

プログラミング言語周りノート

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第 1 章

Preliminaries

1.1 基本的な表記

量子化 (quantifier) の束縛をコンマ (,) で続けて書く。束縛の終わりをピリオド (.) で示す。例えば,

$$\forall x_1 \in X_1, x_2 \in X_2. \exists y_1 \in Y_1, y_2 \in Y_2. x_1 = y_1 \wedge x_2 = y_2$$

は,

$$\forall x_1 \in X_1. \forall x_2 \in X_2. \exists y_1 \in Y_1. \exists y_2 \in Y_2. x_1 = y_1 \wedge x_2 = y_2$$

と等しい。また、量子化の束縛において、*such that* を省略し、コンマ (,) で繋げて書く。例えば,

$$\forall x \in \{0, 1\}, x \neq 0. x = 1$$

は,

$$\forall x \in \{0, 1\}. x \neq 0 \implies x = 1$$

と等しい。また、 \implies , \iff が他の記号と混同する場合、それぞれ *implies*, *iff* を使用する。

集合 (*set*) について、以下の表記を用いる。

- 集合 A について、その濃度 (*cardinality*) を $|A|$ と表記する。なお、 A が有限集合 (*finite set*) の時、濃度とは要素の個数のことである。
- 集合 A について、 $a \in A$ を $a : A$ と表記する。
- 自然数 (*natural number*) の集合を $\mathbb{N} = \{0, 1, \dots\}$ と表記する。また、 n 以上の自然数の集合を $\mathbb{N}_{\geq n} = \{n, n+1, \dots\}$ と表記する。
- 自然数 $n \in \mathbb{N}$ について、 $\{1, \dots, n\}$ を $[n]$ と表記する。
- 集合 A の冪集合 (*power set*) を $\mathcal{P}(A) = \{X \mid X \subseteq A\}$ 、有限冪集合を $\mathcal{P}_{fin}(A) = \{X \in \mathcal{P}(A) \mid X \text{ は有限集合}\}$ と表記する。
- 集合 A_1, \dots, A_n の直積 (*cartesian product*) を $A_1 \times \dots \times A_n = \{(a_1, \dots, a_n) \mid a_1 \in A_1, \dots, a_n \in A_n\}$ と表記する。集合 A の n 直積を $A^n = \underbrace{A \times \dots \times A}_{n \text{ 項}}$ と表記する。特に、 $A^0 = \{\epsilon\}$ である。
- 集合 A_1, \dots, A_n の直和 (*disjoin union*) を $A_1 \uplus \dots \uplus A_n = (A_1 \times \{1\}) \cup \dots \cup (A_n \times \{n\})$ と表記する。なお、文脈から明らかな場合、直和の添字を省略し、 $a \in A_i$ に対して、 $a \in A_1 \uplus \dots \uplus A_n$ と表記する。
- 集合 A の B との差集合 (*relative complement*) を $A \setminus B = \{a \in A \mid a \notin B\}$ と表記する。

集合 Σ について、 $\bigcup_{n \in \mathbb{N}} \Sigma^n$ を Σ^* と表記する。この時、 $\alpha \in \Sigma^*$ を Σ による列 (*sequence*) と呼ぶ。列について、以下の表記を用いる。

- $(\sigma_1, \dots, \sigma_n) \in \Sigma^n$ について、 $(\sigma_1, \dots, \sigma_n)$ を $\sigma_1 \dots \sigma_n$ と表記する。
- 列 $\alpha = \sigma_1 \dots \sigma_n \in \Sigma^*$ について、その長さを $|\alpha| = n$ と表記する。

集合 A, B について、 $R \subseteq A \times B$ を関係 (*relation*) と呼ぶ。また,

$$A \rightarrow B \stackrel{\text{def}}{=} \{R \in \mathcal{P}(A \times B) \mid \forall x \in A, (x, y_1), (x, y_2) \in R. y_1 = y_2\}$$

という表記を導入し、関係 $f : A \rightarrow B$ を A から B への部分関数 (*partial function*) と呼ぶ。さらに,

$$A \rightarrow B \stackrel{\text{def}}{=} \{f : A \rightarrow B \mid \forall x \in A. \exists y \in B. (x, y) \in f\}$$

という表記を導入し、部分関数 $f : A \rightarrow B$ を (全) 関数 (*function*) と呼ぶ。関係について、以下の表記を用いる。

- 関係 $R \subseteq A \times B$ について、 $(a, b) \in R$ を $a R b$ と表記する。
- 関係 $R \subseteq A \times B$ について、定義域 (*domain*) を $\text{dom}(R) = \{a \mid \exists b. (a, b) \in R\}$ 、値域 (*range*) を $\text{cod}(R) = \{b \mid \exists a. (a, b) \in R\}$ と表記する。

- 部分関数 $f : A \rightarrow B$ について, $(a, b) \in f$ を $f(a) = b$ と表記する.
- 関係 $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$ について, その合成 (composition) を $R_1; R_2 = R_2 \circ R_1 = \{(x, z) \in A \times C \mid \exists y \in B. (x, y) \in R_1, (y, z) \in R_2\}$ と表記する.
- 関係 $R \subseteq A \times B$, 集合 $X \subseteq A$ について, R の X による制限 (restriction) を $R \upharpoonright_X = \{(a, b) \in R \mid a \in X\}$ と表記する. 特に関数 $f : A \rightarrow B$ の $X \subseteq A$ による制限は, 関数 $f \upharpoonright_X : X \rightarrow B$ になる.
- $a \in A$, $b \in B$ について, その組を $a \mapsto b = (a, b)$, 関数 $f : A \rightarrow B$ を $f = x \mapsto f(x)$ と表記する.
- 2 項関係 $R \subseteq A^2$ について, その推移閉包 (transitive closure), つまり以下を満たす最小の 2 項関係を $R^+ \subseteq A^2$ と表記する.
 - 任意の $(a, b) \in R$ について, $(a, b) \in R^+$.
 - 任意の $(a, b) \in R^+$, $(b, c) \in R^+$ について, $(a, c) \in R^+$.
- 2 項関係 $R \subseteq A^2$ について, その反射推移閉包 (reflexive transitive closure) を $R^* = R^+ \cup \{(a, a) \mid a \in A\}$ と表記する.

集合 I について, その要素で添字付けられた対象の列 $\{a_i\}_{i \in I}$ を I で添字づけられた族 (indexed family) と呼ぶ. 族について, 以下の表記を用いる.

- 族の集合を $\prod_{i \in I} A_i = \{\{a_i\}_{i \in I} \mid \forall i \in I, a_i \in A_i\}$ と表記する.
- 集合の族 $A = \{A_i\}_{i \in I}$ について, 次の条件を満たす時, A は互いに素 (pairwise disjoint) であるという.

$$\forall i_1, i_2 \in I, i_1 \neq i_2. A_{i_1} \cap A_{i_2} = \emptyset$$

1.2 基本的な定義

定義 1 (ランク付きアルファベット (ranked alphabet)). ランク付きアルファベットとは, 以下の組 (Σ, rank) のこと.

- 集合 Σ .
- 関数 $\text{rank} : \Sigma \rightarrow \mathbb{N}$.

rank が文脈から明らかな時, 単に Σ をランク付きアルファベットと呼ぶ. $f \in \Sigma$ について, $\text{rank}(f) = n$ の時, f は n -変数であるという. これを明示して, $f^{(n)}$ と表記することもある. □

定義 2 (項代数 (term algebra)). 項代数 \mathcal{T} とは, 以下の組 (Σ, X) のこと.

- ランク付きアルファベット Σ .
- 変数の集合 X .

この時, $\llbracket \mathcal{T} \rrbracket$ を以下を満たす最小の集合として定義する.

- $X \subseteq \llbracket \mathcal{T} \rrbracket$.
- $\tau_1, \dots, \tau_n \in \llbracket \mathcal{T} \rrbracket$, $f^{(n)} \in \Sigma$ について, $f(\tau_1, \dots, \tau_n) \in \llbracket \mathcal{T} \rrbracket$.

この時, $\tau \in \llbracket \mathcal{T} \rrbracket$ を \mathcal{T} の項と呼ぶ. □

定義 3 (パス (path)). 項代数 $\mathcal{T} = (\Sigma, X)$ について, $\text{paths} : \llbracket \mathcal{T} \rrbracket \rightarrow \mathcal{P}(\mathbb{N}^*)$ を以下のように定義する.

- $x \in X$ について, $\text{paths}(x) = \{\epsilon\}$.
- $f^{(n)} \in \Sigma$, $f^{(n)}(\tau_1, \dots, \tau_n) \in \llbracket \mathcal{T} \rrbracket$ について, $\text{paths}(f^{(n)}(\tau_1, \dots, \tau_n)) = \{\epsilon\} \cup \bigcup_{i \in [n]} \{i\pi \in \text{paths}(\tau_i)\}$.

この時, $\pi \in \text{paths}(\tau)$ を τ のパスと呼ぶ. □

定義 4 (部分項 (subterm)). 項代数 $\mathcal{T} = (\Sigma, X)$, 項 $\tau \in \llbracket \mathcal{T} \rrbracket$ について, $\text{subterm}_\tau : \text{paths}(\tau) \rightarrow \llbracket \mathcal{T} \rrbracket$ を以下のように定義する.

- $\text{subterm}_\tau(\epsilon) = \tau$.

- $\text{subterm}_{f(n)(\tau_1, \dots, \tau_i, \dots, \tau_n)}(i\pi) = \text{subterm}_{\tau_i}(\pi)$.

この時, $\pi \in \text{paths}(\tau)$ について, $\text{subterm}_{\tau}(\pi)$ を τ の π での部分木と呼ぶ.

□

定義5 (置換 (substitution)). 項代数 $\mathcal{T} = (\Sigma, X)$ について, $\sigma \subseteq \llbracket \mathcal{T} \rrbracket \times \llbracket \mathcal{T} \rrbracket$ が置換とは, 以下を満たすことを言う.

- 任意の $x \in \text{dom}(\sigma)$ について, $(x, y_1), (x, y_2) \in \sigma$ ならば $y_1 = y_2$.
- 任意の $x_1, x_2 \in \text{dom}(\sigma)$ について, $\text{subterm}_{x_1}(\pi) = x_2$ となる $\pi \in \text{paths}(x_1)$ は存在しない.

□

定義6 (出現 (occurrence)). 項代数 $\mathcal{T} = (\Sigma, X)$, 項 $\tau \in \llbracket \mathcal{T} \rrbracket$ について, $\text{occ}_{\tau} : \llbracket \mathcal{T} \rrbracket \rightarrow \mathcal{P}(\text{paths}(\tau))$ を以下のように定義する.

$$\text{occ}_{\tau}(\eta) = \{\pi \in \text{paths}(\tau) \mid \text{subterm}_{\tau}(\pi) = \eta\}$$

この時, $\pi \in \text{occ}_{\tau}(\eta)$ を, η の τ での出現と呼ぶ.

□

定義7 (コンテキスト (context)). 項代数 $\mathcal{T} = (\Sigma, X)$ について, コンテキストとは, $T[] \in \llbracket (\Sigma, X \uplus \{\{\}\}) \rrbracket$ で $[]$ の出現が一意であるもののことを言う. この時, \mathcal{T} のコンテキストの集合を $\mathcal{C}(\mathcal{T})$ と書く.

この時, $\tau \in \llbracket \mathcal{T} \rrbracket$ について, $T[\tau] \in \llbracket \mathcal{T} \rrbracket$ を $T[\tau] = (T[])[[] \leftarrow \tau]$ で定義する.

□

第 2 章

Basic Calculus

2.1 WIP: (Untyped) λ -Calculus

2.2 Simply Typed λ -Calculus

Alias: STLC, λ^\rightarrow [GTL89]

2.2.1 Syntax

e	::=	x	(variable)
		$e e$	(application)
		$\lambda x : \tau. e$	(abstraction)
		c_A	(constant)
τ	::=	A	(atomic type)
		$\tau \rightarrow \tau$	(function type)
Γ	::=	\cdot	(empty)
		$\Gamma, x : \tau$	(cons)

Convention:

$$\tau_1 \rightarrow \tau_2 \rightarrow \cdots \rightarrow \tau_n \stackrel{\text{def}}{=} \tau_1 \rightarrow (\tau_2 \rightarrow (\cdots \rightarrow \tau_n) \cdots)$$

$$e_1 e_2 \cdots e_n \stackrel{\text{def}}{=} (\cdots (e_1 e_2) \cdots) e_n$$

Environment Reference:

$$\boxed{\Gamma(x) = \tau}$$

$$\frac{x = x'}{(\Gamma, x' : \tau)(x) = \tau} \quad \frac{x \neq x' \quad \Gamma(x) = \tau}{(\Gamma, x' : \tau')(x) = \tau}$$

Free Variable:

$$\boxed{fv(e) = \{\bar{x}'\}}$$

$$\frac{}{fv(x) = \{x\}} \quad \frac{fv(e_1) = X_1 \quad fv(e_2) = X_2}{fv(e_1 e_2) = X_1 \cup X_2} \quad \frac{fv(e) = X}{fv(\lambda x : \tau. e) = X \setminus \{x\}} \quad \frac{}{fv(c_A) = \emptyset}$$

Substitution:

部分関数 $\{x_1 \mapsto e_1, \dots, x_n \mapsto e_n\}$ を, $[x_1 \leftarrow e_1, \dots, x_n \leftarrow e_n]$ または $[x_1, \dots, x_n \leftarrow e_1, \dots, e_n]$ と表記する.

$$\boxed{e[x' \leftarrow e'] = e''}$$

$$\frac{[\bar{x}' \leftarrow \bar{e}'](x) = e}{x[\bar{x}' \leftarrow \bar{e}'] = e} \quad \frac{x \notin \text{dom}([\bar{x}' \leftarrow \bar{e}'])}{x[\bar{x}' \leftarrow \bar{e}'] = x}$$

$$\frac{e_1[\bar{x}' \leftarrow \bar{e}'] = e_1'' \quad e_2[\bar{x}' \leftarrow \bar{e}'] = e_2''}{(e_1 e_2)[\bar{x}' \leftarrow \bar{e}'] = e_1'' e_2''} \quad \frac{e([\bar{x}' \leftarrow \bar{e}'] \upharpoonright_{\text{dom}([\bar{x}' \leftarrow \bar{e}']) \setminus \{x\}}) = e''}{(\lambda x : \tau. e)[\bar{x}' \leftarrow \bar{e}'] = \lambda x : \tau. e''} \quad \frac{}{c_A[\bar{x}' \leftarrow \bar{e}'] = c_A}$$

α -Equality:

$$\boxed{e_1 \equiv_\alpha e_2}$$

$$\frac{x_1 = x_2}{x_1 \equiv_\alpha x_2} \quad \frac{x' \notin fv(e_1) \cup fv(e_2) \quad e_1[x_1 \leftarrow x'] \equiv_\alpha e_2[x_2 \leftarrow x']}{\lambda x_1 : \tau. e_1 \equiv_\alpha \lambda x_2 : \tau. e_2} \quad \frac{e_1 \equiv_\alpha e_2 \quad e_1' \equiv_\alpha e_2'}{e_1 e_1' \equiv_\alpha e_2 e_2'} \quad \frac{}{c_A \equiv_\alpha c_A}$$

定理 8 (Correctness of Substitution). 式 e , 置換 $[\bar{x}' \leftarrow \bar{e}']$ について, $X = \text{dom}([\bar{x}' \leftarrow \bar{e}'])$ とした時,

$$fv(e[\bar{x}' \leftarrow \bar{e}']) = (fv(e) \setminus X) \cup \bigcup_{x \in fv(e) \cap X} fv([\bar{x}' \leftarrow \bar{e}'](x)).$$

□

定理 9 (α -Equality Does Not Touch Free Variables). $e_1 \equiv_{\alpha} e_2$ ならば, $fv(e_1) = fv(e_2)$.

□

2.2.2 Typing Semantics

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c} \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{ T-Var} \\ \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \text{ T-Abs} \\ \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \text{ T-App} \\ \frac{}{\Gamma \vdash c_A : A} \text{ T-Const} \end{array}$$

特に, $\cdot \vdash e : \tau$ の時, $e : \tau$ と表記.

2.2.3 Evaluation Semantics (Call-By-Value)

$$\begin{array}{lcl} v & ::= & \lambda x : \tau. e \\ & | & c_A \\ C & ::= & [] \\ & | & C e \\ & | & v C \end{array}$$

Small Step:

$$\boxed{e \Rightarrow e'}$$

$$\begin{array}{c} \overline{(\lambda x : \tau. e) v \Rightarrow e[x \leftarrow v]} \\ \frac{e \Rightarrow e'}{C[e] \Rightarrow C[e']} \end{array}$$

Big Step:

$$\boxed{e \Downarrow v}$$

$$\frac{e_1 \Downarrow \lambda x : \tau. e'_1 \quad e_2 \Downarrow v_2 \quad e'_1[x \leftarrow v_2] \Downarrow v}{e_1 e_2 \Downarrow v}$$

定理 10 (Adequacy of Small Step and Big Step). $e \Rightarrow^* v$ iff $e \Downarrow v$.

□

定理 11 (Type Soundness). $e : \tau$ の時, $e \Rightarrow^* v$, $e \Downarrow v$ となる $v = \text{nf}(\Rightarrow, e)$ が存在し,

- $\tau = \tau_1 \rightarrow \tau_2$ の時, $v \equiv_{\alpha} \lambda x' : \tau_1. e'$ となる $\lambda x' : \tau'. e'$ が存在する.
- $\tau = A$ の時, $v \equiv_{\alpha} c_A$ となる c_A が存在する.

□

2.2.4 Equational Reasoning

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c}
\frac{\Gamma, x : \tau \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\lambda x : \tau. e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau} \text{Eq-}\beta\text{-Lam} \quad \frac{x \notin \text{fv}(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash (\lambda x : \tau_1. e x) \equiv e : \tau_1 \rightarrow \tau_2} \text{Eq-}\eta\text{-Lam} \\
\frac{e_1 \equiv_{\alpha} e_2 \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Refl} \\
\frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-Sym} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau \quad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau} \text{Eq-Trans} \\
\frac{\Gamma, x : \tau \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash \lambda x : \tau. e_1 \equiv \lambda x : \tau. e_2 : \tau \rightarrow \tau'} \text{Eq-Cong-Abs} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \rightarrow \tau \quad \Gamma \vdash e'_1 \equiv e'_2 : \tau'}{\Gamma \vdash e_1 e'_1 \equiv e_2 e'_2 : \tau} \text{Eq-Cong-App}
\end{array}$$

特に, $\cdot \vdash e_1 \equiv e_2 : \tau$ の時, $e_1 \equiv e_2 : \tau$ と表記.

定理 12 (Respect Typing). $\Gamma \vdash e_1 \equiv e_2 : \tau$ ならば, $\Gamma \vdash e_1 : \tau$ かつ $\Gamma \vdash e_2 : \tau$. □

定理 13 (Respect Evaluation). $e_1 \equiv e_2 : \tau$ の時, $e'_1 \Rightarrow^* e_1$, $e_2 \Rightarrow^* e'_2$ ならば $e'_1 \equiv e'_2 : \tau$. □

系 14. $e_1 \equiv e_2 : \tau$ の時, $e_1 \Rightarrow^* e'_1$, $e_2 \Rightarrow^* e'_2$ ならば $e'_1 \equiv e'_2 : \tau$. □

証明. $e_1 \Rightarrow^* e_1$ より, 定理 13 から $e_1 \equiv e'_2 : \tau$. よって, T-Sym から $e'_2 \equiv e_1 : \tau$ であり, $e'_2 \Rightarrow^* e'_2$ より定理 13 から $e'_2 \equiv e'_1 : \tau$. 故に, T-Sym から $e'_1 \equiv e'_2 : \tau$. ■

2.3 WIP: System-T

2.4 WIP: PCF

2.5 System-F

Alias: F, Second Order Typed Lambda Calculus, $\lambda 2$ [GTL89]

2.5.1 Syntax

$e ::= x$	(variable)
$ \lambda x : \tau. e$	(abstraction)
$ e e$	(application)
$ \Lambda t. e$	(universal abstraction)
$ e \tau$	(universal application)
$\tau ::= t$	(type variable)
$ \tau \rightarrow \tau$	(function type)
$ \forall t. \tau$	(polymorphic type)
$\Gamma ::= \cdot$	(empty)
$ \Gamma, x : \tau$	(variable cons)
$ \Gamma, t : \Omega$	(type variable cons)

Convention:

$$\tau_1 \rightarrow \tau_2 \rightarrow \cdots \rightarrow \tau_n \stackrel{\text{def}}{=} \tau_1 \rightarrow (\tau_2 \rightarrow (\cdots \rightarrow \tau_n) \cdots)$$

$$e_1 e_2 \cdots e_n \stackrel{\text{def}}{=} (\cdots (e_1 e_2) \cdots) e_n$$

Environment Reference:

$$\boxed{\Gamma(x) = \tau}$$

$$\frac{x = x'}{(\Gamma, x' : \tau)(x) = \tau} \quad \frac{x \neq x' \quad \Gamma(x) = \tau}{(\Gamma, x' : \tau')(x) = \tau} \quad \frac{\Gamma(x) = \tau}{(\Gamma, t : \Omega)(x) = \tau}$$

$$\frac{t = t'}{(\Gamma, t' : \Omega)(t) = \Omega} \quad \frac{t \neq t' \quad \Gamma(t) = \Omega}{(\Gamma, t' : \Omega')(t) = \Omega} \quad \frac{\Gamma(t) = \Omega}{(\Gamma, x : \tau)(t) = \Omega}$$

Free Variable:

$$\boxed{fv(e) = \{\bar{x}\}}$$

$$\frac{}{fv(x) = \{x\}} \quad \frac{fv(e_1) = X_1 \quad fv(e_2) = X_2}{fv(e_1 e_2) = X_1 \cup X_2} \quad \frac{fv(e) = X}{fv(\lambda x : \tau. e) = X \setminus \{x\}} \quad \frac{fv(e) = X}{fv(e \tau) = X} \quad \frac{fv(e) = X}{fv(\Lambda t. e) = X}$$

Substitution:

部分関数 $\{x_1 \mapsto e_1, \dots, x_n \mapsto e_n\}$ を, $[x_1 \leftarrow e_1, \dots, x_n \leftarrow e_n]$ または $[x_1, \dots, x_n \leftarrow e_1, \dots, e_n]$ と表記する.

$$\boxed{e[\bar{x}' \leftarrow \bar{e}'] = e''}$$

$$\frac{[\bar{x}' \leftarrow \bar{e}'](x) = e}{x[\bar{x}' \leftarrow \bar{e}'] = e} \quad \frac{x \notin \text{dom}([\bar{x}' \leftarrow \bar{e}'])}{x[\bar{x}' \leftarrow \bar{e}'] = x}$$

$$\frac{e_1[\bar{x}' \leftarrow \bar{e}'] = e_1'' \quad e_2[\bar{x}' \leftarrow \bar{e}'] = e_2''}{(e_1 e_2)[\bar{x}' \leftarrow \bar{e}'] = e_1'' e_2''} \quad \frac{e([\bar{x}' \leftarrow \bar{e}'] \upharpoonright_{\text{dom}([\bar{x}' \leftarrow \bar{e}']) \setminus \{x\}}) = e''}{(\lambda x : \tau. e)[\bar{x}' \leftarrow \bar{e}'] = \lambda x : \tau. e''}$$

$$\frac{e[\bar{x}' \leftarrow \bar{e}'] = e''}{(e \tau)[\bar{x}' \leftarrow \bar{e}'] = e'' \tau} \quad \frac{e[\bar{x}' \leftarrow \bar{e}'] = e''}{(\Lambda t. e)[\bar{x}' \leftarrow \bar{e}'] = \Lambda t. e''}$$

Type Free Variable:

$$\boxed{tyfv(e) = \{\bar{x}\}}$$

$$\begin{array}{c} \overline{tyfv(x) = \emptyset} \quad \frac{tyfv(e_1) = T_1 \quad tyfv(e_2) = T_2}{tyfv(e_1 e_2) = T_1 \cup T_2} \quad \frac{tyfv(\tau) = T_1 \quad tyfv(e) = T_2}{tyfv(\lambda x : \tau. e) = T_1 \cup T_2} \\ \frac{tyfv(e) = T_1 \quad tyfv(\tau) = T_2}{tyfv(e \tau) = T_1 \cup T_2} \quad \frac{tyfv(e) = T}{tyfv(\Lambda t. e) = T \setminus \{t\}} \\ \overline{tyfv(t) = \{t\}} \quad \frac{tyfv(\tau_1) = T_1 \quad tyfv(\tau_2) = T_2}{tyfv(\tau_1 \rightarrow \tau_2) = T_1 \cup T_2} \quad \frac{tyfv(\tau) = T}{tyfv(\forall t. \tau) = T \setminus \{t\}} \end{array}$$

Type Substitution:

部分関数 $\{t_1 \mapsto \tau_1, \dots, t_n \mapsto \tau_n\}$ を, $[t_1 \leftarrow \tau_1, \dots, t_n \leftarrow \tau_n]$ または $[t_1, \dots, t_n \leftarrow \tau_1, \dots, \tau_n]$ と表記する.

$$\boxed{e[t \leftarrow \tau] = e'}$$

$$\begin{array}{c} \overline{x[t' \leftarrow \tau'] = x} \quad \frac{e_1[t' \leftarrow \tau'] = e'_1 \quad e_2[t' \leftarrow \tau'] = e'_2}{(e_1 e_2)[t' \leftarrow \tau'] = e'_1 e'_2} \quad \frac{\tau[t' \leftarrow \tau'] = \tau'' \quad e[t' \leftarrow \tau'] = e''}{(\lambda x : \tau. e)[t' \leftarrow \tau'] = \lambda x : \tau''. e''} \\ \frac{e[t' \leftarrow \tau'] = e'' \quad \tau[t' \leftarrow \tau'] = \tau''}{(e \tau)[t' \leftarrow \tau'] = e'' \tau''} \quad \frac{e([t' \leftarrow \tau'] \upharpoonright_{\text{dom}([t' \leftarrow \tau'] \setminus \{t\})}) = e''}{(\Lambda t. e)[t' \leftarrow \tau'] = \Lambda t. e''} \end{array}$$

$$\boxed{\tau[t' \leftarrow \tau'] = \tau''}$$

$$\begin{array}{c} \frac{[t' \leftarrow \tau'](t) = \tau}{t[t' \leftarrow \tau'] = \tau} \quad \frac{t \notin \text{dom}([t' \leftarrow \tau'])}{t[t' \leftarrow \tau'] = t} \quad \frac{\tau_1[t' \leftarrow \tau'] = \tau'_1 \quad \tau_2[t' \leftarrow \tau'] = \tau'_2}{(\tau_1 \rightarrow \tau_2)[t' \leftarrow \tau'] = \tau'_1 \rightarrow \tau'_2} \quad \frac{\tau([t' \leftarrow \tau'] \upharpoonright_{\text{dom}([t' \leftarrow \tau'] \setminus \{t\})}) = \tau''}{(\forall t. \tau)[t' \leftarrow \tau'] = \forall t. \tau''} \end{array}$$

α -Equality:

$$\boxed{e_1 \equiv_{\alpha} e_2}$$

$$\begin{array}{c} \frac{x_1 = x_2}{x_1 \equiv_{\alpha} x_2} \quad \frac{e_1 \equiv_{\alpha} e_2 \quad e'_1 \equiv_{\alpha} e'_2}{e_1 e'_1 \equiv_{\alpha} e_2 e'_2} \quad \frac{\tau_1 \equiv_{\alpha} \tau_2 \quad x' \notin fv(e_1) \cup fv(e_2) \quad e_1[x_1 \leftarrow x'] \equiv_{\alpha} e_2[x_2 \leftarrow x']}{\lambda x_1 : \tau_1. e_1 \equiv_{\alpha} \lambda x_2 : \tau_2. e_2} \\ \frac{e_1 \equiv_{\alpha} e_2 \quad \tau_1 \equiv_{\alpha} \tau_2}{e_1 \tau_1 \equiv_{\alpha} e_2 \tau_2} \quad \frac{t' \notin tyfv(e_1) \cup tyfv(e_2) \quad e_1[t_1 \leftarrow t'] \equiv_{\alpha} e_2[t_2 \leftarrow t']}{\Lambda t_1. e_1 \equiv_{\alpha} \Lambda t_2. e_2} \end{array}$$

$$\boxed{\tau_1 \equiv_{\alpha} \tau_2}$$

$$\frac{t_1 = t_2}{t_1 \equiv_{\alpha} t_2} \quad \frac{\tau_1 \equiv_{\alpha} \tau_2 \quad \tau'_1 \equiv_{\alpha} \tau'_2}{\tau_1 \rightarrow \tau'_1 \equiv_{\alpha} \tau_2 \rightarrow \tau'_2} \quad \frac{t' \notin tyfv(\tau_1) \cup tyfv(\tau_2) \quad \tau_1[t_1 \leftarrow t'] \equiv_{\alpha} \tau_2[t_2 \leftarrow t']}{\forall t_1. \tau_1 \equiv_{\alpha} \forall t_2. \tau_2}$$

定理 15 (Correctness of Substitution). 置換 $[\bar{x}' \leftarrow \bar{e}']$ について, $X = \text{dom}([\bar{x}' \leftarrow \bar{e}'])$ とした時,

$$fv(e[\bar{x}' \leftarrow \bar{e}']) = (fv(e) \setminus X) \cup \bigcup_{x \in fv(e) \cap X} fv([\bar{x}' \leftarrow \bar{e}'](x)).$$

□

定理 16 (Correctness of Type Substitution). 式 e , 型 τ , 型置換 $[\bar{t}' \leftarrow \bar{\tau}']$ について, $T = \text{dom}([\bar{t}' \leftarrow \bar{\tau}'])$ とした時,

$$\begin{array}{c} tyfv(e[\bar{t}' \leftarrow \bar{\tau}']) = (tyfv(e) \setminus T) \cup \bigcup_{t \in tyfv(e) \cap T} tyfv([\bar{t}' \leftarrow \bar{\tau}'](t)) \\ tyfv(\tau[\bar{t}' \leftarrow \bar{\tau}']) = (tyfv(\tau) \setminus T) \cup \bigcup_{t \in tyfv(\tau) \cap T} tyfv([\bar{t}' \leftarrow \bar{\tau}'](t)). \end{array}$$

□

定理 17 (α -Equality Does Not Touch Free Variables).

