

プログラミング言語周りノート

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第 1 章

Preliminaries

1.1 基本的な表記

量子化 (quantifier) の束縛をコンマ (,) で続けて書く. 束縛の終わりをピリオド (.) で示す. 例えば,

$$\forall x_1 \in X_1, x_2 \in X_2. \exists y_1 \in Y_1, y_2 \in Y_2. x_1 = y_1 \wedge x_2 = y_2$$

は,

$$\forall x_1 \in X_1. \forall x_2 \in X_2. \exists y_1 \in Y_1. \exists y_2 \in Y_2. x_1 = y_1 \wedge x_2 = y_2$$

と等しい. また, 量子子の束縛において, *such that* を省略し, コンマ (,) で繋げて書く. 例えば,

$$\forall x \in \{0, 1\}, x \neq 0. x = 1$$

は,

$$\forall x \in \{0, 1\}. x \neq 0 \Rightarrow x = 1$$

と等しい. また, \Rightarrow , \Leftrightarrow が他の記号と混同する場合, それぞれ *implies*, *iff* を使用する.

集合 (set) について, 以下の表記を用いる.

- 集合 A について, その濃度 (cardinality) を $|A|$ と表記する. なお, A が有限集合 (finite set) の時, 濃度とは要素の個数のことである.
- 集合 A について, $a \in A$ を $a : A$ と表記する.
- 自然数 (natural number) の集合を $\mathbb{N} = \{0, 1, \dots\}$ と表記する.
- 自然数 $n \in \mathbb{N}$ について, $\{1, \dots, n\}$ を $[n]$ と表記する.
- 集合 A の冪集合を $\mathcal{P}(A) = \{X \mid X \subseteq A\}$, 有限冪集合を $\mathcal{P}_{fin}(A) = \{X \in \mathcal{P}(A) \mid X \text{ は有限集合}\}$ と表記する.
- 集合 A_1, \dots, A_n の直積 (cartesian product) を $A_1 \times \dots \times A_n = \{(a_1, \dots, a_n) \mid a_1 \in A_1, \dots, a_n \in A_n\}$ と表記する. 集合 A の n 直積を $A^n = \underbrace{A \times \dots \times A}_{n \text{ 項}}$ と表記する. 特に, $A^0 = \{\epsilon\}$ である.
- 集合 A_1, \dots, A_n の直和 (disjoin union) を $A_1 \uplus \dots \uplus A_n = (A_1 \times \{1\}) \cup \dots \cup (A_n \times \{n\})$ と表記する. なお, 文脈から明らかな場合, 直和の添字を省略し, $a \in A_i$ に対して, $a \in A_1 \uplus \dots \uplus A_n$ と表記する.
- 集合 A の B との差集合を $A \setminus B = \{a \in A \mid a \notin B\}$ と表記する.

集合 Σ について, $\bigcup_{n \in \mathbb{N}} \Sigma^n$ を Σ^* と表記する. この時, $\alpha \in \Sigma^*$ を Σ による列 (sequence) と呼ぶ. 列について, 以下の表記を用いる.

- $(\sigma_1, \dots, \sigma_n) \in \Sigma^n$ について, $(\sigma_1, \dots, \sigma_n)$ を $\sigma_1 \dots \sigma_n$ と表記する.
- 列 $\alpha = \sigma_1 \dots \sigma_n \in \Sigma^*$ について, その長さを $|\alpha| = n$ と表記する.

集合 A, B について, $R \subseteq A \times B$ を関係 (relation) と呼ぶ. また,

$$A \rightarrow B \stackrel{\text{def}}{=} \{R \in \mathcal{P}(A \times B) \mid \forall x \in A, (x, y_1), (x, y_2) \in R. y_1 = y_2\}$$

という表記を導入し, 関係 $f : A \rightarrow B$ を A から B への部分関数 (partial function) と呼ぶ. さらに,

$$A \rightarrow B \stackrel{\text{def}}{=} \{f : A \rightarrow B \mid \forall x \in A. \exists y \in B. (x, y) \in f\}$$

という表記を導入し, 部分関数 $f : A \rightarrow B$ を (全) 関数 (function) と呼ぶ. 関係について, 以下の表記を用いる.

- 関係 $R \subseteq A \times B$ について, $(a, b) \in R$ を $a R b$ と表記する.
- 関係 $R \subseteq A \times B$ について, 定義域 (domain) を $\text{dom}(R) = \{a \mid \exists b. (a, b) \in R\}$, 値域 (range) を $\text{cod}(R) = \{b \mid \exists a. (a, b) \in R\}$ と表記する.
- 部分関数 $f : A \rightarrow B$ について, $(a, b) \in f$ を $f(a) = b$ と表記する.

- 関係 $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$ について, その**合成** (*composition*) を $R_1; R_2 = R_2 \circ R_1 = \{(x, z) \in A \times C \mid \exists y \in B. (x, y) \in R_1, (y, z) \in R_2\}$ と表記する.
- 関係 $R \subseteq A \times B$, 集合 $X \subseteq A$ について, R の X による**制限** (*restriction*) を $R \upharpoonright_X = \{(a, b) \in R \mid a \in X\}$ と表記する. 特に関数 $f: A \rightarrow B$ の $X \subseteq A$ による制限は, 関数 $f \upharpoonright_X: X \rightarrow B$ になる.
- $a \in A$, $b \in B$ について, その組を $a \mapsto b = (a, b)$, 関数 $f: A \rightarrow B$ を $f = x \mapsto f(x)$ と表記する.
- 2 項関係 $R \subseteq A^2$ について, その**推移閉包** (*transitive closure*), つまり以下を満たす最小の 2 項関係を $R^+ \subseteq A^2$ と表記する.
 - 任意の $(a, b) \in R$ について, $(a, b) \in R^+$.
 - 任意の $(a, b) \in R^+$, $(b, c) \in R^+$ について, $(a, c) \in R^+$.
- 2 項関係 $R \subseteq A^2$ について, その**反射推移閉包** (*reflexive transitive closure*) を $R^* = R^+ \cup \{(a, a) \mid a \in A\}$ と表記する.

集合 I について, その要素で添字付けられた対象の列 $\{a_i\}_{i \in I}$ を I で添字づけられた**族** (*indexed family*) と呼ぶ. 族について, 以下の表記を用いる.

- 族の集合を $\prod_{i \in I} A_i = \{\{a_i\}_{i \in I} \mid \forall i \in I, a_i \in A_i\}$ と表記する.
- 集合の族 $A = \{A_i\}_{i \in I}$ について, 次の条件を満たす時, A は**互いに素** (*pairwise disjoint*) であるという.

$$\forall i_1, i_2 \in I, i_1 \neq i_2. A_{i_1} \cap A_{i_2} = \emptyset$$

第 2 章

Basic Calculus

2.1 WIP: (Untyped) λ -Calculus

2.2 Simply Typed λ -Calculus

Alias: STLC, $\lambda \rightarrow$ [GTL89]

2.2.1 Syntax

$e ::= x$	(variable)
$ e e$	(application)
$ \lambda x : \tau. e$	(abstraction)
$ c_A$	(constant)
$\tau ::= A$	(atomic type)
$ \tau \rightarrow \tau$	(function type)
$\Gamma ::= \cdot$	(empty)
$ \Gamma, x : \tau$	(cons)

Convention:

$$\tau_1 \rightarrow \tau_2 \rightarrow \cdots \rightarrow \tau_n \stackrel{\text{def}}{=} \tau_1 \rightarrow (\tau_2 \rightarrow (\cdots \rightarrow \tau_n) \cdots)$$

$$e_1 e_2 \cdots e_n \stackrel{\text{def}}{=} (\cdots (e_1 e_2) \cdots) e_n$$

Environment Reference:

$$\boxed{\Gamma(x) = \tau}$$

$$\frac{x = x'}{(\Gamma, x' : \tau)(x) = \tau} \quad \frac{x \neq x' \quad \Gamma(x) = \tau}{(\Gamma, x' : \tau)(x) = \tau}$$

Free Variable:

$$\boxed{fv(e) = \{\bar{x}'\}}$$

$$\frac{}{fv(x) = \{x\}} \quad \frac{fv(e_1) = X_1 \quad fv(e_2) = X_2}{fv(e_1 e_2) = X_1 \cup X_2} \quad \frac{fv(e) = X}{fv(\lambda x : \tau. e) = X \setminus \{x\}} \quad \overline{fv(c_A) = \emptyset}$$

Substitution:

部分関数 $\{x_1 \mapsto e_1, \dots, x_n \mapsto e_n\}$ を, $[x_1 \leftarrow e_1, \dots, x_n \leftarrow e_n]$ または $[x_1, \dots, x_n \leftarrow e_1, \dots, e_n]$ と表記する.

$$\boxed{e[x' \leftarrow e'] = e''}$$

$$\frac{[\bar{x}' \leftarrow \bar{e}'](x) = e}{x[\bar{x}' \leftarrow \bar{e}'] = e} \quad \frac{x \notin \text{dom}([\bar{x}' \leftarrow \bar{e}'])}{x[\bar{x}' \leftarrow \bar{e}'] = x}$$

$$\frac{e_1[\bar{x}' \leftarrow \bar{e}'] = e_1'' \quad e_2[\bar{x}' \leftarrow \bar{e}'] = e_2''}{(e_1 e_2)[\bar{x}' \leftarrow \bar{e}'] = e_1'' e_2''} \quad \frac{e([\bar{x}' \leftarrow \bar{e}'] \upharpoonright_{\text{dom}([\bar{x}' \leftarrow \bar{e}']) \setminus \{x\}}) = e''}{(\lambda x : \tau. e)[\bar{x}' \leftarrow \bar{e}'] = \lambda x : \tau. e''} \quad \overline{c_A[\bar{x}' \leftarrow \bar{e}'] = c_A}$$

α -Equality:

$$\boxed{e_1 \equiv_\alpha e_2}$$

$$\frac{x_1 = x_2}{x_1 \equiv_\alpha x_2} \quad \frac{x' \notin fv(e_1) \cup fv(e_2) \quad e_1[x_1 \leftarrow x'] \equiv_\alpha e_2[x_2 \leftarrow x']}{\lambda x_1 : \tau. e_1 \equiv_\alpha \lambda x_2 : \tau. e_2} \quad \frac{e_1 \equiv_\alpha e_2 \quad e_1' \equiv_\alpha e_2'}{e_1 e_1' \equiv_\alpha e_2 e_2'} \quad \overline{c_A \equiv_\alpha c_A}$$

定理 1 (Correctness of Substitution). 式 e , 置換 $[\bar{x}' \leftarrow \bar{e}']$ について, $X = \text{dom}([\bar{x}' \leftarrow \bar{e}'])$ とした時,

$$fv(e[\bar{x}' \leftarrow \bar{e}']) = (fv(e) \setminus X) \cup \bigcup_{x \in fv(e) \cap X} fv([\bar{x}' \leftarrow \bar{e}'](x)).$$

□

定理 2 (α -Equality Does Not Touch Free Variables). $e_1 \equiv_\alpha e_2$ ならば, $fv(e_1) = fv(e_2)$.

□

2.2.2 Typing Semantics

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c} \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{ T-Var} \\ \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \text{ T-Abs} \\ \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \text{ T-App} \\ \frac{}{\Gamma \vdash c_A : A} \text{ T-Const} \end{array}$$

特に, $\cdot \vdash e : \tau$ の時, $e : \tau$ と表記.

2.2.3 Evaluation Semantics (Call-By-Value)

$$\begin{array}{l} v ::= \lambda x : \tau. e \\ \quad \mid c_A \\ C ::= [] \\ \quad \mid C e \\ \quad \mid v C \end{array}$$

Small Step:

$$\boxed{e \Rightarrow e'}$$

$$\frac{}{(\lambda x : \tau. e) v \Rightarrow e[x \leftarrow v]} \quad \frac{e \Rightarrow e'}{C[e] \Rightarrow C[e']}$$

Big Step:

$$\boxed{e \Downarrow v}$$

$$\frac{e_1 \Downarrow \lambda x : \tau. e'_1 \quad e_2 \Downarrow v_2 \quad e'_1[x \leftarrow v_2] \Downarrow v}{e_1 e_2 \Downarrow v}$$

定理 3 (Adequacy of Small Step and Big Step). $e \Rightarrow^* v$ iff $e \Downarrow v$.

□

定理 4 (Type Soundness). $e : \tau$ の時, $e \Rightarrow^* v$, $e \Downarrow v$ となる $v = \text{nf}(\Rightarrow, e)$ が存在し,

- $\tau = \tau_1 \rightarrow \tau_2$ の時, $v \equiv_\alpha \lambda x' : \tau_1. e'$ となる $\lambda x' : \tau_1. e'$ が存在する.
- $\tau = A$ の時, $v \equiv_\alpha c_A$ となる c_A が存在する.

