

# 1 Dimension

**Intuition:** We often think of “dimension” as the number of spatial “directions”, but this intuition fails past 3D. It also may not be useful when thinking about more abstract vector spaces. Instead, we should think of dimension as “degrees of freedom:” *how many coordinates do we need to uniquely describe any vector in the vector space?* To describe any vector sounds roughly like spanning the vector space. To uniquely do so sounds roughly like linear independence. So, dimension of a space must have something to do with basis sets!

**Theorem:** If a vector space  $V$  has a basis of  $n$  vectors, then every basis of  $V$  has  $n$  vectors.

**Remark:** This theorem tells us that any set which has more than  $n$  vectors must be linearly dependent and any set which has fewer than  $n$  vectors cannot span the vector space.

**Definition:** If  $V$  is spanned by a finite set, then  $V$  is **finite-dimensional** and the dimension of  $V$ , written  $\dim V$  is the number of vectors in any basis for  $V$ . The dimension of  $\{\mathbf{0}\}$  is 0. Any vector space not spanned by a finite set is infinite-dimensional.

**Example 1.**     • What is the dimension of the vector space  $\mathbb{R}^n$ ?

• What is the dimension of the vector space  $\mathbb{P}_n$ ?

**Example 2.** Find the dimensions of the null space and column space of

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

**Solution.** We row reduce the matrix  $A$  to

$$\begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

A basis for the column space will be given by the pivot columns

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \right\}$$

and so the column space is 2 dimensional. A basis for the null space is given by solving  $A\mathbf{x} = \mathbf{0}$ . We see that  $x_2$ ,  $x_4$  and  $x_5$  will be free variables and so the null space will have 3 vectors in its basis and thus the null space is 3 dimensional.