- $\tau_1 \equiv_{\alpha} \tau_2$ ならば $tyfu(\tau_1) = tyfu(\tau_2)$.
- $e_1 \equiv_{\alpha} e_2$ ならば, $fv(e_1) = fv(e_2)$, $tyfu(e_1) = tyfu(e_2)$.

□

2.5.2 Typing Semantics

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c}
 \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{ T-Var} \\
 \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \text{ T-Abs} \\
 \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \text{ T-App} \\
 \frac{\Gamma, t : \Omega \vdash e : \tau}{\Gamma \vdash \Lambda t. e : \forall t. \tau} \text{ T-UnivAbs} \\
 \frac{\Gamma \vdash e : \forall t. \tau_1}{\Gamma \vdash e \tau_2 : \tau_1[t \leftarrow \tau_2]} \text{ T-UnivApp} \\
 \frac{\Gamma \vdash \tau \equiv_{\alpha} \tau' : \Omega \quad \Gamma \vdash e : \tau'}{\Gamma \vdash e : \tau} \text{ T-}\alpha\text{-Equiv}
 \end{array}$$

特に, $\cdot \vdash e : \tau$ の時, $e : \tau$ と表記.

2.5.3 Evaluation Semantics (Call-By-Value)

$$\begin{array}{lcl}
 v & ::= & \lambda x : \tau. e \\
 & | & \Lambda t. e \\
 C & ::= & [] \\
 & | & C e \\
 & | & v C \\
 & | & C \tau
 \end{array}$$

Small Step:

$$\boxed{e \Rightarrow e'}$$

$$\begin{array}{c}
 \overline{(\lambda x : \tau. e) v \Rightarrow e[x \leftarrow v]} \\
 \overline{(\Lambda t. e) \tau \Rightarrow e[t \leftarrow \tau]} \\
 \frac{e \Rightarrow e'}{C[e] \Rightarrow C[e']}
 \end{array}$$

Big Step:

$$\boxed{e \Downarrow v}$$

$$\begin{array}{c}
 \frac{e_1 \Downarrow \lambda x : \tau. e'_1 \quad e_2 \Downarrow v_2 \quad e'_1[x \leftarrow v_2] \Downarrow v}{e_1 e_2 \Downarrow v} \\
 \frac{e \Downarrow \Lambda t. e'_1 \quad e'_1[t \leftarrow \tau] \Downarrow v}{e \tau \Downarrow v}
 \end{array}$$

定理 18 (Adequacy of Small Step and Big Step). $e \Rightarrow^* v$ iff $e \Downarrow v$.

□

定理 19 (Type Soundness). $e : \tau$ の時, $e \Rightarrow^* v$, $e \Downarrow v$ となる $v = \text{nf}(\Rightarrow, e)$ が存在し,

- $\tau = \tau_1 \rightarrow \tau_2$ の時, $v \equiv_\alpha \lambda x' : \tau_1. e'$ となる $\lambda x' : \tau_1. e'$ が存在する.
- $\tau = \forall t. \tau_1$ の時, $v \equiv_\alpha \Lambda t. e'$ となる $\Lambda t. e'$ が存在する.

□

2.5.4 Equational Reasoning

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c} \frac{\Gamma, x : \tau_2 \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\lambda x : \tau_2. e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau} \text{Eq-}\beta\text{-Lam} \quad \frac{x \notin \text{fv}(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash (\lambda x : \tau_1. e x) \equiv e : \tau_1 \rightarrow \tau_2} \text{Eq-}\eta\text{-Lam} \\ \frac{\Gamma, t : \Omega \vdash e : \tau}{\Gamma \vdash (\Lambda t. e) \tau_2 \equiv e[t \leftarrow \tau_2] : \tau[t \leftarrow \tau_2]} \text{Eq-}\beta\text{-UnivLam} \quad \frac{t \notin \text{tyfv}(e) \quad \Gamma \vdash e : \forall t'. \tau}{\Gamma \vdash (\Lambda t. e t) \equiv e : \forall t'. \tau} \text{Eq-}\eta\text{-UnivLam} \\ \frac{e_1 \equiv_\alpha e_2 \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Refl} \quad \frac{\tau \equiv_\alpha \tau' \quad \Gamma \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Type} \\ \frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-Sym} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau \quad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau} \text{Eq-Trans} \\ \frac{\Gamma, x : \tau \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash \lambda x : \tau. e_1 \equiv \lambda x : \tau. e_2 : \tau \rightarrow \tau'} \text{Eq-Cong-Abs} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \rightarrow \tau \quad \Gamma \vdash e'_1 \equiv e'_2 : \tau'}{\Gamma \vdash e_1 e'_1 \equiv e_2 e'_2 : \tau} \text{Eq-Cong-App} \\ \frac{\Gamma, t : \Omega \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \Lambda t. e_1 \equiv \Lambda t. e_2 : \forall(t. \tau)} \text{Eq-Cong-UnivAbs} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \forall t. \tau}{\Gamma \vdash e_1 \tau' \equiv e_2 \tau' : \tau[t \leftarrow \tau']} \text{Eq-Cong-UnivApp} \end{array}$$

特に, $\cdot \vdash e_1 \equiv e_2 : \tau$ の時, $e_1 \equiv e_2 : \tau$ と表記.

定理 20 (Respect Typing). $\Gamma \vdash e_1 \equiv e_2 : \tau$ ならば, $\Gamma \vdash e_1 : \tau$ かつ $\Gamma \vdash e_2 : \tau$.

□

定理 21 (Respect Evaluation). $e_1 \equiv e_2 : \tau$ の時, $e'_1 \Rightarrow^* e_1$, $e_2 \Rightarrow^* e'_2$ ならば $e'_1 \equiv e'_2 : \tau$.

□

系 22. $e_1 \equiv e_2 : \tau$ の時, $e_1 \Rightarrow^* e'_1$, $e_2 \Rightarrow^* e'_2$ ならば $e'_1 \equiv e'_2 : \tau$.

□

証明. $e_1 \Rightarrow^* e_1$ より, 定理 21 から $e_1 \equiv e'_2 : \tau$. よって, T-Sym から $e'_2 \equiv e_1 : \tau$ であり, $e'_2 \Rightarrow^* e'_2$ より定理 21 から $e'_2 \equiv e'_1 : \tau$. 故に, T-Sym から $e'_1 \equiv e'_2 : \tau$. ■

2.5.5 Definability

Product

Product of τ_1 and τ_2 :

$$\begin{aligned} \tau_1 \times \tau_2 &\stackrel{\text{def}}{=} \forall t. (\tau_1 \rightarrow \tau_2 \rightarrow t) \rightarrow t \\ \langle e_1, e_2 \rangle &\stackrel{\text{def}}{=} \Lambda t. \lambda x : \tau_1 \rightarrow \tau_2 \rightarrow t. x e_1 e_2 \\ \pi_1 e &\stackrel{\text{def}}{=} e \tau_1 \lambda x_1. \lambda x_2. x_1 \\ \pi_2 e &\stackrel{\text{def}}{=} e \tau_2 \lambda x_1. \lambda x_2. x_2 \end{aligned}$$

Admissible typing rule:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2} \text{T-Product} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_1 e : \tau_1} \text{T-Proj-1} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_2 e : \tau_2} \text{T-Proj-2}$$

Admissible equality:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c} \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \pi_1 \langle e_1, e_2 \rangle \equiv e_1 : \tau_1} \text{Eq-}\beta\text{-Product-1} \quad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \pi_2 \langle e_1, e_2 \rangle \equiv e_2 : \tau_2} \text{Eq-}\beta\text{-Product-2} \\ \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \langle \pi_1 e, \pi_2 e \rangle \equiv e : \tau_1 \times \tau_2} \text{Eq-}\eta\text{-Product} \end{array}$$

Existential Type

Existence of $\exists t. \tau$:

$$\begin{aligned} \exists t. \tau &\stackrel{\text{def}}{=} \forall t'. (\forall t. \tau \rightarrow t') \rightarrow t' \\ \text{pack} \langle \tau_t, e \rangle &\stackrel{\text{def}}{=} \Lambda t'. \lambda x : (\forall t. \tau \rightarrow t'). x \tau_t e \\ \text{unpack} \langle t, x \rangle = e_1. \tau_2. e_2 &\stackrel{\text{def}}{=} e_1 \tau_2 (\Lambda t. \lambda x : \tau. e_2) \end{aligned}$$

Admissible typing rule:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash e : \tau[t \leftarrow \tau_t]}{\Gamma \vdash \text{pack} \langle \tau_t, e \rangle : \exists t. \tau} \text{T-Pack} \quad \frac{\Gamma \vdash e_1 : \exists t. \tau \quad \Gamma, t : \Omega, x : \tau \vdash e_2 : \tau_2 \quad t \notin \text{tyfu}(\tau_2)}{\Gamma \vdash \text{unpack} \langle t, x \rangle = e_1. \tau_2. e_2 : \tau_2} \text{T-Unpack}$$

Admissible equality:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c} \frac{\Gamma \vdash e_1 : \tau_1[t \leftarrow \tau_t] \quad \Gamma, t : \Omega, x : \tau_1 \vdash e_2 : \tau_2 \quad t \notin \text{tyfu}(\tau_2)}{\Gamma \vdash \text{unpack} \langle t, x \rangle = \text{pack} \langle \tau_t, e_1 \rangle. \tau_2. e_2 \equiv e_2[t \leftarrow \tau_t][x \leftarrow e_1] : \tau_2} \text{Eq-}\beta\text{-Exist} \\ \frac{\Gamma \vdash e : \exists t'. \tau \quad \tau' \equiv_{\alpha} \exists t'. \tau}{\Gamma \vdash \text{unpack} \langle t, x \rangle = e. \tau'. \text{pack} \langle t, x \rangle \equiv e : \exists t'. \tau} \text{Eq-}\eta\text{-Exist} \end{array}$$

2.5.6 Typability

[Wel99]

TODO

2.6 System-F ω

Alias: F ω , $\lambda\omega$ [RRD14]

2.6.1 Syntax

$e ::=$	x	(variable)
	$ \lambda x : \tau. e$	(abstraction)
	$ e e$	(application)
	$ \Lambda t : \kappa. e$	(universal abstraction)
	$ e \tau$	(universal application)
$\tau ::=$	t	(type variable)
	$ \tau \rightarrow \tau$	(function type)
	$ \forall t : \kappa. \tau$	(polymorphic type)
	$ \lambda t : \kappa. \tau$	(type abstraction)
	$ \tau \tau$	(type application)
$\kappa ::=$	Ω	(type kind)
	$ \kappa \rightarrow \kappa$	(arrow kind)
$\Gamma ::=$	\cdot	(empty)
	$ \Gamma, x : \tau$	(variable cons)
	$ \Gamma, t : \kappa$	(type variable cons)

Convention:

$$\begin{aligned} \tau_1 \rightarrow \tau_2 \rightarrow \dots \rightarrow \tau_n &\stackrel{\text{def}}{=} \tau_1 \rightarrow (\tau_2 \rightarrow (\dots \rightarrow \tau_n) \dots) \\ e_1 e_2 \dots e_n &\stackrel{\text{def}}{=} (\dots (e_1 e_2) \dots) e_n \end{aligned}$$

Environment Reference:

$$\boxed{\Gamma(x) = \tau}$$

$$\begin{array}{ccc} \frac{x = x'}{(\Gamma, x' : \tau)(x) = \tau} & \frac{x \neq x' \quad \Gamma(x) = \tau}{(\Gamma, x' : \tau')(x) = \tau} & \frac{\Gamma(x) = \tau}{(\Gamma, t : \kappa)(x) = \tau} \\ \frac{t = t'}{(\Gamma, t' : \kappa)(t) = \kappa} & \frac{t \neq t' \quad \Gamma(t) = \kappa}{(\Gamma, t' : \kappa')(t) = \kappa} & \frac{\Gamma(t) = \kappa}{(\Gamma, x : \tau)(t) = \kappa} \end{array}$$

Free Variable:

$$\boxed{fv(e) = \{\bar{x}\}}$$

$$\begin{array}{ccccc} \overline{fv(x) = \{x\}} & \frac{fv(e) = X}{\overline{fv(\lambda x : \tau. e) = X \setminus \{x\}}} & \frac{fv(e_1) = X_1 \quad fv(e_2) = X_2}{\overline{fv(e_1 e_2) = X_1 \cup X_2}} & \frac{fv(e) = X}{\overline{fv(\Lambda t : \kappa. e) = X}} & \frac{fv(e) = X}{\overline{fv(e \tau) = X}} \end{array}$$

Substitution:

部分関数 $\{x_1 \mapsto e_1, \dots, x_n \mapsto e_n\}$ を, $[x_1 \leftarrow e_1, \dots, x_n \leftarrow e_n]$ または $[x_1, \dots, x_n \leftarrow e_1, \dots, e_n]$ と表記する.

$$\boxed{e[\bar{x}' \leftarrow \bar{e}'] = e''}$$

$$\begin{array}{ccc} \frac{[\bar{x}' \leftarrow \bar{e}'](x) = e}{x[\bar{x}' \leftarrow \bar{e}'] = e} & \frac{x \notin \text{dom}([\bar{x}' \leftarrow \bar{e}'])}{x[\bar{x}' \leftarrow \bar{e}'] = x} & \\ \frac{e([\bar{x}' \leftarrow \bar{e}'] \upharpoonright_{\text{dom}([\bar{x}' \leftarrow \bar{e}']) \setminus \{x\}}) = e''}{(\lambda x : \tau. e)[\bar{x}' \leftarrow \bar{e}'] = \lambda x : \tau. e''} & \frac{e_1[\bar{x}' \leftarrow \bar{e}'] = e'_1 \quad e_2[\bar{x}' \leftarrow \bar{e}'] = e'_2}{(e_1 e_2)[\bar{x}' \leftarrow \bar{e}'] = e'_1 e'_2} & \end{array}$$

$$\frac{e[\overline{x'} \leftarrow \overline{e'}] = e''}{(\Lambda t : \kappa. e)[\overline{x'} \leftarrow \overline{e'}] = \Lambda t : \kappa. e''} \quad \frac{e[\overline{x'} \leftarrow \overline{e'}] = e''}{(e \tau)[\overline{x'} \leftarrow \overline{e'}] = e'' \tau}$$

Type Free Variable:

$$\boxed{tyfv(e) = \{\bar{t}\}}$$

$$\frac{}{tyfv(x) = \emptyset} \quad \frac{tyfv(\tau) = T_1 \quad tyfv(e) = T_2}{tyfv(\lambda x : \tau. e) = T_1 \cup T_2} \quad \frac{tyfv(e_1) = T_1 \quad tyfv(e_2) = T_2}{tyfv(e_1 e_2) = T_1 \cup T_2}$$

$$\frac{tyfv(e) = T}{tyfv(\Lambda t : \kappa. e) = T \setminus \{t\}} \quad \frac{tyfv(e) = T_1 \quad tyfv(\tau) = T_2}{tyfv(e \tau) = T_1 \cup T_2}$$

$$\boxed{tyfv(\tau) = \{\bar{t}\}}$$

$$\frac{}{tyfv(t) = \{t\}} \quad \frac{tyfv(\tau_1) = T_1 \quad tyfv(\tau_2) = T_2}{tyfv(\tau_1 \rightarrow \tau_2) = T_1 \cup T_2} \quad \frac{tyfv(\tau) = T}{tyfv(\forall t : \kappa. \tau) = T \setminus \{t\}}$$

$$\frac{tyfv(\tau) = T}{tyfv(\lambda t : \kappa. \tau) = T \setminus \{t\}} \quad \frac{tyfv(\tau_1) = T_1 \quad tyfv(\tau_2) = T_2}{tyfv(\tau_1 \tau_2) = T_1 \cup T_2}$$

Type Substitution:

部分関数 $\{t_1 \mapsto \tau_1, \dots, t_n \mapsto \tau_n\}$ を, $[t_1 \leftarrow \tau_1, \dots, t_n \leftarrow \tau_n]$ または $[t_1, \dots, t_n \leftarrow t_1, \dots, t_n]$ と表記する.

$$\boxed{e[\bar{t'} \leftarrow \bar{\tau'}] = e'}$$

$$\frac{}{x[\bar{t'} \leftarrow \bar{\tau'}] = x} \quad \frac{e_1[\bar{t'} \leftarrow \bar{\tau'}] = e'_1 \quad e_2[\bar{t'} \leftarrow \bar{\tau'}] = e'_2}{(e_1 e_2)[\bar{t'} \leftarrow \bar{\tau'}] = e'_1 e'_2} \quad \frac{\tau[\bar{t'} \leftarrow \bar{\tau'}] = \tau'' \quad e[\bar{t'} \leftarrow \bar{\tau'}] = e''}{(\lambda x : \tau. e)[\bar{t'} \leftarrow \bar{\tau'}] = \lambda x : \tau''. e''}$$

$$\frac{e[\bar{t'} \leftarrow \bar{\tau'}] = e'' \quad \tau[\bar{t'} \leftarrow \bar{\tau'}] = \tau''}{(e \tau)[\bar{t'} \leftarrow \bar{\tau'}] = e'' \tau''} \quad \frac{e([\bar{t'} \leftarrow \bar{\tau'}] \upharpoonright_{\text{dom}([\bar{t'} \leftarrow \bar{\tau'}]) \setminus \{t\}}) = e''}{(\Lambda t : \kappa. e)[\bar{t'} \leftarrow \bar{\tau'}] = \Lambda t : \kappa. e''}$$

$$\boxed{\tau[\bar{t'} \leftarrow \bar{\tau'}] = \tau''}$$

$$\frac{[\bar{t'} \leftarrow \bar{\tau'}](t) = \tau}{t[\bar{t'} \leftarrow \bar{\tau'}] = \tau} \quad \frac{t \notin \text{dom}([\bar{t'} \leftarrow \bar{\tau'}])}{t[\bar{t'} \leftarrow \bar{\tau'}] = t}$$

$$\frac{\tau_1[\bar{t'} \leftarrow \bar{\tau'}] = \tau'_1 \quad \tau_2[\bar{t'} \leftarrow \bar{\tau'}] = \tau'_2}{(\tau_1 \rightarrow \tau_2)[\bar{t'} \leftarrow \bar{\tau'}] = \tau'_1 \rightarrow \tau'_2} \quad \frac{\tau([\bar{t'} \leftarrow \bar{\tau'}] \upharpoonright_{\text{dom}([\bar{t'} \leftarrow \bar{\tau'}]) \setminus \{t\}}) = \tau''}{(\forall t : \kappa. \tau)[\bar{t'} \leftarrow \bar{\tau'}] = \forall t : \kappa. \tau''}$$

$$\frac{\tau([\bar{t'} \leftarrow \bar{\tau'}] \upharpoonright_{\text{dom}([\bar{t'} \leftarrow \bar{\tau'}]) \setminus \{t\}}) = \tau''}{(\lambda t : \kappa. \tau)[\bar{t'} \leftarrow \bar{\tau'}] = \lambda t : \kappa. \tau''} \quad \frac{\tau_1[\bar{t'} \leftarrow \bar{\tau'}] = \tau'_1 \quad \tau_2[\bar{t'} \leftarrow \bar{\tau'}] = \tau'_2}{(\tau_1 \tau_2)[\bar{t'} \leftarrow \bar{\tau'}] = \tau'_1 \tau'_2}$$

α -Equality:

$$\boxed{e_1 \equiv_{\alpha} e_2}$$

$$\frac{x_1 = x_2}{x_1 \equiv_{\alpha} x_2} \quad \frac{e_1 \equiv_{\alpha} e_2 \quad e'_1 \equiv_{\alpha} e'_2}{e_1 e'_1 \equiv_{\alpha} e_2 e'_2} \quad \frac{\tau_1 \equiv_{\alpha} \tau_2 \quad x' \notin fv(e_1) \cup fv(e_2) \quad e_1[x_1 \leftarrow x'] \equiv_{\alpha} e_2[x_2 \leftarrow x']}{\lambda x_1 : \tau_1. e_1 \equiv_{\alpha} \lambda x_2 : \tau_2. e_2}$$

$$\frac{e_1 \equiv_{\alpha} e_2 \quad \tau_1 \equiv_{\alpha} \tau_2}{e_1 \tau_1 \equiv_{\alpha} e_2 \tau_2} \quad \frac{t' \notin tyfv(e_1) \cup tyfv(e_2) \quad e_1[t_1 \leftarrow t'] \equiv_{\alpha} e_2[t_2 \leftarrow t']}{\Lambda t_1 : \kappa. e_1 \equiv_{\alpha} \Lambda t_2 : \kappa. e_2}$$

$$\boxed{\tau_1 \equiv_{\alpha} \tau_2}$$

$$\frac{t_1 = t_2}{t_1 \equiv_{\alpha} t_2} \quad \frac{\tau_1 \equiv_{\alpha} \tau_2 \quad \tau'_1 \equiv_{\alpha} \tau'_2}{\tau_1 \rightarrow \tau'_1 \equiv_{\alpha} \tau_2 \rightarrow \tau'_2} \quad \frac{t' \notin tyfv(\tau_1) \cup tyfv(\tau_2) \quad \tau_1[t_1 \leftarrow t'] \equiv_{\alpha} \tau_2[t_2 \leftarrow t']}{\forall t_1 : \kappa. \tau_1 \equiv_{\alpha} \forall t_2 : \kappa. \tau_2}$$

$$\frac{t' \notin tyfv(\tau_1) \cup tyfv(\tau_2) \quad \tau_1[t_1 \leftarrow t'] \equiv_{\alpha} \tau_2[t_2 \leftarrow t']}{\lambda t_1 : \kappa. \tau_1 \equiv_{\alpha} \lambda t_2 : \kappa. \tau_2} \quad \frac{\tau_1 \equiv_{\alpha} \tau_2 \quad \tau'_1 \equiv_{\alpha} \tau'_2}{\tau_1 \tau'_1 \equiv_{\alpha} \tau_2 \tau'_2}$$

定理 23 (Correctness of Substitution). 置換 $[\bar{x}' \leftarrow \bar{e}']$ について, $X = \text{dom}([\bar{x}' \leftarrow \bar{e}'])$ とした時,

$$fv(e[\bar{x}' \leftarrow \bar{e}']) = (fv(e) \setminus X) \cup \bigcup_{x \in fv(e) \cap X} fv([\bar{x}' \leftarrow \bar{e}'](x)).$$

□

定理 24 (Correctness of Type Substitution). 式 e , 型 τ , 型置換 $[\bar{t}' \leftarrow \bar{\tau}']$ について, $T = \text{dom}([\bar{t}' \leftarrow \bar{\tau}'])$ とした時,

$$\begin{aligned} tyfv(e[\bar{t}' \leftarrow \bar{\tau}']) &= (tyfv(e) \setminus T) \cup \bigcup_{t \in tyfv(e) \cap T} tyfv([\bar{t}' \leftarrow \bar{\tau}'](t)) \\ tyfv(\tau[\bar{t}' \leftarrow \bar{\tau}']) &= (tyfv(\tau) \setminus T) \cup \bigcup_{t \in tyfv(\tau) \cap T} tyfv([\bar{t}' \leftarrow \bar{\tau}'](t)). \end{aligned}$$

□

定理 25 (α -Equality Does Not Touch Free Variables).

- $\tau_1 \equiv_\alpha \tau_2$ ならば $tyfv(\tau_1) = tyfv(\tau_2)$.
- $e_1 \equiv_\alpha e_2$ ならば, $fv(e_1) = fv(e_2)$, $tyfv(e_1) = tyfv(e_2)$.