□

2.2.4 Equational Reasoning

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c}
\frac{\Gamma, x : \tau \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\lambda x : \tau. e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau} \text{Eq-}\beta\text{-Lam} \quad \frac{x \notin \text{fv}(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash (\lambda x : \tau_1. e x) \equiv e : \tau_1 \rightarrow \tau_2} \text{Eq-}\eta\text{-Lam} \\
\frac{e_1 \equiv_{\alpha} e_2 \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Refl} \\
\frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-Sym} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau \quad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau} \text{Eq-Trans} \\
\frac{\Gamma, x : \tau \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash \lambda x : \tau. e_1 \equiv \lambda x : \tau. e_2 : \tau \rightarrow \tau'} \text{Eq-Cong-Abs} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \rightarrow \tau \quad \Gamma \vdash e'_1 \equiv e'_2 : \tau'}{\Gamma \vdash e_1 e'_1 \equiv e_2 e'_2 : \tau} \text{Eq-Cong-App}
\end{array}$$

特に, $\cdot \vdash e_1 \equiv e_2 : \tau$ の時, $e_1 \equiv e_2 : \tau$ と表記.

定理 5 (Respect Typing). $\Gamma \vdash e_1 \equiv e_2 : \tau$ ならば, $\Gamma \vdash e_1 : \tau$ かつ $\Gamma \vdash e_2 : \tau$. □

定理 6 (Respect Evaluation). $e_1 \equiv e_2 : \tau$ の時, $e'_1 \Rightarrow^* e_1$, $e_2 \Rightarrow^* e'_2$ ならば $e'_1 \equiv e'_2 : \tau$. □

系 7. $e_1 \equiv e_2 : \tau$ の時, $e_1 \Rightarrow^* e'_1$, $e_2 \Rightarrow^* e'_2$ ならば $e'_1 \equiv e'_2 : \tau$. □

証明. $e_1 \Rightarrow^* e_1$ より, 定理 6 から $e_1 \equiv e'_2 : \tau$. よって, T-Sym から $e'_2 \equiv e_1 : \tau$ であり, $e'_2 \Rightarrow^* e'_2$ より定理 6 から $e'_2 \equiv e'_1 : \tau$. 故に, T-Sym から $e'_1 \equiv e'_2 : \tau$. ■

2.3 WIP: System-T

2.4 WIP: PCF

2.5 System-F

Alias: F, Second Order Typed Lambda Calculus, $\lambda 2$ [GTL89]

2.5.1 Syntax

$e ::= x$	(variable)
$ \lambda x : \tau. e$	(abstraction)
$ e e$	(application)
$ \Lambda t. e$	(universal abstraction)
$ e \tau$	(universal application)
$\tau ::= t$	(type variable)
$ \tau \rightarrow \tau$	(function type)
$ \forall t. \tau$	(polymorphic type)
$\Gamma ::= \cdot$	(empty)
$ \Gamma, x : \tau$	(variable cons)
$ \Gamma, t : \Omega$	(type variable cons)

Convention:

$$\tau_1 \rightarrow \tau_2 \rightarrow \cdots \rightarrow \tau_n \stackrel{\text{def}}{=} \tau_1 \rightarrow (\tau_2 \rightarrow (\cdots \rightarrow \tau_n) \cdots)$$

$$e_1 e_2 \cdots e_n \stackrel{\text{def}}{=} (\cdots (e_1 e_2) \cdots) e_n$$

Environment Reference:

$$\boxed{\Gamma(x) = \tau}$$

$$\frac{x = x'}{(\Gamma, x' : \tau)(x) = \tau} \quad \frac{x \neq x' \quad \Gamma(x) = \tau}{(\Gamma, x' : \tau)(x) = \tau} \quad \frac{\Gamma(x) = \tau}{(\Gamma, t : \Omega)(x) = \tau}$$

$$\frac{t = t'}{(\Gamma, t' : \Omega)(t) = \Omega} \quad \frac{t \neq t' \quad \Gamma(t) = \Omega}{(\Gamma, t' : \Omega)(t) = \Omega} \quad \frac{\Gamma(t) = \Omega}{(\Gamma, x : \tau)(t) = \Omega}$$

Free Variable:

$$\boxed{fv(e) = \{\bar{x}\}}$$

$$\frac{}{fv(x) = \{x\}} \quad \frac{fv(e_1) = X_1 \quad fv(e_2) = X_2}{fv(e_1 e_2) = X_1 \cup X_2} \quad \frac{fv(e) = X}{fv(\lambda x : \tau. e) = X \setminus \{x\}} \quad \frac{fv(e) = X}{fv(e \tau) = X} \quad \frac{fv(e) = X}{fv(\Lambda t. e) = X}$$

Substitution:

部分関数 $\{x_1 \mapsto e_1, \dots, x_n \mapsto e_n\}$ を, $[x_1 \leftarrow e_1, \dots, x_n \leftarrow e_n]$ または $[x_1, \dots, x_n \leftarrow e_1, \dots, e_n]$ と表記する.

$$\boxed{e[x' \leftarrow e'] = e''}$$

$$\frac{\frac{\bar{x}' \leftarrow \bar{e}'}{x[\bar{x}' \leftarrow \bar{e}']} = e}{e_1[\bar{x}' \leftarrow \bar{e}'] = e''_1 \quad e_2[\bar{x}' \leftarrow \bar{e}'] = e''_2} \quad \frac{x \notin \text{dom}([\bar{x}' \leftarrow \bar{e}'])}{x[\bar{x}' \leftarrow \bar{e}']} = x$$

$$\frac{}{(e_1 e_2)[\bar{x}' \leftarrow \bar{e}'] = e''_1 e''_2} \quad \frac{e([\bar{x}' \leftarrow \bar{e}']) \upharpoonright_{\text{dom}([\bar{x}' \leftarrow \bar{e}']) \setminus \{x\}} = e''}{(\lambda x : \tau. e)[\bar{x}' \leftarrow \bar{e}'] = \lambda x : \tau. e''}$$

$$\frac{e[\bar{x}' \leftarrow \bar{e}'] = e''}{(e \tau)[\bar{x}' \leftarrow \bar{e}'] = e'' \tau} \quad \frac{e[\bar{x}' \leftarrow \bar{e}'] = e''}{(\Lambda t. e)[\bar{x}' \leftarrow \bar{e}'] = \Lambda t. e''}$$

Type Free Variable:

$$\boxed{tyfv(e) = \{\bar{x}\}}$$

$$\begin{array}{c} \frac{}{tyfv(x) = \emptyset} \quad \frac{tyfv(e_1) = T_1 \quad tyfv(e_2) = T_2}{tyfv(e_1 e_2) = T_1 \cup T_2} \quad \frac{tyfv(\tau) = T_1 \quad tyfv(e) = T_2}{tyfv(\lambda x : \tau. e) = T_1 \cup T_2} \\ \frac{tyfv(e) = T_1 \quad tyfv(\tau) = T_2}{tyfv(e \tau) = T_1 \cup T_2} \quad \frac{tyfv(e) = T}{tyfv(\Lambda t. e) = T \setminus \{t\}} \\ \frac{}{tyfv(t) = \{t\}} \quad \frac{tyfv(\tau_1) = T_1 \quad tyfv(\tau_2) = T_2}{tyfv(\tau_1 \rightarrow \tau_2) = T_1 \cup T_2} \quad \frac{tyfv(\tau) = T}{tyfv(\forall t. \tau) = T \setminus \{t\}} \end{array}$$

Type Substitution:

部分関数 $\{t_1 \mapsto \tau_1, \dots, t_n \mapsto \tau_n\}$ を, $[t_1 \leftarrow \tau_1, \dots, t_n \leftarrow \tau_n]$ または $[t_1, \dots, t_n \leftarrow t_1, \dots, t_n]$ と表記する.

$$\boxed{e[\bar{t} \leftarrow \bar{\tau}] = e'}$$

$$\begin{array}{c} \frac{}{x[\bar{t}' \leftarrow \bar{\tau}'] = x} \quad \frac{e_1[\bar{t}' \leftarrow \bar{\tau}'] = e_1'' \quad e_2[\bar{t}' \leftarrow \bar{\tau}'] = e_2''}{(e_1 e_2)[\bar{t}' \leftarrow \bar{\tau}'] = e_1'' e_2''} \quad \frac{\tau[\bar{t}' \leftarrow \bar{\tau}'] = \tau'' \quad e[\bar{t}' \leftarrow \bar{\tau}'] = e''}{(\lambda x : \tau. e)[\bar{t}' \leftarrow \bar{\tau}'] = \lambda x : \tau''. e''} \\ \frac{e[\bar{t}' \leftarrow \bar{\tau}'] = e'' \quad \tau[\bar{t}' \leftarrow \bar{\tau}'] = \tau''}{(e \tau)[\bar{t}' \leftarrow \bar{\tau}'] = e'' \tau''} \quad \frac{e([\bar{t}' \leftarrow \bar{\tau}'] \upharpoonright_{\text{dom}([\bar{t}' \leftarrow \bar{\tau}'] \setminus \{t\})}) = e''}{(\Lambda t. e)[\bar{t}' \leftarrow \bar{\tau}'] = \Lambda t. e''} \end{array}$$

$$\boxed{\tau[\bar{t}' \leftarrow \bar{\tau}'] = \tau''}$$

$$\begin{array}{c} \frac{[\bar{t}' \leftarrow \bar{\tau}'](t) = \tau}{t[\bar{t}' \leftarrow \bar{\tau}'] = \tau} \quad \frac{t \notin \text{dom}([\bar{t}' \leftarrow \bar{\tau}'])}{t[\bar{t}' \leftarrow \bar{\tau}'] = t} \quad \frac{\tau_1[\bar{t}' \leftarrow \bar{\tau}'] = \tau_1'' \quad \tau_2[\bar{t}' \leftarrow \bar{\tau}'] = \tau_2''}{(\tau_1 \rightarrow \tau_2)[\bar{t}' \leftarrow \bar{\tau}'] = \tau_1'' \rightarrow \tau_2''} \quad \frac{\tau([\bar{t}' \leftarrow \bar{\tau}'] \upharpoonright_{\text{dom}([\bar{t}' \leftarrow \bar{\tau}'] \setminus \{t\})}) = \tau''}{(\forall t. \tau)[\bar{t}' \leftarrow \bar{\tau}'] = \forall t. \tau''} \end{array}$$

α -Equality:

$$\boxed{e_1 \equiv_{\alpha} e_2}$$

$$\begin{array}{c} \frac{x_1 = x_2}{x_1 \equiv_{\alpha} x_2} \quad \frac{e_1 \equiv_{\alpha} e_2 \quad e_1' \equiv_{\alpha} e_2'}{e_1 e_1' \equiv_{\alpha} e_2 e_2'} \quad \frac{\tau_1 \equiv_{\alpha} \tau_2 \quad x' \notin fv(e_1) \cup fv(e_2) \quad e_1[x_1 \leftarrow x'] \equiv_{\alpha} e_2[x_2 \leftarrow x']}{\lambda x_1 : \tau_1. e_1 \equiv_{\alpha} \lambda x_2 : \tau_2. e_2} \\ \frac{e_1 \equiv_{\alpha} e_2 \quad \tau_1 \equiv_{\alpha} \tau_2}{e_1 \tau_1 \equiv_{\alpha} e_2 \tau_2} \quad \frac{t' \notin tyfv(e_1) \cup tyfv(e_2) \quad e_1[t_1 \leftarrow t'] \equiv_{\alpha} e_2[t_2 \leftarrow t']}{\Lambda t_1. e_1 \equiv_{\alpha} \Lambda t_2. e_2} \end{array}$$

$$\boxed{\tau_1 \equiv_{\alpha} \tau_2}$$

$$\frac{t_1 = t_2}{t_1 \equiv_{\alpha} t_2} \quad \frac{\tau_1 \equiv_{\alpha} \tau_2 \quad \tau_1' \equiv_{\alpha} \tau_2'}{\tau_1 \rightarrow \tau_1' \equiv_{\alpha} \tau_2 \rightarrow \tau_2'} \quad \frac{t' \notin tyfv(\tau_1) \cup tyfv(\tau_2) \quad \tau_1[t_1 \leftarrow t'] \equiv_{\alpha} \tau_2[t_2 \leftarrow t']}{\forall t_1. \tau_1 \equiv_{\alpha} \forall t_2. \tau_2}$$

定理 8 (Correctness of Substitution). 置換 $[\bar{x}' \leftarrow \bar{e}']$ について, $X = \text{dom}([\bar{x}' \leftarrow \bar{e}'])$ とした時,

$$fv(e[\bar{x}' \leftarrow \bar{e}']) = (fv(e) \setminus X) \cup \bigcup_{x \in fv(e) \cap X} fv([\bar{x}' \leftarrow \bar{e}'](x)).$$

□

定理 9 (Correctness of Type Substitution). 式 e , 型 τ , 型置換 $[\bar{t}' \leftarrow \bar{\tau}']$ について, $T = \text{dom}([\bar{t}' \leftarrow \bar{\tau}'])$ とした時,

$$tyfv(e[\bar{t}' \leftarrow \bar{\tau}']) = (tyfv(e) \setminus T) \cup \bigcup_{t \in tyfv(e) \cap T} tyfv([\bar{t}' \leftarrow \bar{\tau}'](t))$$

$$tyfv(\tau[\bar{t}' \leftarrow \bar{\tau}']) = (tyfv(\tau) \setminus T) \cup \bigcup_{t \in tyfv(\tau) \cap T} tyfv([\bar{t}' \leftarrow \bar{\tau}'](t)).$$

□

定理 10 (α -Equality Does Not Touch Free Variables).

