Learning Objectives:

1. Calculate null and column spaces of a given matrix.
2. Find a matrix whose column space matches a given subspace.
3. Compare and contrast null and column spaces for a matrix.
4. Extend these concepts to more general linear transformations on vector spaces.

1 Null space of a matrix

Definition: The **null space** of an $m \times n$ matrix A, written $\operatorname{Nul} A$ is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation:

$$\operatorname{Nul} A = \{\mathbf{x} \colon A\mathbf{x} = \mathbf{0}\}.$$

- Remember that the null space is the same thing as the solution set here. By calling it the null space we are emphasizing the fact that it is a subspace.
- To describe the null space we can row reduce to find a spanning set of vectors.

Theorem: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

Example 1. **T/F**: *Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates (a, b, c, d) satisfy*

$$\begin{aligned} a - 2b + 5c &= d \\ c - a &= b. \end{aligned}$$

Then H a subspace of $\mathbb{R}^4$.

Example 2. **T/F:** *Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates (a, b, c, d) satisfy*

$$\begin{aligned} a - 2b + 5c - d &= 4 \\ c - a - b &= 1. \end{aligned}$$

Then H a subspace of $\mathbb{R}^4$.

Example 3. *What is the relationship between pivots and the number of vectors in the spanning set of* $\operatorname{Nul} A$*?*

2 Column space of a matrix

Definition: The **column space** of an $m \times n$ matrix A, written $\operatorname{Col} A$, is the set of all linear combinations of the columns of A. That is, if $A = (\mathbf{a}_1 \ \cdots \mathbf{a}_n)$ then

$$\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}.$$

Theorem. The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only

Example 4. *Find a matrix A so that $W = \operatorname{Col} A$ where*

$$W = \left\{ \begin{pmatrix} 6a - b \\ a + b \\ -7a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

3 Contrasting the null and column spaces

Example 5. *Let*

$$A = \begin{pmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{pmatrix}.$$

In which $\mathbb{R}^k$ *do* Col A *and* Nul A *live?*

Contrast Between Nul *A* and Col *A* for an *m* x *n* Matrix *A*

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

4 Linear transformations

Definition: A **linear transformation** T from vectors space V to W is a rule assigning to each $\mathbf{x} \in V$ a unique vector $T(\mathbf{x}) \in W$ so that

Definition: The **kernel** of T is the set of all $\mathbf{u} \in V$ so that _______________. The **range** of T is the set of all vectors in W ____________________________________.

Analogy: Null space $\leftrightarrow$ ____________________

Column space $\leftrightarrow$ ____________________

Theorem: The kernel and range of a linear transformation $T\colon V \to W$ are subspaces of V and W, respectively.

Example 6. *Define* $T\colon \mathbb{P}_2 \to \mathbb{R}^2$ *by* $T(\mathbf{p}) = \begin{pmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{pmatrix}$. *Show that* T *is a linear transformation. Moreover, find a polynomial* $\mathbf{p}$ *that spans the kernel of* T*, and find the range of* T.