Learning Objectives:

1. Identify linear and non-linear systems of equations
2. Describe the types of solution sets which can arise from linear systems
3. Translate between systems of equations and their matrix forms
4. Use row operations to solve systems of equations

1 Motivation

Linear algebra is possibly **the most important** collection of ideas and techniques in mathematics. It is used in almost all pure and applied mathematical fields and is central to physics, engineering, computer science. Some applications include:

1. data science/machine learning
2. image processing
3. cryptography
4. social media suggestions (aka how TikTok becomes addictive)

2 Systems of linear equations

Definition: A *linear equation* is an equation which can be written in the form

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b,$$

where b and the coefficients a_i are real or complex numbers, and the unknown variables are the x_i. A collection of linear equations is called a *linear system.*

Example 1. *The equation*

$$2x_1 - \pi x_2 + 3x_3 - 4 = 0$$

is linear since we can write it as

$$2x_1 - \pi x_2 + 3x_3 = 4.$$

The system

$$\begin{aligned} 2x_1 &= 4 \\ 5\sqrt{x_1} - x_3x_2 &= 0 \end{aligned}$$

is non-linear. Although the first equation is linear, the second equation is non-linear due to the square-root term, as well as the product x_3x_2.

Question 1. *Which of the following is a linear equation?*

A. $x_1 - 4\sqrt{x_2} - 3x_3 = 0$

B. $x_1 + 2x_2 = \pi^2 x_3$

C. $2x_1^2 + x_2 - x_3 = 0$

D. $x_1x_2 + 4x_3 = 0$

Solution. A. The term $\sqrt{x_2}$ is non-linear.
B. We see that we can write this in the form $x_1 + 2x_2 - \pi^2 x_3 = 0$. There is no problem with π^2 since it is a coefficient.
C. The term x_1^2 is non-linear.
D. The term x_1x_2 is non-linear.
Thus, the solution is **B.**

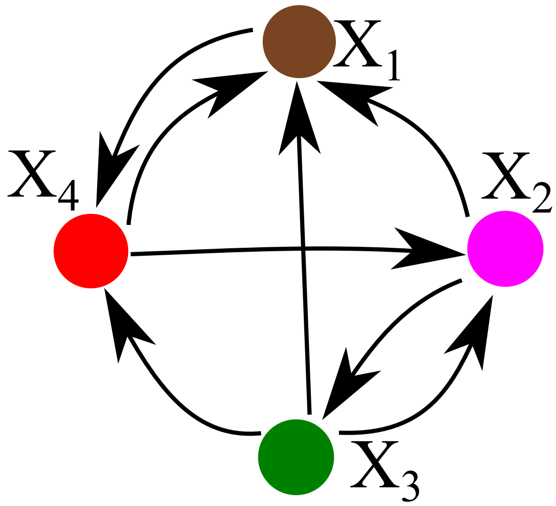
A primary goal will be to **solve** systems of linear equations, which means to find values of $x_1,\ x_2,\ ,\ldots, x_n$ which make all equations simultaneously true.

Definition: We say that a system is *consistent* if it has at least one solution. Otherwise we say it is *inconsistent*.

The set of all possible solutions to a system is the *solution set*, and two linear systems are called *equivalent* if they have the same solution set.

Example 2. Google PageRank: *How does Google determine the pages which appear at the top of your search? One tool they use is the PageRank algorithm, which determines the most "important" website.*

"PageRank works by counting the number and quality of links to a page to determine a rough estimate of how important the website is. The underlying assumption is that more important websites are likely to receive more links from other websites."



As an example, let's pretend the internet has 4 pages X_1, X_2, X_3, X_4, *with arrows indicating a link between websites.*

We need to give each page a rank, denoted x_1, x_2, x_3, *and* x_4, *respectively. We will want each* x_i *to be between* 0 *and* 1, *with* 1 *being the most important website possible.*

We want our own rank to be higher if we receive links from other websites with high ranks... but this almost seems circular and impossible to figure out!