- $\tau_1 \equiv_\alpha \tau_2$ ならば $tyfv(\tau_1) = tyfv(\tau_2)$.
- $e_1 \equiv_\alpha e_2$ ならば, $fv(e_1) = fv(e_2)$, $tyfv(e_1) = tyfv(e_2)$.

□

2.5.2 Typing Semantics

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c} \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{ T-Var} \\ \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \text{ T-Abs} \\ \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \text{ T-App} \\ \frac{\Gamma, t : \Omega \vdash e : \tau}{\Gamma \vdash \Lambda t. e : \forall t. \tau} \text{ T-UnivAbs} \\ \frac{\Gamma \vdash e : \forall t. \tau_1}{\Gamma \vdash e \tau_2 : \tau_1[t \leftarrow \tau_2]} \text{ T-UnivApp} \\ \frac{\Gamma \vdash \tau \equiv_\alpha \tau' : \Omega \quad \Gamma \vdash e : \tau'}{\Gamma \vdash e : \tau} \text{ T-}\alpha\text{-Equiv} \end{array}$$

特に, $\cdot \vdash e : \tau$ の時, $e : \tau$ と表記.

2.5.3 Evaluation Semantics (Call-By-Value)

$$\begin{array}{l} v ::= \lambda x : \tau. e \\ \quad | \Lambda t. e \\ C ::= [] \\ \quad | C e \\ \quad | v C \\ \quad | C \tau \end{array}$$

Small Step:

$$\boxed{e \Rightarrow e'}$$

$$\begin{array}{c} \overline{(\lambda x : \tau. e) v \Rightarrow e[x \leftarrow v]} \\ \overline{(\Lambda t. e) \tau \Rightarrow e[t \leftarrow \tau]} \\ \frac{e \Rightarrow e'}{C[e] \Rightarrow C[e']} \end{array}$$

Big Step:

$$\boxed{e \Downarrow v}$$

$$\frac{e_1 \Downarrow \lambda x : \tau. e'_1 \quad e_2 \Downarrow v_2 \quad e'_1[x \leftarrow v_2] \Downarrow v}{e_1 e_2 \Downarrow v}$$

$$\frac{e \Downarrow \Lambda t. e'_1 \quad e'_1[t \leftarrow \tau] \Downarrow v}{e \tau \Downarrow v}$$

定理 11 (Adequacy of Small Step and Big Step). $e \Rightarrow^* v$ iff $e \Downarrow v$. □

定理 12 (Type Soundness). $e : \tau$ の時, $e \Rightarrow^* v$, $e \Downarrow v$ となる $v = \text{nf}(\Rightarrow, e)$ が存在し,

- $\tau = \tau_1 \rightarrow \tau_2$ の時, $v \equiv_\alpha \lambda x' : \tau_1. e'$ となる $\lambda x' : \tau_1. e'$ が存在する.
- $\tau = \forall t. \tau_1$ の時, $v \equiv_\alpha \Lambda t. e'$ となる $\Lambda t. e'$ が存在する.

□

2.5.4 Equational Reasoning

$\Gamma \vdash e_1 \equiv e_2 : \tau$

$$\frac{\Gamma, x : \tau_2 \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\lambda x : \tau_2. e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau} \text{Eq-}\beta\text{-Lam} \quad \frac{x \notin \text{fv}(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash (\lambda x : \tau_1. e \ x) \equiv e : \tau_1 \rightarrow \tau_2} \text{Eq-}\eta\text{-Lam}$$

$$\frac{\Gamma, t : \Omega \vdash e : \tau}{\Gamma \vdash (\Lambda t. e) \tau_2 \equiv e[t \leftarrow \tau_2] : \tau[t \leftarrow \tau_2]} \text{Eq-}\beta\text{-UnivLam} \quad \frac{t \notin \text{tyfv}(e) \quad \Gamma \vdash e : \forall t'. \tau}{\Gamma \vdash (\Lambda t. e \ t) \equiv e : \forall t'. \tau} \text{Eq-}\eta\text{-UnivLam}$$

$$\frac{e_1 \equiv_\alpha e_2 \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Refl} \quad \frac{\tau \equiv_\alpha \tau' \quad \Gamma \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Type}$$

$$\frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-Sym} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau \quad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau} \text{Eq-Trans}$$

$$\frac{\Gamma, x : \tau \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash \lambda x : \tau. e_1 \equiv \lambda x : \tau. e_2 : \tau \rightarrow \tau'} \text{Eq-Cong-Abs} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \rightarrow \tau \quad \Gamma \vdash e'_1 \equiv e'_2 : \tau'}{\Gamma \vdash e_1 e'_1 \equiv e_2 e'_2 : \tau} \text{Eq-Cong-App}$$

$$\frac{\Gamma, t : \Omega \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \Lambda t. e_1 \equiv \Lambda t. e_2 : \forall (t. \tau)} \text{Eq-Cong-UnivAbs} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \forall t. \tau}{\Gamma \vdash e_1 \tau' \equiv e_2 \tau' : \tau[t \leftarrow \tau']} \text{Eq-Cong-UnivApp}$$

特に, $\cdot \vdash e_1 \equiv e_2 : \tau$ の時, $e_1 \equiv e_2 : \tau$ と表記.

定理 13 (Respect Typing). $\Gamma \vdash e_1 \equiv e_2 : \tau$ ならば, $\Gamma \vdash e_1 : \tau$ かつ $\Gamma \vdash e_2 : \tau$. □

定理 14 (Respect Evaluation). $e_1 \equiv e_2 : \tau$ の時, $e'_1 \Rightarrow^* e_1$, $e_2 \Rightarrow^* e'_2$ ならば $e'_1 \equiv e'_2 : \tau$. □

系 15. $e_1 \equiv e_2 : \tau$ の時, $e_1 \Rightarrow^* e'_1$, $e_2 \Rightarrow^* e'_2$ ならば $e'_1 \equiv e'_2 : \tau$. □

証明. $e_1 \Rightarrow^* e_1$ より, 定理 14 から $e_1 \equiv e'_2 : \tau$. よって, T-Sym から $e'_2 \equiv e_1 : \tau$ であり, $e'_2 \Rightarrow^* e'_2$ より定理 14 から $e'_2 \equiv e'_1 : \tau$. 故に, T-Sym から $e'_1 \equiv e'_2 : \tau$. ■

2.5.5 Definability

Product

Product of τ_1 and τ_2 :

$$\tau_1 \times \tau_2 \stackrel{\text{def}}{=} \forall t. (\tau_1 \rightarrow \tau_2 \rightarrow t) \rightarrow t$$

$$\langle e_1, e_2 \rangle \stackrel{\text{def}}{=} \Lambda t. \lambda x : \tau_1 \rightarrow \tau_2 \rightarrow t. x \ e_1 \ e_2$$

$$\pi_1 e \stackrel{\text{def}}{=} e \tau_1 \lambda x_1. \lambda x_2. x_1$$

$$\pi_2 e \stackrel{\text{def}}{=} e \tau_2 \lambda x_1. \lambda x_2. x_2$$

Admissible typing rule:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2} \text{T-Product} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_1 e : \tau_1} \text{T-Proj-1} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_2 e : \tau_2} \text{T-Proj-2}$$

Admissible equality:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \pi_1 \langle e_1, e_2 \rangle \equiv e_1 : \tau_1} \text{Eq-}\beta\text{-Product-1} \quad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \pi_2 \langle e_1, e_2 \rangle \equiv e_2 : \tau_2} \text{Eq-}\beta\text{-Product-2}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \langle \pi_1 e, \pi_2 e \rangle \equiv e : \tau_1 \times \tau_2} \text{Eq-}\eta\text{-Product}$$

Existential Type

Existence of $\exists t. \tau$:

$$\exists t. \tau \stackrel{\text{def}}{=} \forall t'. (\forall t. \tau \rightarrow t') \rightarrow t'$$

$$\text{pack} \langle \tau_t, e \rangle \stackrel{\text{def}}{=} \Lambda t'. \lambda x : (\forall t. \tau \rightarrow t'). x \tau_t e$$

$$\text{unpack} \langle t, x \rangle = e_1. \tau_2. e_2 \stackrel{\text{def}}{=} e_1 \tau_2 (\Lambda t. \lambda x : \tau. e_2)$$

Admissible typing rule:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash e : \tau[t \leftarrow \tau_t]}{\Gamma \vdash \text{pack} \langle \tau_t, e \rangle : \exists t. \tau} \text{T-Pack} \quad \frac{\Gamma \vdash e_1 : \exists t. \tau \quad \Gamma, t : \Omega, x : \tau \vdash e_2 : \tau_2 \quad t \notin \text{tyfv}(\tau_2)}{\Gamma \vdash \text{unpack} \langle t, x \rangle = e_1. \tau_2. e_2 : \tau_2} \text{T-Unpack}$$

Admissible equality:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1[t \leftarrow \tau_t] \quad \Gamma, t : \Omega, x : \tau_1 \vdash e_2 : \tau_2 \quad t \notin \text{tyfv}(\tau_2)}{\Gamma \vdash \text{unpack} \langle t, x \rangle = \text{pack} \langle \tau_t, e_1 \rangle. \tau_2. e_2 \equiv e_2[t \leftarrow \tau_t][x \leftarrow e_1] : \tau_2} \text{Eq-}\beta\text{-Exist}$$

$$\frac{\Gamma \vdash e : \exists t'. \tau \quad \tau' \equiv_\alpha \exists t'. \tau}{\Gamma \vdash \text{unpack} \langle t, x \rangle = e. \tau'. \text{pack} \langle t, x \rangle \equiv e : \exists t'. \tau} \text{Eq-}\eta\text{-Exist}$$

2.6 System-F ω

Alias: F ω , $\lambda\omega$ [RRD14]

2.6.1 Syntax

$e ::= x$	(variable)
$\quad \lambda x : \tau. e$	(abstraction)
$\quad e e$	(application)
$\quad \Lambda t : \kappa. e$	(universal abstraction)
$\quad e \tau$	(universal application)
$\tau ::= t$	(type variable)
$\quad \tau \rightarrow \tau$	(function type)
$\quad \forall t : \kappa. \tau$	(polymorphic type)
$\quad \lambda t : \kappa. \tau$	(type abstraction)
$\quad \tau \tau$	(type application)
$\kappa ::= \Omega$	(type kind)
$\quad \kappa \rightarrow \kappa$	(arrow kind)
$\Gamma ::= \cdot$	(empty)
$\quad \Gamma, x : \tau$	(variable cons)
$\quad \Gamma, t : \kappa$	(type variable cons)

Convention:

$$\begin{aligned} \tau_1 \rightarrow \tau_2 \rightarrow \cdots \rightarrow \tau_n &\stackrel{\text{def}}{=} \tau_1 \rightarrow (\tau_2 \rightarrow (\cdots \rightarrow \tau_n) \cdots) \\ e_1 e_2 \cdots e_n &\stackrel{\text{def}}{=} (\cdots (e_1 e_2) \cdots) e_n \end{aligned}$$

Environment Reference:

$$\boxed{\Gamma(x) = \tau}$$

$$\begin{array}{ccc} \frac{x = x'}{(\Gamma, x' : \tau)(x) = \tau} & \frac{x \neq x' \quad \Gamma(x) = \tau}{(\Gamma, x' : \tau)(x) = \tau} & \frac{\Gamma(x) = \tau}{(\Gamma, t : \kappa)(x) = \tau} \\ \frac{t = t'}{(\Gamma, t' : \kappa)(t) = \kappa} & \frac{t \neq t' \quad \Gamma(t) = \kappa}{(\Gamma, t' : \kappa')(t) = \kappa} & \frac{\Gamma(t) = \kappa}{(\Gamma, x : \tau)(t) = \kappa} \end{array}$$

Free Variable:

$$\boxed{fv(e) = \{\bar{x}\}}$$

$$\begin{array}{ccccc} \frac{}{fv(x) = \{x\}} & \frac{fv(e) = X}{fv(\lambda x : \tau. e) = X \setminus \{x\}} & \frac{fv(e_1) = X_1 \quad fv(e_2) = X_2}{fv(e_1 e_2) = X_1 \cup X_2} & \frac{fv(e) = X}{fv(\Lambda t : \kappa. e) = X} & \frac{fv(e) = X}{fv(e \tau) = X} \end{array}$$

Substitution:

部分関数 $\{x_1 \mapsto e_1, \dots, x_n \mapsto e_n\}$ を, $[x_1 \leftarrow e_1, \dots, x_n \leftarrow e_n]$ または $[x_1, \dots, x_n \leftarrow e_1, \dots, e_n]$ と表記する.

