Learning Objectives:

1. Use the characteristic equation to find the eigenvalues of a matrix.

2. Define when two matrices are similar.

1 Characteristic polynomial

Recall: So far we have seen how to find eigenvectors if we are given eigenvalues. So far we do not know how to find the eigenvalues themselves.

Example 1. *How do we find the eigenvalues of a matrix A?*

Solution. In order to determine the eigenvalues of a matrix A, we must determine for which values λ

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

has non-trivial solutions.

Note that $A - \lambda I$ has non-trivial solutions if and only if it is non-invertible (by the Invertible Matrix Theorem). Thus, an equivalent criteria for $A - \lambda I$ to have a non-trivial null space is that

$$\det(A - \lambda I) = 0.$$

Now, the key idea: If we compute $\det(A - \lambda I)$ then we will get a polynomial in the variable λ! So solving

$$\det(A - \lambda I) = 0$$

will simply mean to solve for the roots of some polynomial equation.

We call this equation the *characteristic equation*. We often write the polynomial $p_A(\lambda) = \det(A - \lambda I)$ to signify the fact that the determinant is a polynomial of the variable λ. We call $p_A(\lambda)$ the **characteristic polynomial**.

Example 2. *Find the eigenvalues of*

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}.$$

Solution. Since

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix},$$

We can compute the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - (-1)(1) = 8 - 6\lambda + \lambda^2 + 1 = \lambda^2 - 6\lambda + 9.$$

Thus setting $\det(A - \lambda I) = 0$ yields

$$\begin{aligned} \lambda^2 - 6\lambda + 9 &= 0 \\ (\lambda - 3)^2 &= 0 \end{aligned}$$

Thus $\lambda = 3$ is the only eigenvalue. Since it is a repeated root, we say that $\lambda = 3$ has **algebraic multiplicity** 2.

The other big topic of the section is called "similarity."

Definition: Two matrices A and B are **similar** if there exists an invertible matrix P so that $A = PBP^{-1}$. Of course, this equation also means that $B = P^{-1}AP$, so A is similar to B if and only if B is similar to A.

Idea: In a lot of applications, it will be hard to compute eigenvalues for a matrix A, but oftentimes moving to a similar matrix B will make computing eigenvalues much easier. Luckily, it turns out that the eigenvalues of both matrices will always be the same!