□

2.6.2 Typing Semantics

Kinding:

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\begin{aligned} & \frac{\Gamma(t) = \kappa}{\Gamma \vdash t : \kappa} \text{K-Var} \\ & \frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma \vdash \tau_2 : \Omega}{\Gamma \vdash \tau_1 \rightarrow \tau_2 : \Omega} \text{K-Arrow} \\ & \frac{\Gamma, t : \kappa \vdash \tau : \Omega}{\Gamma \vdash \forall t : \kappa. \tau : \Omega} \text{K-Forall} \\ & \frac{\Gamma, t : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda t : \kappa_1. \tau : \kappa_1 \rightarrow \kappa_2} \text{K-Abs} \\ & \frac{\Gamma \vdash \tau_1 : \kappa_2 \rightarrow \kappa \quad \Gamma \vdash \tau_2 : \kappa_2}{\Gamma \vdash \tau_1 \tau_2 : \kappa} \text{K-App} \end{aligned}$$

Type equivalence:

$$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$$

$$\begin{aligned} & \frac{\Gamma, t : \kappa_2 \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa_2}{\Gamma \vdash (\lambda t : \kappa_2. \tau_1) \tau_2 \equiv \tau_1[t \leftarrow \tau_2] : \kappa} \text{T-Eq-}\beta\text{-Lam} & \frac{t \notin tyfv(\tau) \quad \Gamma \vdash \tau : \kappa_1 \rightarrow \kappa_2}{\Gamma \vdash (\lambda t : \kappa_1. \tau) t \equiv \tau : \kappa_1 \rightarrow \kappa_2} \text{T-Eq-}\eta\text{-Lam} \\ & \frac{\tau_1 \equiv_\alpha \tau_2 \quad \Gamma \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa} \text{T-Eq-}\alpha\text{-Refl} \\ & \frac{\Gamma \vdash \tau_2 \equiv \tau_1 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa} \text{T-Eq-Sym} & \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa \quad \Gamma \vdash \tau_2 \equiv \tau_3 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_3 : \kappa} \text{T-Eq-Trans} \\ & \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \Omega}{\Gamma \vdash \tau_1 \rightarrow \tau'_1 \equiv \tau_2 \rightarrow \tau'_2 : \Omega} \text{T-Eq-Cong-Arrow} & \frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash \forall t : \kappa. \tau_1 \equiv \forall t : \kappa. \tau_2 : \Omega} \text{Eq-Cong-Forall} \\ & \frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \kappa'}{\Gamma \vdash \lambda t : \kappa. \tau_1 \equiv \lambda t : \kappa. \tau_2 : \kappa \rightarrow \kappa'} \text{T-Eq-Cong-Abs} & \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa' \rightarrow \kappa \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \kappa'}{\Gamma \vdash \tau_1 \tau'_1 \equiv \tau_2 \tau'_2 : \kappa} \text{Eq-Cong-App} \end{aligned}$$

定理 26 (Respect Kinding). $\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa$ ならば, $\Gamma \vdash \tau_1 : \kappa$ かつ $\Gamma \vdash \tau_2 : \kappa$.

□

Typing:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c} \frac{\Gamma \vdash \tau : \Omega \quad \Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{ T-Var} \\ \frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \text{ T-Abs} \\ \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \text{ T-App} \\ \frac{\Gamma, t : \kappa \vdash e : \tau}{\Gamma \vdash \Lambda t : \kappa. e : \forall t : \kappa. \tau} \text{ T-UnivAbs} \\ \frac{\Gamma \vdash e : \forall t : \kappa. \tau_1 \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash e \tau_2 : \tau_1[t \leftarrow \tau_2]} \text{ T-UnivApp} \\ \frac{\Gamma \vdash \tau \equiv \tau' : \Omega \quad \Gamma \vdash e : \tau'}{\Gamma \vdash e : \tau} \text{ T-Equiv} \end{array}$$

特に, $\cdot \vdash e : \tau$ の時, $e : \tau$ と表記.

定理 27 (Respect Type Kind). $\Gamma \vdash e : \tau$ ならば, $\Gamma \vdash \tau : \Omega$.

□

2.6.3 Evaluation Semantics (Call-By-Value)

$$\begin{array}{lcl} v & ::= & \lambda x : \tau. e \\ & | & \Lambda t : \kappa. e \\ C & ::= & [] \\ & | & C e \\ & | & v C \\ & | & C \tau \end{array}$$

Small Step:

$$\boxed{e \Rightarrow e'}$$

$$\begin{array}{c} \overline{(\lambda x : \tau. e) v \Rightarrow e[x \leftarrow v]} \\ \overline{(\Lambda t : \kappa. e) \tau \Rightarrow e[t \leftarrow \tau]} \\ \frac{e \Rightarrow e'}{C[e] \Rightarrow C[e']} \end{array}$$

Big Step:

$$\boxed{e \Downarrow v}$$

$$\begin{array}{c} \frac{e_1 \Downarrow \lambda x : \tau. e'_1 \quad e_2 \Downarrow v_2 \quad e'_1[x \leftarrow v_2] \Downarrow v}{e_1 e_2 \Downarrow v} \\ \frac{e \Downarrow \Lambda t : \kappa. e'_1 \quad e'_1[t \leftarrow \tau] \Downarrow v}{e \tau \Downarrow v} \end{array}$$

定理 28 (Adequacy of Small Step and Big Step). $e \Rightarrow^* v$ iff $e \Downarrow v$.

□

定理 29 (Type Soundness). $e : \tau$ の時, $e \Rightarrow^* v$, $e \Downarrow v$ となる $v = \text{nf}(\Rightarrow, e)$ が存在し,

- $\tau = \tau_1 \rightarrow \tau_2$ の時, $v \equiv_{\alpha} \lambda x' : \tau_1. e'$ となる $\lambda x' : \tau_1. e'$ が存在する.
- $\tau = \forall t : \kappa. \tau_1$ の時, $v \equiv_{\alpha} \Lambda t : \kappa. e'$ となる $\Lambda t : \kappa. e'$ が存在する.

□

2.6.4 Equational Reasoning

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c}
\frac{\Gamma, x : \tau_2 \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\lambda x : \tau_2. e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau} \text{Eq-}\beta\text{-Lam} \quad \frac{x \notin \text{fv}(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash (\lambda x : \tau_1. e) x \equiv e : \tau_1 \rightarrow \tau_2} \text{Eq-}\eta\text{-Lam} \\
\frac{\Gamma, t : \kappa \vdash e : \tau}{\Gamma \vdash (\Lambda t : \kappa. e) \tau_2 \equiv e[t \leftarrow \tau_2] : \tau[t \leftarrow \tau_2]} \text{Eq-}\beta\text{-UnivLam} \quad \frac{t \notin \text{tyfv}(e) \quad \Gamma \vdash e : \forall t : \kappa. \tau}{\Gamma \vdash (\Lambda t : \kappa. e) t \equiv e : \forall t : \kappa. \tau} \text{Eq-}\eta\text{-UnivLam} \\
\frac{e_1 \equiv_{\alpha} e_2 \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Refl} \quad \frac{\tau \equiv_{\alpha} \tau' \quad \Gamma \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Type} \\
\frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-Sym} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau \quad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau} \text{Eq-Trans} \\
\frac{\Gamma, x : \tau \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash \lambda x : \tau. e_1 \equiv \lambda x : \tau. e_2 : \tau \rightarrow \tau'} \text{Eq-Cong-Abs} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \rightarrow \tau \quad \Gamma \vdash e'_1 \equiv e'_2 : \tau'}{\Gamma \vdash e_1 e'_1 \equiv e_2 e'_2 : \tau} \text{Eq-Cong-App} \\
\frac{\Gamma, t : \kappa \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \Lambda t : \kappa. e_1 \equiv \Lambda t : \kappa. e_2 : (\forall t : \kappa. \tau)} \text{Eq-Cong-UnivAbs} \\
\frac{\Gamma \vdash e_1 \equiv e_2 : \forall t : \kappa. \tau \quad \Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash e_1 \tau_1 \equiv e_2 \tau_2 : \tau[t \leftarrow \tau_1]} \text{Eq-Cong-UnivApp}
\end{array}$$

特に, $\cdot \vdash e_1 \equiv e_2 : \tau$ の時, $e_1 \equiv e_2 : \tau$ と表記.

定理 30 (Respect Typing). $\Gamma \vdash e_1 \equiv e_2 : \tau$ ならば, $\Gamma \vdash e_1 : \tau$ かつ $\Gamma \vdash e_2 : \tau$. □

定理 31 (Respect Evaluation). $e_1 \equiv e_2 : \tau$ の時, $e'_1 \Rightarrow^* e_1$, $e_2 \Rightarrow^* e'_2$ ならば $e'_1 \equiv e'_2 : \tau$. □

系 32. $e_1 \equiv e_2 : \tau$ の時, $e_1 \Rightarrow^* e'_1$, $e_2 \Rightarrow^* e'_2$ ならば $e'_1 \equiv e'_2 : \tau$. □

証明. $e_1 \Rightarrow^* e_1$ より, 定理 21 から $e_1 \equiv e'_2 : \tau$. よって, T-Sym から $e'_2 \equiv e_1 : \tau$ であり, $e'_2 \Rightarrow^* e'_2$ より定理 21 から $e'_2 \equiv e'_1 : \tau$. 故に, T-Sym から $e'_1 \equiv e'_2 : \tau$. ■

2.6.5 Definability

Product

Product of τ_1 and τ_2 :

$$\begin{aligned}
\tau_1 \times \tau_2 &\stackrel{\text{def}}{=} \forall t : \Omega. (\tau_1 \rightarrow \tau_2 \rightarrow t) \rightarrow t \\
\langle e_1, e_2 \rangle &\stackrel{\text{def}}{=} \Lambda t : \Omega. \lambda x : \tau_1 \rightarrow \tau_2 \rightarrow t. x e_1 e_2 \\
\pi_1 e &\stackrel{\text{def}}{=} e \tau_1 \lambda x_1. \lambda x_2. x_1 \\
\pi_2 e &\stackrel{\text{def}}{=} e \tau_2 \lambda x_1. \lambda x_2. x_2
\end{aligned}$$

Admissible kinding:

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma \vdash \tau_2 : \Omega}{\Gamma \vdash \tau_1 \times \tau_2 : \Omega} \text{T-Product}$$

Admissible type equality:

$$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$$

$$\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \Omega}{\Gamma \vdash \tau_1 \times \tau'_1 \equiv \tau_2 \times \tau'_2 : \Omega} \text{ T-Eq-Product}$$

Admissible typing:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2} \text{ T-Product} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_1 e : \tau_1} \text{ T-Proj-1} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_2 e : \tau_2} \text{ T-Proj-2}$$

Admissible equality:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \pi_1 \langle e_1, e_2 \rangle \equiv e_1 : \tau_1} \text{ Eq-}\beta\text{-Product-1} \quad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \pi_2 \langle e_1, e_2 \rangle \equiv e_2 : \tau_2} \text{ Eq-}\beta\text{-Product-2}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \langle \pi_1 e, \pi_2 e \rangle \equiv e : \tau_1 \times \tau_2} \text{ Eq-}\eta\text{-Product}$$

Existential Type

Existence of $\exists t : \kappa. \tau$:

$$\begin{aligned} \exists t : \kappa. \tau &\stackrel{\text{def}}{=} \forall t' : \Omega. (\forall t : \kappa. \tau \rightarrow t') \rightarrow t' \\ \text{pack}\langle \tau_t, e \rangle_{\exists t : \kappa. \tau} &\stackrel{\text{def}}{=} \Lambda t' : \Omega. \lambda x : (\forall t : \kappa. \tau \rightarrow t'). x \tau_t e \\ \text{unpack}\langle t : \kappa, x : \tau \rangle &= e_1. \tau_2. e_2 \stackrel{\text{def}}{=} e_1 \tau_2 (\Lambda t : \kappa. \lambda x : \tau. e_2) \end{aligned}$$

Admissible kinding:

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\frac{\Gamma, t : \kappa \vdash \tau : \Omega}{\Gamma \vdash \exists t : \kappa. \tau : \Omega} \text{ T-Exist}$$

Admissible type equality:

$$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$$

$$\frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash \exists t : \kappa. \tau_1 \equiv \exists t : \kappa. \tau_2 : \Omega} \text{ T-Eq-Cong-Exist}$$

Admissible typing rule:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma, t : \kappa \vdash \tau : \Omega \quad \Gamma \vdash \tau_t : \kappa \quad \Gamma \vdash e : \tau[t \leftarrow \tau_t]}{\Gamma \vdash \text{pack}\langle \tau_t, e \rangle_{\exists t : \kappa. \tau} : \exists t : \kappa. \tau} \text{ T-Pack}$$

$$\frac{\Gamma \vdash e_1 : \exists t : \kappa. \tau \quad \Gamma, t : \kappa, x : \tau \vdash e_2 : \tau_2 \quad t \notin \text{tyfv}(\tau_2)}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau \rangle = e_1. \tau_2. e_2 : \tau_2} \text{ T-Unpack}$$

Admissible equality:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau_t : \kappa \quad \Gamma \vdash e_1 : \tau_1[t \leftarrow \tau_t] \quad \Gamma, t : \kappa, x : \tau_1 \vdash e_2 : \tau_2 \quad t \notin \text{tyfv}(\tau_2)}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau_1 \rangle = \text{pack}\langle \tau_t, e_1 \rangle_{\exists t : \kappa. \tau_1}. \tau_2. e_2 \equiv e_2[t \leftarrow \tau_t][x \leftarrow e_1] : \tau_2} \text{Eq-}\beta\text{-Exist} \\
\frac{\Gamma \vdash e : (\exists t : \kappa. \tau) \quad \tau' \equiv \exists t : \kappa. \tau}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau \rangle = e. \tau'. \text{pack}\langle t, x \rangle_{\exists t : \kappa. \tau} \equiv e : (\exists t : \kappa. \tau)} \text{Eq-}\eta\text{-Exist}
\end{array}$$

2.7 $\lambda \mu$ -Calculus

Alias: $\lambda \mu$ [Sel01][Roc05]

2.7.1 Syntax

$\tau ::=$	t	(type variable)
	\top	(top type)
	$\tau \times \tau$	(product type)
	$\tau \rightarrow \tau$	(function type)
	\perp	(bottom type)
$e ::=$	x	(variable)
	$\langle \rangle$	(top value)
	$\langle e, e \rangle$	(product)
	$\pi_1 e$	(left projection)
	$\pi_2 e$	(right projection)
	$\lambda x : \tau. e$	(abstraction)
	$e e$	(application)
	$[\alpha]e$	(naming)
	$\mu \alpha : \tau. e$	(un-naming)
$\Gamma ::=$	\cdot	
	$\Gamma, x : \tau$	
$\Delta ::=$	\cdot	
	$\alpha : \tau, \Delta$	

Environment Reference:

$$\boxed{\Gamma(x) = \tau}$$

$$\frac{x = x'}{(\Gamma, x' : \tau)(x) = \tau} \quad \frac{x \neq x' \quad \Gamma(x) = \tau}{(\Gamma, x' : \tau')(x) = \tau}$$

$$\boxed{\Delta(\alpha) = \tau}$$

$$\frac{\alpha = \alpha'}{(\alpha' : \tau, \Delta)(\alpha) = \tau} \quad \frac{\alpha \neq \alpha' \quad \Delta(\alpha) = \tau}{(\alpha' : \tau', \Delta)(\alpha) = \tau}$$

2.7.2 Typing Semantics

$$\boxed{\Gamma \vdash e : \tau \mid \Delta}$$

$$\begin{array}{c} \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau \mid \Delta} \text{ T-Var} \\ \frac{}{\Gamma \vdash \langle \rangle : \top \mid \Delta} \text{ T-Top} \\ \frac{\Gamma \vdash e_1 : \tau_1 \mid \Delta \quad \Gamma \vdash e_2 : \tau_2 \mid \Delta}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2 \mid \Delta} \text{ T-Product} \\ \frac{\Gamma \vdash e : \tau_1 \times \tau_2 \mid \Delta}{\Gamma \vdash \pi_1 e : \tau_1 \mid \Delta} \text{ T-Proj-1} \\ \frac{\Gamma \vdash e : \tau_1 \times \tau_2 \mid \Delta}{\Gamma \vdash \pi_2 e : \tau_2 \mid \Delta} \text{ T-Proj-2} \\ \frac{\Gamma, x : \tau_1 \vdash e : \tau_2 \mid \Delta}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2 \mid \Delta} \text{ T-Abs} \end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \mid \Delta \quad \Gamma \vdash e_2 : \tau_2 \mid \Delta}{\Gamma \vdash e_1 e_2 : \tau \mid \Delta} \text{ T-App} \\
\frac{\Delta(\alpha) = \tau \quad \Gamma \vdash e : \tau \mid \Delta}{\Gamma \vdash [\alpha]e : \perp \mid \Delta} \text{ T-Name} \\
\frac{\Gamma \vdash e : \perp \mid \alpha : \tau, \Delta}{\Gamma \vdash (\mu\alpha : \tau. e) : \tau \mid \Delta} \text{ T-Unname}
\end{array}$$

2.7.3 Equivalence

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau \mid \Delta}$$

$$\begin{array}{c}
\frac{\Gamma, x : \tau_2 \vdash e_1 : \tau \mid \Delta \quad \Gamma \vdash e_2 : \tau_2 \mid \Delta}{\Gamma \vdash (\lambda x : \tau_2. e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau \mid \Delta} \text{ Eq-}\beta\text{-Lam} \\
\frac{x \notin fv(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2 \mid \Delta}{\Gamma \vdash (\lambda x : \tau_1. e x) \equiv e : \tau_1 \rightarrow \tau_2 \mid \Delta} \text{ Eq-}\eta\text{-Lam} \\
\frac{\Gamma \vdash e : \top \mid \Delta}{\Gamma \vdash \langle \rangle \equiv e : \top \mid \Delta} \text{ Eq-}\eta\text{-Top} \\
\frac{\Gamma \vdash e_1 : \tau_1 \mid \Delta \quad \Gamma \vdash e_2 : \tau_2 \mid \Delta}{\Gamma \vdash \pi_1 \langle e_1, e_2 \rangle \equiv e_1 : \tau_1 \mid \Delta} \text{ Eq-}\beta\text{-Product-1} \\
\frac{\Gamma \vdash e_1 : \tau_1 \mid \Delta \quad \Gamma \vdash e_2 : \tau_2 \mid \Delta}{\Gamma \vdash \pi_2 \langle e_1, e_2 \rangle \equiv e_2 : \tau_2 \mid \Delta} \text{ Eq-}\beta\text{-Product-2} \\
\frac{\Gamma \vdash e : \tau_1 \times \tau_2 \mid \Delta}{\Gamma \vdash \langle \pi_1 e, \pi_2 e \rangle \equiv e : \tau_1 \times \tau_2 \mid \Delta} \text{ Eq-}\eta\text{-Product} \\
\frac{\alpha_1 \notin fv(e) \quad \Gamma \vdash e : \perp \mid \alpha : \tau_1 \times \tau_2, \Delta}{\Gamma \vdash \pi_1(\mu\alpha : \tau_1 \times \tau_2. e) \equiv \mu\alpha_1 : \tau_1. e[[\alpha](-) \leftarrow [\alpha_1](\pi_1(-))]] : \tau_1 \mid \Delta} \text{ Eq-}\zeta\text{-Product-1} \\
\frac{\alpha_2 \notin fv(e) \quad \Gamma \vdash e : \perp \mid \alpha : \tau_1 \times \tau_2, \Delta}{\Gamma \vdash \pi_2(\mu\alpha : \tau_1 \times \tau_2. e) \equiv \mu\alpha_2 : \tau_2. e[[\alpha](-) \leftarrow [\alpha_2](\pi_2(-))]] : \tau_2 \mid \Delta} \text{ Eq-}\zeta\text{-Product-2} \\
\frac{\Gamma \vdash e : \perp \mid \alpha_2 : \tau_\alpha, \Delta}{\Gamma \vdash [\alpha_1](\mu\alpha_2 : \tau_\alpha. e) \equiv e[\alpha_2 \leftarrow \alpha_1] : \perp \mid \Delta} \text{ Eq-}\beta\text{-Mu} \\
\frac{\Gamma \vdash e : \tau \mid \Delta}{\Gamma \vdash (\mu\alpha : \tau. [\alpha]e) \equiv e : \tau \mid \Delta} \text{ Eq-}\eta\text{-Mu} \\
\frac{\alpha_2 \notin fv(e_1) \cup fv(e_2) \quad \Gamma \vdash e_1 : \perp \mid \alpha : \tau_1 \rightarrow \tau_2, \Delta \quad \Gamma \vdash e_2 : \tau_2 \mid \Delta}{\Gamma \vdash (\mu\alpha : \tau_1 \rightarrow \tau_2. e_1) e_2 \equiv \mu\alpha_2 : \tau_2. e_1[[\alpha](-) \leftarrow [\alpha_2]((-) e_2)]] : \tau_2 \mid \Delta} \text{ Eq-}\zeta\text{-Mu}
\end{array}$$

2.7.4 Elaboration (Call-By-Value)

$$\boxed{\Gamma \vdash e : \tau \rightsquigarrow e'}$$

$$\begin{array}{c}
\frac{\Gamma(x_{x_0}) = V_\tau}{\Gamma \vdash x_0 : \tau \rightsquigarrow \lambda x_k : K_\tau. x_k x_{x_0}} \\
\frac{}{\Gamma \vdash \langle \rangle : \top \rightsquigarrow \lambda x_k : K_\top. x_k \langle \rangle} \\
\frac{\Gamma \vdash e_1 : \tau_1 \rightsquigarrow e'_1 \quad \Gamma \vdash e_2 : \tau_2 \rightsquigarrow e'_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2 \rightsquigarrow \lambda x_k : K_{\tau_1 \times \tau_2}. e'_1 (\lambda x_1 : V_{\tau_1}. e'_2 (\lambda x_2 : V_{\tau_2}. x_k \langle x_1, x_2 \rangle))} \\
\frac{\Gamma \vdash e : \tau_1 \times \tau_2 \rightsquigarrow e'}{\Gamma \vdash \pi_1 e : \tau_1 \rightsquigarrow \lambda x_k : K_{\tau_1}. e' (\lambda x : V_{\tau_1} \times V_{\tau_2}. x_k (\pi_1 x))} \\
\frac{\Gamma \vdash e : \tau_1 \times \tau_2 \rightsquigarrow e'}{\Gamma \vdash \pi_2 e : \tau_2 \rightsquigarrow \lambda x_k : K_{\tau_2}. e' (\lambda x : V_{\tau_1} \times V_{\tau_2}. x_k (\pi_2 x))} \\
\frac{\Gamma, x_{x_0} : V_{\tau_1} \vdash e : \tau_2 \rightsquigarrow e'}{\Gamma \vdash (\lambda x_0 : \tau_1. e) : \tau_1 \rightarrow \tau_2 \rightsquigarrow \lambda x_k : K_{\tau_1 \rightarrow \tau_2}. x_k (\lambda x : V_{\tau_1} \times K_{\tau_2}. (\lambda x_{x_0} : V_{\tau_1}. e') (\pi_1 x) (\pi_2 x))}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \rightsquigarrow e'_1 \quad \Gamma \vdash e_2 : \tau_2 \rightsquigarrow e'_2}{\Gamma \vdash e_1 e_2 : \tau \rightsquigarrow \lambda x_k : K_\tau. e'_1 (\lambda x_1 : V_{\tau_2 \rightarrow \tau}. e'_2 (\lambda x_2 : V_{\tau_2}. x_1 \langle x_2, x_k \rangle))} \\
\frac{\Gamma, x_\alpha : K_\tau \vdash e : \perp \rightsquigarrow e'}{\Gamma \vdash (\mu \alpha : \tau. e) : \tau \rightsquigarrow \lambda x_\alpha : K_\tau. e' (\lambda x : \perp. \text{case } x \{ \})} \\
\frac{\Gamma(x_\alpha) = K_\tau \quad \Gamma \vdash e : \tau \rightsquigarrow e'}{\Gamma \vdash [\alpha]e : \tau \rightsquigarrow \lambda x_k : K_\perp. e' x_\alpha}
\end{array}$$

$$\boxed{V_\tau = \tau'}$$

$$\begin{array}{c}
\overline{V_\tau = \tau} \\
\frac{V_{\tau_1} = \tau'_1 \quad V_{\tau_2} = \tau'_2}{V_{\tau_1 \times \tau_2} = V_{\tau'_1} \times V_{\tau'_2}} \\
\frac{V_{\tau_1} = \tau'_1 \quad K_{\tau_2} = \tau'_2}{V_{\tau_1 \rightarrow \tau_2} = \tau'_1 \times \tau'_2 \rightarrow R} \\
\overline{V_\perp = \perp}
\end{array}$$

Abbreviation:

$$\begin{array}{l}
K_\tau \stackrel{\text{def}}{=} V_\tau \rightarrow R \\
C_\tau \stackrel{\text{def}}{=} K_\tau \rightarrow R
\end{array}$$

定理 33. $\Gamma \vdash e : \tau \rightsquigarrow e'$ ならば, $\Gamma \vdash e' : C_\tau$. □

定理 34. $\Gamma \vdash e : \tau \mid \Delta \iff V(\Gamma), K(\Delta) \vdash e : \tau \rightsquigarrow e'$. ただし,

$$\begin{array}{l}
V(\Gamma) \stackrel{\text{def}}{=} \begin{cases} V(\Gamma'), x_{x'} : V_{\tau'} & (\Gamma = \Gamma', x' : \tau') \\ . & (\Gamma = \cdot) \end{cases} \\
K(\Delta) \stackrel{\text{def}}{=} \begin{cases} x_\alpha : K_\tau, K(\Delta') & (\Delta = \alpha : \tau, \Delta') \\ . & (\Delta = \cdot) \end{cases} .
\end{array}$$

□

2.7.5 Elaboration (Call-By-Name)

$$\boxed{\Gamma \vdash e : \tau \rightsquigarrow e'}$$

$$\begin{array}{c}
\frac{\Gamma(x_{x_0}) = C_\tau}{\Gamma \vdash x_0 : \tau \rightsquigarrow \lambda x_k : K_\tau. x_{x_0} x_k} \\
\frac{\Gamma \vdash \langle \rangle : \top \rightsquigarrow \lambda x_k : \perp. \text{case } x_k \{ \}}{\Gamma \vdash e_1 : \tau_1 \rightsquigarrow e'_1 \quad \Gamma \vdash e_2 : \tau_2 \rightsquigarrow e'_2} \\
\frac{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2 \rightsquigarrow \lambda x_k : K_{\tau_1} + K_{\tau_2}. \text{case } x_k \{ x_{k_1}. e'_1 x_{k_1} \mid x_{k_2}. e'_2 x_{k_2} \}}{\Gamma \vdash e : \tau_1 \times \tau_2 \rightsquigarrow e'} \\
\frac{\Gamma \vdash \pi_1 e : \tau_1 \rightsquigarrow \lambda x_k : K_{\tau_1}. e' (i_1 x_k)}{\Gamma, x_{x_1} : C_{\tau_1} \vdash e : \tau_2 \rightsquigarrow e'} \\
\frac{\Gamma \vdash (\lambda x_1 : \tau_1. e) : \tau_1 \rightarrow \tau_2 \rightsquigarrow \lambda x_k : C_{\tau_1} \times K_{\tau_2}. e' [x_{x_1} \leftarrow \pi_1 x_k] (\pi_2 x_k)}{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \rightsquigarrow e'_1 \quad \Gamma \vdash e_2 : \tau_2 \rightsquigarrow e'_2} \\
\frac{\Gamma \vdash e_1 e_2 : \tau \rightsquigarrow \lambda x_k : K_\tau. e'_1 \langle e'_2, x_k \rangle}{\Gamma(x_\alpha) = K_\tau \quad \Gamma \vdash e : \tau \rightsquigarrow e'} \\
\frac{\Gamma \vdash [\alpha]e : \perp \rightsquigarrow \lambda x_k : K_\perp. e' x_\alpha}{\Gamma, x_\alpha : K_\tau \vdash e : \perp \rightsquigarrow e'} \\
\frac{\Gamma, x_\alpha : K_\tau \vdash e : \perp \rightsquigarrow e'}{\Gamma \vdash (\mu \alpha : \tau. e) : \tau \rightsquigarrow \lambda x_\alpha : K_\tau. e' \langle \rangle}
\end{array}$$

$$\boxed{K_\tau = \tau'}$$

$$\begin{array}{c} \overline{K_\top = \perp} \\ \frac{K_{\tau_1} = \tau'_1 \quad K_{\tau_2} = \tau'_2}{K_{\tau_1 \times \tau_2} = \tau'_1 + \tau'_2} \\ \frac{C_{\tau_1} = \tau'_1 \quad K_{\tau_2} = \tau'_2}{K_{\tau_1 \rightarrow \tau_2} = \tau'_1 \times \tau'_2} \\ \overline{K_\perp = \top} \end{array}$$