$$\boxed{e[\overline{x'} \leftarrow \overline{e'}] = \overline{e''}}$$

$$\begin{array}{cc} \frac{[\overline{x'} \leftarrow \overline{e'}](x) = e}{x[\overline{x'} \leftarrow \overline{e'}] = e} & \frac{x \notin \text{dom}([\overline{x'} \leftarrow \overline{e'}])}{x[\overline{x'} \leftarrow \overline{e'}] = x} \end{array}$$

$$\begin{array}{c}
\frac{e(\overline{x'} \leftarrow \overline{e'}) \upharpoonright_{\text{dom}(\overline{x'} \leftarrow \overline{e'}) \setminus \{x\}} = e''}{(\lambda x : \tau. e)[\overline{x'} \leftarrow \overline{e'}] = \lambda x : \tau. e''} \quad \frac{e_1[\overline{x'} \leftarrow \overline{e'}] = e_1'' \quad e_2[\overline{x'} \leftarrow \overline{e'}] = e_2''}{(e_1 e_2)[\overline{x'} \leftarrow \overline{e'}] = e_1'' e_2''} \\
\frac{e[\overline{x'} \leftarrow \overline{e'}] = e''}{(\Lambda t : \kappa. e)[\overline{x'} \leftarrow \overline{e'}] = \Lambda t : \kappa. e''} \quad \frac{e[\overline{x'} \leftarrow \overline{e'}] = e''}{(e \tau)[\overline{x'} \leftarrow \overline{e'}] = e'' \tau}
\end{array}$$

Type Free Variable:

$$\boxed{\text{tyfv}(e) = \{\bar{t}\}}$$

$$\begin{array}{c}
\frac{}{\text{tyfv}(x) = \emptyset} \quad \frac{\text{tyfv}(\tau) = T_1 \quad \text{tyfv}(e) = T_2}{\text{tyfv}(\lambda x : \tau. e) = T_1 \cup T_2} \quad \frac{\text{tyfv}(e_1) = T_1 \quad \text{tyfv}(e_2) = T_2}{\text{tyfv}(e_1 e_2) = T_1 \cup T_2} \\
\frac{\text{tyfv}(e) = T}{\text{tyfv}(\Lambda t : \kappa. e) = T \setminus \{t\}} \quad \frac{\text{tyfv}(e) = T_1 \quad \text{tyfv}(\tau) = T_2}{\text{tyfv}(e \tau) = T_1 \cup T_2}
\end{array}$$

$$\boxed{\text{tyfv}(\tau) = \{\bar{t}\}}$$

$$\begin{array}{c}
\frac{}{\text{tyfv}(t) = \{t\}} \quad \frac{\text{tyfv}(\tau_1) = T_1 \quad \text{tyfv}(\tau_2) = T_2}{\text{tyfv}(\tau_1 \rightarrow \tau_2) = T_1 \cup T_2} \quad \frac{\text{tyfv}(\tau) = T}{\text{tyfv}(\forall t : \kappa. \tau) = T \setminus \{t\}} \\
\frac{\text{tyfv}(\tau) = T}{\text{tyfv}(\lambda t : \kappa. \tau) = T \setminus \{t\}} \quad \frac{\text{tyfv}(\tau_1) = T_1 \quad \text{tyfv}(\tau_2) = T_2}{\text{tyfv}(\tau_1 \tau_2) = T_1 \cup T_2}
\end{array}$$

Type Substitution:

部分関数 $\{t_1 \mapsto \tau_1, \dots, t_n \mapsto \tau_n\}$ を, $[t_1 \leftarrow \tau_1, \dots, t_n \leftarrow \tau_n]$ または $[t_1, \dots, t_n \leftarrow t_1, \dots, t_n]$ と表記する.

$$\boxed{e[\overline{t'} \leftarrow \overline{\tau'}] = e'}$$

$$\begin{array}{c}
\frac{}{x[\overline{t'} \leftarrow \overline{\tau'}] = x} \quad \frac{e_1[\overline{t'} \leftarrow \overline{\tau'}] = e_1'' \quad e_2[\overline{t'} \leftarrow \overline{\tau'}] = e_2''}{(e_1 e_2)[\overline{t'} \leftarrow \overline{\tau'}] = e_1'' e_2''} \quad \frac{\tau[\overline{t'} \leftarrow \overline{\tau'}] = \tau'' \quad e[\overline{t'} \leftarrow \overline{\tau'}] = e''}{(\lambda x : \tau. e)[\overline{t'} \leftarrow \overline{\tau'}] = \lambda x : \tau''. e''} \\
\frac{e[\overline{t'} \leftarrow \overline{\tau'}] = e'' \quad \tau[\overline{t'} \leftarrow \overline{\tau'}] = \tau''}{(e \tau)[\overline{t'} \leftarrow \overline{\tau'}] = e'' \tau''} \quad \frac{e([\overline{t'} \leftarrow \overline{\tau'}] \upharpoonright_{\text{dom}([\overline{t'} \leftarrow \overline{\tau'}] \setminus \{t\})}) = e''}{(\Lambda t : \kappa. e)[\overline{t'} \leftarrow \overline{\tau'}] = \Lambda t : \kappa. e''}
\end{array}$$

$$\boxed{\tau[\overline{t'} \leftarrow \overline{\tau'}] = \tau''}$$

$$\begin{array}{c}
\frac{\overline{[t' \leftarrow \tau']}(t) = \tau}{t[\overline{t'} \leftarrow \overline{\tau'}] = \tau} \quad \frac{t \notin \text{dom}([\overline{t'} \leftarrow \overline{\tau'}])}{t[\overline{t'} \leftarrow \overline{\tau'}] = t} \\
\frac{\tau_1[\overline{t'} \leftarrow \overline{\tau'}] = \tau_1'' \quad \tau_2[\overline{t'} \leftarrow \overline{\tau'}] = \tau_2''}{(\tau_1 \rightarrow \tau_2)[\overline{t'} \leftarrow \overline{\tau'}] = \tau_1'' \rightarrow \tau_2''} \quad \frac{\tau([\overline{t'} \leftarrow \overline{\tau'}] \upharpoonright_{\text{dom}([\overline{t'} \leftarrow \overline{\tau'}] \setminus \{t\})}) = \tau''}{(\forall t : \kappa. \tau)[\overline{t'} \leftarrow \overline{\tau'}] = \forall t : \kappa. \tau''} \\
\frac{\tau([\overline{t'} \leftarrow \overline{\tau'}] \upharpoonright_{\text{dom}([\overline{t'} \leftarrow \overline{\tau'}] \setminus \{t\})}) = \tau''}{(\lambda t : \kappa. \tau)[\overline{t'} \leftarrow \overline{\tau'}] = \lambda t : \kappa. \tau''} \quad \frac{\tau_1[\overline{t'} \leftarrow \overline{\tau'}] = \tau_1'' \quad \tau_2[\overline{t'} \leftarrow \overline{\tau'}] = \tau_2''}{(\tau_1 \tau_2)[\overline{t'} \leftarrow \overline{\tau'}] = \tau_1'' \tau_2''}
\end{array}$$

α -Equality:

$$\boxed{e_1 \equiv_{\alpha} e_2}$$

$$\begin{array}{c}
\frac{x_1 = x_2}{x_1 \equiv_{\alpha} x_2} \quad \frac{e_1 \equiv_{\alpha} e_2 \quad e_1' \equiv_{\alpha} e_2'}{e_1 e_1' \equiv_{\alpha} e_2 e_2'} \quad \frac{\tau_1 \equiv_{\alpha} \tau_2 \quad x' \notin \text{fv}(e_1) \cup \text{fv}(e_2) \quad e_1[x_1 \leftarrow x'] \equiv_{\alpha} e_2[x_2 \leftarrow x']}{\lambda x_1 : \tau_1. e_1 \equiv_{\alpha} \lambda x_2 : \tau_2. e_2} \\
\frac{e_1 \equiv_{\alpha} e_2 \quad \tau_1 \equiv_{\alpha} \tau_2}{e_1 \tau_1 \equiv_{\alpha} e_2 \tau_2} \quad \frac{t' \notin \text{tyfv}(e_1) \cup \text{tyfv}(e_2) \quad e_1[t_1 \leftarrow t'] \equiv_{\alpha} e_2[t_2 \leftarrow t']}{\Lambda t_1 : \kappa. e_1 \equiv_{\alpha} \Lambda t_2 : \kappa. e_2}
\end{array}$$

$$\boxed{\tau_1 \equiv_{\alpha} \tau_2}$$

$$\frac{t_1 = t_2}{t_1 \equiv_\alpha t_2} \quad \frac{\tau_1 \equiv_\alpha \tau_2 \quad \tau'_1 \equiv_\alpha \tau'_2}{\tau_1 \rightarrow \tau'_1 \equiv_\alpha \tau_2 \rightarrow \tau'_2} \quad \frac{t' \notin \text{tyfv}(\tau_1) \cup \text{tyfv}(\tau_2) \quad \tau_1[t_1 \leftarrow t'] \equiv_\alpha \tau_2[t_2 \leftarrow t']}{\forall t_1 : \kappa. \tau_1 \equiv_\alpha \forall t_2 : \kappa. \tau_2}$$

$$\frac{t' \notin \text{tyfv}(\tau_1) \cup \text{tyfv}(\tau_2) \quad \tau_1[t_1 \leftarrow t'] \equiv_\alpha \tau_2[t_2 \leftarrow t']}{\lambda t_1 : \kappa. \tau_1 \equiv_\alpha \lambda t_2 : \kappa. \tau_2} \quad \frac{\tau_1 \equiv_\alpha \tau_2 \quad \tau'_1 \equiv_\alpha \tau'_2}{\tau_1 \tau'_1 \equiv_\alpha \tau_2 \tau'_2}$$

定理 16 (Correctness of Substitution). 置換 $[\bar{x}' \leftarrow \bar{e}']$ について, $X = \text{dom}([\bar{x}' \leftarrow \bar{e}'])$ とした時,

$$fv(e[\bar{x}' \leftarrow \bar{e}']) = (fv(e) \setminus X) \cup \bigcup_{x \in fv(e) \cap X} fv([\bar{x}' \leftarrow \bar{e}'](x)).$$

□

定理 17 (Correctness of Type Substitution). 式 e , 型 τ , 型置換 $[\bar{t}' \leftarrow \bar{\tau}']$ について, $T = \text{dom}([\bar{t}' \leftarrow \bar{\tau}'])$ とした時,

$$\text{tyfv}(e[\bar{t}' \leftarrow \bar{\tau}']) = (\text{tyfv}(e) \setminus T) \cup \bigcup_{t \in \text{tyfv}(e) \cap T} \text{tyfv}([\bar{t}' \leftarrow \bar{\tau}'](t))$$

$$\text{tyfv}(\tau[\bar{t}' \leftarrow \bar{\tau}']) = (\text{tyfv}(\tau) \setminus T) \cup \bigcup_{t \in \text{tyfv}(\tau) \cap T} \text{tyfv}([\bar{t}' \leftarrow \bar{\tau}'](t)).$$

□

定理 18 (α -Equality Does Not Touch Free Variables).

- $\tau_1 \equiv_\alpha \tau_2$ ならば $\text{tyfv}(\tau_1) = \text{tyfv}(\tau_2)$.
- $e_1 \equiv_\alpha e_2$ ならば, $fv(e_1) = fv(e_2)$, $\text{tyfv}(e_1) = \text{tyfv}(e_2)$.

□

2.6.2 Typing Semantics

Kinding:

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\frac{\Gamma(t) = \kappa}{\Gamma \vdash t : \kappa} \text{ K-Var}$$

$$\frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma \vdash \tau_2 : \Omega}{\Gamma \vdash \tau_1 \rightarrow \tau_2 : \Omega} \text{ K-Arrow}$$

$$\frac{\Gamma, t : \kappa \vdash \tau : \Omega}{\Gamma \vdash \forall t : \kappa. \tau : \Omega} \text{ K-Forall}$$

$$\frac{\Gamma, t : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda t : \kappa_1. \tau : \kappa_1 \rightarrow \kappa_2} \text{ K-Abs}$$

$$\frac{\Gamma \vdash \tau_1 : \kappa_2 \rightarrow \kappa \quad \Gamma \vdash \tau_2 : \kappa_2}{\Gamma \vdash \tau_1 \tau_2 : \kappa} \text{ K-App}$$

Type equivalence:

$$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$$

$$\frac{\Gamma, t : \kappa_2 \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa_2}{\Gamma \vdash (\lambda t : \kappa_2. \tau_1) \tau_2 \equiv \tau_1[t \leftarrow \tau_2] : \kappa} \text{ T-Eq-}\beta\text{-Lam} \quad \frac{t \notin \text{tyfv}(\tau) \quad \Gamma \vdash \tau : \kappa_1 \rightarrow \kappa_2}{\Gamma \vdash (\lambda t : \kappa_1. \tau t) \equiv \tau : \kappa_1 \rightarrow \kappa_2} \text{ T-Eq-}\eta\text{-Lam}$$

$$\frac{\tau_1 \equiv_\alpha \tau_2 \quad \Gamma \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa} \text{ T-Eq-}\alpha\text{-Refl}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau_2 \equiv \tau_1 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa} \text{T-Eq-Sym} \quad \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa \quad \Gamma \vdash \tau_2 \equiv \tau_3 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_3 : \kappa} \text{T-Eq-Trans} \\
\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \Omega}{\Gamma \vdash \tau_1 \rightarrow \tau'_1 \equiv \tau_2 \rightarrow \tau'_2 : \Omega} \text{T-Eq-Cong-Arrow} \quad \frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash \forall t : \kappa. \tau_1 \equiv \forall t : \kappa. \tau_2 : \Omega} \text{Eq-Cong-Forall} \\
\frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \kappa'}{\Gamma \vdash \lambda t : \kappa. \tau_1 \equiv \lambda t : \kappa. \tau_2 : \kappa \rightarrow \kappa'} \text{T-Eq-Cong-Abs} \quad \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa' \rightarrow \kappa \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \kappa'}{\Gamma \vdash \tau_1 \tau'_1 \equiv \tau_2 \tau'_2 : \kappa} \text{Eq-Cong-App}
\end{array}$$

定理 19 (Respect Kinding). $\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa$ ならば, $\Gamma \vdash \tau_1 : \kappa$ かつ $\Gamma \vdash \tau_2 : \kappa$. □

Typing:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau : \Omega \quad \Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{T-Var} \\
\frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \text{T-Abs} \\
\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \text{T-App} \\
\frac{\Gamma, t : \kappa \vdash e : \tau}{\Gamma \vdash \Lambda t : \kappa. e : \forall t : \kappa. \tau} \text{T-UnivAbs} \\
\frac{\Gamma \vdash e : \forall t : \kappa. \tau_1 \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash e \tau_2 : \tau_1[t \leftarrow \tau_2]} \text{T-UnivApp} \\
\frac{\Gamma \vdash \tau \equiv \tau' : \Omega \quad \Gamma \vdash e : \tau'}{\Gamma \vdash e : \tau} \text{T-Equiv}
\end{array}$$

特に, $\cdot \vdash e : \tau$ の時, $e : \tau$ と表記.

