Learning Objectives:

1. Describe solution sets of homogeneous and non-homogeneous systems in parametric forms.
2. Relate solution sets of homogeneous and non-homogeneous systems as geometric translations of each other.

We know how to row reduce to determine whether a system is consistent. And if there are non-unique solutions, we can find solutions by plugging in particular values of free variables. In this section we learn how to describe **all** solutions to a system.

1 Homogeneous linear systems

We call a system of linear equations **homogeneous** if $\mathbf{b} = \mathbf{0}$. That is, a homogeneous system is of the form

$$A\mathbf{x} = \mathbf{0}.$$

Example 1. *When is the homogeneous system $A\mathbf{x} = \mathbf{0}$ solvable?*

Solution. Always! We always have the **trivial** solution $\mathbf{x} = \mathbf{0}$.

We are interested in the existence of *non-trivial solutions*. Remembering the existence/uniqueness theorem:

> The homogeneous system $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions if and only if the system has free variables.

Example 2. *Determine the solution set of the system*

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0. \end{aligned}$$

Solution. We solve this system by row reducing. After some computations, the augmented matrix is

$$\begin{pmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have solutions

$$\begin{cases} x_1 = 4/3x_3 \\ x_2 = 0 \\ x_3 \text{ is free} \end{cases}$$

We write this in vector form to describe all solutions.

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4/3x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}.$$

By choosing different values of x_3, we recover different solutions. Geometrically, the solution set is a line. Using span, we see that the solution set is Span $\left\{ \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Example 3. *Determine the solution set of* $A\mathbf{x} = \mathbf{0}$ *where*

$$A = \begin{pmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution. We first reduce A to RREF:

$$\begin{pmatrix} 1 & 5 & 2 & -6 & 9 & 0 \\ 0 & 0 & 1 & -7 & 4 & -8 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & 0 & 8 & 1 & 0 \\ 0 & 0 & 1 & -7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, we have basic variables x_1, x_3, x_6 and free variables x_2, x_4, x_5. We can write the solution set

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -5x_2 - 8x_4 - x_5 \\ x_2 \\ 7x_4 - 4x_5 \\ x_4 \\ x_5 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

So, the solution set is

$$\text{Span} \left\{ \begin{pmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Take-away: We have seen that the solution set of a homogeneous system can always be written as the span of a collection of vectors:

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

This is called a **parametric vector equation** or **parametric vector form** of the solution since we can get all the solutions by simply plugging in different scalars c_i.

2 Non-homogeneous linear systems

Whenever $\mathbf{b} \neq \mathbf{0}$ then we have a *non-homogeneous* system. Luckily, the same approach we just used still works! There will just be one small difference.

Example 4. *Find all solutions of $A\mathbf{x} = \mathbf{b}$ where*

$$A = \begin{pmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix}.$$

Solution. We find solutions by row reducing the associated augmented matrix:

$$\begin{pmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{pmatrix} \sim \begin{pmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 3 & 0 & -4 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is consistent, so it has solutions. Also, note that x_3 is a free variable. We thus can write

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 + 4/3x_3 \\ 2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}.$$

(Be careful of signs!) So, writing $\mathbf{p} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4/3 \\ 0 \\ 1 \end{pmatrix}$ we see that all solutions are of the form $\mathbf{x} = \mathbf{p} + t\mathbf{v}$, where t is any real number.

> **Take-away:** If the non-homogeneous system $A\mathbf{x} = \mathbf{b}$ is solvable, then its solution set is of the form
> $$\mathbf{x} = \mathbf{p} + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$
> where c_i are scalars.

Example 5. *Suppose that* $\mathbf{p}$ *is a* particular solution *of* $A\mathbf{x} = \mathbf{b}$*, so that* $A\mathbf{p} = \mathbf{b}$*. If* $\mathbf{v}_h$ *is a solution of the homogeneous equation* $A\mathbf{x} = \mathbf{0}$*, then what equation does* $\mathbf{p} + \mathbf{v}_h$ *solve?*

Solution. We can use properties of matrix multiplication to compute

$$A(\mathbf{p} + \mathbf{v}_h) = A\mathbf{p} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

So $\mathbf{p} + \mathbf{v}_h$ is also a solution of the non-homogeneous system.

> If $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and $\mathbf{p}$ is some vector satisfying $A\mathbf{p} = \mathbf{b}$, then the solution set of $A\mathbf{x} = \mathbf{b}$ consists of all vectors of the form $\mathbf{x} = \mathbf{p} + \mathbf{v}$ where $\mathbf{v}$ is a solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Geometrically: We can think of solutions of homogeneous equations as say a line or plane through the origin (since we know $\mathbf{0}$ must be a solution). Then, solutions of the non-homogeneous system are the same line or plane but translated or shifted by the vector $\mathbf{p}$.

Overall, we have the following algorithm to determine the solution set of a matrix equation:

> **Algorithm to determine the solution set of a matrix equation:**
>
> 1. Row reduce the associated augmented matrix to RREF.
> 2. Express the basic variables in terms of free variables.
> 3. Write the solutions $\mathbf{x}$ as a vector whose entries depend on the free variables.
> 4. Decompose and factor $\mathbf{x}$ into a linear combination of vectors with weights comprised of free variables.

Example 6. *Suppose A is a 3×3 matrix and $\mathbf{y} \in \mathbb{R}^3$ is such that $A\mathbf{x} = \mathbf{y}$ does not have a solution. Does there exist a vector $\mathbf{z} \in \mathbb{R}^3$ so that $A\mathbf{x} = \mathbf{z}$ has a unique solution?*

Solution. No. Since $A\mathbf{x} = \mathbf{y}$ does not have a solution, then A cannot have 3 pivots. So, either $A\mathbf{x} = \mathbf{z}$ has no solution, or else it has at least one free variable and so it has infinitely many solutions.

Example 7. *The following equations describe planes in $\mathbb{R}^3$. Describe their intersection.*

$$\begin{aligned} x_1 + 4x_2 - 5x^3 &= 0 \\ 2x_1 - x_2 + 8x_3 &= 9. \end{aligned}$$

Solution. Row reduce to get

$$\begin{pmatrix} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & -5 & 0 \\ 0 & 1 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{pmatrix}.$$

Thus the intersection is given by

$$\mathbf{x} = \begin{pmatrix} -3x_3 + 4 \\ 2x_3 - 1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}.$$

This is the parametric form of a line.