Abbreviation:

$$C_\tau \stackrel{\text{def}}{=} K_\tau \rightarrow R$$

定理 35. $\Gamma \vdash e : \tau \rightsquigarrow e'$ ならば, $\Gamma \vdash e' : C_\tau$. □

定理 36. $\Gamma \vdash e : \tau \mid \Delta \iff C(\Gamma), K(\Delta) \vdash e : \tau \rightsquigarrow e'$. ただし,

$$\begin{aligned} C(\Gamma) &\stackrel{\text{def}}{=} \begin{cases} C(\Gamma'), x_{x'} : C_{\tau'} & (\Gamma = \Gamma', x' : \tau') \\ \cdot & (\Gamma = \cdot) \end{cases} \\ K(\Delta) &\stackrel{\text{def}}{=} \begin{cases} x_\alpha : K_\tau, K(\Delta') & (\Delta = \alpha : \tau, \Delta') \\ \cdot & (\Delta = \cdot) \end{cases} . \end{aligned}$$

□

2.8 WIP: Lambda Bar Mu Mu Tilde Calculus

$\bar{\lambda} \mu \tilde{\mu}$ -Calculus

2.9 WIP: π -Calculus

第 3 章

Basic Algorithms

3.1 Martelli-Montanari Algorithm

[MM82]

$$\begin{array}{c}
 \overline{\mathcal{U}(x, x) = \emptyset} \\
 \overline{x_1 \neq x_2} \\
 \overline{\mathcal{U}(x_1, x_2) = \{x_1 \mapsto x_2\}} \\
 \overline{\mathcal{U}(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) = \bigcup_{1 \leq i \leq n} \mathcal{U}(a_i, b_i)} \\
 \overline{x \notin \text{fv}(f(a_1, \dots, a_n))} \\
 \overline{\mathcal{U}(x, f(a_1, \dots, a_n)) = \{x \mapsto f(a_1, \dots, a_n)\}} \\
 \overline{x \notin \text{fv}(f(a_1, \dots, a_n))} \\
 \overline{\mathcal{U}(f(a_1, \dots, a_n), x) = \{x \mapsto f(a_1, \dots, a_n)\}}
 \end{array}$$

第 4 章

Modules and Phase Distinction

4.1 Light-Weight F-ing modules

[RRD14]

4.1.1 Internal Language

Having same power as System F ω

Syntax:

$$\begin{aligned}
 \kappa &::= \Omega \mid \kappa \rightarrow \kappa \\
 \tau &::= t \mid \tau \rightarrow \tau \mid \overline{\{l : \tau\}} \mid \forall t : \kappa. \tau \mid \exists t : \kappa. \tau \mid \lambda t : \kappa. \tau \mid \tau \tau \\
 e &::= x \mid \lambda x : \tau. e \mid e e \mid \overline{\{l = e\}} \mid e.l \mid \Lambda t : \kappa. e \mid e \tau \mid \text{pack}\langle \tau, e \rangle_\tau \mid \text{unpack}\langle t : \kappa, x : \tau \rangle = e \text{ in } e \\
 \Gamma &::= \cdot \mid \Gamma, t : \kappa \mid \Gamma, x : \tau
 \end{aligned}$$

Abbreviation:

$$\begin{aligned}
 \Sigma.l &\stackrel{\text{def}}{=} \begin{cases} (\Sigma.l).\bar{l}' & (\bar{l} = l \bar{l}') \\ \Sigma & (\bar{l} = \epsilon) \end{cases} \\
 \bar{\tau}_1 \rightarrow \bar{\tau}_2 &\stackrel{\text{def}}{=} \begin{cases} \tau_1 \rightarrow (\bar{\tau}_1' \rightarrow \tau_2) & (\bar{\tau}_1 = \tau_1 \bar{\tau}_1') \\ \tau_2 & (\bar{\tau}_1 = \epsilon) \end{cases} \\
 \lambda \overline{x : \tau}. e &\stackrel{\text{def}}{=} \begin{cases} \lambda x : \tau. \lambda \overline{x' : \tau'}. e & (\overline{x : \tau} = x : \tau \overline{x' : \tau'}) \\ e & (\overline{x : \tau} = \epsilon) \end{cases} \\
 e_0 \bar{e}_1 &\stackrel{\text{def}}{=} \begin{cases} e_0 e_1 \bar{e}_1' & (\bar{e}_1 = e_1 \bar{e}_1') \\ e_0 & (\bar{e}_1 = \epsilon) \end{cases} \\
 \forall \overline{t : \kappa}. \tau &\stackrel{\text{def}}{=} \begin{cases} \forall t : \kappa. \forall \overline{t' : \kappa'}. \tau & (\overline{t : \kappa} = t : \kappa \overline{t' : \kappa'}) \\ \tau & (\overline{t : \kappa} = \epsilon) \end{cases} \\
 \Lambda \overline{t : \kappa}. e &\stackrel{\text{def}}{=} \begin{cases} \Lambda t : \kappa. \Lambda \overline{t' : \kappa'}. e & (\overline{t : \kappa} = t : \kappa \overline{t' : \kappa'}) \\ e & (\overline{t : \kappa} = \epsilon) \end{cases} \\
 e \bar{\tau} &\stackrel{\text{def}}{=} \begin{cases} e \tau \bar{\tau}' & (\bar{\tau} = \tau \bar{\tau}') \\ e & (\bar{\tau} = \epsilon) \end{cases} \\
 \text{let } \overline{x : \tau} = \bar{e}_1 \overline{t : \kappa} = \bar{\tau} \text{ in } e_2 &\stackrel{\text{def}}{=} (\lambda \overline{x : \tau}. \Lambda \overline{t : \kappa}. e_2) \bar{e}_1 \bar{\tau} \\
 \exists \overline{t : \kappa}. \tau &\stackrel{\text{def}}{=} \begin{cases} \exists t : \kappa. \exists \overline{t' : \kappa'}. \tau & (\overline{t : \kappa} = t : \kappa \overline{t' : \kappa'}) \\ \tau & (\overline{t : \kappa} = \epsilon) \end{cases} \\
 \text{pack}\langle \bar{\tau}, e \rangle_{\exists \overline{t : \kappa}. \tau_0} &\stackrel{\text{def}}{=} \begin{cases} \text{pack}\langle \tau, \text{pack}\langle \bar{\tau}', e \rangle_{\exists \overline{t' : \kappa'}. \tau_0} \rangle_{\exists \overline{t : \kappa}. \tau_0} & (\bar{\tau} = \tau \bar{\tau}', \overline{t : \kappa} = t : \kappa \overline{t' : \kappa'}) \\ e & (\bar{\tau} = \epsilon, \overline{t : \kappa} = \epsilon) \end{cases} \\
 (\text{unpack}\langle \overline{t : \kappa}, x : \tau \rangle = e_1 \text{ in } e_2) &\stackrel{\text{def}}{=} \begin{cases} \text{unpack}\langle t : \kappa, x_1 : \exists \overline{t' : \kappa'}. \tau \rangle = e_1 \text{ in } & (\overline{t : \kappa} = t : \kappa \overline{t' : \kappa'}) \\ \text{unpack}\langle \overline{t' : \kappa'}, x_2 : \tau \rangle = x_1 \text{ in } e_2 & \\ \text{let } x : \tau = e_1 \text{ in } e_2 & (\overline{t : \kappa} = \epsilon) \end{cases}
 \end{aligned}$$

Kinding:

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\begin{array}{c}
 \frac{\Gamma(t) = \kappa}{\Gamma \vdash t : \kappa} \quad \frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma \vdash \tau_2 : \Omega}{\Gamma \vdash \tau_1 \rightarrow \tau_2 : \Omega} \quad \frac{\bigwedge_l \Gamma \vdash \tau_l : \Omega}{\Gamma \vdash \{\bar{l} : \tau_l\} : \Omega} \\
 \frac{\Gamma, t : \kappa \vdash \tau : \Omega}{\Gamma \vdash \forall t : \kappa. \tau : \Omega} \quad \frac{\Gamma, t : \kappa \vdash \tau : \Omega}{\Gamma \vdash \exists t : \kappa. \tau : \Omega} \quad \frac{\Gamma, t : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda t : \kappa_1. \tau : \kappa_1 \rightarrow \kappa_2} \quad \frac{\Gamma \vdash \tau_1 : \kappa_2 \rightarrow \kappa \quad \Gamma \vdash \tau_2 : \kappa_2}{\Gamma \vdash \tau_1 \tau_2 : \kappa}
 \end{array}$$

Type equivalence:

$$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$$

$$\frac{\Gamma, t : \kappa_2 \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa_2}{\Gamma \vdash (\lambda t : \kappa_2. \tau_1) \tau_2 \equiv \tau_1[t \leftarrow \tau_2] : \kappa} \quad \frac{t \notin \text{tyfv}(\tau) \quad \Gamma \vdash \tau : \kappa_1 \rightarrow \kappa_2}{\Gamma \vdash (\lambda t : \kappa_1. \tau t) \equiv \tau : \kappa_1 \rightarrow \kappa_2}$$

$$\frac{\tau_1 \equiv_{\alpha} \tau_2 \quad \Gamma \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa} \quad \frac{\Gamma \vdash \tau_2 \equiv \tau_1 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa} \quad \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa \quad \Gamma \vdash \tau_2 \equiv \tau_3 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_3 : \kappa}$$

$$\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \Omega}{\Gamma \vdash \tau_1 \rightarrow \tau'_1 \equiv \tau_2 \rightarrow \tau'_2 : \Omega} \quad \frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash \forall t : \kappa. \tau_1 \equiv \forall t : \kappa. \tau_2 : \Omega}$$

$$\frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \kappa'}{\Gamma \vdash \lambda t : \kappa. \tau_1 \equiv \lambda t : \kappa. \tau_2 : \kappa \rightarrow \kappa'} \quad \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa' \rightarrow \kappa \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \kappa'}{\Gamma \vdash \tau_1 \tau'_1 \equiv \tau_2 \tau'_2 : \kappa}$$

Typing:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash \tau : \Omega \quad \Gamma(x) = \tau}{\Gamma \vdash x : \tau} \quad \frac{\Gamma \vdash \tau \equiv \tau' : \Omega \quad \Gamma \vdash e : \tau'}{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \quad \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau}$$

$$\frac{\bigwedge_l \Gamma \vdash e_l : \tau_l}{\Gamma \vdash \{\overline{l = e_l}\} : \{\overline{l = \tau_l}\}} \quad \frac{\Gamma \vdash e : \{\overline{l' = \tau_{l'}}\}}{\Gamma \vdash e.l : \tau_l}$$

$$\frac{\Gamma, t : \kappa \vdash e : \tau}{\Gamma \vdash \Lambda t : \kappa. e : (\forall t : \kappa. \tau)} \quad \frac{\Gamma \vdash e : (\forall t : \kappa. \tau_1) \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash e \tau_2 : \tau_1[t \leftarrow \tau_2]}$$

$$\frac{\Gamma, t : \kappa \vdash \tau : \Omega \quad \Gamma \vdash \tau_t : \kappa \quad \Gamma \vdash e : \tau[t \leftarrow \tau_t]}{\Gamma \vdash \text{pack}\langle \tau_t, e \rangle_{\exists t : \kappa. \tau} : (\exists t : \kappa. \tau)} \quad \frac{\Gamma \vdash e_1 : (\exists t : \kappa. \tau_1) \quad \Gamma, t : \kappa, x : \tau_1 \vdash e_2 : \tau}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau_1 \rangle = e_1 \text{ in } e_2 : \tau}$$

Reduction:

$$\begin{aligned} v &::= \lambda x : \tau. e \mid \{\overline{l = e}\} \mid \Lambda t : \kappa. e \mid \text{pack}\langle \tau_t, e \rangle_{\exists t : \kappa. \tau} \\ C &::= [] \mid C e \mid v C \mid \{\overline{l = v}, l = C, l = e\} \mid C.l \mid C \tau \mid \text{pack}\langle \tau, C \rangle_{\tau} \mid \text{unpack}\langle t : \kappa, x : \tau \rangle = C \text{ in } e \end{aligned}$$

$$\boxed{e \Rightarrow e'}$$

$$\frac{}{(\lambda x : \tau. e)v \Rightarrow e[x \leftarrow v]} \quad \frac{}{\{\overline{l' = v_{l'}}\}.l \Rightarrow v_l} \quad \frac{}{(\Lambda t : \kappa. e)\tau \Rightarrow e[t \leftarrow \tau]}$$

$$\frac{}{\text{unpack}\langle t : \kappa, x : \tau \rangle = \text{pack}\langle \tau_t, v \rangle_{\tau_{\exists}} \text{ in } e \Rightarrow e[t \leftarrow \tau_t][x \leftarrow v]} \quad \frac{e \Rightarrow e'}{C[e] \Rightarrow C[e']}$$

Equivalence:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\frac{\Gamma, x : \tau_2 \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\lambda x : \tau_2. e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau} \quad \frac{x \notin \text{fv}(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash (\lambda x : \tau_1. e x) \equiv e : \tau_1 \rightarrow \tau_2}$$

$$\frac{\bigwedge_{l'} \Gamma \vdash e_{l'} : \tau_{l'}}{\Gamma \vdash \{\overline{l' = e_{l'}}\}.l \equiv e_l : \tau_l} \quad \frac{\Gamma \vdash e : \{\overline{l : \tau_l}\}}{\Gamma \vdash \{\overline{l = e.l}\} \equiv e : \{\overline{l : \tau_l}\}}$$

$$\frac{\Gamma, t : \kappa \vdash e : \tau}{\Gamma \vdash (\Lambda t : \kappa. e) \tau_2 \equiv e[t \leftarrow \tau_2] : \tau[t \leftarrow \tau_2]} \quad \frac{t \notin \text{tyfv}(e) \quad \Gamma \vdash e : \forall t : \kappa. \tau}{\Gamma \vdash (\Lambda t : \kappa. e t) \equiv e : \forall t : \kappa. \tau}$$

$$\frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau'_1 : \Omega \quad \Gamma \vdash \tau_t : \kappa \quad \Gamma \vdash e_1 : \tau_1[t \leftarrow \tau_t] \quad \Gamma, t : \kappa, x : \tau_1 \vdash e_2 : \tau}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau'_1 \rangle = \text{pack}\langle \tau_t, e_1 \rangle_{\exists t : \kappa. \tau_1} \text{ in } e_2 \equiv e_2[t \leftarrow \tau_t][x \leftarrow e_1] : \tau}$$

$$\frac{\Gamma \vdash e : \exists t : \kappa. \tau \quad \Gamma, t : \kappa \vdash \tau \equiv \tau' : \Omega}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau' \rangle = e \text{ in } \text{pack}\langle t, x \rangle_{\exists t : \kappa. \tau} \equiv e : (\exists t : \kappa. \tau)}$$

$$\begin{array}{c}
\frac{e_1 \equiv_{\alpha} e_2 \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \quad \frac{\Gamma \vdash \tau \equiv \tau' : \Omega \quad \Gamma \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash e_1 \equiv e_2 : \tau} \\
\frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau \quad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau} \\
\frac{\Gamma, x : \tau \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash \lambda x : \tau. e_1 \equiv \lambda x : \tau. e_2 : \tau \rightarrow \tau'} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \rightarrow \tau \quad \Gamma \vdash e'_1 \equiv e'_2 : \tau'}{\Gamma \vdash e_1 e'_1 \equiv e_2 e'_2 : \tau} \\
\frac{\bigwedge_l \Gamma \vdash e_{l,1} \equiv e_{l,2} : \tau_l}{\Gamma \vdash \{l = e_{l,1}\} \equiv \{l = e_{l,2}\} : \{\overline{l} : \tau_l\}} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \{l : \tau_l, \overline{l'} : \tau'\}}{\Gamma \vdash e_1.l \equiv e_2.l : \tau_l} \\
\frac{\Gamma, t : \kappa \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \Lambda t : \kappa. e_1 \equiv \Lambda t : \kappa. e_2 : (\forall t : \kappa. \tau)} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \forall t : \kappa. \tau \quad \Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash e_1 \tau_1 \equiv e_2 \tau_2 : \tau[t \leftarrow \tau_1]} \\
\frac{\Gamma \vdash \tau'_1 \equiv \tau'_2 : \kappa \quad \Gamma \vdash e_1 \equiv e_2 : \tau_1[t \leftarrow \tau'_1] \quad \Gamma, t : \kappa. \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash \text{pack}\langle \tau'_1, e_1 \rangle_{\exists t : \kappa. \tau_1} \equiv \text{pack}\langle \tau'_2, e_2 \rangle_{\exists t : \kappa. \tau_2} : (\exists t : \kappa. \tau_1)} \\
\frac{\Gamma, t : \kappa \vdash \tau'_1 \equiv \tau'_2 : \Omega \quad \Gamma \vdash e'_1 \equiv e'_2 : (\exists t : \kappa. \tau'_1) \quad \Gamma, t : \kappa, x : \tau'_1 \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau'_1 \rangle = e'_1 \text{ in } e_1 \equiv \text{unpack}\langle t : \kappa, x : \tau'_2 \rangle = e'_2 \text{ in } e_2 : \tau}
\end{array}$$

4.1.2 Syntax

$X ::= \dots$	(identifier)
$K ::= \dots$	(kind)
$T ::= \dots \mid P$	(type)
$E ::= \dots \mid P$	(expression)
$P ::= M$	(path)
$M ::= X$	(identifier)
$\mid \{B\}$	(bindings)
$\mid M.X$	(projection)
$B ::= \text{val } X = E$	(value binding)
$\mid \text{type } X = T$	(type binding)
$\mid \text{module } X = M$	(module binding)
$\mid \text{signature } X = S$	(signature binding)
$\mid \text{include } M$	(module including)
$\mid \epsilon$	(empty binding)
$\mid B; B$	(binding concatenation)
$S ::= P$	(signature path)
$\mid \{D\}$	(declarations)
$D ::= \text{val } X : T$	(value declaration)
$\mid \text{type } X = T$	(type binding)
$\mid \text{module } X : S$	(module declaration)
$\mid \text{signature } X = S$	(signature binding)
$\mid \text{include } S$	(signature including)
$\mid \epsilon$	(empty declaration)
$\mid D; D$	(declaration concatenation)

4.1.3 Signature

$\Sigma ::= [\tau]$	(anonymous value declaration)
$\mid [= \tau : \kappa]$	(anonymous type declaration)
$\mid [= \Sigma]$	(anonymous signature declaration)
$\mid \{\overline{l_X} : \Sigma\}$	(structural signature)

Atomic Signature:

$$[\tau] \stackrel{\text{def}}{=} \{\text{val} : \tau\}$$

$$\begin{aligned}
[e] &\stackrel{\text{def}}{=} \{\text{val} = e\} \\
[= \tau : \kappa] &\stackrel{\text{def}}{=} \{\text{type} : \forall t : (\kappa \rightarrow \Omega). t \tau \rightarrow t \tau\} \\
[\tau : \kappa] &\stackrel{\text{def}}{=} \{\text{type} = \Lambda t : (\kappa \rightarrow \Omega). \lambda x : (t \tau). x\} \\
[= \Sigma] &\stackrel{\text{def}}{=} \{\text{sig} : \Sigma \rightarrow \Sigma\} \\
[\Sigma] &\stackrel{\text{def}}{=} \{\text{sig} = \lambda x : \Sigma. x\}
\end{aligned}$$

NotAtomic(Σ)

$$\overline{\text{NotAtomic}(\{l_X : \Sigma\})}$$

Admissible kinding:

$\Gamma \vdash \tau : \kappa$

$$\begin{aligned}
&\frac{\Gamma \vdash \tau : \Omega}{\Gamma \vdash [\tau] : \Omega} \text{K-A-Val} \\
&\frac{\Gamma \vdash \tau : \kappa}{\Gamma \vdash [= \tau : \kappa] : \Omega} \text{K-A-Typ} \\
&\frac{\Gamma \vdash \Sigma : \Omega}{\Gamma \vdash [= \Sigma] : \Omega} \text{K-A-Sig}
\end{aligned}$$

Admissible type equivalence:

$\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa$

$$\begin{aligned}
&\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash [\tau_1] \equiv [\tau_2] : \Omega} \text{T-Eq-Cong-A-Val} \\
&\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash [= \tau_1 : \kappa] \equiv [= \tau_2 : \kappa] : \Omega} \text{T-Eq-Cong-A-Typ} \\
&\frac{\Gamma \vdash \Sigma_1 \equiv \Sigma_2 : \Omega}{\Gamma \vdash [= \Sigma_1] \equiv [= \Sigma_2] : \Omega} \text{T-Eq-Cong-A-Sig}
\end{aligned}$$

Admissible typing:

$\Gamma \vdash e : \tau$

$$\begin{aligned}
&\frac{\Gamma \vdash e : \tau}{\Gamma \vdash [e] : [\tau]} \text{T-A-Val} \\
&\frac{\Gamma \vdash \tau : \kappa}{\Gamma \vdash [\tau : \kappa] : [= \tau : \kappa]} \text{T-A-Typ} \\
&\frac{\Gamma \vdash \Sigma : \Omega}{\Gamma \vdash [\Sigma] : [= \Sigma]} \text{T-A-Sig}
\end{aligned}$$

Admissible equivalence:

$\Gamma \vdash e_1 \equiv e_2 : \tau$

$$\begin{aligned}
&\frac{\Gamma \vdash e : \tau}{\Gamma \vdash [e].\text{val} \equiv e : \tau} \text{Eq-}\beta\text{-A-Val} \quad \frac{\Gamma \vdash e : [\tau]}{\Gamma \vdash [e.\text{val}] \equiv e : [\tau]} \text{Eq-}\eta\text{-A-Val} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash [e_1] \equiv [e_2] : [\tau]} \text{Eq-Cong-A-Val} \\
&\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash [\tau_1 : \kappa] \equiv [\tau_2 : \kappa] : [= \tau_1 : \kappa]} \text{Eq-Cong-A-Typ} \\
&\frac{\Gamma \vdash \Sigma_1 \equiv \Sigma_2 : \Omega}{\Gamma \vdash [\Sigma_1] \equiv [\Sigma_2] : [= \Sigma_1]} \text{Eq-Cong-A-Sig}
\end{aligned}$$

4.1.4 Elaboration

Signature:

$$\boxed{\Gamma \vdash S \rightsquigarrow \Sigma}$$

$$\frac{\Gamma \vdash P : [= \Sigma] \rightsquigarrow e}{\Gamma \vdash P \rightsquigarrow \Sigma} \text{ S-Path}$$

$$\frac{\Gamma \vdash D \rightsquigarrow \Sigma}{\Gamma \vdash \{D\} \rightsquigarrow \Sigma} \text{ S-Struct}$$

Declarations:

$$\boxed{\Gamma \vdash D \rightsquigarrow \Sigma}$$

$$\frac{\Gamma \vdash T : \Omega \rightsquigarrow \tau}{\Gamma \vdash \text{val } X : T \rightsquigarrow \{l_X : [\tau]\}} \text{ D-Val}$$

$$\frac{\Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash \text{type } X = T \rightsquigarrow \{l_X : [= \tau : \kappa]\}} \text{ D-Typ-Eq}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Sigma}{\Gamma \vdash \text{module } X : S \rightsquigarrow \{l_X : \Sigma\}} \text{ D-Mod}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Sigma}{\Gamma \vdash \text{signature } X = S \rightsquigarrow \{l_X : [= \Sigma]\}} \text{ D-Sig-Eq}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \{\overline{l_X : \Sigma}\}}{\Gamma \vdash \text{include } S \rightsquigarrow \{\overline{l_X : \Sigma}\}} \text{ D-Incl}$$

$$\frac{\Gamma \vdash \epsilon \rightsquigarrow \{\}}{\Gamma \vdash \epsilon \rightsquigarrow \{\}} \text{ D-Emt}$$

$$\frac{\{\overline{l_{X_1}}\} \cap \{\overline{l_{X_2}}\} = \emptyset \quad \Gamma \vdash D_1 \rightsquigarrow \{\overline{l_{X_1} : \Sigma_1}\} \quad \Gamma, \overline{x_{X_1} : \Sigma_1} \vdash D_2 \rightsquigarrow \{\overline{l_{X_2} : \Sigma_2}\}}{\Gamma \vdash D_1; D_2 \rightsquigarrow \{\overline{l_{X_1} : \Sigma_1}, \overline{l_{X_2} : \Sigma_2}\}} \text{ D-Seq}$$

Module:

$$\boxed{\Gamma \vdash M : \Sigma \rightsquigarrow e}$$

$$\frac{\Gamma(x_X) = \Sigma}{\Gamma \vdash X : \Sigma \rightsquigarrow x_X} \text{ M-Var}$$

$$\frac{\Gamma \vdash B : \Sigma \rightsquigarrow e}{\Gamma \vdash \{B\} : \Sigma \rightsquigarrow e} \text{ M-Struct}$$

$$\frac{\Gamma \vdash M : \{l_X : \Sigma, \overline{l_{X'} : \Sigma'}\} \rightsquigarrow e}{\Gamma \vdash M.X : \Sigma \rightsquigarrow e.l_X} \text{ M-Dot}$$

Bindings:

$$\boxed{\Gamma \vdash B : \Sigma \rightsquigarrow e}$$

$$\frac{\Gamma \vdash E : \tau \rightsquigarrow e}{\Gamma \vdash \text{val } X = E : \{l_X : [\tau]\} \rightsquigarrow \{l_X = [e]\}} \text{ B-Val}$$

$$\frac{\Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash \text{type } X = T : \{l_X : [= \tau : \kappa]\} \rightsquigarrow \{l_X = [\tau : \kappa]\}} \text{ B-Typ}$$

$$\frac{\Gamma \vdash M : \Sigma \rightsquigarrow e \quad \text{NotAtomic}(\Sigma)}{\Gamma \vdash \text{module } X = M : \{l_X : \Sigma\} \rightsquigarrow \{l_X = e\}} \text{ B-Mod}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Sigma}{\Gamma \vdash \text{signature } X = S : \{l_X : [= \Sigma]\} \rightsquigarrow \{l_X = [\Sigma]\}} \text{ B-Sig}$$

$$\frac{\Gamma \vdash M : \{\overline{l_X : \Sigma}\} \rightsquigarrow e}{\Gamma \vdash \text{include } M : \{\overline{l_X : \Sigma}\} \rightsquigarrow e} \text{ B-Incl}$$

$$\begin{array}{c}
\overline{\Gamma \vdash \epsilon : \{\} \rightsquigarrow \{\}} \quad \text{B-Emt} \\
\frac{\overline{l'_{X_1} = l_{X_1} \setminus l_{X_2}} \quad \overline{l'_{X_1} : \Sigma'_1 \subseteq l_{X_1} : \Sigma_1} \quad \overline{\Gamma \vdash B_1 : \{l_{X_1} : \Sigma_1\} \rightsquigarrow e_1} \quad \overline{\Gamma, x_{X_1} : \Sigma_1 \vdash B_2 : \{l_{X_2} : \Sigma_2\} \rightsquigarrow e_2}}{\overline{\Sigma = \{l'_{X_1} : \Sigma'_1, l_{X_2} : \Sigma_2\}} \quad \Gamma \vdash B_1; B_2 : \Sigma \rightsquigarrow} \quad \text{B-Seq} \\
\text{let } x_1 = e_1 \text{ in } \frac{\overline{\Gamma \vdash B_1; B_2 : \Sigma \rightsquigarrow} \quad \overline{\text{let } x_2 = (\text{let } x_{X_1} : \Sigma_1 = x_1.l_{X_1} \text{ in } e_2) \text{ in } \{l'_{X_1} = x_1.l'_{X_1}, l_{X_2} = x_2.l_{X_2}\}}}{\Gamma \vdash B_1; B_2 : \Sigma \rightsquigarrow}
\end{array}$$

Path:

$$\boxed{\Gamma \vdash P : \Sigma \rightsquigarrow e}$$

Use M-Dot.

$$\boxed{\Gamma \vdash T : \kappa \rightsquigarrow \tau}$$

$$\frac{\Gamma \vdash P : [= \tau : \kappa] \rightsquigarrow e}{\Gamma \vdash P : \kappa \rightsquigarrow \tau} \quad \text{T-Elab-Path}$$

$$\boxed{\Gamma \vdash E : \tau \rightsquigarrow e}$$

$$\frac{\Gamma \vdash P : [\tau] \rightsquigarrow e}{\Gamma \vdash P : \tau \rightsquigarrow e.\text{val}} \quad \text{E-Path}$$

4.2 F-ing modules

[RRD14]

4.2.1 Internal Language

See 第 4.1.1 小節.