定理 20 (Respect Type Kind). $\Gamma \vdash e : \tau$ ならば, $\Gamma \vdash \tau : \Omega$. □

2.6.3 Evaluation Semantics (Call-By-Value)

$$\begin{array}{l}
v ::= \lambda x : \tau. e \\
\quad | \Lambda t : \kappa. e \\
C ::= [] \\
\quad | C e \\
\quad | v C \\
\quad | C \tau
\end{array}$$

Small Step:

$$\boxed{e \Rightarrow e'}$$

$$\begin{array}{c}
\overline{(\lambda x : \tau. e) v \Rightarrow e[x \leftarrow v]} \\
\overline{(\Lambda t : \kappa. e) \tau \Rightarrow e[t \leftarrow \tau]} \\
\frac{e \Rightarrow e'}{C[e] \Rightarrow C[e']}
\end{array}$$

Big Step:

$$\boxed{e \Downarrow v}$$

$$\frac{e_1 \Downarrow \lambda x : \tau. e'_1 \quad e_2 \Downarrow v_2 \quad e'_1[x \leftarrow v_2] \Downarrow v}{e_1 e_2 \Downarrow v}$$

$$\frac{e \Downarrow \Lambda t : \kappa.e'_1 \quad e'_1[t \leftarrow \tau] \Downarrow v}{e \tau \Downarrow v}$$

定理 21 (Adequacy of Small Step and Big Step). $e \Rightarrow^* v$ iff $e \Downarrow v$. □

定理 22 (Type Soundness). $e : \tau$ の時, $e \Rightarrow^* v$, $e \Downarrow v$ となる $v = \text{nf}(\Rightarrow, e)$ が存在し,

- $\tau = \tau_1 \rightarrow \tau_2$ の時, $v \equiv_\alpha \lambda x' : \tau_1.e'$ となる $\lambda x' : \tau_1.e'$ が存在する.
- $\tau = \forall t : \kappa.\tau_1$ の時, $v \equiv_\alpha \Lambda t : \kappa.e'$ となる $\Lambda t : \kappa.e'$ が存在する.

□

2.6.4 Equational Reasoning

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c} \frac{\Gamma, x : \tau_2 \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\lambda x : \tau_2.e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau} \text{Eq-}\beta\text{-Lam} \quad \frac{x \notin fv(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash (\lambda x : \tau_1.e x) \equiv e : \tau_1 \rightarrow \tau_2} \text{Eq-}\eta\text{-Lam} \\ \frac{\Gamma, t : \kappa \vdash e : \tau}{\Gamma \vdash (\Lambda t : \kappa.e) \tau_2 \equiv e[t \leftarrow \tau_2] : \tau[t \leftarrow \tau_2]} \text{Eq-}\beta\text{-UnivLam} \quad \frac{t \notin tyfv(e) \quad \Gamma \vdash e : \forall t : \kappa.\tau}{\Gamma \vdash (\Lambda t : \kappa.e t) \equiv e : \forall t : \kappa.\tau} \text{Eq-}\eta\text{-UnivLam} \\ \frac{e_1 \equiv_\alpha e_2 \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Refl} \quad \frac{\tau \equiv_\alpha \tau' \quad \Gamma \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Type} \\ \frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-Sym} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau \quad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau} \text{Eq-Trans} \\ \frac{\Gamma, x : \tau \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash \lambda x : \tau.e_1 \equiv \lambda x : \tau.e_2 : \tau \rightarrow \tau'} \text{Eq-Cong-Abs} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \rightarrow \tau \quad \Gamma \vdash e'_1 \equiv e'_2 : \tau'}{\Gamma \vdash e_1 e'_1 \equiv e_2 e'_2 : \tau} \text{Eq-Cong-App} \\ \frac{\Gamma, t : \kappa \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \Lambda t : \kappa.e_1 \equiv \Lambda t : \kappa.e_2 : (\forall t : \kappa.\tau)} \text{Eq-Cong-UnivAbs} \\ \frac{\Gamma \vdash e_1 \equiv e_2 : \forall t : \kappa.\tau \quad \Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash e_1 \tau_1 \equiv e_2 \tau_2 : \tau[t \leftarrow \tau_1]} \text{Eq-Cong-UnivApp} \end{array}$$

特に, $\cdot \vdash e_1 \equiv e_2 : \tau$ の時, $e_1 \equiv e_2 : \tau$ と表記.

定理 23 (Respect Typing). $\Gamma \vdash e_1 \equiv e_2 : \tau$ ならば, $\Gamma \vdash e_1 : \tau$ かつ $\Gamma \vdash e_2 : \tau$. □

定理 24 (Respect Evaluation). $e_1 \equiv e_2 : \tau$ の時, $e'_1 \Rightarrow^* e_1$, $e_2 \Rightarrow^* e'_2$ ならば $e'_1 \equiv e'_2 : \tau$. □

系 25. $e_1 \equiv e_2 : \tau$ の時, $e_1 \Rightarrow^* e'_1$, $e_2 \Rightarrow^* e'_2$ ならば $e'_1 \equiv e'_2 : \tau$. □

証明. $e_1 \Rightarrow^* e_1$ より, 定理 14 から $e_1 \equiv e'_2 : \tau$. よって, T-Sym から $e'_2 \equiv e_1 : \tau$ であり, $e'_2 \Rightarrow^* e'_2$ より定理 14 から $e'_2 \equiv e'_1 : \tau$. 故に, T-Sym から $e'_1 \equiv e'_2 : \tau$. ■

2.6.5 Definability

Product

Product of τ_1 and τ_2 :

$$\begin{aligned} \tau_1 \times \tau_2 &\stackrel{\text{def}}{=} \forall t : \Omega. (\tau_1 \rightarrow \tau_2 \rightarrow t) \rightarrow t \\ \langle e_1, e_2 \rangle &\stackrel{\text{def}}{=} \Lambda t : \Omega. \lambda x : \tau_1 \rightarrow \tau_2 \rightarrow t. x e_1 e_2 \\ \pi_1 e &\stackrel{\text{def}}{=} e \tau_1 \lambda x_1. \lambda x_2. x_1 \\ \pi_2 e &\stackrel{\text{def}}{=} e \tau_2 \lambda x_1. \lambda x_2. x_2 \end{aligned}$$

Admissible kinding:

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma \vdash \tau_2 : \Omega}{\Gamma \vdash \tau_1 \times \tau_2 : \Omega} \text{ T-Product}$$

Admissible type equality:

$$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$$

$$\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \Omega}{\Gamma \vdash \tau_1 \times \tau'_1 \equiv \tau_2 \times \tau'_2 : \Omega} \text{ T-Eq-Product}$$

Admissible typing:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2} \text{ T-Product} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_1 e : \tau_1} \text{ T-Proj-1} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_2 e : \tau_2} \text{ T-Proj-2}$$

Admissible equality:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \pi_1 \langle e_1, e_2 \rangle \equiv e_1 : \tau_1} \text{ Eq-}\beta\text{-Product-1} \quad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \pi_2 \langle e_1, e_2 \rangle \equiv e_2 : \tau_2} \text{ Eq-}\beta\text{-Product-2}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \langle \pi_1 e, \pi_2 e \rangle \equiv e : \tau_1 \times \tau_2} \text{ Eq-}\eta\text{-Product}$$

Existential Type

Existence of $\exists t : \kappa. \tau$:

$$\begin{aligned} \exists t : \kappa. \tau &\stackrel{\text{def}}{=} \forall t' : \Omega. (\forall t : \kappa. \tau \rightarrow t') \rightarrow t' \\ \text{pack}\langle \tau_t, e \rangle_{\exists t : \kappa. \tau} &\stackrel{\text{def}}{=} \Lambda t' : \Omega. \lambda x : (\forall t : \kappa. \tau \rightarrow t'). x \tau_t e \\ \text{unpack}\langle t : \kappa, x : \tau \rangle &= e_1. \tau_2. e_2 \stackrel{\text{def}}{=} e_1 \tau_2 (\Lambda t : \kappa. \lambda x : \tau. e_2) \end{aligned}$$

Admissible kinding:

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\frac{\Gamma, t : \kappa \vdash \tau : \Omega}{\Gamma \vdash \exists t : \kappa. \tau : \Omega} \text{ T-Exist}$$

Admissible type equality:

$$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$$

$$\frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash \exists t : \kappa. \tau_1 \equiv \exists t : \kappa. \tau_2 : \Omega} \text{ T-Eq-Cong-Exist}$$

Admissible typing rule:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c}
\frac{\Gamma, t : \kappa \vdash \tau : \Omega \quad \Gamma \vdash \tau_t : \kappa \quad \Gamma \vdash e : \tau[t \leftarrow \tau_t]}{\Gamma \vdash \text{pack}\langle \tau_t, e \rangle_{\exists t : \kappa. \tau} : \exists t : \kappa. \tau} \text{ T-Pack} \\
\frac{\Gamma \vdash e_1 : \exists t : \kappa. \tau \quad \Gamma, t : \kappa, x : \tau \vdash e_2 : \tau_2 \quad t \notin \text{tyfv}(\tau_2)}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau \rangle = e_1. \tau_2. e_2 : \tau_2} \text{ T-Unpack}
\end{array}$$

Admissible equality:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau_t : \kappa \quad \Gamma \vdash e_1 : \tau_1[t \leftarrow \tau_t] \quad \Gamma, t : \kappa, x : \tau_1 \vdash e_2 : \tau_2 \quad t \notin \text{tyfv}(\tau_2)}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau_1 \rangle = \text{pack}\langle \tau_t, e_1 \rangle_{\exists t : \kappa. \tau_1}. \tau_2. e_2 \equiv e_2[t \leftarrow \tau_t][x \leftarrow e_1] : \tau_2} \text{ Eq-}\beta\text{-Exist} \\
\frac{\Gamma \vdash e : (\exists t : \kappa. \tau) \quad \tau' \equiv \exists t : \kappa. \tau}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau \rangle = e. \tau'. \text{pack}\langle t, x \rangle_{\exists t : \kappa. \tau} \equiv e : (\exists t : \kappa. \tau)} \text{ Eq-}\eta\text{-Exist}
\end{array}$$

2.7 WIP: λ μ -Calculus

2.8 WIP: Lambda Bar Mu Mu Tilde Calculus

$\tilde{\lambda} \mu$ -Calculus

2.9 WIP: π -Calculus

第 3 章

Modules and Phase Distinction

3.1 F-ing modules

[RRD14]

3.1.1 Syntax

$X ::= \dots$	(identifier)
$K ::= \dots$	(kind)
$T ::= \dots \mid P$	(type)
$E ::= \dots \mid P$	(expression)
$P ::= M$	(path)
$M ::= X$	(identifier)
$\mid \{B\}$	(bindings)
$\mid M.X$	(projection)
$\mid \text{fun } X : S \Rightarrow M$	(functor)
$\mid X X$	(functor application)
$\mid X :> S$	(sealing)
$B ::= \text{val } X = E$	(value binding)
$\mid \text{type } X = T$	(type binding)
$\mid \text{module } X = M$	(module binding)
$\mid \text{signature } X = S$	(signature binding)
$\mid \text{include } M$	(module including)
$\mid \epsilon$	(empty binding)
$\mid B; B$	(binding concatenation)
$S ::= P$	(signature path)
$\mid \{D\}$	(declarations)
$\mid (X : S) \rightarrow S$	((generative) functor signature)
$\mid S \text{ where type } \overline{X} = T$	(bounded signature)
$D ::= \text{val } X : T$	(value declaration)
$\mid \text{type } X = T$	(type binding)
$\mid \text{type } X : K$	(type declaration)
$\mid \text{module } X : S$	(module declaration)
$\mid \text{signature } X = S$	(signature binding)
$\mid \text{include } S$	(signature including)
$\mid \epsilon$	(empty declaration)
$\mid D; D$	(declaration concatenation)

