Motivation: We have so far discussed operations between two vectors, and operations between matrices and vectors. In order to more efficiently solve linear algebra problems, as well as solve more complex questions, we need to introduce new operations between two matrices.

1 Matrix operations

Assume A is an $m \times n$ matrix. Remember that we can think of A as n column vectors from $\mathbb{R}^m$:

$$A = (\mathbf{a}_1 \ \ldots \ \mathbf{a}_n).$$

We can also denote the entry in the ith row and jth column of A by a_{ij} (typically, we use lower case letters for entries). We call this the (i, j)-entry of A (think, i-down, j-across). So

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Sometimes people will write $A = (a_{ij})$ as an alternative notation representing the matrix by all of its entries. If the matrix A is $n \times n$, then it is called a *square matrix.*

Definition: Given the matrix $A = (a_{ij})$, We call the entries $a_{11}, a_{22}, a_{33}, \ldots$ the **diagonal entries** of A. A **diagonal matrix** is an $n \times n$ (square) matrix whose only non-zero entries are on the diagonal.

Example 1. *The matrix*

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 & 1 \\ 1 & 1 & 8 & 1 & 1 \\ 1 & 1 & 1 & 13 & 1 \end{pmatrix}$$

is 1 except on its diagonal entries. The identity matrix $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$ *is a diagonal square matrix.*

If A and B are matrices of the same size (same number of rows and columns), then their **sum** $A+B$ is the matrix obtained by summing the entries of A and B component-wise.

Example 2. *Let*

$$A = \begin{pmatrix} 1 & 3 & 10 \\ 2 & \pi & 3 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 0 & -3 \\ 0 & 3 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$A + B = \begin{pmatrix} 1+4 & 3+0 & 10-3 \\ 2+0 & \pi+3 & 3+1 \end{pmatrix} = \begin{pmatrix} 5 & 3 & 7 \\ 2 & \pi+3 & 4 \end{pmatrix}.$$

However, the sum $A+C$ is not defined because the sizes of A and C are different.

Given a matrix A and a scalar r, their **scalar multiplication** is the matrix rA obtained by multiplying each entry in A by r.

Example 3. *Using the same matrices from Example 2:*

$$A-2B = \begin{pmatrix} 1 & 3 & 10 \\ 2 & \pi & 3 \end{pmatrix} - 2\begin{pmatrix} 4 & 0 & -3 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 10 \\ 2 & \pi & 3 \end{pmatrix} + \begin{pmatrix} -8 & 0 & 6 \\ 0 & -6 & -2 \end{pmatrix} = \begin{pmatrix} -7 & 3 & 16 \\ 2 & \pi-6 & 1 \end{pmatrix}$$

Theorem: Let A, B, C be matrices of the same size, and r, s be scalars. Then

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $A + 0 = A$ (here, 0 represents the zero matrix: the matrix whose entries are all zero.)
4. $r(A + B) = rA + rB$
5. $(r + s)A = rA + sA$
6. $r(sA) = (rs)A$

Definition: Given a matrix A of size $m \times n$, the matrix A^T is called the **transpose** and is the $n \times m$ matrix whose columns are formed from the rows of A.

Example 4. *Given* $B = \begin{pmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}$, *what is* B^T*?*

Solution. We swap the rows and columns of B, so that

$$B^T = \begin{pmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{pmatrix}.$$

We will see later why transposes are so useful in linear algebra!