4.2.2 Syntax

X	$::=$	\dots	(identifier)
K	$::=$	\dots	(kind)
T	$::=$	$\dots \mid P$	(type)
E	$::=$	$\dots \mid P$	(expression)
P	$::=$	M	(path)
M	$::=$	X	(identifier)
		$\mid \{B\}$	(bindings)
		$\mid M.X$	(projection)
		$\mid \text{fun } X : S \Rightarrow M$	(functor)
		$\mid X X$	(functor application)
		$\mid X :> S$	(sealing)
B	$::=$	$\text{val } X = E$	(value binding)
		$\mid \text{type } X = T$	(type binding)
		$\mid \text{module } X = M$	(module binding)
		$\mid \text{signature } X = S$	(signature binding)
		$\mid \text{include } M$	(module including)
		$\mid \epsilon$	(empty binding)
		$\mid B; B$	(binding concatenation)
S	$::=$	P	(signature path)
		$\mid \{D\}$	(declarations)
		$\mid (X : S) \rightarrow S$	((generative) functor signature)
		$\mid S \text{ where type } \bar{X} = T$	(bounded signature)
D	$::=$	$\text{val } X : T$	(value declaration)
		$\mid \text{type } X = T$	(type binding)
		$\mid \text{type } X : K$	(type declaration)
		$\mid \text{module } X : S$	(module declaration)
		$\mid \text{signature } X = S$	(signature binding)
		$\mid \text{include } S$	(signature including)
		$\mid \epsilon$	(empty declaration)
		$\mid D; D$	(declaration concatenation)

4.2.3 Signature

Ξ	$::=$	$\overline{\exists t : \kappa. \Sigma}$	(abstract signature)
Σ	$::=$	$[\tau]$	(atomic value declaration)
		$\mid [= \tau : \kappa]$	(atomic type declaration)
		$\mid [= \Xi]$	(atomic signature declaration)
		$\mid \{\overline{l_X : \Sigma}\}$	(structure signature)
		$\mid \forall t : \kappa. \Sigma \rightarrow \Xi$	(functor signature)

Atomic Signature:

$$[\tau] \stackrel{\text{def}}{=} \{\text{val} : \tau\}$$

$$\begin{aligned}
[e] &\stackrel{\text{def}}{=} \{\text{val} = e\} \\
[= \tau : \kappa] &\stackrel{\text{def}}{=} \{\text{type} : \forall t : (\kappa \rightarrow \Omega). t \tau \rightarrow t \tau\} \\
[\tau : \kappa] &\stackrel{\text{def}}{=} \{\text{type} = \Lambda t : (\kappa \rightarrow \Omega). \lambda x : (t \tau). x\} \\
[= \Xi] &\stackrel{\text{def}}{=} \{\text{sig} : \Xi \rightarrow \Xi\} \\
[\Xi] &\stackrel{\text{def}}{=} \{\text{sig} = \lambda x : \Xi. x\}
\end{aligned}$$

$\text{NotAtomic}(\Sigma)$

$$\frac{}{\text{NotAtomic}(\{l_X : \Sigma\})} \quad \frac{}{\text{NotAtomic}(\forall t : \kappa. \Sigma \rightarrow \Xi)}$$

Admissible kinding:

$\Gamma \vdash \tau : \kappa$

$$\begin{aligned}
&\frac{\Gamma \vdash \tau : \Omega}{\Gamma \vdash [\tau] : \Omega} \text{ K-A-Val} \\
&\frac{\Gamma \vdash \tau : \kappa}{\Gamma \vdash [= \tau : \kappa] : \Omega} \text{ K-A-Typ} \\
&\frac{\Gamma \vdash \Xi : \Omega}{\Gamma \vdash [= \Xi] : \Omega} \text{ K-A-Sig}
\end{aligned}$$

Admissible type equivalence:

$\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa$

$$\begin{aligned}
&\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash [\tau_1] \equiv [\tau_2] : \Omega} \text{ T-Eq-Cong-A-Val} \\
&\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash [= \tau_1 : \kappa] \equiv [= \tau_2 : \kappa] : \Omega} \text{ T-Eq-Cong-A-Typ} \\
&\frac{\Gamma \vdash \Xi_1 \equiv \Xi_2 : \Omega}{\Gamma \vdash [= \Xi_1] \equiv [= \Xi_2] : \Omega} \text{ T-Eq-Cong-A-Sig}
\end{aligned}$$

Admissible typing:

$\Gamma \vdash e : \tau$

$$\begin{aligned}
&\frac{\Gamma \vdash e : \tau}{\Gamma \vdash [e] : [\tau]} \text{ T-A-Val} \\
&\frac{\Gamma \vdash \tau : \kappa}{\Gamma \vdash [\tau : \kappa] : [= \tau : \kappa]} \text{ T-A-Typ} \\
&\frac{\Gamma \vdash \Xi : \Omega}{\Gamma \vdash [\Xi] : [= \Xi]} \text{ T-A-Sig}
\end{aligned}$$

Admissible equivalence:

$\Gamma \vdash e_1 \equiv e_2 : \tau$

$$\begin{aligned}
&\frac{\Gamma \vdash e : \tau}{\Gamma \vdash [e]. \text{val} \equiv e : \tau} \text{ Eq-}\beta\text{-A-Val} \quad \frac{\Gamma \vdash e : [\tau]}{\Gamma \vdash [e. \text{val}] \equiv e : [\tau]} \text{ Eq-}\eta\text{-A-Val} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash [e_1] \equiv [e_2] : [\tau]} \text{ Eq-Cong-A-Val} \\
&\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash [\tau_1 : \kappa] \equiv [\tau_2 : \kappa] : [= \tau_1 : \kappa]} \text{ Eq-Cong-A-Typ} \\
&\frac{\Gamma \vdash \Xi_1 \equiv \Xi_2 : \Omega}{\Gamma \vdash [\Xi_1] \equiv [\Xi_2] : [= \Xi_1]} \text{ Eq-Cong-A-Sig}
\end{aligned}$$

4.2.4 (Generative) Elaboration

Signature:

$$\boxed{\Gamma \vdash S \rightsquigarrow \Xi}$$

$$\begin{array}{c} \frac{\Gamma \vdash P : [= \Xi] \rightsquigarrow e}{\Gamma \vdash P \rightsquigarrow \Xi} \text{ S-Path} \\ \frac{\Gamma \vdash D \rightsquigarrow \Xi}{\Gamma \vdash \{D\} \rightsquigarrow \Xi} \text{ S-Struct} \\ \frac{\Gamma \vdash S_1 \rightsquigarrow \exists \overline{t} : \overline{\kappa}. \Sigma \quad \Gamma, \overline{t} : \overline{\kappa}, x_X : \Sigma \vdash S_2 \rightsquigarrow \Xi}{\Gamma \vdash (X : S_1) \rightarrow S_2 \rightsquigarrow \forall \overline{t} : \overline{\kappa}. \Sigma \rightarrow \Xi} \text{ S-Funct} \\ \frac{\Gamma \vdash S \rightsquigarrow \exists \overline{t}_1 : \overline{\kappa}_1 \overline{t} : \overline{\kappa}_2 \overline{\kappa}_2. \Sigma \quad \Sigma. \overline{l}_X = [= \overline{t} : \overline{\kappa}] \quad \Gamma \vdash T : \overline{\kappa} \rightsquigarrow \tau}{\Gamma \vdash S \text{ where type } \overline{X} = T \rightsquigarrow \exists \overline{t}_1 : \overline{\kappa}_1 \overline{t}_2 : \overline{\kappa}_2. \Sigma[\overline{t} \leftarrow \tau]} \text{ S-Where-Typ} \end{array}$$

Declarations:

$$\boxed{\Gamma \vdash D \rightsquigarrow \Xi}$$

$$\begin{array}{c} \frac{\Gamma \vdash T : \Omega \rightsquigarrow \tau}{\Gamma \vdash \text{val } X : T \rightsquigarrow \{l_X : [\tau]\}} \text{ D-Val} \\ \frac{\Gamma \vdash T : \overline{\kappa} \rightsquigarrow \tau}{\Gamma \vdash \text{type } X = T \rightsquigarrow \{l_X : [= \tau : \overline{\kappa}]\}} \text{ D-Typ-Eq} \\ \frac{\Gamma \vdash K \rightsquigarrow \overline{\kappa}}{\Gamma \vdash \text{type } X : K \rightsquigarrow \exists \overline{t} : \overline{\kappa}. \{l_X : [= \overline{t} : \overline{\kappa}]\}} \text{ D-Typ} \\ \frac{\Gamma \vdash S \rightsquigarrow \exists \overline{t} : \overline{\kappa}. \Sigma}{\Gamma \vdash \text{module } X : S \rightsquigarrow \exists \overline{t} : \overline{\kappa}. \{l_X : \Sigma\}} \text{ D-Mod} \\ \frac{\Gamma \vdash S \rightsquigarrow \Xi}{\Gamma \vdash \text{signature } X = S \rightsquigarrow \{l_X : [= \Xi]\}} \text{ D-Sig-Eq} \\ \frac{\Gamma \vdash S \rightsquigarrow \exists \overline{t} : \overline{\kappa}. \{\overline{l}_X : \Sigma\}}{\Gamma \vdash \text{include } S \rightsquigarrow \exists \overline{t} : \overline{\kappa}. \{\overline{l}_X : \Sigma\}} \text{ D-Incl} \\ \frac{}{\Gamma \vdash \epsilon \rightsquigarrow \{\}} \text{ D-Emt} \\ \frac{\{\overline{l}_{X_1}\} \cap \{\overline{l}_{X_2}\} = \emptyset \quad \Gamma \vdash D_1 \rightsquigarrow \exists \overline{t}_1 : \overline{\kappa}_1. \{\overline{l}_{X_1} : \Sigma_1\} \quad \Gamma, \overline{t}_1 : \overline{\kappa}_1, x_{X_1} : \Sigma_1 \vdash D_2 \rightsquigarrow \exists \overline{t}_2 : \overline{\kappa}_2. \{\overline{l}_{X_2} : \Sigma_2\}}{\Gamma \vdash D_1; D_2 \rightsquigarrow \exists \overline{t}_1 : \overline{\kappa}_1 \overline{t}_2 : \overline{\kappa}_2. \{\overline{l}_{X_1} : \Sigma_1 \overline{l}_{X_2} : \Sigma_2\}} \text{ D-Seq} \end{array}$$

Matching:

$$\boxed{\Gamma \vdash \Sigma_1 \leq \exists \overline{t} : \overline{\kappa}. \Sigma_2 \uparrow \overline{\tau} \rightsquigarrow e}$$

$$\frac{\Gamma \vdash \Sigma_1 \leq \Sigma_2[\overline{t} \leftarrow \overline{\tau}_t] \rightsquigarrow e \quad \bigwedge_t \Gamma \vdash \tau_t : \overline{\kappa}_t}{\Gamma \vdash \Sigma_1 \leq \exists \overline{t} : \overline{\kappa}_t. \Sigma_2 \uparrow \overline{\tau}_t \rightsquigarrow e} \text{ U-Match}$$

Subtyping:

$$\boxed{\Gamma \vdash \Xi_1 \leq \Xi_2 \rightsquigarrow e}$$

$$\begin{array}{c} \frac{\Gamma \vdash \tau_1 \leq \tau_2 \rightsquigarrow e}{\Gamma \vdash [\tau_1] \leq [\tau_2] \rightsquigarrow \lambda x : [\tau_1]. [e(x.\text{val})]} \text{ U-Val} \\ \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \overline{\kappa}}{\Gamma \vdash [= \tau_1 : \overline{\kappa}] \leq [= \tau_2 : \overline{\kappa}] \rightsquigarrow \lambda x : [= \tau_1 : \overline{\kappa}]. x} \text{ U-Typ} \\ \frac{\Gamma \vdash \Xi_1 \leq \Xi_2 \rightsquigarrow e_1 \quad \Gamma \vdash \Xi_2 \leq \Xi_1 \rightsquigarrow e_2}{\Gamma \vdash [= \Xi_1] \leq [= \Xi_2] \rightsquigarrow \lambda x : [= \Xi_1]. [\Xi_2]} \text{ U-Sig} \end{array}$$

$$\begin{array}{c}
\frac{\bigwedge_l \Gamma \vdash \Sigma_{l_1} \leq \Sigma_{l_2} \rightsquigarrow e_l}{\Gamma \vdash \{\overline{l} : \Sigma_{l_1}, \overline{l'} : \Sigma'\} \leq \{\overline{l} : \Sigma_{l_2}\} \rightsquigarrow \lambda x : \{\overline{l} : \Sigma_{l_1}, \overline{l'} : \Sigma'\}. \{\overline{l} = e_l(x.l)\}} \text{U-Struct} \\
\frac{\Gamma, \overline{t_2} : \overline{\kappa_2} \vdash \Sigma_2 \leq \exists \overline{t_1} : \overline{\kappa_1}. \Sigma_1 \uparrow \overline{\tau} \rightsquigarrow e_1 \quad \Gamma, \overline{t_2} : \overline{\kappa_2} \vdash \Sigma_1[\overline{t_1} \leftarrow \overline{\tau}] \leq \Sigma_2 \rightsquigarrow e_2}{\Gamma \vdash \forall \overline{t_1} : \overline{\kappa_1}. \Sigma_1 \rightarrow \Sigma_2 \leq \exists \overline{t_2} : \overline{\kappa_2}. \Sigma_2 \rightarrow \Sigma_2 \rightsquigarrow \lambda x_1 : (\forall \overline{t_1} : \overline{\kappa_1}. \Sigma_1 \rightarrow \Sigma_2). \lambda x_2 : \Sigma_2. e_2(x_1 \overline{\tau}(e_1 x_2))} \text{U-Funct} \\
\frac{\Gamma, \overline{t_1} : \overline{\kappa_1} \vdash \Sigma_1 \leq \exists \overline{t_2} : \overline{\kappa_2}. \Sigma_2 \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash \exists \overline{t_1} : \overline{\kappa_1}. \Sigma_1 \leq \exists \overline{t_2} : \overline{\kappa_2}. \Sigma_2 \rightsquigarrow \lambda x_1 : (\exists \overline{t_1} : \overline{\kappa_1}. \Sigma_1). \text{unpack}\langle \overline{t_1} : \overline{\kappa_1}, x'_1 : \Sigma_1 \rangle = x_1 \text{ in pack}\langle \overline{\tau}, e \ x'_1 \rangle_{\exists \overline{t_2} : \overline{\kappa_2}. \Sigma_2}} \text{U-Abs}
\end{array}$$

Module:

$$\boxed{\Gamma \vdash M : \Xi \rightsquigarrow e}$$

$$\begin{array}{c}
\frac{\Gamma(x_X) = \Sigma}{\Gamma \vdash X : \Sigma \rightsquigarrow x_X} \text{M-Var} \\
\frac{\Gamma \vdash B : \Xi \rightsquigarrow e}{\Gamma \vdash \{B\} : \Xi \rightsquigarrow e} \text{M-Struct} \\
\frac{\Gamma \vdash M : \exists \overline{t} : \overline{\kappa}. \{\overline{l_X} : \Sigma, \overline{l_{X'}} : \Sigma'\} \rightsquigarrow e}{\Gamma \vdash M.X : \exists \overline{t} : \overline{\kappa}. \Sigma \rightsquigarrow \text{unpack}\langle \overline{t} : \overline{\kappa}, x : \{\overline{l_X} : \Sigma, \overline{l_{X'}} : \Sigma'\} \rangle = e \text{ in pack}\langle \overline{t}, x.l_X \rangle_{\exists \overline{t} : \overline{\kappa}. \Sigma}} \text{M-Dot} \\
\frac{\Sigma \vdash S \rightsquigarrow \exists \overline{t} : \overline{\kappa}. \Sigma \quad \Gamma, \overline{t} : \overline{\kappa}, x_X : \Sigma \vdash M : \Xi \rightsquigarrow e}{\Gamma \vdash \text{fun } X : S \Rightarrow M : \forall \overline{t} : \overline{\kappa}. \Sigma \rightarrow \Xi \rightsquigarrow \Lambda \overline{t} : \overline{\kappa}. \lambda x_X : \Sigma. e} \text{M-Funct} \\
\frac{\Gamma(x_{X_1}) = \forall \overline{t} : \overline{\kappa}. \Sigma' \rightarrow \Xi \quad \Gamma(x_{X_2}) = \Sigma \quad \Gamma \vdash \Sigma \leq \exists \overline{t} : \overline{\kappa}. \Sigma' \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash X_1 X_2 : \Xi[\overline{t} \leftarrow \overline{\tau}] \rightsquigarrow x_{X_1} \overline{\tau}(e x_{X_2})} \text{M-App} \\
\frac{\Gamma(x_X) = \Sigma \quad \Gamma \vdash S \rightsquigarrow \exists \overline{t} : \overline{\kappa}. \Sigma' \quad \Gamma \vdash \Sigma \leq \exists \overline{t} : \overline{\kappa}. \Sigma' \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash X : > S : \exists \overline{t} : \overline{\kappa}. \Sigma' \rightsquigarrow \text{pack}\langle \overline{\tau}, e \ x_X \rangle_{\exists \overline{t} : \overline{\kappa}. \Sigma'}} \text{M-Seal}
\end{array}$$

Bindings:

$$\boxed{\Gamma \vdash B : \Xi \rightsquigarrow e}$$

$$\begin{array}{c}
\frac{\Gamma \vdash E : \tau \rightsquigarrow e}{\Gamma \vdash \text{val } X = E : \{\overline{l_X} : [\tau]\} \rightsquigarrow \{\overline{l_X} = [e]\}} \text{B-Val} \\
\frac{\Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash \text{type } X = T : \{\overline{l_X} : [= \tau : \kappa]\} \rightsquigarrow \{\overline{l_X} = [\tau : \kappa]\}} \text{B-Typ} \\
\frac{\Gamma \vdash M : \exists \overline{t} : \overline{\kappa}. \Sigma \rightsquigarrow e \quad \text{NotAtomic}(\Sigma)}{\Gamma \vdash \text{module } X = M : \exists \overline{t} : \overline{\kappa}. \{\overline{l_X} : \Sigma\} \rightsquigarrow \text{unpack}\langle \overline{t} : \overline{\kappa}, x : \Sigma \rangle = e \text{ in pack}\langle \overline{t}, \{\overline{l_X} = x \} \rangle_{\exists \overline{t} : \overline{\kappa}. \{\overline{l_X} : \Sigma\}}} \text{B-Mod} \\
\frac{\Gamma \vdash S \rightsquigarrow \Xi}{\Gamma \vdash \text{signature } X = S : \{\overline{l_X} : [= \Xi]\} \rightsquigarrow \{\overline{l_X} = [\Xi]\}} \text{B-Sig} \\
\frac{\Gamma \vdash M : \exists \overline{t} : \overline{\kappa}. \{\overline{l_X} : \Sigma\} \rightsquigarrow e}{\Gamma \vdash \text{include } M : \exists \overline{t} : \overline{\kappa}. \{\overline{l_X} : \Sigma\} \rightsquigarrow e} \text{B-Incl} \\
\frac{}{\Gamma \vdash \epsilon : \{\} \rightsquigarrow \{\}} \text{B-Emt} \\
\frac{\overline{l'_{X_1}} = \overline{l_{X_1}} \setminus \overline{l_{X_2}} \quad \overline{l'_{X_1}} : \Sigma'_1 \subseteq \overline{l_{X_1}} : \Sigma_1 \quad \Gamma \vdash B_1 : \exists \overline{t_1} : \overline{\kappa_1}. \{\overline{l_{X_1}} : \Sigma_1\} \rightsquigarrow e_1 \quad \Sigma = \{\overline{l'_{X_1}} : \Sigma'_1, \overline{l_{X_2}} : \Sigma_2\} \quad \Gamma, \overline{t_1} : \overline{\kappa_1}, x_{X_1} : \Sigma_1 \vdash B_2 : \exists \overline{t_2} : \overline{\kappa_2}. \{\overline{l_{X_2}} : \Sigma_2\} \rightsquigarrow e_2}{\Gamma \vdash B_1; B_2 : \exists \overline{t_1} : \overline{\kappa_1} \overline{t_2} : \overline{\kappa_2}. \Sigma \rightsquigarrow \text{unpack}\langle \overline{t_1} : \overline{\kappa_1}, x_1 \rangle = e_1 \text{ in } \text{unpack}\langle \overline{t_2} : \overline{\kappa_2}, x_2 \rangle = (\text{let } x_{X_1} : \Sigma_1 = x_1.l_{X_1} \text{ in } e_2) \text{ in } \text{pack}\langle \overline{t_1} \overline{t_2}, \{\overline{l'_{X_1}} = x_1.l'_{X_1}, \overline{l_{X_2}} = x_2.l_{X_2}\} \rangle_{\exists \overline{t_1} : \overline{\kappa_1} \overline{t_2} : \overline{\kappa_2}. \Sigma}} \text{B-Seq}
\end{array}$$

Path:

$$\boxed{\Gamma \vdash P : \Sigma \rightsquigarrow e}$$

$$\frac{\Gamma \vdash P : \exists \overline{t} : \overline{\kappa}. \Sigma \quad \Gamma \vdash \Sigma : \Omega}{\Gamma \vdash P : \Sigma \rightsquigarrow \text{unpack}\langle \overline{t} : \overline{\kappa}, x \rangle = e \text{ in } x} \text{P-Mod}$$

$$\boxed{\Gamma \vdash T : \kappa \rightsquigarrow \tau}$$

$$\frac{\Gamma \vdash P : [= \tau : \kappa] \rightsquigarrow e}{\Gamma \vdash P : \kappa \rightsquigarrow \tau} \text{ T-Elab-Path}$$

$$\boxed{\Gamma \vdash E : \tau \rightsquigarrow e}$$

$$\frac{\Gamma \vdash P : [\tau] \rightsquigarrow e}{\Gamma \vdash P : \tau \rightsquigarrow e.\text{val}} \text{ E-Path}$$

4.2.5 Modules as First-Class Values

$$\begin{aligned} T &::= \dots \mid \text{pack } S \\ E &::= \dots \mid \text{pack } M : S \\ M &::= \dots \mid \text{unpack } E : S \end{aligned}$$

Rootedness:

$$\boxed{t : \kappa \text{ rooted in } \Sigma \text{ at } \overline{l}_X}$$

$$\frac{t = \tau'}{t : \kappa \text{ rooted in } [= \tau : \kappa] \text{ at } \epsilon} \quad \frac{t : \kappa \text{ rooted in } \{\overline{l}_X : \Sigma\}.l \text{ at } \overline{l}'}{t : \kappa \text{ rooted in } \{\overline{l}_X : \Sigma\} \text{ at } l \overline{l}'}$$

Rooted ordering:

$$t_1 : \kappa_1 \leq_{\Sigma} t_2 : \kappa_2 \iff \min\{\bar{l} \mid t_1 : \kappa_1 \text{ rooted in } \Sigma \text{ at } \bar{l}\} \leq \min\{\bar{l} \mid t_2 : \kappa_2 \text{ rooted in } \Sigma \text{ at } \bar{l}\}$$

Signature normalization:

$$\begin{aligned} &\frac{\text{norm}_0(\tau) = \tau'}{\text{norm}([\tau]) = [\tau']} \\ &\frac{}{\text{norm}([= \tau : \kappa]) = [= \tau : \kappa]} \\ &\frac{\text{norm}(\Xi) = \Xi'}{\text{norm}([= \Xi]) = [= \Xi']} \\ &\frac{\bigwedge_X \text{norm}(\Sigma_X) = \Sigma'_X}{\text{norm}(\{\overline{l}_X : \Sigma_X\}) = \{\overline{l}_X : \Sigma'_X\}} \\ &\frac{\text{sort}_{\leq_{\Sigma'}}(\overline{t : \kappa}) = \overline{t' : \kappa'} \quad \text{norm}(\Sigma) = \Sigma' \quad \text{norm}(\Xi) = \Xi'}{\text{norm}(\forall \overline{t : \kappa}. \Sigma \rightarrow \Xi) = \forall \overline{t' : \kappa'}. \Sigma' \rightarrow \Xi'} \\ &\frac{\text{sort}_{\leq_{\Sigma'}}(\overline{t : \kappa}) = \overline{t' : \kappa'} \quad \text{norm}(\Sigma) = \Sigma'}{\text{norm}(\exists \overline{t : \kappa}. \Sigma) = \exists \overline{t' : \kappa'}. \Sigma'} \end{aligned}$$

Type:

$$\boxed{\Gamma \vdash T : \kappa \rightsquigarrow \tau}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Xi}{\Gamma \vdash \text{pack } S : \Omega \rightsquigarrow \text{norm}(\Xi)} \text{ T-Pack}$$