3.1.2 Internal Language

Having same power as System F ω

Syntax:

$\kappa ::= \Omega \mid \kappa \rightarrow \kappa$
$\tau ::= t \mid \tau \rightarrow \tau \mid \overline{\{l : \tau\}} \mid \forall t : \kappa. \tau \mid \exists t : \kappa. \tau \mid \lambda t : \kappa. \tau \mid \tau \tau$
$e ::= x \mid \lambda x : \tau. e \mid e e \mid \overline{\{l = e\}} \mid e.l \mid \Lambda t : \kappa. e \mid e \tau \mid \text{pack}\langle \tau, e \rangle_\tau \mid \text{unpack}\langle t : \kappa, x : \tau \rangle = e \text{ in } e$

$$\Gamma ::= \cdot \mid \Gamma, t : \kappa \mid \Gamma, x : \tau$$

Abbreviation:

$$\begin{aligned} \Sigma.\bar{l} &\stackrel{\text{def}}{=} \begin{cases} (\Sigma.l).\bar{l}' & (\bar{l} = l \bar{l}') \\ \Sigma & (\bar{l} = \epsilon) \end{cases} \\ \bar{\tau}_1 \rightarrow \bar{\tau}_2 &\stackrel{\text{def}}{=} \begin{cases} \tau_1 \rightarrow (\tau'_1 \rightarrow \tau_2) & (\bar{\tau}_1 = \tau_1 \bar{\tau}'_1) \\ \tau_2 & (\bar{\tau}_1 = \epsilon) \end{cases} \\ \lambda \bar{x} : \bar{\tau}. e &\stackrel{\text{def}}{=} \begin{cases} \lambda x : \tau. \lambda x' : \tau'. e & (\bar{x} : \bar{\tau} = x : \tau \bar{x}' : \tau') \\ e & (\bar{x} : \bar{\tau} = \epsilon) \end{cases} \\ e_0 \bar{e}_1 &\stackrel{\text{def}}{=} \begin{cases} e_0 e_1 \bar{e}'_1 & (\bar{e}_1 = e_1 \bar{e}'_1) \\ e_0 & (\bar{e}_1 = \epsilon) \end{cases} \\ \forall \bar{t} : \bar{\kappa}. \tau &\stackrel{\text{def}}{=} \begin{cases} \forall t : \kappa. \forall t' : \kappa'. \tau & (\bar{t} : \bar{\kappa} = t : \kappa \bar{t}' : \kappa') \\ \tau & (\bar{t} : \bar{\kappa} = \epsilon) \end{cases} \\ \Lambda \bar{t} : \bar{\kappa}. e &\stackrel{\text{def}}{=} \begin{cases} \Lambda t : \kappa. \Lambda t' : \kappa'. e & (\bar{t} : \bar{\kappa} = t : \kappa \bar{t}' : \kappa') \\ e & (\bar{t} : \bar{\kappa} = \epsilon) \end{cases} \\ e \bar{\tau} &\stackrel{\text{def}}{=} \begin{cases} e \tau \bar{\tau}' & (\bar{\tau} = \tau \bar{\tau}') \\ e & (\bar{\tau} = \epsilon) \end{cases} \\ \text{let } \bar{x} : \bar{\tau} = e_1 \bar{t} : \bar{\kappa} = \bar{\tau} \text{ in } e_2 &\stackrel{\text{def}}{=} (\lambda \bar{x} : \bar{\tau}. \Lambda \bar{t} : \bar{\kappa}. e_2) \bar{e}_1 \bar{\tau} \\ \exists \bar{t} : \bar{\kappa}. \tau &\stackrel{\text{def}}{=} \begin{cases} \exists t : \kappa. \exists t' : \kappa'. \tau & (\bar{t} : \bar{\kappa} = t : \kappa \bar{t}' : \kappa') \\ \tau & (\bar{t} : \bar{\kappa} = \epsilon) \end{cases} \\ \text{pack}(\bar{\tau}, e)_{\exists \bar{t} : \bar{\kappa}. \tau_0} &\stackrel{\text{def}}{=} \begin{cases} \text{pack}(\tau, \text{pack}(\bar{\tau}', e)_{\exists \bar{t}' : \bar{\kappa}'. \tau_0})_{\exists \bar{t} : \bar{\kappa}. \tau_0} & (\bar{\tau} = \tau \bar{\tau}', \bar{t} : \bar{\kappa} = t : \kappa \bar{t}' : \kappa') \\ e & (\bar{\tau} = \epsilon, \bar{t} : \bar{\kappa} = \epsilon) \end{cases} \\ (\text{unpack}(\bar{t} : \bar{\kappa}, x : \tau) = e_1 \text{ in } e_2) &\stackrel{\text{def}}{=} \begin{cases} \text{unpack}(t : \kappa, x_1 : \exists \bar{t}' : \bar{\kappa}'. \tau) = e_1 \text{ in} & (\bar{t} : \bar{\kappa} = t : \kappa \bar{t}' : \kappa') \\ \text{unpack}(\bar{t}' : \bar{\kappa}', x_2 : \tau) = x_1 \text{ in } e_2 & \\ \text{let } x : \tau = e_1 \text{ in } e_2 & (\bar{t} : \bar{\kappa} = \epsilon) \end{cases} \end{aligned}$$

Kinding:

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\begin{array}{c} \frac{\Gamma(t) = \kappa}{\Gamma \vdash t : \kappa} \quad \frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma \vdash \tau_2 : \Omega}{\Gamma \vdash \tau_1 \rightarrow \tau_2 : \Omega} \quad \frac{\bigwedge_l \Gamma \vdash \tau_l : \Omega}{\Gamma \vdash \{\bar{l} : \bar{\tau}_l\} : \Omega} \\ \frac{\Gamma, t : \kappa \vdash \tau : \Omega}{\Gamma \vdash \forall t : \kappa. \tau : \Omega} \quad \frac{\Gamma, t : \kappa \vdash \tau : \Omega}{\Gamma \vdash \exists t : \kappa. \tau : \Omega} \quad \frac{\Gamma, t : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda t : \kappa_1. \tau : \kappa_1 \rightarrow \kappa_2} \quad \frac{\Gamma \vdash \tau_1 : \kappa_2 \rightarrow \kappa \quad \Gamma \vdash \tau_2 : \kappa_2}{\Gamma \vdash \tau_1 \tau_2 : \kappa} \end{array}$$

Type equivalence:

$$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$$

$$\begin{array}{c} \frac{\Gamma, t : \kappa_2 \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa_2}{\Gamma \vdash (\lambda t : \kappa_2. \tau_1) \tau_2 \equiv \tau_1[t \leftarrow \tau_2] : \kappa} \quad \frac{t \notin \text{tyfu}(\tau) \quad \Gamma \vdash \tau : \kappa_1 \rightarrow \kappa_2}{\Gamma \vdash (\lambda t : \kappa_1. \tau t) \equiv \tau : \kappa_1 \rightarrow \kappa_2} \\ \frac{\tau_1 \equiv_\alpha \tau_2 \quad \Gamma \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa} \quad \frac{\Gamma \vdash \tau_2 \equiv \tau_1 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa} \quad \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa \quad \Gamma \vdash \tau_2 \equiv \tau_3 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_3 : \kappa} \\ \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \Omega}{\Gamma \vdash \tau_1 \rightarrow \tau'_1 \equiv \tau_2 \rightarrow \tau'_2 : \Omega} \quad \frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash \forall t : \kappa. \tau_1 \equiv \tau_2 : \Omega} \\ \frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \kappa'}{\Gamma \vdash \lambda t : \kappa. \tau_1 \equiv \lambda t : \kappa. \tau_2 : \kappa \rightarrow \kappa'} \quad \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa' \rightarrow \kappa \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \kappa'}{\Gamma \vdash \tau_1 \tau'_1 \equiv \tau_2 \tau'_2 : \kappa} \end{array}$$

Typing:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau : \Omega \quad \Gamma(x) = \tau}{\Gamma \vdash x : \tau} \quad \frac{\Gamma \vdash \tau \equiv \tau' : \Omega \quad \Gamma \vdash e : \tau'}{\Gamma \vdash e : \tau} \\
\frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \quad \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \\
\frac{\bigwedge_l \Gamma \vdash e_l : \tau_l}{\Gamma \vdash \{\overline{l = e_l}\} : \{\overline{l = \tau_l}\}} \quad \frac{\Gamma \vdash e : \{\overline{l' = \tau_{l'}}\}}{\Gamma \vdash e.l : \tau_l} \\
\frac{\Gamma, t : \kappa \vdash e : \tau}{\Gamma \vdash \Lambda t : \kappa. e : (\forall t : \kappa. \tau)} \quad \frac{\Gamma \vdash e : (\forall t : \kappa. \tau_1) \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash e \tau_2 : \tau_1[t \leftarrow \tau_2]} \\
\frac{\Gamma, t : \kappa \vdash \tau : \Omega \quad \Gamma \vdash \tau_t : \kappa \quad \Gamma \vdash e : \tau[t \leftarrow \tau_t]}{\Gamma \vdash \text{pack}\langle \tau_t, e \rangle_{\exists t : \kappa. \tau} : (\exists t : \kappa. \tau)} \quad \frac{\Gamma \vdash e_1 : (\exists t : \kappa. \tau_1) \quad \Gamma, t : \kappa, x : \tau_1 \vdash e_2 : \tau}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau_1 \rangle = e_1 \text{ in } e_2 : \tau}
\end{array}$$

Reduction:

$$v ::= \lambda x : \tau. e \mid \overline{\{l = e\}} \mid \Lambda t : \kappa. e \mid \text{pack}\langle \tau_t, e \rangle_{\exists t : \kappa. \tau}$$

$$C ::= [] \mid C e \mid v C \mid \overline{\{l = v, l = C, \overline{l = e}\}} \mid C.l \mid C \tau \mid \text{pack}\langle \tau, C \rangle_\tau \mid \text{unpack}\langle t : \kappa, x : \tau \rangle = C \text{ in } e$$

$$\boxed{e \Rightarrow e'}$$

$$\begin{array}{c}
\overline{(\lambda x : \tau. e)v \Rightarrow e[x \leftarrow v]} \quad \overline{\{\overline{l' = v_{l'}}\}.l \Rightarrow v_l} \quad \overline{(\Lambda t : \kappa. e)\tau \Rightarrow e[t \leftarrow \tau]} \\
\overline{\text{unpack}\langle t : \kappa, x : \tau \rangle = \text{pack}\langle \tau_t, v \rangle_{\tau_\exists} \text{ in } e \Rightarrow e[t \leftarrow \tau_t][x \leftarrow v]} \quad \frac{e \Rightarrow e'}{C[e] \Rightarrow C[e']}
\end{array}$$

Equivalence:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c}
\frac{\Gamma, x : \tau_2 \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\lambda x : \tau_2. e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau} \quad \frac{x \notin \text{fv}(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash (\lambda x : \tau_1. e x) \equiv e : \tau_1 \rightarrow \tau_2} \\
\frac{\bigwedge_{l'} \Gamma \vdash e_{l'} : \tau_{l'}}{\Gamma \vdash \{\overline{l' = e_{l'}}\}.l \equiv e_l : \tau_l} \quad \frac{\Gamma \vdash e : \{\overline{l : \tau_l}\}}{\Gamma \vdash \{\overline{l = e.l}\} \equiv e : \{\overline{l : \tau_l}\}} \\
\frac{\Gamma, t : \kappa \vdash e : \tau}{\Gamma \vdash (\Lambda t : \kappa. e) \tau_2 \equiv e[t \leftarrow \tau_2] : \tau[t \leftarrow \tau_2]} \quad \frac{t \notin \text{tyfv}(e) \quad \Gamma \vdash e : \forall t : \kappa. \tau}{\Gamma \vdash (\Lambda t : \kappa. e t) \equiv e : \forall t : \kappa. \tau} \\
\frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau'_1 : \Omega \quad \Gamma \vdash \tau_t : \kappa \quad \Gamma \vdash e_1 : \tau_1[t \leftarrow \tau_t] \quad \Gamma, t : \kappa, x : \tau_1 \vdash e_2 : \tau}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau'_1 \rangle = \text{pack}\langle \tau_t, e_1 \rangle_{\exists t : \kappa. \tau_1} \text{ in } e_2 \equiv e_2[t \leftarrow \tau_t][x \leftarrow e_1] : \tau} \\
\frac{\Gamma \vdash e : \exists t : \kappa. \tau \quad \Gamma, t : \kappa \vdash \tau \equiv \tau' : \Omega}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau' \rangle = e \text{ in } \text{pack}\langle t, x \rangle_{\exists t : \kappa. \tau} \equiv e : (\exists t : \kappa. \tau)} \\
\frac{e_1 \equiv_\alpha e_2 \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \quad \frac{\Gamma \vdash \tau \equiv \tau' : \Omega \quad \Gamma \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash e_1 \equiv e_2 : \tau} \\
\frac{\Gamma \vdash e_2 \equiv e_1 : \tau \quad \Gamma \vdash e_1 \equiv e_2 : \tau \quad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau} \\
\frac{\Gamma, x : \tau \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash \lambda x : \tau. e_1 \equiv \lambda x : \tau. e_2 : \tau \rightarrow \tau'} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \rightarrow \tau \quad \Gamma \vdash e'_1 \equiv e'_2 : \tau'}{\Gamma \vdash e_1 e'_1 \equiv e_2 e'_2 : \tau} \\
\frac{\bigwedge_l \Gamma \vdash e_{l,1} \equiv e_{l,2} : \tau_l}{\Gamma \vdash \{\overline{l = e_{l,1}}\} \equiv \{\overline{l = e_{l,2}}\} : \{\overline{l : \tau_l}\}} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \{\overline{l : \tau_l, l' : \tau'_l}\}}{\Gamma \vdash e_1.l \equiv e_2.l : \tau_l} \\
\frac{\Gamma, t : \kappa \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \Lambda t : \kappa. e_1 \equiv \Lambda t : \kappa. e_2 : (\forall t : \kappa. \tau)} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \forall t : \kappa. \tau \quad \Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash e_1 \tau_1 \equiv e_2 \tau_2 : \tau[t \leftarrow \tau_1]}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau'_1 \equiv \tau'_2 : \kappa \quad \Gamma \vdash e_1 \equiv e_2 : \tau_1[t \leftarrow \tau'_1] \quad \Gamma, t : \kappa. \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash \text{pack}(\tau'_1, e_1)_{\exists t : \kappa. \tau_1} \equiv \text{pack}(\tau'_2, e_2)_{\exists t : \kappa. \tau_2} : (\exists t : \kappa. \tau_1)} \\
\frac{\Gamma, t : \kappa \vdash \tau'_1 \equiv \tau'_2 : \Omega \quad \Gamma \vdash e'_1 \equiv e'_2 : (\exists t : \kappa. \tau'_1) \quad \Gamma, t : \kappa, x : \tau'_1 \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \text{unpack}(t : \kappa, x : \tau'_1) = e'_1 \text{ in } e_1 \equiv \text{unpack}(t : \kappa, x : \tau'_2) = e'_2 \text{ in } e_2 : \tau}
\end{array}$$

