An Introduction to Knot Theory

Joy Chang, Mizuki Shitomi
University of California - Santa Barbara

What is Knot Theory?

Knot theory is a sub-field of Topology that deals with the study of mathematical knots. It allows us to classify knots based on their properties and helps us explain how knots are transformed within space. Although there are many definitions of a knot, in knot theory, a **knot** is the cross-section of a single point. To help us study knots, knots are drawn into **projections** where we can clearly see the crossing. There are several ways to project a knot whether that be through sticks, tricolorability, or planar graph (mentioned later on). These knot projections help us classify the types of knots such as the unknot/trivial knot which is the simplest knot (Figure 1.1) or trefoil knot which is the simplest non-trivial knot (figure 1.2).

Figure 1.1

Figure 1.2

Reidemeister Moves

More often than knot (pun intended), a knot is not as simple as the figure above. When knots are more complicated, it is harder to determine the types of know they are including whether it is a knot or an unknot. Hence, the **Reidemeister moves** is a method used to help us classify whether a knot is an unknot. It allows us to alter the knot without changing its properties. There are three Reidemeister moves.

- The **first** Reidemeister move allows us to put in or take out a twist in the knot (Figure 2.1).

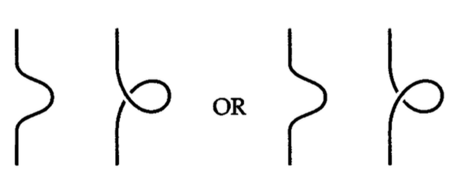


Figure 2.1

- The **second** Reidemeister move allows us to either add two crossings or remove two crossings (Figure 2.2).

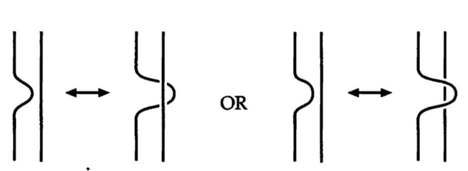


Figure 2.2

- The **third** Reidemeister move allows us to slide a strand of the knot from one side of a crossing to the other side of the crossing (Figure 2.3).

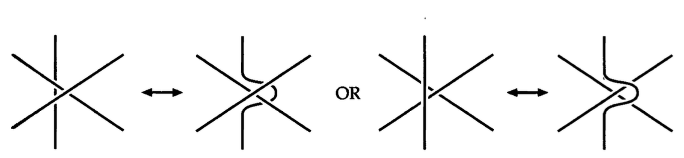
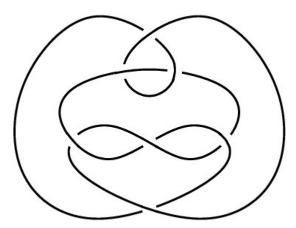


Figure 2.3

Try It Yourself

Prove that the knot below is an unknot using the Reidemeister moves.



Planar Graphs

A **planar graph** explains itself in its name - a graph that lies in the plane. It can be created from a projection of a knot or link in the following steps. A link is a set of knots all tangled up together.

- Shade every other region of the link projection (Figure 3.1).
- Put a vertex at the center of each shaded region and connect with an edge any two vertices that are in regions that share a crossing (Figure 3.2).
- Define crossings to be positive or negative (Figure 3.4).
- The result is a **signed planar graph** (Figure 3.3).

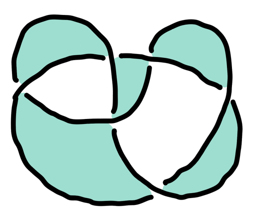


Figure 3.1

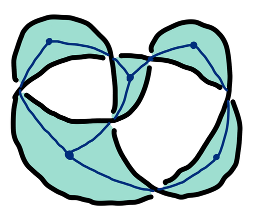


Figure 3.2

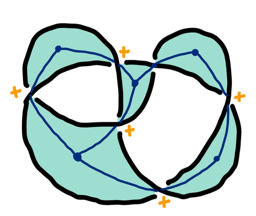


Figure 3.3

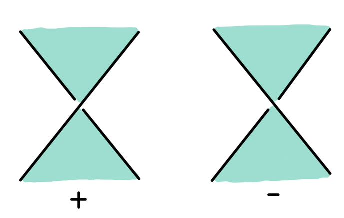


Figure 3.4

We can also go in the other direction - turning a signed planar graph into a knot projection. Just follow these steps:

- Put an X across each edge in the singed planar graph (Figure 4.1).
- Connect the edges formed by X inside each region (Figure 4.2).
- Shade the areas that contain a vertex (Figure 4.3).
- At each of the X's, put in a crossing corresponding to whether the edge is a + or a - edge (Figure 4.4).