Expression:

$$\boxed{\Gamma \vdash E : \tau \rightsquigarrow e}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Xi \quad \Gamma \vdash \Xi' \leq \text{norm}(\Xi) \rightsquigarrow e_1 \quad \Gamma \vdash M : \Xi' \rightsquigarrow e_2}{\Gamma \vdash (\text{pack } M : S) : \text{norm}(\Xi) \rightsquigarrow e_1 e_2} \text{ E-Pack}$$

Module:

$$\boxed{\Gamma \vdash M : \Xi \rightsquigarrow e}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Xi \quad \Gamma \vdash E : \text{norm}(\Xi) \rightsquigarrow e}{\Gamma \vdash (\text{unpack } E : S) : \text{norm}(\Xi) \rightsquigarrow e} \text{ M-Unpack}$$

4.2.6 Elaboration with Applicative Functor

$$\begin{aligned} S &::= \dots \\ &| (X : S) \Rightarrow S \quad (\text{applicative functor signature}) \end{aligned}$$

$$\begin{aligned} \varphi &::= I && (\text{impure effect}) \\ &| P && (\text{pure effect}) \\ \Sigma &::= \dots \\ &| \overline{\{l_X : \Sigma\}} \\ &| \forall t : \kappa. \Sigma \rightarrow_I \Xi \quad (\text{generative functor signature}) \\ &| \forall t : \kappa. \Sigma \rightarrow_P \Sigma \quad (\text{applicative functor signature}) \end{aligned}$$

Abbreviation:

$$\begin{aligned} \tau_1 \rightarrow_\varphi \tau_2 &\stackrel{\text{def}}{=} \tau_1 \rightarrow \{l_\varphi : \tau_2\} \\ \lambda_\varphi x : \tau. e &\stackrel{\text{def}}{=} \lambda x : \tau. \{l_\varphi = e\} \\ (e_1 e_2)_\varphi &\stackrel{\text{def}}{=} (e_1 e_2).l_\varphi \\ \Gamma^\varphi &\stackrel{\text{def}}{=} \begin{cases} \cdot & (\varphi = I) \\ \Gamma & (\varphi = P) \end{cases} \\ \text{tyenv}(\Gamma) &\stackrel{\text{def}}{=} \begin{cases} \text{tyenv}(\Gamma') \, t : \kappa & (\Gamma = \Gamma', t : \kappa) \\ \text{tyenv}(\Gamma') & (\Gamma = \Gamma', x : \tau) \\ \epsilon & (\Gamma = \cdot) \end{cases} \\ \forall_P \Gamma. \tau_0 &\stackrel{\text{def}}{=} \begin{cases} \forall_P \Gamma'. \forall t : \kappa. \tau_0 & (\Gamma = \Gamma', t : \kappa) \\ \forall_P \Gamma'. \tau \rightarrow_P \tau_0 & (\Gamma = \Gamma', x : \tau) \\ \tau_0 & (\Gamma = \cdot) \end{cases} \\ \Lambda_P \Gamma. e &\stackrel{\text{def}}{=} \begin{cases} \Lambda_P \Gamma'. \Lambda t : \kappa. e & (\Gamma = \Gamma', t : \kappa) \\ \Lambda_P \Gamma'. \lambda_P x : \tau. e & (\Gamma = \Gamma', x : \tau) \\ e & (\Gamma = \cdot) \end{cases} \\ (e \, \Gamma)_P &\stackrel{\text{def}}{=} \begin{cases} (e \, \Gamma')_P \, t & (\Gamma = \Gamma', t : \kappa) \\ ((e \, \Gamma')_P \, x)_P & (\Gamma = \Gamma', x : \tau) \\ e & (\Gamma = \cdot) \end{cases} \end{aligned}$$

Effect combining:

$$\boxed{\varphi_1 \vee \varphi_2 = \varphi}$$

$$\overline{\varphi \vee \varphi} = \varphi \quad \overline{I \vee P} = I \quad \overline{P \vee I} = I$$

Subeffects:

$$\varphi_1 \leq \varphi_2$$

$$\overline{\varphi \leq \varphi} \text{ F-Refl} \quad \overline{P \leq I} \text{ F-Sub}$$

Signature:

$$\Gamma \vdash S \rightsquigarrow \Xi$$

$$\frac{\Gamma \vdash S_1 \rightsquigarrow \exists \overline{t_1 : \kappa_1}. \Sigma \quad \Gamma, \overline{t_1 : \kappa_1}, x_X : \Sigma \vdash S_2 \rightsquigarrow \Xi}{\Gamma \vdash (X : S_1) \rightarrow S_2 \rightsquigarrow \forall \overline{t_1 : \kappa_1}. \Sigma \rightarrow_I \Xi} \text{ S-Funct-I}$$

$$\frac{\Gamma \vdash S_1 \rightsquigarrow \exists \overline{t_1 : \kappa_1}. \Sigma_1 \quad \Gamma, \overline{t_1 : \kappa_1}, x_X : \Sigma_1 \vdash S_2 \rightsquigarrow \exists \overline{t_2 : \kappa_2}. \Sigma_2}{\Gamma \vdash (X : S_1) \Rightarrow S_2 \rightsquigarrow \exists \overline{t'_2 : \kappa_1 \rightarrow \kappa_2}. \forall \overline{t_1 : \kappa_1}. \Sigma_1 \rightarrow_P \Sigma_2[t_2 \leftarrow t'_2 \overline{t_1}]} \text{ S-Funct-P}$$

Subtyping:

$$\Gamma \vdash \Xi_1 \leq \Xi_2 \rightsquigarrow e$$

$$\frac{\Gamma, \overline{t_2 : \kappa_2} \vdash \Sigma_2 \leq \exists \overline{t_1 : \kappa_1}. \Sigma_1 \uparrow \overline{\tau} \rightsquigarrow e_1 \quad \Gamma, \overline{t_2 : \kappa_2} \vdash \Xi_1[\overline{t_1 \leftarrow \tau}] \leq \Xi_2 \rightsquigarrow e_2 \quad \varphi_1 \leq \varphi_2}{\Gamma \vdash (\forall \overline{t_1 : \kappa_1}. \Sigma_1 \rightarrow_{\varphi_1} \Xi_1) \leq (\forall \overline{t_2 : \kappa_2}. \Sigma_2 \rightarrow_{\varphi_2} \Xi_2) \rightsquigarrow \frac{\lambda x_1 : (\forall \overline{t_1 : \kappa_1}. \Sigma_1 \rightarrow_{\varphi_1} \Xi_1). \Lambda \overline{t_2 : \kappa_2}. \lambda_{\varphi_2} x_2 : \Sigma_2. e_2 (x_1 \overline{\tau} (e_1 x_2))}{\varphi_1}} \text{ U-Funct}$$

Module:

$$\Gamma \vdash M :_{\varphi} \Xi \rightsquigarrow e$$

$$\frac{\Gamma(x_X) = \Sigma}{\Gamma \vdash X :_P \Sigma \rightsquigarrow \Lambda_P \Gamma. x_X} \text{ M-Var}$$

$$\frac{\Gamma \vdash B :_{\varphi} \Xi \rightsquigarrow e}{\Gamma \vdash \{B\} :_{\varphi} \Xi \rightsquigarrow e} \text{ M-Struct}$$

$$\frac{\Gamma \vdash M :_{\varphi} \exists \overline{t : \kappa}. \{l_X : \Sigma, \overline{l_{X'}} : \Sigma'\} \rightsquigarrow e}{\Gamma \vdash M.X :_{\varphi} \exists \overline{t : \kappa}. \Sigma \rightsquigarrow \text{unpack} \langle \overline{t : \kappa}, x \rangle = e \text{ in } \text{pack} \langle \overline{t}, \Lambda_P \Gamma^{\varphi}. (x \Gamma^{\varphi})_P. l_X \rangle} \text{ M-Dot}$$

$$\frac{\Sigma \vdash S \rightsquigarrow \exists \overline{t : \kappa}. \Sigma \quad \Gamma, \overline{t : \kappa}, x_X : \Sigma \vdash M :_I \Xi \rightsquigarrow e}{\Gamma \vdash \text{fun } X : S \Rightarrow M :_P \forall \overline{t : \kappa}. \Sigma \rightarrow_I \Xi \rightsquigarrow \Lambda_P \Gamma. \Lambda \overline{t : \kappa}. \lambda_1 x_X : \Sigma. e} \text{ M-Funct-I}$$

$$\frac{\Sigma \vdash S \rightsquigarrow \exists \overline{t : \kappa}. \Sigma \quad \Gamma, \overline{t : \kappa}, x_X : \Sigma \vdash M :_P \exists \overline{t_2 : \kappa_2}. \Sigma_2 \rightsquigarrow e}{\Gamma \vdash \text{fun } X : S \Rightarrow M :_P \exists \overline{t_2 : \kappa_2}. \forall \overline{t : \kappa}. \Sigma \rightarrow_P \Sigma_2 \rightsquigarrow e} \text{ M-Funct-P}$$

$$\frac{\Gamma(x_{X_1}) = \forall \overline{t : \kappa}. \Sigma' \rightarrow_{\varphi} \Xi \quad \Gamma(x_{X_2}) = \Sigma \quad \Gamma \vdash \Sigma \leq \exists \overline{t : \kappa}. \Sigma' \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash X_1 X_2 :_{\varphi} \Xi[\overline{t \leftarrow \tau}] \rightsquigarrow \Lambda_P \Gamma^{\varphi}. (x_{X_1} \overline{\tau} (e x_{X_2}))_{\varphi}} \text{ M-App}$$

$$\frac{\overline{t_{\Gamma} : \kappa_{\Gamma}} = \text{tyenv}(\Gamma) \quad \Gamma(x_X) = \Sigma \quad \Gamma \vdash S \rightsquigarrow \exists \overline{t : \kappa}. \Sigma' \quad \Gamma \vdash \Sigma \leq \exists \overline{t : \kappa}. \Sigma' \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash X :> S :_P \exists \overline{t' : \overline{t_{\Gamma} : \kappa_{\Gamma}} \rightarrow \kappa}. \Sigma'[t \leftarrow t' \overline{t_{\Gamma}}] \rightsquigarrow \text{pack} \langle \lambda \overline{t_{\Gamma} : \kappa_{\Gamma}}. \tau, \Lambda_P \Gamma. e x_X \rangle} \text{ M-Seal}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Xi \quad \Gamma \vdash E : \text{norm}(\Xi) \rightsquigarrow e}{\Gamma \vdash (\text{unpack } E : S) :_I \text{norm}(\Xi) \rightsquigarrow e} \text{ M-Unpack}$$

定理 37 (Typing for module elaboration).

- $\Gamma \vdash M :_I \Xi \rightsquigarrow e$ ならば, $\Gamma \vdash e : \Xi$.
- $\Gamma \vdash M :_P \exists \overline{t : \kappa}. \Sigma \rightsquigarrow e$ ならば, $\cdot \vdash e : \exists \overline{t : \kappa}. \forall_P \Gamma. \Sigma$.

□

Bindings:

$$\Gamma \vdash B :_{\varphi} \Xi \rightsquigarrow e$$

$$\begin{array}{c}
\frac{\Gamma \vdash E : \tau \rightsquigarrow e}{\Gamma \vdash \text{val } X = E :_{\mathbf{P}} \{l_X : [\tau]\} \rightsquigarrow \Lambda_{\mathbf{P}}\Gamma.\{l_X = e\}} \text{ B-Val} \\
\frac{\Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash \text{type } X = T :_{\mathbf{P}} \{l_X : [\tau : \kappa]\} \rightsquigarrow \Lambda_{\mathbf{P}}\Gamma.\{l_X = [\tau : \kappa]\}} \text{ B-Typ} \\
\frac{\Gamma \vdash M :_{\varphi} \exists \overline{t} : \overline{\kappa}. \Sigma \rightsquigarrow e \quad \text{NotAtomic}(\Sigma)}{\Gamma \vdash \text{module } X = M :_{\varphi} \exists \overline{t} : \overline{\kappa}. \{l_X : \Sigma\} \rightsquigarrow \text{unpack}\langle \overline{t} : \overline{\kappa}, x \rangle = e \text{ in pack}\langle \overline{t}, \Lambda_{\mathbf{P}}\Gamma^{\varphi}.\{l_X = x \Gamma^{\varphi}\} \rangle} \text{ B-Mod} \\
\frac{\Gamma \vdash S \rightsquigarrow \Xi}{\Gamma \vdash \text{signature } X = S :_{\mathbf{P}} \{l_X : [\Xi]\} \rightsquigarrow \Lambda_{\mathbf{P}}\Gamma.\{l_X = [\Xi]\}} \text{ B-Sig} \\
\frac{\Gamma \vdash M :_{\varphi} \exists \overline{t} : \overline{\kappa}. \{l_X : \Sigma\} \rightsquigarrow e}{\Gamma \vdash \text{include } M :_{\varphi} \exists \overline{t} : \overline{\kappa}. \{l_X : \Sigma\} \rightsquigarrow e} \text{ B-Incl} \\
\frac{}{\Gamma \vdash \epsilon :_{\mathbf{P}} \{\} \rightsquigarrow \Lambda_{\mathbf{P}}\Gamma.\{\}} \text{ B-Emt} \\
\frac{\overline{l'_{X_1}} = \overline{l_{X_1}} \setminus \overline{l_{X_2}} \quad \overline{l'_{X_1}} : \Sigma'_1 \subseteq \overline{l_{X_1}} : \Sigma_1 \quad \Gamma \vdash B_1 :_{\varphi_1} \exists \overline{t_1} : \overline{\kappa_1}. \{\overline{l_{X_1}} : \Sigma_1\} \rightsquigarrow e_1 \quad \Sigma = \{\overline{l'_{X_1}} : \Sigma'_1, \overline{l_{X_2}} : \Sigma_2\} \quad \Gamma, \overline{t_1} : \overline{\kappa_1}, \overline{x_{X_1}} : \overline{\Sigma_1} \vdash B_2 :_{\varphi_2} \exists \overline{t_2} : \overline{\kappa_2}. \{\overline{l_{X_2}} : \Sigma_2\} \rightsquigarrow e_2}{\Gamma \vdash B_1; B_2 :_{\varphi_1 \vee \varphi_2} \exists \overline{t_1} : \overline{\kappa_1} \overline{t_2} : \overline{\kappa_2}. \Sigma \rightsquigarrow \text{unpack}\langle \overline{t_1} : \overline{\kappa_1}, x_1 \rangle = e_1 \text{ in } \text{unpack}\langle \overline{t_2} : \overline{\kappa_2}, x_2 \rangle = (\text{let } x_{X_1} = \Lambda_{\mathbf{P}}\Gamma^{\varphi_1 \vee \varphi_2}. (x_1 \Gamma^{\varphi_1})_{\mathbf{P}}. l_{X_1} \text{ in } e_2) \text{ in } \text{pack}\langle \overline{t_1} \overline{t_2}, \Lambda_{\mathbf{P}}\Gamma^{\varphi_1 \vee \varphi_2}. \text{let } x_{X_1} = (x_1 \Gamma^{\varphi_1})_{\mathbf{P}}. l_{X_1} \text{ in } \{\overline{l'_{X_1}} = (x_1 \Gamma^{\varphi_1})_{\mathbf{P}}. l'_{X_1}, \overline{l_{X_2}} = (x_2 (\Gamma, \overline{t_1} : \overline{\kappa_1}, \overline{x_{X_1}} : \overline{\Sigma_1})^{\varphi_2})_{\mathbf{P}}. l_{X_2} \} \rangle} \text{ B-Seq}
\end{array}$$

Path:

$$\boxed{\Gamma \vdash P : \Sigma \rightsquigarrow e}$$

$$\frac{\Gamma \vdash P :_{\varphi} \exists \overline{t} : \overline{\kappa}. \Sigma \quad \Gamma \vdash \Sigma : \Omega}{\Gamma \vdash P : \Sigma \rightsquigarrow \text{unpack}\langle \overline{t} : \overline{\kappa}, x \rangle = e \text{ in } (x \Gamma^{\varphi})_{\mathbf{P}}} \text{ P-Mod}$$

Expression:

$$\boxed{\Gamma \vdash E : \tau \rightsquigarrow e}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Xi \quad \Gamma \vdash \exists \overline{t} : \overline{\kappa}. \Sigma \leq \text{norm}(\Xi) \rightsquigarrow e_1 \quad \Gamma \vdash M :_{\varphi} \exists \overline{t} : \overline{\kappa}. \Sigma \rightsquigarrow e_2}{\Gamma \vdash (\text{pack } M : S) : \text{norm}(\Xi) \rightsquigarrow e_1 \text{ (unpack}\langle \overline{t} : \overline{\kappa}, x \rangle = e_2 \text{ in pack}\langle \overline{t} : \overline{\kappa}, (x \Gamma^{\varphi})_{\mathbf{P}} \rangle \text{)}} \text{ E-Unpack}$$

第 5 章

Control Operators

第 6 章

Implicit Parameters and Coherence

第 7 章

Records and Polymorphism

第 8 章

Type Checking and Inference

8.1 Hindley/Milner Type System

[LY98]

8.1.1 Language

$$X = \{x, y, z, \dots\}, \mathcal{A} = \{\alpha, \beta, \dots\}$$

E

$$\begin{aligned} e ::= & () \\ & | x \\ & | \lambda x. e \\ & | e e \\ & | \mathbf{let} \ x = e \ \mathbf{in} \ e \\ & | \mathbf{fix} \ f \ \lambda x. e \end{aligned}$$

T

$$\begin{aligned} \tau ::= & \mathbf{unit} \\ & | \alpha \\ & | \tau \rightarrow \tau \end{aligned}$$

Σ

$$\sigma ::= \forall \vec{\alpha}. \sigma$$

$$\Gamma = \mathcal{A} \xrightarrow{\text{fin}} \Sigma$$

8.1.2 Type System

$$\forall \vec{\alpha}. \tau_1 > \tau_2 \iff \exists S. S(\tau_1) = \tau_2 \wedge \text{dom}(S)$$

$$\text{Gen}(\Gamma, \tau) = \forall \vec{\alpha}. \tau$$

$$(\vec{\alpha} = \text{ftv}(\tau) \setminus \text{ftv}(\Gamma))$$

$$\begin{aligned} & \overline{\Gamma \vdash () : \mathbf{unit}} \\ & \frac{\Gamma(x) > \tau}{\Gamma \vdash x : \tau} \\ & \frac{\Gamma + x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x. e : \tau_1 \rightarrow \tau_2} \\ & \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \\ & \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma + x : \text{Gen}(\Gamma, \tau_1) \vdash e_2 : \tau}{\Gamma \vdash \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 : \tau} \\ & \frac{\Gamma + f : \tau \vdash \lambda x. e : \tau}{\Gamma \vdash \mathbf{fix} \ f \ \lambda x. e : \tau} \end{aligned}$$

第 3.1 節

定理 38. $\mathcal{U}(\tau_1, \tau_2) = S$ ならば, $S(\tau_1) = S(\tau_2)$.

□

8.1.3 Algorithm W

定理 39. 以下は同値

- $\mathcal{U}(\tau_1, \tau_2) = S$ を満たす S が存在する.
- $S(\tau_1) = S(\tau_2)$ を満たす S が存在する.

□

$$\begin{array}{c}
\overline{\mathcal{W}(\Gamma, ()) = (\emptyset, \mathbf{unit})} \\
\frac{\Gamma(x) = \forall \vec{\alpha}. \tau \quad \text{fresh } \vec{\beta}}{\mathcal{W}(\Gamma, x) = (\emptyset, [\vec{\alpha} \leftarrow \vec{\beta}]\tau)} \\
\frac{\text{fresh } \beta \quad \mathcal{W}(\Gamma + x : \beta, e) = (S_1, \tau_1)}{\mathcal{W}(\Gamma, \lambda x. e) = (S_1, S_1(\beta) \rightarrow \tau_1)} \\
\frac{\mathcal{W}(\Gamma, e_1) = (S_1, \tau_1) \quad \mathcal{W}(S_1(\Gamma), e_2) = (S_2, \tau_2) \quad \text{fresh } \beta \quad \mathcal{U}(S_2(\tau_1), \tau_2 \rightarrow \beta) = S_3}{\mathcal{W}(\Gamma, e_1 e_2) = (S_3 S_2 S_1, S_3(\beta))} \\
\frac{\mathcal{W}(\Gamma, e_1) = (S_1, \tau_1) \quad \Gamma_1 = S_1(\Gamma) \quad \mathcal{W}(\Gamma_1 + x : \text{Gen}(\Gamma_1, \tau_1), e_2) = (S_2, \tau_2)}{\mathcal{W}(\Gamma, \mathbf{let } x = e_1 \mathbf{ in } e_2) = (S_2 S_1, \tau_2)} \\
\frac{\text{fresh } \beta \quad \mathcal{W}(\Gamma + f : \beta, \lambda x. e) = (S_1, \tau_1) \quad \mathcal{U}(S_1(\beta), \tau_1) = S_2}{\mathcal{W}(\Gamma, \mathbf{fix } f \lambda x. e) = (S_2 S_1, S_2(\tau_1))}
\end{array}$$

定理 40. 以下は同値

- $\mathcal{W}(\Gamma_0, e) = (S, \tau_0)$, $S(\Gamma_0) = \Gamma$, $S(\tau_0) = \tau$ を満たす S , Γ_0 , τ_0 が存在する.
- $\Gamma \vdash e : \tau$.

□

8.1.4 Algorithm M

$$\begin{array}{c}
\frac{\mathcal{U}(\rho, \mathbf{unit}) = S}{\mathcal{M}(\Gamma, (), \rho) = S} \\
\frac{\mathcal{U}(\rho, [\vec{\beta} \leftarrow \vec{\alpha}]\tau) = S \quad \Gamma(x) = \forall \vec{\alpha}. \tau \quad \text{fresh } \vec{\beta}}{\mathcal{M}(\Gamma, x, \rho) = S} \\
\frac{\mathcal{U}(\rho, \beta_1 \rightarrow \beta_2) = S_1 \quad \text{fresh } \beta_1, \beta_2 \quad \mathcal{M}(S_1(\Gamma) + x : S_1(\beta_1), e, S_1(\beta_2)) = S_2}{\mathcal{M}(\Gamma, \lambda x. e, \rho) = S_2 S_1} \\
\frac{\mathcal{M}(\Gamma, e_1, \beta \rightarrow \rho) = S_1 \quad \text{fresh } \beta \quad \mathcal{M}(S_1(\Gamma), e_2, S_1(\beta)) = S_2}{\mathcal{M}(\Gamma, e_1 e_2, \rho) = S_2 S_1} \\
\frac{\mathcal{M}(\Gamma, e_1, \beta) = S_1 \quad \text{fresh } \beta \quad \mathcal{M}(S_1(\Gamma) + x : \text{Gen}(\Gamma, S_1(\beta)), e_2, S_1(\rho)) = S_2}{\mathcal{M}(\Gamma, \mathbf{let } x = e_1 \mathbf{ in } e_2, \rho) = S_2 S_1} \\
\frac{\mathcal{M}(\Gamma + f : \rho, \lambda x. e, \rho) = S}{\mathcal{M}(\Gamma, \mathbf{fix } f \lambda x. e, \rho) = S}
\end{array}$$

定理 41. 以下は同値

- $\mathcal{M}(\Gamma_0, e, \rho) = S$, $S(\Gamma_0) = \Gamma$, $S(\rho) = \tau$ を満たす S , Γ_0 , ρ が存在する.
- $\Gamma \vdash e : \tau$.

□

8.1.5 Alternative Type System

$$\begin{array}{c}
\overline{\Gamma \vdash () : \mathbf{unit}} \\
\frac{\Gamma(x) = \sigma}{\Gamma \vdash x : \sigma} \\
\frac{\Gamma + x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x. e : \tau_1 \rightarrow \tau_2} \\
\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \\
\frac{\Gamma \vdash e_1 : \sigma_1 \quad \Gamma + x : \sigma_1 \vdash e_2 : \tau}{\Gamma \vdash \mathbf{let} x = e_1 \mathbf{in} e_2 : \tau} \\
\frac{\Gamma + f : \tau \vdash \lambda x. e : \tau}{\Gamma \vdash \mathbf{fix} f \lambda x. e : \tau} \\
\frac{\Gamma \vdash e : \tau \quad \vec{\alpha} \notin \text{ftv}(\tau)}{\Gamma \vdash e : \forall \vec{\alpha}. \tau} \\
\frac{\Gamma \vdash e : \forall \vec{\alpha}. \tau}{\Gamma \vdash e : [\vec{\alpha} \leftarrow \vec{\tau}'] \tau}
\end{array}$$

8.2 HM(X): HM Type System with Constraint System

[OSW99]

8.2.1 制約システム

定義 42 (単純制約システム (simple constraint system)). 単純制約システムとは、以下の組 (Ω, \Vdash) のこと。

- 非空のアルファベット Ω .
- 関係 $(\Vdash) \subseteq \mathcal{P}(\Omega) \times \Omega$ で、以下を満たすもの。
 - 任意の $C \in \mathcal{P}(\Omega)$, $P \in \Omega$ について, $C \Vdash P$.
 - 任意の $C, D \in \mathcal{P}(\Omega)$, $Q \in \Omega$ について, $(\forall P \in D. C \Vdash P)$ かつ $D \Vdash Q$ ならば $C \Vdash Q$.

この時, $C \in \mathcal{P}(\Omega)$ を制約 (*constraint*) と呼ぶ。また, $(\Vdash) \subseteq (\mathcal{P}(\Omega))^2$ への拡張を, $C \Vdash D \stackrel{\text{def}}{\iff} \forall P \in D. C \Vdash P$ と定義する。 $C \Vdash D$ かつ $D \Vdash C$ の時, $C \dashv\vdash D$ と表記する。さらに, $C \wedge D = C \cup D$ と表記する。 \square

命題 43. 単純制約システム (Ω, \Vdash) は、以下を admissible にする。

$$\frac{\overline{C \Vdash C}}{C_1 \Vdash C_2 \quad C_2 \Vdash C_3} \quad \frac{C_1 \Vdash C_3}{C \Vdash D} \quad \frac{C \Vdash D}{C \wedge C' \Vdash D}$$

 \square

証明.

$$\begin{aligned} C \Vdash C &\iff \forall P \in C. C \Vdash P \\ C_1 \Vdash C_2 \wedge C_2 \Vdash C_3 &\implies \forall Q \in C_3. C_1 \Vdash C_2 \wedge C_2 \Vdash Q \\ &\implies \forall Q \in C_3. (\forall P \in C_2. C_1 \Vdash P) \wedge C_2 \Vdash Q \\ &\implies \forall Q \in C_3. C_1 \Vdash Q \\ &\implies C_1 \Vdash C_3 \\ \forall P \in C \wedge C'. C \in P &\implies C \wedge C' \Vdash C \\ C \Vdash D &\implies C \wedge C' \Vdash C \wedge C \Vdash D \implies C \wedge C' \Vdash D \end{aligned} \quad (\because \text{単純制約システムの公理})$$

より明らか。 \blacksquare

定義 44 (Cylindric 制約システム (cylindric constraint system)). Cylindric 制約システムとは、以下の組 $(\Omega, \Vdash, \mathcal{A}, \exists)$ のこと。

- 単純制約システム (Ω, \Vdash) .
- 変数の無限集合 \mathcal{A} .
- 関数の族 $\{\exists\alpha\}_{\alpha \in \mathcal{A}} \in \prod_{\alpha \in \mathcal{A}} \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ で以下を満たすもの。
 - 任意の $C \in \mathcal{P}(\Omega)$, $\alpha \in \mathcal{A}$ について, $C \Vdash \exists\alpha. C$.
 - 任意の $C, D \in \mathcal{P}(\Omega)$, $\alpha \in \mathcal{A}$ について, $C \Vdash D$ ならば, $\exists\alpha. C \Vdash \exists\alpha. D$.
 - 任意の $C, D \in \mathcal{P}(\Omega)$, $\alpha \in \mathcal{A}$ について, $\exists\alpha. (C \wedge \exists\alpha. C) \dashv\vdash (\exists\alpha. C) \wedge (\exists\alpha. D)$.
 - 任意の $C \in \mathcal{P}(\Omega)$, $\alpha, \beta \in \mathcal{A}$ について, $\exists\alpha. \exists\beta. C \dashv\vdash \exists\beta. \exists\alpha. C$.