3.1.3 Signature

$$\begin{array}{ll}
\Xi ::= \exists t : \kappa. \Sigma & \text{(abstract signature)} \\
\Sigma ::= [\tau] & \text{(atomic value declaration)} \\
\quad | [= \tau : \kappa] & \text{(atomic type declaration)} \\
\quad | [= \Xi] & \text{(atomic signature declaration)} \\
\quad | \{\overline{l_X : \Sigma}\} & \text{(structure signature)} \\
\quad | \forall t : \kappa. \Sigma \rightarrow \Xi & \text{(functor signature)}
\end{array}$$

Atomic Signature:

$$\begin{aligned}
[\tau] &\stackrel{\text{def}}{=} \{\text{val} : \tau\} \\
[e] &\stackrel{\text{def}}{=} \{\text{val} = e\} \\
[= \tau : \kappa] &\stackrel{\text{def}}{=} \{\text{type} : \forall t : (\kappa \rightarrow \Omega). t \tau \rightarrow t \tau\} \\
[\tau : \kappa] &\stackrel{\text{def}}{=} \{\text{type} = \Lambda t : (\kappa \rightarrow \Omega). \lambda x : (t \tau). x\} \\
[= \Xi] &\stackrel{\text{def}}{=} \{\text{sig} : \Xi \rightarrow \Xi\} \\
[\Xi] &\stackrel{\text{def}}{=} \{\text{sig} = \lambda x : \Xi. x\}
\end{aligned}$$

NotAtomic(Σ)

$$\frac{}{\text{NotAtomic}(\{\overline{l_X : \Sigma}\})} \quad \frac{}{\text{NotAtomic}(\forall t : \kappa. \Sigma \rightarrow \Xi)}$$

Admissible kinding:

$\Gamma \vdash \tau : \kappa$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau : \Omega}{\Gamma \vdash [\tau] : \Omega} \text{ K-A-Val} \\
\frac{\Gamma \vdash \tau : \kappa}{\Gamma \vdash [= \tau : \kappa] : \Omega} \text{ K-A-Typ} \\
\frac{\Gamma \vdash \Xi : \Omega}{\Gamma \vdash [= \Xi] : \Omega} \text{ K-A-Sig}
\end{array}$$

Admissible type equivalence:

$\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash [\tau_1] \equiv [\tau_2] : \Omega} \text{ T-Eq-Cong-A-Val} \\
\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash [= \tau_1 : \kappa] \equiv [= \tau_2 : \kappa] : \Omega} \text{ T-Eq-Cong-A-Typ} \\
\frac{\Gamma \vdash \Xi_1 \equiv \Xi_2 : \Omega}{\Gamma \vdash [= \Xi_1] \equiv [= \Xi_2] : \Omega} \text{ T-Eq-Cong-A-Sig}
\end{array}$$

Admissible typing:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c} \frac{\Gamma \vdash e : \tau}{\Gamma \vdash [e] : [\tau]} \text{ T-A-Val} \\ \frac{\Gamma \vdash \tau : \kappa}{\Gamma \vdash [\tau : \kappa] : [= \tau : \kappa]} \text{ T-A-Typ} \\ \frac{\Gamma \vdash \Xi : \Omega}{\Gamma \vdash [\Xi] : [= \Xi]} \text{ T-A-Sig} \end{array}$$

Admissible equivalence:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c} \frac{\Gamma \vdash e : \tau}{\Gamma \vdash [e].\text{val} \equiv e : \tau} \text{ Eq-}\beta\text{-A-Val} \quad \frac{\Gamma \vdash e : [\tau]}{\Gamma \vdash [e.\text{val}] \equiv e : [\tau]} \text{ Eq-}\eta\text{-A-Val} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash [e_1] \equiv [e_2] : [\tau]} \text{ Eq-Cong-A-Val} \\ \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash [\tau_1 : \kappa] \equiv [\tau_2 : \kappa] : [= \tau_1 : \kappa]} \text{ Eq-Cong-A-Typ} \\ \frac{\Gamma \vdash \Xi_1 \equiv \Xi_2 : \Omega}{\Gamma \vdash [\Xi_1] \equiv [\Xi_2] : [= \Xi_1]} \text{ Eq-Cong-A-Sig} \end{array}$$

3.1.4 (Generative) Elaboration

Signature:

$$\boxed{\Gamma \vdash S \rightsquigarrow \Xi}$$

$$\begin{array}{c} \frac{\Gamma \vdash P : [= \Xi] \rightsquigarrow e}{\Gamma \vdash P \rightsquigarrow \Xi} \text{ S-Path} \\ \frac{\Gamma \vdash D \rightsquigarrow \Xi}{\Gamma \vdash \{D\} \rightsquigarrow \Xi} \text{ S-Struct} \\ \frac{\Gamma \vdash S_1 \rightsquigarrow \overline{\exists t : \kappa. \Sigma} \quad \Gamma, \overline{t : \kappa}, x_X : \Sigma \vdash S_2 \rightsquigarrow \Xi}{\Gamma \vdash (X : S_1) \rightarrow S_2 \rightsquigarrow \forall \overline{t : \kappa. \Sigma} \rightarrow \Xi} \text{ S-Funct} \\ \frac{\Gamma \vdash S \rightsquigarrow \overline{\exists t_1 : \kappa_1} \quad \overline{t : \kappa_2. \Sigma} \quad \overline{\Sigma.l_X = [t : \kappa]} \quad \Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash S \text{ where type } \overline{X} = T \rightsquigarrow \overline{\exists t_1 : \kappa_1} \quad \overline{t_2 : \kappa_2. \Sigma} [t \leftarrow \tau]} \text{ S-Where-Typ} \end{array}$$

Declarations:

$$\boxed{\Gamma \vdash D \rightsquigarrow \Xi}$$

$$\begin{array}{c} \frac{\Gamma \vdash T : \Omega \rightsquigarrow \tau}{\Gamma \vdash \text{val } X : T \rightsquigarrow \{l_X : [\tau]\}} \text{ D-Val} \\ \frac{\Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash \text{type } X = T \rightsquigarrow \{l_X : [= \tau : \kappa]\}} \text{ D-Typ-Eq} \\ \frac{\Gamma \vdash K \rightsquigarrow \kappa}{\Gamma \vdash \text{type } X : K \rightsquigarrow \exists t : \kappa. \{l_X : [= t : \kappa]\}} \text{ D-Typ} \\ \frac{\Gamma \vdash S \rightsquigarrow \overline{\exists t : \kappa. \Sigma}}{\Gamma \vdash \text{module } X : S \rightsquigarrow \overline{\exists t : \kappa. \{l_X : \Sigma\}}} \text{ D-Mod} \\ \frac{\Gamma \vdash S \rightsquigarrow \Xi}{\Gamma \vdash \text{signature } X = S \rightsquigarrow \{l_X : [= \Xi]\}} \text{ D-Sig-Eq} \\ \frac{\Gamma \vdash S \rightsquigarrow \overline{\exists t : \kappa. \{l_X : \Sigma\}}}{\Gamma \vdash \text{include } S \rightsquigarrow \overline{\exists t : \kappa. \{l_X : \Sigma\}}} \text{ D-Incl} \end{array}$$

$$\begin{array}{c}
\overline{\Gamma \vdash \epsilon \rightsquigarrow \{\}} \quad \text{D-Emt} \\
\frac{\{\overline{l_{X_1}}\} \cap \{\overline{l_{X_2}}\} = \emptyset \quad \Gamma \vdash D_1 \rightsquigarrow \exists \overline{t_1 : \kappa_1}. \{\overline{l_{X_1} : \Sigma_1}\} \quad \Gamma, \overline{t_1 : \kappa_1}, \overline{x_{X_1} : \Sigma_1} \vdash D_2 \rightsquigarrow \exists \overline{t_2 : \kappa_2}. \{\overline{l_{X_2} : \Sigma_2}\}}{\Gamma \vdash D_1; D_2 \rightsquigarrow \exists \overline{t_1 : \kappa_1} \overline{t_2 : \kappa_2}. \{\overline{l_{X_1} : \Sigma_1} \overline{l_{X_2} : \Sigma_2}\}} \quad \text{D-Seq}
\end{array}$$

Matching:

$$\boxed{\Gamma \vdash \Sigma_1 \leq \exists \overline{t : \kappa}. \Sigma_2 \uparrow \overline{\tau} \rightsquigarrow e}$$

$$\frac{\Gamma \vdash \Sigma_1 \leq \Sigma_2[\overline{t \leftarrow \tau_t}] \rightsquigarrow e \quad \bigwedge_t \Gamma \vdash \tau_t : \kappa_t}{\Gamma \vdash \Sigma_1 \leq \exists \overline{t : \kappa_t}. \Sigma_2 \uparrow \overline{\tau_t} \rightsquigarrow e} \quad \text{U-Match}$$

Subtyping:

$$\boxed{\Gamma \vdash \Xi_1 \leq \Xi_2 \rightsquigarrow e}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau_1 \leq \tau_2 \rightsquigarrow e}{\Gamma \vdash [\tau_1] \leq [\tau_2] \rightsquigarrow \lambda x : [\tau_1]. [e (x. \text{val})]} \quad \text{U-Val} \\
\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash [= \tau_1 : \kappa] \leq [= \tau_2 : \kappa] \rightsquigarrow \lambda x : [= \tau_1 : \kappa]. x} \quad \text{U-Typ} \\
\frac{\Gamma \vdash \Xi_1 \leq \Xi_2 \rightsquigarrow e_1 \quad \Gamma \vdash \Xi_2 \leq \Xi_1 \rightsquigarrow e_2}{\Gamma \vdash [= \Xi_1] \leq [= \Xi_2] \rightsquigarrow \lambda x : [= \Xi_1]. [\Xi_2]} \quad \text{U-Sig} \\
\frac{\bigwedge_l \Gamma \vdash \Sigma_{l_1} \leq \Sigma_{l_2} \rightsquigarrow e_l}{\Gamma \vdash \{\overline{l : \Sigma_{l_1}}, \overline{l' : \Sigma_{l_2}}\} \leq \{\overline{l : \Sigma_{l_2}}, \overline{l' : \Sigma_{l_1}}\} \rightsquigarrow \lambda x : \{\overline{l : \Sigma_{l_1}}, \overline{l' : \Sigma_{l_2}}\}. \{\overline{l = e_l (x.l)}\}} \quad \text{U-Struct} \\
\frac{\Gamma, \overline{t_2 : \kappa_2} \vdash \Sigma_2 \leq \exists \overline{t_1 : \kappa_1}. \Sigma_1 \uparrow \overline{\tau} \rightsquigarrow e_1 \quad \Gamma, \overline{t_2 : \kappa_2} \vdash \Xi_1[\overline{t_1 \leftarrow \tau}] \leq \Xi_2 \rightsquigarrow e_2}{\Gamma \vdash \forall \overline{t_1 : \kappa_1}. \Sigma_1 \rightarrow \Xi_1 \leq \forall \overline{t_2 : \kappa_2}. \Sigma_2 \rightarrow \Xi_2 \rightsquigarrow \lambda x_1 : (\forall \overline{t_1 : \kappa_1}. \Sigma_1 \rightarrow \Xi_1). \lambda x_2 : \Sigma_2. e_2 (x_1 \overline{\tau} (e_1 x_2))} \quad \text{U-Funct} \\
\frac{\Gamma, \overline{t_1 : \kappa_1} \vdash \Sigma_1 \leq \exists \overline{t_2 : \kappa_2}. \Sigma_2 \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash \exists \overline{t_1 : \kappa_1}. \Sigma_1 \leq \exists \overline{t_2 : \kappa_2}. \Sigma_2 \rightsquigarrow \lambda x_1 : (\exists \overline{t_1 : \kappa_1}. \Sigma_1). \text{unpack} \langle \overline{t_1 : \kappa_1}, x_1' : \Sigma_1 \rangle = x_1 \text{ in pack} \langle \overline{\tau}, e x_1' \rangle_{\exists \overline{t_2 : \kappa_2}. \Sigma_2}} \quad \text{U-Abs}
\end{array}$$

