MAT205 **Supplementary Notes**

Section 6.3 Orthogonal Projections

Learning Objectives:

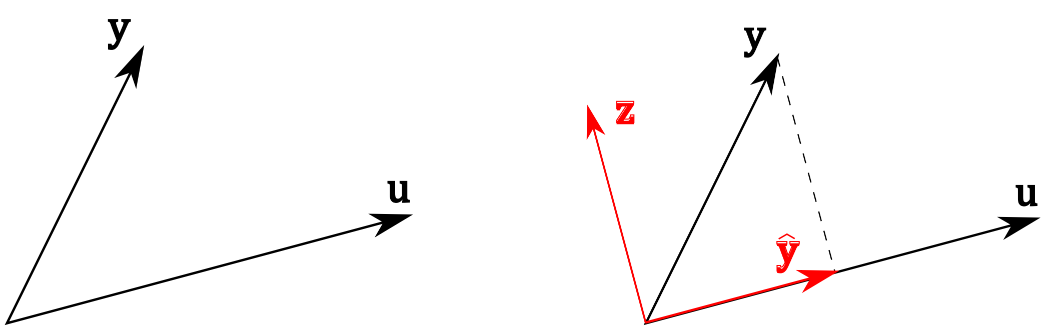
1. Calculate the orthogonal projection of a vector onto a subspace.

2. Interpret the orthogonal projection geometrically.

1 Orthogonal projections to lines

Example 1. *Suppose we are given two vectors* $\mathbf{u}$ *and* $\mathbf{y}$*. The* **orthogonal projection** *of* $\mathbf{y}$ *onto* $\mathbf{u}$ *is the vector* $\hat{\mathbf{y}}$ *that is the "shadow" of* $\mathbf{y}$ *cast down in the direction of* $\mathbf{u}$*.*

To make this more rigorous, the picture below shows the essential ideas: starting with $\mathbf{u}$ *and* $\mathbf{y}$*, if we drop a line segment from the tip of* $\mathbf{y}$ *perpendicular to* $\mathbf{u}$ *then we get the projection* $\hat{\mathbf{y}}$*. The vector* $\mathbf{z}$ *is the same as the perpendicular dotted line (just translated).*



So, we have

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

where $\hat{\mathbf{y}}$ *is some scalar multiple of* $\mathbf{u}$ *and* $\mathbf{z}$ *and* $\hat{\mathbf{y}}$ *are orthogonal. Is there a nice formula for these different vectors? Yes!*

Remark: The orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$ can be interpreted in the following equivalent way. Let $L = \text{span}\{\mathbf{u}\}$. That is, L is the line spanned by $\mathbf{u}$. Then $\hat{\mathbf{y}}$ is the point on the line L that is closest to $\hat{\mathbf{y}}$. Some people use the notation $\text{proj}_L(\mathbf{y})$ for this, so we have

$$\hat{\mathbf{y}} = \text{proj}_L(\mathbf{y}).$$

Also, notice that since $\mathbf{z}$ is perpendicular to L, it must be in the orthogonal complement, $L^\perp$.

Example 2. *Find a formula for* $\hat{\mathbf{y}}$ *from the example above.*

Solution. From our picture we want $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some value of α. To determine the value of α we need, take the dot product of both sides of the equation with $\mathbf{u}$. Then

$$\mathbf{y} \cdot \mathbf{u} = \alpha\mathbf{u} \cdot \mathbf{u} + \mathbf{z} \cdot \mathbf{u}.$$

From the above, we also want $\mathbf{z}$ to be perpendicular to $\mathbf{u}$ so we must have $\mathbf{z} \cdot \mathbf{u} = 0$! Thus we can solve our equation for α:

$$\alpha = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}.$$

That is, we have determined a formula for the projection. The **orthogonal projection** of $\mathbf{y}$ onto the vector $\mathbf{u}$ is given by

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}.$$

Once we know $\hat{\mathbf{y}}$ then the orthogonal component is easy to compute:

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \mathbf{y} - \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}.$$

Take-away: Given vectors $\mathbf{y}$ and $\mathbf{u}$, the projection of $\mathbf{y}$ onto $\mathbf{u}$ is the point on the line spanned by $\mathbf{u}$, called L, that is closest to $\mathbf{y}$. We have

$$\hat{\mathbf{y}} = \operatorname{proj}_L(\mathbf{y}) = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u},$$

and the orthogonal component $\mathbf{z}$ is $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

Example 3. *Let* $\mathbf{y} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ *and* $\mathbf{u} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$. *Compute the orthogonal projection of* $\mathbf{y}$ *onto* $\mathbf{u}$. *In addition, compute the distance from* $\mathbf{y}$ *to the line spanned by* $\mathbf{u}$.

