of D is easy:

$$D^3 = \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 125 & 0 \\ 0 & 81 \end{pmatrix}$$

and in general

$$D^k = \begin{pmatrix} 5^k & 0 \\ 0 & 3^k \end{pmatrix}.$$

Example 2. The plot twist: *It turns out that A and D from the last example are* **similar matrices**. *Indeed, take*

$$P = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then

$$P^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

and one can check that

$$A = PDP^{-1}.$$

Is this observation at all useful?

Solution. It is! Notice that another way of calculating A^2 is

$$A^2 = AA = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}. \qquad (*)$$

Let's first check that this works.

$$\begin{aligned} PD^2P^{-1} &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 25 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 25 & 25 \\ 9 & 18 \end{pmatrix} \\ &= \begin{pmatrix} 41 & 32 \\ -16 & -7 \end{pmatrix}. \end{aligned}$$

This matches our work from above! OK, this was actually more work than just computing A^2 normally, *but* what if we wanted to compute A^{100}?

Notice that in the line labeled $(*)$ there is cancellation resulting in the "inner" P^{-1} and P matrices to disappear. If we kept multiplying more (PDP^{-1}) terms, then this cancellation would always happen, resulting in just higher powers of the diagonal matrix D (perhaps you should check yourself what happens for A^3 to really confirm what happens). So we have found a nice formula for A^k:

$$A^k = PD^kP^{-1}.$$

Take a second to appreciate how much easier this would be to calculate: D^k is easy to compute no matter how big k is (from the previous example), and once we know that we just need to multiply the three matrices together, rather than multiply A together k times!

Definition: We say that a square matrix A is **diagonalizable** if it is similar to a diagonal matrix.

Problem: The examples above should convince us that if a matrix A is diagonalizable, then it is easier to use it for calculations. *However*, we have two problems:

- How do we know when a matrix is diagonalizable?
- Even if we know it is diagonalizable, how do we find the matrix P that transforms A to the diagonal matrix?

Hint: Using the same matrices A and D from above, what are the eigenvalues of A? Can you calculate the corresponding eigenvectors of A?

Example 3. *(a). The eigenvectors of* $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$ *are*

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \text{ and } \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

corresponding to the eigenvalues 2 *and* 1 *respectively. Let* P *be the matrix whose columns are these eigenvectors. Compute* $P^{-1}AP$*. What does this show?*

(b). The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ *has its* 1*-eigenspace spanned by*

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and its 2*-eigenspace spanned by*

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In this case A *is not diagonalizable. Why do you think not?*

Solution. (a). We compute

$$P^{-1}AP = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

so A is diagonalizable with $A = PDP^{-1}$. The columns of P are the eigenvectors of the matrix A!

(b). In this case, since we cannot make a square matrix P of eigenvectors, we believe A is not diagonalizable.

Theorem: An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors. If A is diagonalizable with $A = PDP^{-1}$, then P is formed by the linearly independent eigenvectors of A, and the entries of D are the eigenvalues.

Remark: Another ways of interpreting this Theorem is to say: A is diagonalizable if and only if the n eigenvectors for A form a basis of $\mathbb{R}^n$ (since they should be linearly independent).

Theorem: Let A be $n \times n$ with eigenvalues $\lambda_1, \ldots, \lambda_p$. Then, the matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n. This happens if and only if the characteristic polynomial factors into linear factors and the dimension of the eigenspace for each λ_k is the multiplicity of λ_k.

Example 4. *Let*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\lambda = 1$ is the only eigenvalue of A, and we see that $(A - I)$ has null space spanned only by the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. So, A is not diagonalizable.

Example 5. *Diagonalize*

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}.$$

We first compute the eigenvalues:

$$\det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2.$$

So, the eigenvalues are $\lambda = 1$ and $\lambda = -2$ (with multiplicity 2). Next, we need to find corresponding eigenvectors. This amounts to solving

$$(A - I)\mathbf{x} = \mathbf{0},$$

and

$$(A + 2I)\mathbf{x} = \mathbf{0},$$

using row-reduction. After some work we compute a basis for the $\lambda = 1$ *eigenspace to be* $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ *and a basis for the* $\lambda = -2$ *eigenspace to be* $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. *Then, we can see that the set of these eigenvectors is linearly independent, and thus a basis of* $\mathbb{R}^3$.

So define

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Recall that a triangular matrix has as its eigenvalues the diagonal elements. Also, remember that eigenvectors corresponding to distinct eigenvalues are linearly independent. So,

> **Theorem:** An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Question 1. *Diagonalize the following matrix:*

$$A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{pmatrix}.$$

We see immediately that the eigenvalues are $\lambda = 5$ and $\lambda = -3$. Then

$$\mathbf{v}_1 = \begin{pmatrix} -8 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -16 \\ 4 \\ 0 \\ 1 \end{pmatrix}$$

form a basis for the $\lambda = 5$ eigenspace and

$$\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{v}_4 = \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

form a basis for the -3-eigenspace.

Example 6. T/F: *The following matrix is diagonalizable:*

$$A = \begin{pmatrix} 5 & 8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 7. T/F: *A 5×5 matrix has only two eigenvalues and is diagonalizable. One of the eigenspaces has odd dimension.*