To solve this, think of ranks as points which can be distributed to all other websites. For example, website X_2 *sends half of its points to website* X_1 *and the other half to* X_3. *Website* X_3 *sends* $1/3$ *of its points to all other sites. The total points (rank) of* X_1 *are the sum of all points it receives:*

$$x_1 = \frac{1}{2}x_2 + \frac{1}{3}x_3 + \frac{1}{2}x_4.$$

Likewise we can determine

$$\begin{aligned} x_2 &= \frac{1}{3}x_3 + \frac{1}{2}x_4 \\ x_3 &= \frac{1}{2}x_2 \\ x_4 &= x_1 + \frac{1}{3}x_3. \end{aligned}$$

This is a linear system, as we can check by re-writing:

$$\begin{aligned} -x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 + \frac{1}{2}x_4 &= 0 \\ -x_2 + \frac{1}{3}x_3 + \frac{1}{2}x_4 &= 0 \\ \frac{1}{2}x_2 - x_3 &= 0 \\ x_1 + \frac{1}{3}x_3 - x_4 &= 0. \end{aligned}$$

One could check by plugging in that the following values comprise a solution:

$$x_1 = .3214286$$
$$x_2 = .2142857$$
$$x_3 = .1071429$$
$$x_4 = .3571429$$

Thus, website X_4 is the most important one! This may be surprising since it only has 2 websites linked to it, while X_1 has 3 websites linked to it. (Can you explain why this may happen?)

3 Solving systems using matrices and Gaussian elimination

A *matrix* is an array of numbers which represents the information of a linear system. For example, given the linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

we can write the *coefficient matrix* consisting of coefficients:

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{pmatrix}$$

Note that we must include 0 entries in the matrix for equations where x_i do not appear. The *augmented matrix* of the system includes the right hand side of the equations:

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{pmatrix}$$

The size of a matrix is the number of rows and columns it has. For example, the above coefficient matrix is 3×3 ("three by three") and the augmented matrix is 3×4. The mathematical convention is that the size of a matrix is "rows $\times$ columns."

Solving a linear system using its matrix

To solve the system we use a technique called *Gaussian elimination*. The idea is to use the first equation to eliminate x_1 from the rest of the system. Then we will eliminate x_2 from the other equations and so on until we have a very simple system which can be solved easily.

Example 3.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned} \qquad \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{pmatrix}$$

We want the first column of the matrix to have 0 under the 1 representing x_1. To do this we will **replace:** *we multiply the first equation by -5 and then add the first and third equations together. We use that result to replace the third equation.*

$$\begin{aligned} -5x_1 + 10x_2 - 5x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned} \qquad \begin{pmatrix} -5 & 10 & -5 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{pmatrix}$$

$$\begin{aligned} -5x_1 + 10x_2 - 5x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{pmatrix} -5 & 10 & -5 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{pmatrix}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{pmatrix}$$

Next divide or **scale** *the second equation by 2 to get a coefficient of* 1 *for x_2:*

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ 10x_2 - 10x_3 &= 10 \end{aligned} \qquad \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{pmatrix}$$

Like before, we want 0*s below the* 1 *for x_2, so we will* **replace**.

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
-10x_2 + 40x_3 &= -40 \\
10x_2 - 10x_3 &= 10
\end{aligned}
\qquad
\begin{pmatrix}
1 & -2 & 1 & 0 \\
0 & -10 & 40 & -40 \\
0 & 10 & -10 & 10
\end{pmatrix}
$$

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
30x_3 &= -30
\end{aligned}
\qquad
\begin{pmatrix}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & 0 & 30 & -30
\end{pmatrix}
$$

And another **scaling:**

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
x_3 &= -1
\end{aligned}
\qquad
\begin{pmatrix}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & 0 & 1 & -1
\end{pmatrix}
$$