Module:

$$\boxed{\Gamma \vdash M : \Xi \rightsquigarrow e}$$

$$\begin{array}{c}
\frac{\Gamma(x_X) = \Sigma}{\Gamma \vdash X : \Sigma \rightsquigarrow x_X} \quad \text{M-Var} \\
\frac{\Gamma \vdash B : \Xi \rightsquigarrow e}{\Gamma \vdash \{B\} : \Xi \rightsquigarrow e} \quad \text{M-Struct} \\
\frac{\Gamma \vdash M : \exists \overline{t : \kappa}. \{\overline{l_X : \Sigma}, \overline{l_{X'} : \Sigma'}\} \rightsquigarrow e}{\Gamma \vdash M.X : \exists \overline{t : \kappa}. \Sigma \rightsquigarrow \text{unpack} \langle \overline{t : \kappa}, x : \{\overline{l_X : \Sigma}, \overline{l_{X'} : \Sigma'}\} \rangle = e \text{ in pack} \langle \overline{t}, x.l_X \rangle_{\exists \overline{t : \kappa}. \Sigma}} \quad \text{M-Dot} \\
\frac{\Sigma \vdash S \rightsquigarrow \exists \overline{t : \kappa}. \Sigma \quad \Gamma, \overline{t : \kappa}, x_X : \Sigma \vdash M : \Xi \rightsquigarrow e}{\Gamma \vdash \text{fun } X : S \Rightarrow M : \forall \overline{t : \kappa}. \Sigma \rightarrow \Xi \rightsquigarrow \Lambda \overline{t : \kappa}. \lambda x_X : \Sigma. e} \quad \text{M-Funct} \\
\frac{\Gamma(x_{X_1}) = \forall \overline{t : \kappa}. \Sigma' \rightarrow \Xi \quad \Gamma(x_{X_2}) = \Sigma \quad \Gamma \vdash \Sigma \leq \exists \overline{t : \kappa}. \Sigma' \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash X_1 X_2 : \Xi[\overline{t \leftarrow \tau}] \rightsquigarrow x_{X_1} \overline{\tau} (e x_{X_2})} \quad \text{M-App} \\
\frac{\Gamma(x_X) = \Sigma \quad \Gamma \vdash S \rightsquigarrow \exists \overline{t : \kappa}. \Sigma' \quad \Gamma \vdash \Sigma \leq \exists \overline{t : \kappa}. \Sigma' \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash X : > S : \exists \overline{t : \kappa}. \Sigma' \rightsquigarrow \text{pack} \langle \overline{\tau}, e x_X \rangle_{\exists \overline{t : \kappa}. \Sigma'}} \quad \text{M-Seal}
\end{array}$$

Bindings:

$$\boxed{\Gamma \vdash B : \Xi \rightsquigarrow e}$$

$$\frac{\Gamma \vdash E : \tau \rightsquigarrow e}{\Gamma \vdash \text{val } X = E : \{\overline{l_X : [\tau]}\} \rightsquigarrow \{\overline{l_X = e}\}} \quad \text{B-Val}$$

$$\begin{array}{c}
\frac{\Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash \text{type } X = T : \{\overline{l_X} : [\tau : \kappa]\} \rightsquigarrow \{\overline{l_X} : [\tau : \kappa]\}} \text{B-Typ} \\
\frac{\Gamma \vdash M : \exists \overline{t} : \kappa. \overline{\Sigma} \rightsquigarrow e \quad \text{NotAtomic}(\Sigma)}{\Gamma \vdash \text{module } X = M : \exists \overline{t} : \kappa. \{\overline{l_X} : \Sigma\} \rightsquigarrow \text{unpack}\langle \overline{t} : \kappa, x : \Sigma \rangle = e \text{ in pack}\langle \overline{t}, \{\overline{l_X} = x \} \rangle_{\exists \overline{t} : \kappa. \{\overline{l_X} : \Sigma\}}} \text{B-Mod} \\
\frac{\Gamma \vdash S \rightsquigarrow \Xi}{\Gamma \vdash \text{signature } X = S : \{\overline{l_X} : [\Xi]\} \rightsquigarrow \{\overline{l_X} : [\Xi]\}} \text{B-Sig} \\
\frac{\Gamma \vdash M : \exists \overline{t} : \kappa. \{\overline{l_X} : \Sigma\} \rightsquigarrow e}{\Gamma \vdash \text{include } M : \exists \overline{t} : \kappa. \{\overline{l_X} : \Sigma\} \rightsquigarrow e} \text{B-Incl} \\
\frac{}{\Gamma \vdash \epsilon : \{\} \rightsquigarrow \{\}} \text{B-Emt} \\
\frac{\overline{l'_{X_1}} = \overline{l_{X_1}} \setminus \overline{l_{X_2}} \quad \overline{l'_{X_1}} : \Sigma'_1 \subseteq \overline{l_{X_1}} : \Sigma_1 \quad \Gamma \vdash B_1 : \exists \overline{t_1} : \kappa_1. \{\overline{l_{X_1}} : \Sigma_1\} \rightsquigarrow e_1 \quad \Sigma = \{\overline{l'_{X_1}} : \Sigma'_1, \overline{l_{X_2}} : \Sigma_2\} \quad \Gamma, \overline{t_1} : \kappa_1, \overline{x_{X_1}} : \Sigma_1 \vdash B_2 : \exists \overline{t_2} : \kappa_2. \{\overline{l_{X_2}} : \Sigma_2\} \rightsquigarrow e_2}{\Gamma \vdash B_1; B_2 : \exists \overline{t_1} : \kappa_1 \overline{t_2} : \kappa_2. \overline{\Sigma} \rightsquigarrow \text{unpack}\langle \overline{t_1} : \kappa_1, x_1 \rangle = e_1 \text{ in } \text{unpack}\langle \overline{t_2} : \kappa_2, x_2 \rangle = (\text{let } \overline{x_{X_1}} : \Sigma_1 = \overline{x_1}.\overline{l_{X_1}} \text{ in } e_2) \text{ in } \text{pack}\langle \overline{t_1} \overline{t_2}, \{\overline{l'_{X_1}} = x_1.\overline{l'_{X_1}}, \overline{l_{X_2}} = x_2.\overline{l_{X_2}} \} \rangle_{\exists \overline{t_1} : \kappa_1 \overline{t_2} : \kappa_2. \overline{\Sigma}}} \text{B-Seq}
\end{array}$$

Path:

$$\boxed{\Gamma \vdash P : \Sigma \rightsquigarrow e}$$

$$\frac{\Gamma \vdash P : \exists \overline{t} : \kappa. \overline{\Sigma} \quad \Gamma \vdash \Sigma : \Omega}{\Gamma \vdash P : \Sigma \rightsquigarrow \text{unpack}\langle \overline{t} : \kappa, x \rangle = e \text{ in } x} \text{P-Mod}$$

$$\boxed{\Gamma \vdash T : \kappa \rightsquigarrow \tau}$$

$$\frac{\Gamma \vdash P : [\tau : \kappa] \rightsquigarrow e}{\Gamma \vdash P : \kappa \rightsquigarrow \tau} \text{T-Elab-Path}$$

$$\boxed{\Gamma \vdash E : \tau \rightsquigarrow e}$$

$$\frac{\Gamma \vdash P : [\tau] \rightsquigarrow e}{\Gamma \vdash P : \tau \rightsquigarrow e.\text{val}} \text{E-Path}$$

3.1.5 Modules as First-Class Values

3.1.6 Elaboration with Applicative Functor

$$\begin{array}{l}
S ::= \dots \\
\quad | (X : S) \Rightarrow S \quad \text{(applicative functor signature)}
\end{array}$$

$$\begin{array}{l}
\varphi ::= I \quad \text{(impure effect)} \\
\quad | P \quad \text{(pure effect)}
\end{array}$$

$$\begin{array}{l}
\Sigma ::= \dots \\
\quad | \{\overline{l_X} : \Sigma\} \\
\quad | \forall \overline{t} : \kappa. \Sigma \rightarrow_1 \Xi \quad \text{(generative functor signature)} \\
\quad | \forall \overline{t} : \kappa. \Sigma \rightarrow_p \Sigma \quad \text{(applicative functor signature)}
\end{array}$$

Abbreviation:

$$\begin{aligned}
\tau_1 \rightarrow_{\varphi} \tau_2 &\stackrel{\text{def}}{=} \tau_1 \rightarrow \{l_{\varphi} : \tau_2\} \\
\lambda_{\varphi} x : \tau. e &\stackrel{\text{def}}{=} \lambda x : \tau. \{l_{\varphi} = e\} \\
(e_1 \ e_2)_{\varphi} &\stackrel{\text{def}}{=} (e_1 \ e_2).l_{\varphi}
\end{aligned}$$

Effect combining:

$$\boxed{\varphi_1 \vee \varphi_2 = \varphi}$$

$$\overline{\varphi \vee \varphi} = \varphi \quad \overline{I \vee P} = I \quad \overline{P \vee I} = I$$

Subeffects:

$$\boxed{\varphi_1 \leq \varphi_2}$$

$$\overline{\varphi \leq \varphi} \text{ F-Refl} \quad \overline{P \leq I} \text{ F-Sub}$$

Signature:

$$\boxed{\Gamma \vdash S \rightsquigarrow \Xi}$$

$$\begin{aligned}
&\frac{\Gamma \vdash S_1 \rightsquigarrow \overline{\exists t_1 : \kappa_1. \Sigma} \quad \Gamma, \overline{t_1 : \kappa_1}, x_X : \Sigma \vdash S_2 \rightsquigarrow \Xi}{\Gamma \vdash (X : S_1) \rightarrow S_2 \rightsquigarrow \overline{\forall t_1 : \kappa_1. \Sigma \rightarrow_1 \Xi}} \text{ S-Funct-I} \\
&\frac{\Gamma \vdash S_1 \rightsquigarrow \overline{\exists t_1 : \kappa_1. \Sigma_1} \quad \Gamma, \overline{t_1 : \kappa_1}, x_X : \Sigma_1 \vdash S_2 \rightsquigarrow \overline{\exists t_2 : \kappa_2. \Sigma_2}}{\Gamma \vdash (X : S_1) \Rightarrow S_2 \rightsquigarrow \overline{\exists t'_2 : \kappa_1 \rightarrow \kappa_2. \forall t_1 : \kappa_1. \Sigma_1 \rightarrow_P \Sigma_2[t_2 \leftarrow t'_2 \ \overline{t_1}]}} \text{ S-Funct-P}
\end{aligned}$$

Subtyping:

$$\boxed{\Gamma \vdash \Xi_1 \leq \Xi_2 \rightsquigarrow e}$$

$$\frac{\Gamma, \overline{t_2 : \kappa_2} \vdash \Sigma_2 \leq \overline{\exists t_1 : \kappa_1. \Sigma_1} \uparrow \overline{\tau} \rightsquigarrow e_1 \quad \Gamma, \overline{t_2 : \kappa_2} \vdash \Xi_1[\overline{t_1 \leftarrow \tau}] \leq \Xi_2 \rightsquigarrow e_2 \quad \varphi_1 \leq \varphi_2}{\Gamma \vdash (\overline{\forall t_1 : \kappa_1. \Sigma_1} \rightarrow_{\varphi_1} \Xi_1) \leq (\overline{\forall t_2 : \kappa_2. \Sigma_2} \rightarrow_{\varphi_2} \Xi_2) \rightsquigarrow \frac{\lambda x_1 : (\overline{\forall t_1 : \kappa_1. \Sigma_1} \rightarrow_{\varphi_1} \Xi_1). \Lambda t_2 : \kappa_2. \lambda_{\varphi_2} x_2 : \Sigma_2. e_2 (x_1 \ \overline{\tau} (e_1 \ x_2))}{\varphi_1}} \text{ U-Funct}$$

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参考文献

- [GTL89] Jean-Yves Girard, Paul Taylor, and Yves Lafont, *Proofs and Types*, Cambridge University Press, apr 1989.
- [RRD14] Andreas Rossberg, Claudio Russo, and Derek Dreyer, *F-ing modules*, Journal of Functional Programming **24** (2014), no. 5, 529–607.