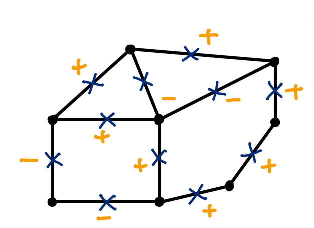


Figure 4.1

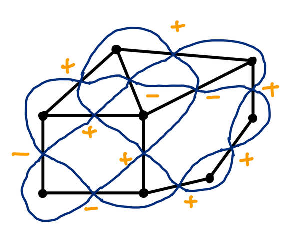


Figure 4.2

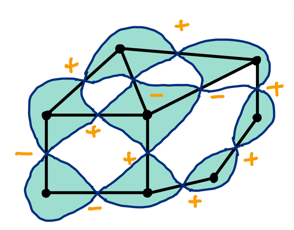


Figure 4.3

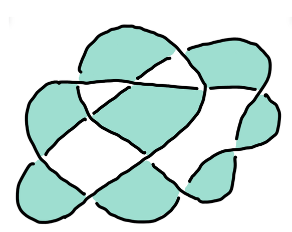


Figure 4.4

Try it yourself: Turn the signed planar graph in Figure 4.5 into the corresponding link projection.

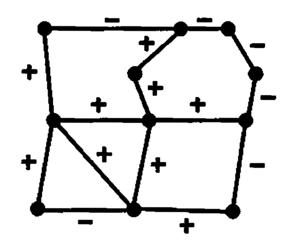


Figure 4.5

Knots vs. Graphs

Why do we want to convert knot projections into planar graphs and vice versa? Sometimes the problems in knot theory can be easier to solve if we turn the knot projections into signed planar graphs. For instance, there is an open problem that aims to find a practical algorithm for determining if a projection is a projection of the unknot. This is equivalent to asking if there is a sequence of Reidemeister moves that can convert the given projection to the projection of the unknot. By turning knot projections into signed planar graphs, this problem becomes determining the induced Reidemeister moves in the signed planar graph.

The planar graphs also have real-world applications in fields such as Chemistry, machine learning, statistical mechanics, and hydraulic engineering. Here we discuss a mathematical model of ferromagnetism in statistical mechanics known as the **Ising model**. It models a system where particles only interact with nearby ones. Two particles that are not neighbors have no effect on one another. Take the magnetization of a metal as an example: each molecule of the metal is considered to be a vertex of a graph. The interactions between adjacent molecules are represented by the edges of the planar graph. Only two molecules connected by an edge can interact. **Lattice** is a particular type of the Ising model, where the vertices and edges form a regular repeating in two-dimensional space (Figure 5.1). To relate this concept to the real world, metals consist of molecules that are at the vertices of a lattice in three-dimensional space (Figure 5.2).

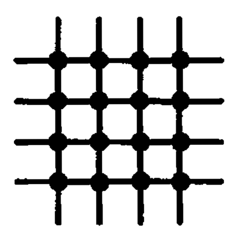


Figure 5.1

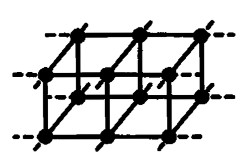


Figure 5.2

Conclusion

Although we only covered Reidmeister moves and planar graphs, the world of knot theory is endless. There are several other methods to understand how knots interact with space that we unfortunately cannot cover today. Moreover, knot theory can apply to graph theory, quantum theory, DNA modeling, and in everyday interactions. So the next time you are tying up your shoelaces or your charging cords, try thinking through the lenses of a topologist.

References

"The Knot Book" by Colin C. Adams

We thank our mentor Mychelle Parker for making this project possible as well as the UCSB Directed Reading Program.