Rather than plugging things back in, we can work backwards using the same **replacement** *technique from before. This time, we will make all the terms above the 1 term for x_3 equal to 0:*

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
x_3 &= -1
\end{aligned}
\qquad
\begin{pmatrix}
1 & -2 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{pmatrix}
$$

$$
\begin{aligned}
x_1 - 2x_2 &= 1 \\
x_2 &= 0 \\
x_3 &= -1
\end{aligned}
\qquad
\begin{pmatrix}
1 & -2 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1
\end{pmatrix}
$$

Last step: use **replacement** *to make all the terms above the 1 for the x_2 term equal* 0*:*

$$
\begin{aligned}
x_1 &= 1 \\
x_2 &= 0 \\
x_3 &= -1
\end{aligned}
$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

We are done! We have the solution $(x_1, x_2, x_3) = (1, 0, -1)$.

Moving forward we want to use only the matrix to solve the system, and eliminate the need for cumbersome systems of equations. In the previous example we saw that we could perform certain operations when row reducing the matrix, called *elementary row operations*

Elementary row operations:

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Scaling) Multiply all entries in a row by a nonzero constant.
3. (Interchange) Interchange two rows.

Main conclusion: Each of the elementary row operations do not change the solution set of a system of equations. We say that two matrices are *row equivalent* if they can be transformed to one another by elementary row operations. **Matrices that are row equivalent have the same solution sets**.

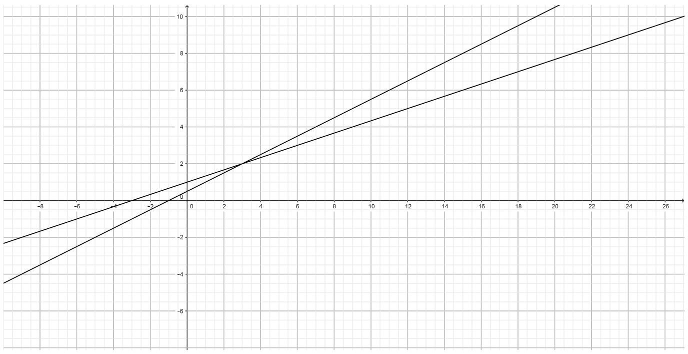
4 Understanding solution sets

Example 4. *Consider the linear system*

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3. \end{aligned}$$

The solution is $(x_1, x_2) = (3, 2)$. *How would you interpret this geometrically?*

Solution. The solutions of each individual linear equation forms a line. The solution to system is the point where the lines intersect.



Example 5. *A linear system of 2 equations and 2 unknowns in general has the form*

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2.$$

By choosing the coefficients differently, could we have 0 solutions? 2 solutions? Infinitely many?

What about if we had 3 equations and 3 unknowns?

Solution. The two lines could be parallel and not intersect (0 solutions), intersect at one point (1 solution), or be the same line (infinitely many solutions).

If we had 3 equations and 3 unknowns then the same happens! The main difference is that if there are infinitely many solutions the planes could all intersect creating a line of solutions or possibly creating an entire plane of solutions.

Theorem: A linear system of equations has either zero, one, or infinite solutions.

This is the heart of the two **fundamental questions of linear algebra:** do solutions exist (is the system consistent)? If so, are they unique?

Example 6. *The following system*

$$x_2 - 4x_3 = 8$$
$$2x_1 - 3x_2 + 2x_3 = 1$$
$$4x_1 - 8x_2 + 12x_3 = 1$$

can be row reduced to

$$\begin{pmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{pmatrix} \sim \begin{pmatrix} 4 & -8 & 12 & 1 \\ 0 & -2 & 8 & -1 \\ 0 & 0 & 0 & 15 \end{pmatrix}.$$

Is it consistent?

Solution. We notice here that the final equation reads $0 = 15$, which is a clear contradiction. No values of (x_1, x_2, x_3) will ever make this equation true, so this system is *inconsistent*: there are no solutions.