ただし, $\exists\alpha. C = (\exists\alpha)(C)$ である。

 \square

定義 45 (自由変数). Cylindric 制約システム $(\Omega, \Vdash, \mathcal{A}, \exists)$, 制約 $C \in \mathcal{P}(\Omega)$ について, 自由変数の集合を $\text{fv}(C) = \{\alpha \mid \exists\alpha. C \dashv\vdash C\}$ とおく。 \square

定義 46 (充足可能 (satisfiable)). Cylindric 制約システム $(\Omega, \Vdash, \mathcal{A}, \exists)$, 制約 $C \in \mathcal{P}(\Omega)$ について, $\Vdash \exists \text{fv}(C). C$ の時, C は充足可能であるという. \square

補題 47. Cylindric 制約システム $(\Omega, \Vdash, \mathcal{A}, \exists)$, 制約 $C \in \mathcal{P}(\Omega)$ について, 以下は同値.

- C は充足可能.
- $\exists \alpha. C$ は充足可能.

\square

定義 48 (項制約システム (term constraint system)). 項制約システムとは,

- 項代数 (Σ, X) .
- 述語のランク付きアルファベット P .
- Cylindric 制約システム $(\Omega, \Vdash, X, \exists)$, ただし, $\Omega = \{p(\tau_1, \dots, \tau_n) \mid p^{(n)} \in P, \tau_1, \dots, \tau_n \in \llbracket (\Sigma, X) \rrbracket\}$.

の組 $(\Sigma, P, \Omega, \Vdash, X, \exists)$ で, 以下を満たすもの.

- 任意の $\alpha \in X$ について, $\Vdash \alpha = \alpha$.
- 任意の $\alpha_1, \alpha_2 \in X$ について, $(\alpha_1 = \alpha_2) \Vdash (\alpha_2 = \alpha_1)$.
- 任意の $\alpha_1, \alpha_2, \alpha_3 \in X$ について, $(\alpha_1 = \alpha_2) \wedge (\alpha_2 = \alpha_3) \Vdash (\alpha_1 = \alpha_3)$.
- 任意の $\alpha_1, \alpha_2 \in X$, $C \in \mathcal{P}(\Omega)$ について, $(\alpha_1 = \alpha_2) \wedge \exists \alpha_1. (C \wedge (\alpha_1 = \alpha_2)) \Vdash C$.
- 任意のコンテキスト $T[\] \in \mathcal{C}(\mathcal{T})$, $\tau_1, \tau_2 \in \llbracket (\Sigma, X) \rrbracket$ について, $(\tau_1 = \tau_2) \Vdash (T[\tau_1] = T[\tau_2])$.
- 任意の $P \in \Omega$, $\tau \in \llbracket (\Sigma, X) \rrbracket$, $\alpha \in X$, $\alpha \notin \text{fv}(\tau)$ について, $P[\alpha \leftarrow \tau] \Vdash \exists \alpha. (P \wedge (\alpha = \tau))$.

\square

定義 49 (置換の拡張). $(P_1 \wedge \dots \wedge P_n)[\vec{\alpha} \leftarrow \vec{\tau}] = P_1[\vec{\alpha} \leftarrow \vec{\tau}] \wedge \dots \wedge P_n[\vec{\alpha} \leftarrow \vec{\tau}]$ と表記する. \square

補題 50 (改名 (renaming)). 項制約システム $(\Sigma, P, \Omega, \Vdash, X, \exists)$, $C \in \mathcal{P}(\Omega)$, $\alpha_1, \alpha_2 \in X$ について, α_2 が C に出現しない時, $\exists \alpha_1. C \Vdash \exists \alpha_2. C[\alpha_1 \leftarrow \alpha_2]$. \square

補題 51 (正規形 (normal form)). 項制約システム $(\Sigma, P, \Omega, \Vdash, X, \exists)$, $C \in \mathcal{P}(\Omega)$ について, 以下が成り立つ.

$$C[\alpha_1 \leftarrow \tau_1, \dots, \alpha_n \leftarrow \tau_n] \Vdash \exists \alpha_1, \dots, \alpha_n. C \wedge (\alpha_1 = \tau_1) \wedge \dots \wedge (\alpha_n = \tau_n)$$

\square

補題 52 (置換 (substitution)). 項制約システム $(\Sigma, P, \Omega, \Vdash, X, \exists)$, $C, D \in \mathcal{P}(\Omega)$, 置換 ϕ について, 以下が成り立つ.

$$C \Vdash D \implies \phi C \Vdash \phi D$$

\square

8.2.2 型システム

定義 53 (包含 (subsumption)). 項制約システム $(\Sigma, P, \Omega, \Vdash, X, \exists)$ について, 包含付きであるとは, $\lesssim \in P^{(2)}$ で以下を満たすことを言う.

$$\begin{array}{c} \hline (\alpha_1 = \alpha_2) \Vdash (\alpha_1 \lesssim \alpha_2) \wedge (\alpha_2 \lesssim \alpha_1) \\ \hline (\alpha_1 \lesssim \alpha_2) \wedge (\alpha_2 \lesssim \alpha_1) \Vdash (\alpha_1 = \alpha_2) \\ \hline D \Vdash (\alpha_1 \lesssim \alpha_2) \quad D \Vdash (\alpha_2 \lesssim \alpha_3) \\ \hline D \Vdash (\alpha_1 \lesssim \alpha_3) \\ \hline D \Vdash (\alpha_1 \lesssim \alpha_2) \quad D \Vdash (\beta_1 \lesssim \beta_2) \\ \hline D \Vdash (\alpha_1 \rightarrow \beta_1 \lesssim \alpha_2 \rightarrow \beta_2) \end{array}$$

□

定義 54 (型システム). 包含付き項制約システム $(\Sigma, P, \Omega, \Vdash, X, \exists)$ について、制約 $C \in \mathcal{P}(\Omega)$ 、環境 Γ 、式 e 、型スキーム σ の型判定 $C, \Gamma \vdash e : \sigma$ を以下のように定義する。

$$\begin{array}{c}
\frac{x : \sigma \in \Gamma}{C, \Gamma \vdash x : \sigma} \\
\frac{C, \Gamma \vdash e : \tau_1 \quad C \Vdash \tau_1 \lesssim \tau_2}{C, \Gamma \vdash e : \tau_2} \\
\frac{C, \Gamma + x : \tau_1 \vdash e : \tau_2}{C, \Gamma \vdash \lambda x. e : \tau_1 \rightarrow \tau_2} \\
\frac{C, \Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad C, \Gamma \vdash e_2 : \tau_1}{C, \Gamma \vdash e_1 e_2 : \tau_2} \\
\frac{C, \Gamma \vdash e_1 : \sigma_1 \quad C, \Gamma + x : \sigma_1 \vdash e_2 : \tau_2}{C, \Gamma \vdash \text{let } x = e_1 \text{ in } e_2 : \tau_2} \\
\frac{C \wedge D, \Gamma \vdash e : \tau \quad \vec{\alpha} \notin \text{fv}(C) \cup \text{fv}(\Gamma)}{C \wedge \exists \vec{\alpha}. D, \Gamma \vdash e : \forall \vec{\alpha}. D \Rightarrow \tau} \\
\frac{C, \Gamma \vdash e : \forall \vec{\alpha}. D \Rightarrow \tau' \quad C \Vdash D[\vec{\alpha} \leftarrow \vec{\tau}]}{C, \Gamma \vdash e : \tau'[\vec{\alpha} \leftarrow \vec{\tau}]}
\end{array}$$

□

8.2.3 推論アルゴリズム

定義 55. 変数の集合 U 、置換 ϕ 、 $x \in U$ について、 $\phi|_U$ を以下のようにおく。

$$\phi|_U(x) = \begin{cases} \sigma & (x : \sigma \in \phi) \\ x & (\text{otherwise}) \end{cases}$$

また、

$$\begin{aligned}
& \Vdash \psi =_U \phi \stackrel{\text{def}}{\iff} \forall x \in U. \Vdash \psi|_U(x) = \phi|_U(x) \\
& \Vdash \psi \leq_U^\chi \phi \stackrel{\text{def}}{\iff} \Vdash \chi \circ \psi =_U \phi \\
& \Vdash \psi \leq_U \phi \stackrel{\text{def}}{\iff} \exists \chi. \Vdash \psi \leq_U^\chi \phi
\end{aligned}$$

と表記する。

□

定義 56 (正規形). 項制約システム $(\Sigma, P, \Omega, \Vdash, X, \exists)$ 、制約 $C, D \in \mathcal{P}(\Omega)$ 、置換 ϕ, ψ について、 (C, ψ) が (D, ϕ) の正規形とは、 $\phi \leq \psi$ 、 $C \Vdash \psi D$ 、 $\psi C = C$ を満たすことを言う。

□

定義 57 (制約付き Algorithm W). 項制約システム $(\Sigma, P, \Omega, \Vdash, X, \exists)$ について、 $norm$ を制約 $C \in \mathcal{P}(\Omega)$ 、置換 ψ において $norm(C, \psi) = (D, \phi)$ が (C, ψ) の正規形になる関数とする。また、 gen を制約 $C \in \mathcal{P}(\Omega)$ 、環境 Γ 、型スキーム σ 、変数列 $\vec{\alpha} = (\text{fv}(\sigma) \cup \text{fv}(C)) \setminus \text{fv}(\Gamma)$ 、 $C \dashv \vdash C' \wedge D$ 、 $\text{fv}(D) \wedge \vec{\alpha} = \emptyset$ を満たす制約 $C', D \in \mathcal{P}(\Omega)$ について、

$$gen(C, \Gamma, \sigma) = (D \wedge \exists \vec{\alpha}. C', \forall \vec{\alpha}. C' \Rightarrow \sigma)$$

を満たす関数とする。この時、置換 ψ 、 $C \in \mathcal{P}(\Omega)$ 、環境 Γ 、式 e 、型スキーム σ について、判定 $\psi, C, \Gamma \vdash^W e : \sigma$ を以下のように定義する。

$$\begin{array}{c}
\frac{x : \forall \vec{\alpha}. D \Rightarrow \tau \in \Gamma \quad \text{fresh } \vec{\beta} \quad norm(D, [\vec{\alpha} \leftarrow \vec{\beta}]) = (C, \psi)}{\psi|_{\text{fv}(\Gamma)}, C, \Gamma \vdash^W x : \psi \tau} \\
\frac{\psi, C, \Gamma + x : \alpha \vdash^W e : \tau \quad \text{fresh } \alpha}{\psi|_{\{\alpha\}}, C, \Gamma \vdash^W \lambda x. e : \psi(\alpha) \rightarrow \tau} \\
\frac{\psi_1, C_1, \Gamma \vdash^W e_1 : \tau_1 \quad \psi_2, C_2, \Gamma \vdash^W e_2 : \tau_2 \quad D = C_1 \wedge C_2 \wedge \tau_1 \lesssim \tau_2 \rightarrow \alpha \quad \text{fresh } \alpha \quad norm(D, \psi_1 \sqcup \psi_2) = (C, \psi)}{\psi|_{\text{fv}(\Gamma)}, C, \Gamma \vdash^W e_1 e_2 : \psi(\alpha)}
\end{array}$$

$$\frac{\psi_1, C_1, \Gamma \vdash^W e_1 : \tau_1 \quad (C_2, \sigma) = \text{gen}(C_1, \psi_1 \Gamma, \tau_1) \quad \psi_2, C_3, \Gamma + x : \sigma \vdash^W e_2 : \tau_2 \quad \text{norm}(C_2 \wedge C_3, \psi_1 \sqcup \psi_2) = (C, \psi)}{\psi|_{fv(\Gamma)}, C, \Gamma \vdash^W \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 : \psi\tau_2}$$

□

8.2.4 自由構成

構文:

$$\begin{aligned} T &::= \rightarrow \mid \dots \\ D &::= \simeq \mid \lesssim \mid \dots \\ Q &::= \epsilon \\ &\quad \mid Q_1 \wedge Q_2 \\ &\quad \mid D\vec{\tau} \\ C &::= Q \\ \tau &::= \alpha \\ &\quad \mid T\vec{\tau} \\ \sigma &::= \forall \vec{\alpha}. Q \Rightarrow \tau \\ e &::= x \\ &\quad \mid \lambda x. e \\ &\quad \mid e_1 e_2 \\ &\quad \mid \mathbf{let} \ x = e_1 \ \mathbf{in} \ e_2 \end{aligned}$$

制約推論:

$$\begin{aligned} &\frac{}{(D\vec{\tau}) \in (D\vec{\tau})} \quad \frac{(D\vec{\tau}) \in C_1}{(D\vec{\tau}) \in C_1 \wedge C_2} \quad \frac{(D\vec{\tau}) \in C_2}{(D\vec{\tau}) \in C_1 \wedge C_2} \\ &\frac{(D\vec{\tau}) \in C}{C \Vdash D\vec{\tau}} \quad \frac{C \Vdash Q_1 \quad C \Vdash Q_2}{C \Vdash Q_1 \wedge Q_2} \\ &\frac{}{C \Vdash \tau \simeq \tau} \quad \frac{C \Vdash \tau_2 \simeq \tau_1}{C \Vdash \tau_1 \simeq \tau_2} \quad \frac{C \Vdash \tau_1 \simeq \tau_2 \quad C \Vdash \tau_2 \simeq \tau_3}{C \Vdash \tau_1 \simeq \tau_3} \\ &\frac{C \Vdash \tau_1 \simeq \tau_2}{C \Vdash \tau_1 \lesssim \tau_2} \\ &\frac{\frac{C \Vdash \tau_1 \simeq \tau_2}{C \Vdash T\vec{\tau}_1 \simeq T\vec{\tau}_2} \quad \frac{C \Vdash T\vec{\tau}_1 \simeq T\vec{\tau}_2}{C \Vdash \tau_1 \simeq \tau_2}}{C \Vdash D\vec{\tau}_1 \quad C \Vdash \tau_1 \simeq \tau_2} \\ &\frac{C \Vdash D\vec{\tau}_1 \quad C \Vdash \tau_1 \simeq \tau_2}{C \Vdash D\vec{\tau}_2} \end{aligned}$$

制約解決:

$$\frac{\begin{array}{l} C \vdash \text{flat}(Q_1) \rightarrow^* W_2 \not\vdash \\ \theta_2 = [\alpha_2 \leftarrow \theta_2 \tau_2 \mid (\alpha_2 \simeq \tau_2) \in W_2] \quad Q_2 = \bigwedge \{D\vec{\tau} \mid (D\vec{\tau}) \in W_2\} \\ \vec{\alpha}_3 = \text{fv}(Q_2) \quad \theta_3 = [\alpha_3 \leftarrow \vec{\tau}_3] \quad C \Vdash \theta_3 \theta_2 Q_2 \end{array}}{\text{solv}(C, Q_1) = \theta_3 \theta_2}$$

$$\begin{aligned} &\overline{\text{flat}(\epsilon) = \emptyset} \\ &\frac{\text{flat}(Q_1) = W_1 \quad \text{flat}(Q_2) = W_2}{\text{flat}(Q_1 \wedge Q_2) = W_1 \cup W_2} \\ &\overline{\text{flat}(D\vec{\tau}) = \{D\vec{\tau}\}} \end{aligned}$$

$$\begin{aligned} &\frac{\alpha \leq \beta (\text{lexicographically})}{\alpha < \beta} \\ &\overline{\alpha < T\vec{\tau}} \end{aligned}$$

$$\frac{\alpha < \tau \quad \alpha \notin \text{ftv}(\tau)}{\alpha \sim \tau}$$

$$\begin{array}{c} \frac{Q = (\tau \simeq \tau) \in W}{C \vdash W \rightarrow W \setminus \{Q\}} \\ \frac{Q = (T\vec{\tau}_1 \simeq T\vec{\tau}_2) \in W}{C \vdash W \rightarrow (W \setminus \{Q\}) \cup \{\tau_1 \simeq \tau_2\}} \\ \frac{(T\vec{\tau}_1 \simeq S\vec{\tau}_2) \in W \quad T \neq S}{C \vdash W \rightarrow \perp} \\ \frac{(\beta \simeq \tau) \in W \quad \beta \in \text{ftv}(\tau)}{C \vdash W \rightarrow \perp} \\ \frac{Q = (\tau_1 \simeq \tau_2) \in W \quad \tau_2 < \tau_1}{C \vdash W \rightarrow (W \setminus \{Q\}) \cup \{\tau_2 \simeq \tau_1\}} \\ \frac{\{\beta \simeq \tau_1, \beta \simeq \tau_2\} \subseteq W \quad \tau_1 \neq \tau_2 \quad \beta \sim \tau_1 \quad \beta \sim \tau_2}{C \vdash W \rightarrow (W \setminus \{\beta \simeq \tau_2\}) \cup \{\tau_1 \simeq \tau_2\}} \\ \frac{\{\beta_1 \simeq \tau_1, \beta_2 \simeq \tau_2\} \subseteq W \quad \beta_1 \in \text{ftv}(\tau_2) \quad \beta_1 \sim \tau_1 \quad \beta_2 \sim \tau_2}{C \vdash W \rightarrow (W \setminus \{\beta_2 \simeq \tau_2\}) \cup \{\beta_2 \simeq \tau_2[\beta_1 \leftarrow \tau_1]\}} \\ \frac{\{\beta_1 \simeq \tau_1, D\vec{\tau}_2\} \subseteq W \quad \beta_1 \in \text{ftv}(\vec{\tau}_2) \quad \beta_1 \sim \tau_1}{C \vdash W \rightarrow (W \setminus \{D\vec{\tau}_2\}) \cup \{(D\vec{\tau}_2)[\beta_1 \leftarrow \tau_1]\}} \\ \frac{Q = D\vec{\tau} \in W \quad D\vec{\tau} \in C}{C \vdash W \rightarrow W \setminus \{Q\}} \end{array}$$

補題 58. $C \vdash \text{flat}(Q_1) \rightarrow^* W_2 \not\vdash$ の時, $W_3 = \{\tau_1 \simeq \tau_2 \mid \tau_1 \simeq \tau_2 \mid W_2\}$, $W_4 = W_2 \setminus W_3$ とすると, 以下が成り立つ:

- $\tau_1 \simeq \tau_2 \in W_3$ について, $\tau_1 = \alpha$.
- $\alpha_1 \simeq \tau_2 \in W_3$ について, $\alpha_1 \notin \text{ftv}(\tau_2)$.
- $\alpha_1 \simeq \tau_2 \in W_3$ について, $\tau_2 = \alpha_2$ ならば, $\alpha_1 \not\leq \alpha_2$.
- $\alpha \simeq \tau_1, \alpha \simeq \tau_2 \in W_3$ について, $\tau_1 = \tau_2$.
- $Q \in W_2$, $\alpha \in \text{ftv}(Q)$ について, $\alpha \simeq \tau \in W_3$ となる τ は存在しない.
- $D\vec{\tau} \in W_4$ について, $D\vec{\tau} \notin C$.

□

8.3 OutsiderIn(X): Modular Type Inference with Local Assumptions

[VJSS11]

8.3.1 Syntax

x, y, z, f, g, h 変数

α, β, γ 型変数

K コンストラクタ

T 型コンストラクタ

D 制約コンストラクタ

F 型関数

$$\begin{aligned}
 P &::= \epsilon \\
 &| f = e, P \\
 &| f : \sigma = e, P \\
 \nu &::= x \mid K \\
 e &::= \nu \\
 &| \lambda x. e \\
 &| e_1 e_2 \\
 &| \text{case}(e, K\vec{x} \mapsto e) \\
 &| \text{let}(x : \sigma = e_1, e_2) \\
 &| \text{let}(x = e_1, e_2) \\
 \sigma &::= \forall \vec{\alpha}. Q \Rightarrow \tau \\
 P &::= \tau_1 \simeq \tau_2 \\
 &| D\vec{\tau} \\
 Q &::= \epsilon \\
 &| Q_1 \wedge Q_2 \\
 &| P \\
 \mathcal{J} &::= \alpha \\
 &| \rightarrow \\
 &| T \\
 \tau, \nu &::= \alpha \\
 &| \tau_1 \rightarrow \tau_2 \\
 &| T\vec{\tau} \\
 &| F\vec{\tau} \\
 \Gamma &::= \epsilon \\
 &| \nu : \sigma, \Gamma \\
 \mathcal{Q} &::= Q \\
 &| Q \wedge Q \\
 &| \forall \vec{\alpha}. Q \Rightarrow Q \\
 &| \forall \vec{\alpha}. F\vec{\tau}_1 \simeq \tau_2
 \end{aligned}$$

8.3.2 Entailment

Concrete:

$$\begin{array}{c}
 \frac{Q \Vdash Q_1 \quad Q \Vdash Q_2}{Q \Vdash Q_1 \wedge Q_2} \\
 \frac{Q \Vdash \tau \simeq \tau \quad Q \Vdash \tau_2 \simeq \tau_1 \quad Q \Vdash \tau_1 \simeq \tau_2 \quad Q \Vdash \tau_2 \simeq \tau_3}{Q \Vdash \tau_1 \simeq \tau_3} \\
 \frac{Q \Vdash T\vec{\tau}_1 \simeq T\vec{\tau}_2 \quad Q \Vdash \bigwedge \overline{\tau_1 \simeq \tau_2} \quad Q \Vdash \bigwedge \overline{\tau_1 \simeq \tau_2}}{Q \Vdash \bigwedge \overline{\tau_1 \simeq \tau_2}} \quad \frac{Q \Vdash \bigwedge \overline{\tau_1 \simeq \tau_2} \quad Q \Vdash T\vec{\tau}_1 \simeq T\vec{\tau}_2 \quad Q \Vdash F\vec{\tau}_1 \simeq F\vec{\tau}_2}{Q \Vdash T\vec{\tau}_1 \simeq T\vec{\tau}_2} \quad \frac{Q \Vdash \bigwedge \overline{\tau_1 \simeq \tau_2} \quad Q \Vdash F\vec{\tau}_1 \simeq F\vec{\tau}_2}{Q \Vdash F\vec{\tau}_1 \simeq F\vec{\tau}_2}
 \end{array}$$

$$\frac{(\forall \vec{\alpha}. Q_1 \Rightarrow Q_2) \in \mathcal{Q} \quad \mathcal{Q} \Vdash Q_1[\vec{\alpha} \leftarrow \vec{\tau}]}{\mathcal{Q} \Vdash Q_2[\vec{\alpha} \leftarrow \vec{\tau}]}$$

$$\frac{\mathcal{Q} \Vdash D\vec{\tau}_1 \quad \mathcal{Q} \Vdash \bigwedge \vec{\tau}_1 \simeq \vec{\tau}_2}{\mathcal{Q} \Vdash D\vec{\tau}_2}$$

- projection って必要ないん？

Requirements:

$$\frac{}{\mathcal{Q} \wedge Q \Vdash Q} \quad \frac{\mathcal{Q} \wedge Q_1 \Vdash Q_2 \quad \mathcal{Q} \wedge Q_2 \Vdash Q_3}{\mathcal{Q} \wedge Q_1 \Vdash Q_3} \quad \frac{\mathcal{Q} \Vdash Q}{\mathcal{Q} \Vdash Q[\alpha \leftarrow \tau]}$$

$$\frac{\mathcal{Q} \Vdash \tau \simeq \tau}{\mathcal{Q} \Vdash \tau \simeq \tau} \quad \frac{\mathcal{Q} \Vdash \tau_2 \simeq \tau_1 \quad \mathcal{Q} \Vdash \tau_1 \simeq \tau_2}{\mathcal{Q} \Vdash \tau_1 \simeq \tau_2} \quad \frac{\mathcal{Q} \Vdash \tau_1 \simeq \tau_2 \quad \mathcal{Q} \Vdash \tau_2 \simeq \tau_3}{\mathcal{Q} \Vdash \tau_1 \simeq \tau_3}$$

$$\frac{\mathcal{Q} \Vdash Q_1 \quad \mathcal{Q} \Vdash Q_2}{\mathcal{Q} \Vdash Q_1 \wedge Q_2}$$

$$\frac{\mathcal{Q} \Vdash \tau_1 \simeq \tau_2}{\mathcal{Q} \Vdash \tau[\alpha \leftarrow \tau_1] \simeq \tau[\alpha \leftarrow \tau_2]}$$

8.3.3 Type System

$$\frac{(\nu : \forall \vec{\alpha}. Q_1 \Rightarrow \tau_1) \in \Gamma \quad \mathcal{Q} \Vdash Q_1[\vec{\alpha} \leftarrow \vec{\tau}_2]}{\mathcal{Q}; \Gamma \vdash \nu : \tau_1[\vec{\alpha} \leftarrow \vec{\tau}_2]}$$

$$\frac{\mathcal{Q}; \Gamma \vdash e : \tau_1 \quad \mathcal{Q} \Vdash \tau_1 \simeq \tau_2}{\mathcal{Q}; \Gamma \vdash e : \tau_2}$$

$$\frac{\mathcal{Q}; \Gamma, x : \tau_1 \vdash e : \tau_2}{\mathcal{Q}; \Gamma \vdash \lambda x. e : \tau_1 \rightarrow \tau_2}$$

$$\frac{\mathcal{Q}; \Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad \mathcal{Q}; \Gamma \vdash e_2 : \tau_1}{\mathcal{Q}; \Gamma \vdash e_1 e_2 : \tau_2}$$

$$\frac{\mathcal{Q}; \Gamma \vdash e_1 : \tau_1 \quad \mathcal{Q}; \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\mathcal{Q}; \Gamma \vdash \mathbf{let}(x = e_1, e_2) : \tau_2}$$

$$\frac{Q \wedge Q_1; \Gamma \vdash e_1 : \tau_1 \quad \vec{\alpha} \wedge (fiv(Q) \cup fiv(\Gamma)) = \emptyset \quad \mathcal{Q}; \Gamma, x : \forall \vec{\alpha}. Q_1 \Rightarrow \tau_1 \vdash e_2 : \tau_2}{\mathcal{Q}; \Gamma \vdash \mathbf{let}(x : \forall \vec{\alpha}. Q_1 \Rightarrow \tau_1 = e_1, e_2) : \tau_2}$$

$$\frac{\begin{array}{l} \mathcal{Q}; \Gamma \vdash e : T\vec{\tau}_1 \\ \bigwedge_i (K_i : \forall \vec{\alpha}_i \vec{\beta}. Q_i \Rightarrow \vec{v}_i \rightarrow T\vec{\alpha}_i) \in \Gamma \\ \vec{\beta} \wedge (fiv(Q) \cup fiv(\Gamma) \cup \overrightarrow{fiv(\tau_1)} \cup \overrightarrow{fiv(\tau_2)}) = \emptyset \\ \bigwedge_i Q \wedge Q_i[\vec{\alpha}_i \leftarrow \vec{\tau}]; \Gamma, x_i : \nu_i[\vec{\alpha}_i \leftarrow \vec{\tau}] \vdash e_i : \tau_2 \end{array}}{\mathcal{Q}; \Gamma \vdash \mathbf{case}(e, \overrightarrow{K_i \vec{x}_i \mapsto e_i}) : \tau_2}$$

$$\frac{(fiv(\Gamma) \cup fiv(\mathcal{Q})) = \emptyset}{\mathcal{Q}; \Gamma \vdash \epsilon}$$

$$\frac{Q \wedge Q_1 \Vdash Q_2 \quad \mathcal{Q}_2; \Gamma \vdash e : \tau \quad \vec{\alpha} = fiv(Q_1) \cup fiv(\tau) \quad \mathcal{Q}; \Gamma, (f : \forall \vec{\alpha}. Q_1 \Rightarrow \tau) \vdash P}{\mathcal{Q}; \Gamma \vdash f = e, P}$$

$$\frac{Q \wedge Q_1 \Vdash Q_2 \quad \mathcal{Q}_2; \Gamma \vdash e : \tau \quad \vec{\alpha} = fiv(Q_1) \cup fiv(\tau) \quad \mathcal{Q}; \Gamma, (f : \forall \vec{\alpha}. Q_1 \Rightarrow \tau) \vdash P}{\mathcal{Q}; \Gamma \vdash f : \forall \vec{\alpha}. Q_1 \Rightarrow \tau = e, P}$$

