Learning Objectives:

1. Combine and review the various notions seen thus far in the course.
2. Translate notions of invertible matrices to invertible linear transformations.

1 The Theorem

The Invertible Matrix Theorem. Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, they are all either true or all false.

a. A is an invertible matrix

b. A is row equivalent to the $n \times n$ identity matrix

c. A has n pivot positions

d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

e. The columns of A form a linearly independent set

f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one

g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$

h. The columns of A span $\mathbb{R}^n$

i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

j. There is an $n \times n$ matrix C such that $CA = I$

k. There is an $n \times n$ matrix D such that $AD = I$

l. A^T is an invertible matrix

Example 1. *Show that (j) implies (d).*

Solution. Suppose that there exists C so that $CA = I_n$. Suppose that $A\mathbf{x} = \mathbf{0}$. We will show that $\mathbf{x} = \mathbf{0}$. Multiplying both sides of the equation by C we get $CA\mathbf{x} = C\mathbf{0}$. Since $CA = I$ the left hand side simplifies $CA\mathbf{x} = I\mathbf{x} = \mathbf{x}$. The right hand side must be $\mathbf{0}$ and so we have $\mathbf{x} = \mathbf{0}$.

Example 2. **T/F:** *The matrix*

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{pmatrix}$$

is invertible.

A. Yes, and I am confident.

B. Yes, but I am not confident.

C. No, but I am not confident.

D. No, and I am confident.

Example 3. *Suppose A is 3×4 and that $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^3$. Does $A\mathbf{x} = \mathbf{0}$ have a unique solution?*

Solution. No. There must be a column without a pivot and so there is a free variable. As such there are non-unique solutions. This theorem does not apply because A is non-square.

Example 4. *An $n \times n$* **upper triangular matrix** *is one whose entries below the diagonal are 0's. When is a square upper triangular matrix invertible?*

Solution. If an entry on the diagonal is 0 then the matrix cannot have pivots in all columns, so it cannot be invertible. On the other hand, if each diagonal entry is non-zero, then the matrix has pivots in all columns so it is invertible.

2 Invertible Linear Transformations

Definition: A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there exists a function $S: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = T(S(\mathbf{x})) = \mathbf{x}.$$

In fact, Theorem 9 from the book shows that if such an S exists then it must be unique and a linear transformation. So, we may write $S = T^{-1}$.

Example 5. *Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation defined by*

$$T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2).$$

Show that T is invertible and find a formula for T^{-1}. (We wrote the column vectors here as row vectors for convenience, but the math does not change!)

Solution. The standard matrix for T is

$$A = \begin{pmatrix} -5 & 9 \\ 4 & -7 \end{pmatrix}.$$

The inverse matrix is

$$A^{-1} = \frac{1}{35-36} \begin{pmatrix} -7 & -9 \\ -4 & -5 \end{pmatrix} = \begin{pmatrix} 7 & 9 \\ 4 & 5 \end{pmatrix}.$$

So the inverse map is

$$T^{-1}(x_1, x_2) = (7x_1 + 9x_2, 4x_1 + 5x_2).$$

Example 6. *Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation that is one-to-one. Show that T is onto.*

Solution. Let A be the standard matrix for T. Since A is one-to-one then $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$. Thus T is onto. That is to say, T is bijective.

Remark. For those who have seen discrete math, notice the distinction here: if a linear transformation T is one-to-one then it is invertible. This is not true for all functions!