Solution. From the formulas above, the orthogonal projection is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{30}{100}\mathbf{u} = \frac{3}{10}\begin{pmatrix} 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 12/5 \\ 9/5 \end{pmatrix}.$$

The distance from $\mathbf{y}$ to the line spanned by $\mathbf{u}$ can be found by first finding the orthogonal vector

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 12/5 \\ 9/5 \end{pmatrix} = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}.$$

The distance between $\mathbf{y}$ and the line spanned by $\mathbf{u}$ is simply the length of this vector $\mathbf{z}$:

$$\|\mathbf{z}\| = \sqrt{(3/5)^2 + (-4/5)^2} = \sqrt{1} = 1.$$

Example 4. *Let* $\mathbf{y} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$ *and* $\mathbf{u} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. *Find the projection of* $\mathbf{y}$ *onto* $\mathbf{u}$ *and the orthogonal component* $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

Solution.

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

and the component of y orthogonal to u is

$$y - \hat{y} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

2 Orthogonal projection to more general subspaces

Our previous discussion can generalize quite nicely to other subspaces!

Example 5. *Suppose that* W *is a plane in* $\mathbb{R}^3$ *spanned by the orthogonal basis* $\{\mathbf{u}_1, \mathbf{u}_2\}$. *If* $\mathbf{y} \in \mathbb{R}^3$, *then write the orthogonal projection of* $\mathbf{y}$ *onto the space* W.

Solution. Since the basis is orthogonal, we can write a representation of $\hat{\mathbf{y}}$ easily:

$$\hat{\mathbf{y}} = \text{proj}_W(\mathbf{y}) = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2.$$

As before, this points to the point on the plane closest to the vector $\mathbf{y}$. Then

$$\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} \in W^{\perp}.$$

As before, $\|\mathbf{z}\|$ is the distance from $\mathbf{y}$ to W.

Theorem: (The Orthogonal Decomposition Theorem) Let W be a subspace of $\mathbb{R}^n$. Then each $y \in \mathbb{R}^n$ can be written

$$y = \hat{y} + z$$

where $\hat{y} \in W$ and $z \in W^{\perp}$. In fact, letting $\{u_1, \dots, u_p\}$ be any orthogonal basis of W then

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p,$$

and $z = y - \hat{y}$.

Example 6. *Let* $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ *and* $\mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. *Then,* $\{\mathbf{u}_1, \mathbf{u}_2\}$ *is an orthogonal set. Letting* $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, *find the distance from* $\mathbf{y}$ *to* W.

Solution. The orthogonal projection of y onto W is

$$proj_W(y) = \frac{9}{30}u_1 + \frac{15}{30}u_2 = \begin{pmatrix} -2/5 \\ 2 \\ 1/5 \end{pmatrix}.$$

Then the component of y orthogonal to W is

$$y - proj_W(y) = \begin{pmatrix} 7/5 \\ 0 \\ 14/5 \end{pmatrix}.$$

So the distance is

$$\|y - proj_W(y)\| = \sqrt{49/25 + 196/25} = \sqrt{245/25} = \sqrt{49/5}.$$

Example 7. T/F: *If W has orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\mathbf{z}$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$, then $\mathbf{z} \in W^\perp$.*

Example 8. *Confirm that if $W \subseteq \mathbb{R}^n$ has orthogonal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ and $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ with $\hat{\mathbf{y}} \in W$ the orthogonal projection, then $\mathbf{z} \in W^\perp$.*

Solution. Since $\hat{\mathbf{y}} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p$ we can compute

$$(y - \hat{y}) \cdot u_i = 0$$

for each i. Thus $\mathbf{z}$ is orthogonal to each basis vector of W and thus is in $W^\perp$.