8.3.4 Type Inference

$$\begin{array}{c}
C ::= Q \\
| C_1 \wedge C_2 \\
| \exists \vec{\alpha}. (Q \supset C)
\end{array}$$

$$\begin{array}{c}
\frac{\text{fresh } \vec{\beta} \quad (\nu : \forall \vec{\alpha}. Q \Rightarrow \tau) \in \Gamma}{\Gamma \triangleright \nu \rightsquigarrow Q[\vec{\alpha} \leftarrow \vec{\beta}] \Rightarrow \tau[\vec{\alpha} \leftarrow \vec{\beta}]} \\
\frac{\text{fresh } \beta \quad \Gamma, x : \beta \triangleright e \rightsquigarrow C \Rightarrow \tau}{\Gamma \triangleright \lambda x. e \rightsquigarrow C \Rightarrow \beta \rightarrow \tau} \\
\frac{\Gamma \triangleright e_1 \rightsquigarrow C_1 \Rightarrow \tau_1 \quad \Gamma \triangleright e_2 \rightsquigarrow C_2 \Rightarrow \tau_2 \quad \text{fresh } \beta}{\Gamma \triangleright e_1 e_2 \rightsquigarrow C_1 \wedge C_2 \wedge (\tau_1 \simeq (\tau_2 \rightarrow \beta)) \Rightarrow \beta} \\
\frac{\Gamma \triangleright e_1 \rightsquigarrow C_1 \Rightarrow \tau_1 \quad \Gamma, x : \tau_1 \triangleright e_2 \rightsquigarrow C_2 \Rightarrow \tau_2}{\Gamma \triangleright \mathbf{let}(x = e_1, e_2) \rightsquigarrow C_1 \wedge C_2 \Rightarrow \tau_2} \\
\frac{\Gamma \triangleright e_1 \rightsquigarrow C'_1 \Rightarrow \tau'_1 \quad \vec{\beta}_1 = (\text{ftv}(\tau'_1) \cup \text{ftv}(C'_1)) \setminus \text{ftv}(\Gamma) \quad \Gamma, x : \forall \vec{\alpha}_1. Q_1 \Rightarrow \tau_1 \triangleright e_2 \rightsquigarrow C_2 \Rightarrow \tau_2}{\Gamma \triangleright \mathbf{let}(x : \forall \vec{\alpha}_1. Q_1 \Rightarrow \tau_1 = e_1, e_2) \rightsquigarrow (\exists \vec{\beta}_1. Q_1 \supset C'_1 \wedge \tau_1 \simeq \tau'_1) \wedge C_2 \Rightarrow \tau_2} \\
\frac{\Gamma \triangleright e \rightsquigarrow C \Rightarrow \tau \quad \text{fresh } \beta, \vec{\gamma} \quad \bigwedge_i \text{fresh } \vec{\beta}_i \quad \bigwedge_i (K_i : \forall \vec{\alpha}_i \vec{\beta}_i. Q_i \Rightarrow \vec{v}_i \rightarrow T\vec{\alpha}_i) \in \Gamma \quad \bigwedge_i \Gamma, (x_i : v_i[\vec{\alpha}_i \leftarrow \vec{\gamma}]) \triangleright e_i \rightsquigarrow C_i \Rightarrow \tau_i \quad \bigwedge_i \delta_i = (\text{ftv}(\tau_i) \cup \text{ftv}(C_i)) \setminus (\text{ftv}(\Gamma) \cup \bigcup \text{ftv}(\vec{\gamma}))}{\Gamma \triangleright \mathbf{case}(e, \overrightarrow{K_i \vec{x}_i \mapsto e_i}) \rightsquigarrow C \wedge (\tau \simeq T\vec{\gamma}) \wedge (\bigwedge_i \exists \delta_i. Q_i[\vec{\alpha} \leftarrow \vec{\gamma}] \supset C_i \wedge \tau_i \simeq \beta) \Rightarrow \beta}
\end{array}$$

制約解決 $Q; Q; \vec{\alpha} \vdash C_1 \rightsquigarrow^{solv} Q_2 \mid \theta$ については、後述する。

$$\begin{array}{c}
\overline{Q; \Gamma \triangleright \epsilon \rightsquigarrow \top} \\
\Gamma \triangleright e \rightsquigarrow C \Rightarrow \tau \\
Q; \epsilon; \text{ftv}(\tau) \cup \text{ftv}(C) \vdash C \rightsquigarrow^{solv} Q \mid \theta \\
\vec{\alpha} = \text{ftv}(\theta\tau) \cup \text{ftv}(Q) \\
\text{fresh } \vec{\beta} \\
\frac{Q; \Gamma, f : \forall \vec{\beta}. (Q \Rightarrow \theta\tau)[\vec{\alpha} \leftarrow \vec{\beta}] \triangleright P \rightsquigarrow \top}{Q; \Gamma \triangleright f = e, P \rightsquigarrow \top} \\
\Gamma \triangleright e \rightsquigarrow C' \Rightarrow \tau' \\
Q; Q; \text{ftv}(\tau') \cup \text{ftv}(C') \vdash C' \wedge (\tau \simeq \tau') \rightsquigarrow^{solv} \epsilon \mid \theta \\
Q; \Gamma, f : \forall \vec{\alpha}. Q \Rightarrow \tau \triangleright P \rightsquigarrow \top \\
\hline
Q; \Gamma \triangleright f : \forall \vec{\alpha}. Q \Rightarrow \tau = e, P \rightsquigarrow \top
\end{array}$$

8.3.5 Constraint Solving

$$\begin{array}{c}
\overline{\text{split}(Q) = \langle Q, \emptyset \rangle} \\
\text{split}(C_1) = \langle Q_1, I_1 \rangle \quad \text{split}(C_2) = \langle Q_2, I_2 \rangle \\
\hline
\text{split}(C_1 \wedge C_2) = \langle Q_1 \wedge Q_2, I_1 \cup I_2 \rangle
\end{array}$$

$$\overline{\text{split}(\exists \vec{\alpha}. Q \supset C) = \langle \epsilon, \{\exists \vec{\alpha}. Q \supset C\} \rangle}$$

$$\frac{\begin{array}{c} \text{split}(C_1) = \langle Q_1, I_1 \rangle \\ Q; Q; \vec{\alpha} \vdash Q_1 \xrightarrow{\text{simpl}} Q_2 \mid \theta \\ \bigwedge_{(\exists \vec{\alpha}', Q' \supset C') \in \theta_{I_1}} Q; Q \wedge Q_2 \wedge Q'; \vec{\alpha}' \vdash C' \xrightarrow{\text{solv}} \epsilon \mid \theta' \end{array}}{Q; Q; \vec{\alpha} \vdash C_1 \xrightarrow{\text{solv}} Q_2 \mid \theta}$$

Simplification:

$$\begin{array}{c} \text{canon}_g(P_1) = \langle \vec{\alpha}_2, \theta_2, W_2 \rangle \quad \text{dom}(\theta_1) \cap \text{dom}(\theta_2) = \emptyset \\ \hline Q \vdash \langle \vec{\alpha}_1, \theta_1, W_g \uplus \{P_1\}, W_w \rangle \rightarrow \langle \vec{\alpha}_1 \vec{\alpha}_2, \theta_1 \cup \theta_2, W_g \cup W_2, W_w \rangle \\ \text{canon}_w(P_1) = \langle \vec{\alpha}_2, \theta_2, W_2 \rangle \quad \text{dom}(\theta_1) \cap \text{dom}(\theta_2) = \emptyset \\ \hline Q \vdash \langle \vec{\alpha}_1, \theta_1, W_g, W_w \uplus \{P_1\} \rangle \rightarrow \langle \vec{\alpha}_1 \vec{\alpha}_2, \theta_1 \cup \theta_2, W_g, W_w \cup W_2 \rangle \\ \text{interact}_g(P_1, P_2) = W_3 \\ \hline Q \vdash \langle \vec{\alpha}, \theta, W_g \uplus \{P_1, P_2\}, W_w \rangle \rightarrow \langle \vec{\alpha}, \theta, W_g \cup W_3, W_w \rangle \\ \text{interact}_w(P_1, P_2) = W_3 \\ \hline Q \vdash \langle \vec{\alpha}, \theta, W_g, W_w \uplus \{P_1, P_2\} \rangle \rightarrow \langle \vec{\alpha}, \theta, W_g, W_w \cup W_3 \rangle \\ \text{simplify}(P, P_1) = W_2 \\ \hline Q \vdash \langle \vec{\alpha}, \theta, W_g \uplus \{P\}, W_w \uplus \{P_1\} \rangle \rightarrow \langle \vec{\alpha}, \theta, W_g \uplus \{P\}, W_w \cup W_2 \rangle \\ \text{topreact}_g(Q, P_1) = \langle \epsilon, W_2 \rangle \\ \hline Q \vdash \langle \vec{\alpha}, \theta, W_g \uplus \{P_1\}, W_w \rangle \rightarrow \langle \vec{\alpha}, \theta, W_g \cup W_2, W_w \rangle \\ \text{topreact}_w(Q, P_1) = \langle \vec{\alpha}_2, W_2 \rangle \\ \hline Q \vdash \langle \vec{\alpha}_1, \theta, W_g, W_w \uplus \{P_1\} \rangle \rightarrow \langle \vec{\alpha}_1 \vec{\alpha}_2, \theta, W_g, W_w \cup W_2 \rangle \end{array}$$

$$\begin{array}{c} \beta_1 \in \vec{\alpha} \quad \beta_1 \notin \text{ftv}(\tau_2) \\ \hline \text{extract}(\beta_1 \simeq \tau_2, \vec{\alpha}) = \langle \epsilon, \{\beta_1 \mapsto \tau_2\} \rangle \\ \beta_2 \in \vec{\alpha} \quad \beta_2 \notin \text{ftv}(\tau_1) \\ \hline \text{extract}(\tau_1 \simeq \beta_2, \vec{\alpha}) = \langle \epsilon, \{\beta_2 \mapsto \tau_1\} \rangle \\ (\tau_1 \notin \vec{\alpha} \vee \tau_1 \in \text{ftv}(\tau_2)) \quad (\tau_2 \notin \vec{\alpha} \vee \tau_2 \in \text{ftv}(\tau_1)) \\ \hline \text{extract}(\tau_1 \simeq \tau_2, \vec{\alpha}) = \langle \tau_1 \simeq \tau_2, \emptyset \rangle \\ \hline \text{extract}(D\vec{\tau}, \vec{\alpha}) = \langle D\vec{\tau}, \emptyset \rangle \end{array}$$

$$\begin{array}{c} \overline{\text{flat}(\epsilon) = \emptyset} \\ \text{flat}(Q_1) = W_1 \quad \text{flat}(Q_2) = W_2 \\ \hline \text{flat}(Q_1 \wedge Q_2) = W_1 \cup W_2 \\ \hline \overline{\text{flat}(\tau_1 \simeq \tau_2) = \{\tau_1 \simeq \tau_2\}} \\ \hline \overline{\text{flat}(D\vec{\tau}) = \{D\vec{\tau}\}} \end{array}$$

$$\begin{array}{c} Q \vdash \langle \vec{\alpha}, \emptyset, \text{flat}(Q), \text{flat}(Q_1) \rangle \rightarrow^* \langle \vec{\alpha}', \theta', W', W_2' \rangle \not\vdash \\ W_2 = \bigcup \{W \mid P'_2 \in W_2', \text{extract}(\theta' P'_2, \vec{\alpha}') = \langle W, R \rangle\} \\ R_2 = \bigcup \{R \mid P'_2 \in W_2', \text{extract}(\theta' P'_2, \vec{\alpha}') = \langle W, R \rangle\} \\ \theta = \{\beta \mapsto \tau \mid \beta \in \text{dom}(R_2), \forall \beta \mapsto \tau' \in R_2. \tau = \theta \tau'\} \\ \hline Q; Q; \vec{\alpha} \vdash Q_1 \xrightarrow{\text{simpl}} \theta \wedge W_2 \mid \theta \end{array}$$

Canonicalization:

$$\overline{\text{canon}_l(\mathcal{J}\vec{\tau}_1 \simeq \mathcal{J}\vec{\tau}_2) = \langle \epsilon, \emptyset, \{\tau_1 \simeq \tau_2 \in \overline{\tau_1 \simeq \tau_2} \mid \tau_1 \neq \tau_2\} \rangle}$$

$$\overline{\text{canon}_l(F\vec{\tau} \simeq F\vec{\tau}) = \langle \epsilon, \emptyset, \emptyset \rangle}$$

8.4 ML Type Inference by HM(X)

[EL04]

8.5 Bidirectional Type Checking for System-F

[DK13][JVWS07]

8.5.1 Language

Syntax:

Type Variables	$\alpha ::= \dots$
Variables	$x ::= \dots$
Mono Types	$\tau ::= \alpha$
	$\mid \tau_1 \rightarrow \tau_2$
Types	$\sigma ::= \alpha$
	$\mid \sigma_1 \rightarrow \sigma_2$
	$\mid \forall \alpha. \sigma$
Terms	$e ::= x$
	$\mid \lambda x. e$
	$\mid \lambda x : \sigma. e$
	$\mid e_1 e_2$
	$\mid e : \sigma$
Contexts	$\Gamma ::= \epsilon$
	$\mid \Gamma_1 + \Gamma_2$
	$\mid x : \sigma$
	$\mid \alpha$

Context Member:

$$\frac{}{x : \sigma \in x : \sigma} \quad \frac{x : \sigma \in \Gamma_1}{x : \sigma \in \Gamma_1 + \Gamma_2} \quad \frac{x : \sigma \in \Gamma_2}{x : \sigma \in \Gamma_1 + \Gamma_2}$$

$$\frac{}{\alpha \in \alpha} \quad \frac{\alpha \in \Gamma_1}{\alpha \in \Gamma_1 + \Gamma_2} \quad \frac{\alpha \in \Gamma_2}{\alpha \in \Gamma_1 + \Gamma_2}$$

Type Validity:

$$\frac{\alpha \in \Gamma}{\Gamma \vdash \alpha}$$

$$\frac{\Gamma \vdash \sigma_1 \quad \Gamma \vdash \sigma_2}{\Gamma \vdash \sigma_1 \rightarrow \sigma_2}$$

$$\frac{\Gamma, \alpha \vdash \sigma}{\Gamma \vdash \forall \alpha. \sigma}$$

Term Typing (predicative):

$$\frac{x : \sigma \in \Gamma}{\Gamma \vdash x : \sigma} \text{ Var}$$

$$\frac{\Gamma \vdash \sigma_1 \quad \Gamma, x : \sigma_1 \vdash e : \sigma_2}{\Gamma \vdash \lambda x. e : \sigma_1 \rightarrow \sigma_2} \text{ Abs}$$

$$\frac{\Gamma \vdash \sigma_1 \quad \Gamma, x : \sigma_1 \vdash e : \sigma_2}{\Gamma \vdash \lambda x : \sigma_1. e : \sigma_1 \rightarrow \sigma_2} \text{ AnnAbs}$$

$$\frac{\Gamma \vdash e_1 : \sigma_2 \rightarrow \sigma \quad \Gamma \vdash e_2 : \sigma_2}{\Gamma \vdash e_1 e_2 : \sigma} \text{ App}$$

$$\frac{\Gamma, \alpha \vdash e : \sigma}{\Gamma \vdash e : \forall \alpha. \sigma} \text{ Gen}$$

$$\frac{\Gamma \vdash e : \forall \alpha. \sigma \quad \Gamma \vdash \tau}{\Gamma \vdash e : \sigma[\alpha \leftarrow \tau]} \text{ Inst}$$

8.5.2 Bidirectional Typing

Bidirectional Typing:

$$\begin{array}{c}
\frac{\Gamma \vdash e \Rightarrow \sigma_1 \quad \Gamma \vdash \sigma_1 \leq \sigma_2}{\Gamma \vdash e \Leftarrow \sigma_2} \text{Sub} \\
\frac{x : \sigma \in \Gamma}{\Gamma \vdash x \Rightarrow \sigma} \text{Var} \\
\frac{\Gamma \vdash \sigma \quad \Gamma \vdash e \Leftarrow \sigma}{\Gamma \vdash (e : \sigma) \Rightarrow \sigma} \text{Ann} \\
\frac{\Gamma, \alpha \vdash e \Leftarrow \sigma}{\Gamma \vdash e \Leftarrow \forall \alpha. \sigma} \text{TyAbs} \\
\frac{\Gamma, x : \sigma_1 \vdash e \Leftarrow \sigma_2}{\Gamma \vdash \lambda x. e \Leftarrow \sigma_1 \rightarrow \sigma_2} \text{Abs} \\
\frac{\Gamma \vdash \tau_1 \rightarrow \tau_2 \quad \Gamma, x : \tau_1 \vdash e \Leftarrow \tau_2}{\Gamma \vdash \lambda x. e \Rightarrow \tau_1 \rightarrow \tau_2} \text{AbsSyn} \\
\frac{\Gamma, x : \sigma_1 \vdash e \Leftarrow \sigma_2}{\Gamma \vdash \lambda x : \sigma_1. e \Leftarrow \sigma_1 \rightarrow \sigma_2} \text{AnnAbs} \\
\frac{\Gamma \vdash \sigma_1 \rightarrow \tau_2 \quad \Gamma, x : \sigma_1 \vdash e \Leftarrow \tau_2}{\Gamma \vdash \lambda x : \sigma_1. e \Rightarrow \sigma_1 \rightarrow \tau_2} \text{AnnAbsSyn} \\
\frac{\Gamma \vdash e_1 \Rightarrow \sigma_1 \quad \Gamma \vdash \sigma_1 \leq \sigma_2 \rightarrow \sigma \quad \Gamma \vdash e_2 \Leftarrow \sigma_2}{\Gamma \vdash e_1 e_2 \Rightarrow \sigma} \text{App}
\end{array}$$

Subtyping:

$$\begin{array}{c}
\frac{\alpha \in \Gamma}{\Gamma \vdash \alpha \leq \alpha} \text{Var} \\
\frac{\Gamma \vdash \sigma'_1 \leq \sigma_1 \quad \Gamma \vdash \sigma_2 \leq \sigma'_2}{\Gamma \vdash \sigma_1 \rightarrow \sigma_2 \leq \sigma'_1 \rightarrow \sigma'_2} \text{Arrow} \\
\frac{\Gamma \vdash \tau_1 \quad \Gamma \vdash \sigma_1[\alpha_1 \leftarrow \tau_1] \leq \sigma_2}{\Gamma \vdash \forall \alpha_1. \sigma_1 \leq \sigma_2} \text{Spec} \\
\frac{\Gamma, \alpha_2 \vdash \sigma_1 \leq \sigma_2}{\Gamma \vdash \sigma_1 \leq \forall \alpha_2. \sigma_2} \text{Skol}
\end{array}$$

Subsumption:

$$\begin{array}{c}
\frac{}{\Gamma \vdash \sigma \leq \sigma} \text{Refl} \\
\frac{\Gamma \vdash \tau_1 \quad \Gamma \vdash \sigma_1[\alpha_1 \leftarrow \tau_1] \leq \sigma_2}{\Gamma \vdash \forall \alpha_1. \sigma_1 \leq \sigma_2} \text{Spec} \\
\frac{\alpha \notin \text{ftv}(\sigma_1)}{\Gamma \vdash \forall \alpha. \sigma_1 \rightarrow \sigma_2 \leq \sigma_1 \rightarrow \forall \alpha. \sigma_2} \text{WeakSpec}
\end{array}$$

8.5.3 Algorithmic Type Inference

Algorithmic context:

$$\begin{array}{lcl}
\Gamma & ::= & \epsilon \\
& | & \Gamma, \alpha \\
& | & \Gamma, x : \sigma \\
& | & \Gamma, \hat{\alpha} \\
& | & \Gamma, \hat{\alpha} = \tau \\
& | & \Gamma, \alpha \mapsto \hat{\alpha}
\end{array}$$

Substitution:

$$\overline{[\Gamma]\alpha = \alpha}$$

$$\begin{array}{c}
\frac{\hat{\alpha} = \tau \in \Gamma}{[\Gamma]\hat{\alpha} = \tau} \\
\frac{[\Gamma](\sigma_1) = \sigma'_1 \quad [\Gamma](\sigma_2) = \sigma'_2}{[\Gamma](\sigma_1 \rightarrow \sigma_2) = \sigma'_1 \rightarrow \sigma'_2} \\
\frac{[\Gamma]\sigma = \sigma'}{[\Gamma](\forall \alpha. \sigma) = \forall \alpha. \sigma'}
\end{array}$$

Bidirectional typing:

$$\begin{array}{c}
\frac{\Gamma \vdash e \Rightarrow \sigma_1 \mid \Theta \quad \Theta \vdash [\Theta]\sigma_1 \leq [\Theta]\sigma_2 \mid \Delta}{\Gamma \vdash e \Leftarrow \sigma_2 \mid \Delta} \text{ Sub} \\
\frac{x : \sigma \in \Gamma}{\Gamma \vdash x \Rightarrow \sigma \mid \Gamma} \text{ Var} \\
\frac{\Gamma \vdash e \Leftarrow \sigma \mid \Delta}{\Gamma \vdash e : \sigma \Rightarrow \sigma \mid \Delta} \text{ Ann} \\
\frac{\Gamma, \alpha \vdash e \Leftarrow \sigma \mid \Delta, \alpha, \Theta}{\Gamma \vdash e \Leftarrow \forall \alpha. \sigma \mid \Delta} \text{ TyAbs} \\
\frac{\Gamma, x : \sigma_1 \vdash e \Leftarrow \sigma_2 \mid \Delta, x : \sigma_1, \Theta}{\Gamma \vdash \lambda x. e \Leftarrow \sigma_1 \rightarrow \sigma_2 \mid \Delta} \text{ Abs} \\
\frac{\Gamma, \hat{\alpha}_1, \hat{\alpha}_2, x : \hat{\alpha}_1 \vdash e \Leftarrow \hat{\alpha}_2 \mid \Delta, x : \hat{\alpha}_1, \Theta}{\Gamma \vdash \lambda x. e \Rightarrow \hat{\alpha}_1 \rightarrow \hat{\alpha}_2 \mid \Delta} \text{ AbsSyn} \\
\frac{\Gamma, x : \sigma_1 \vdash e \Leftarrow \sigma_2 \mid \Delta, x : \sigma_1, \Theta}{\Gamma \vdash \lambda x : \sigma_1. e \Leftarrow \sigma_1 \rightarrow \sigma_2 \mid \Delta} \text{ AnnAbs} \\
\frac{\Gamma, \hat{\alpha}_2, x : \sigma_1 \vdash e \Leftarrow \hat{\alpha}_2 \mid \Delta, x : \sigma_1, \Theta}{\Gamma \vdash \lambda x : \sigma_1. e \Rightarrow \sigma_1 \rightarrow \hat{\alpha}_2 \mid \Delta} \text{ AnnAbsSyn} \\
\frac{\Gamma \vdash e_1 \Rightarrow \sigma_1 \mid \Theta_1 \quad \Theta_1 \vdash [\Theta_1]\sigma_1 \leq \sigma_2 \rightarrow \sigma \mid \Theta_2 \quad \Theta_2 \vdash e_2 \Leftarrow [\Theta_2]\sigma_2 \mid \Delta}{\Gamma \vdash e_1 e_2 \Rightarrow \sigma \mid \Delta} \text{ App}
\end{array}$$

Subtyping:

$$\begin{array}{c}
\frac{\alpha \in \Gamma}{\Gamma \vdash \alpha \leq \alpha \mid \Gamma} \\
\frac{\hat{\alpha} \in \Gamma}{\Gamma \vdash \hat{\alpha} \leq \hat{\alpha} \mid \Gamma} \\
\frac{\Gamma \vdash \sigma_1 \leq \sigma'_1 \mid \Theta \quad \Theta \vdash [\Theta]\sigma_2 \leq [\Theta]\sigma'_2 \mid \Delta}{\Gamma \vdash \sigma_1 \rightarrow \sigma_2 \leq \sigma'_1 \rightarrow \sigma'_2 \mid \Delta} \\
\frac{\Gamma, \alpha \mapsto \hat{\alpha}, \hat{\alpha} \vdash \sigma_1[\alpha \leftarrow \hat{\alpha}] \leq \sigma_2 \mid \Delta, \alpha \mapsto \hat{\alpha}, \Theta}{\Gamma \vdash \forall \alpha. \sigma_1 \leq \sigma_2 \mid \Delta} \\
\frac{\Gamma, \alpha \vdash \sigma_1 \leq \sigma_2 \mid \Delta, \alpha, \Theta}{\Gamma \vdash \sigma_1 \leq \forall \alpha. \sigma_2 \mid \Delta} \\
\frac{\hat{\alpha}_1 \notin \text{ftv}(\sigma_2) \quad \hat{\alpha}_1 \in \Gamma \quad \Gamma \vdash \hat{\alpha}_1 \simeq \sigma_2 \mid \Delta}{\Gamma \vdash \hat{\alpha}_1 \leq \sigma_2 \mid \Delta} \\
\frac{\hat{\alpha}_2 \notin \text{ftv}(\sigma_1) \quad \hat{\alpha}_2 \in \Gamma \quad \Gamma \vdash \sigma_1 \simeq \hat{\alpha}_2 \mid \Delta}{\Gamma \vdash \sigma_1 \leq \hat{\alpha}_2 \mid \Delta}
\end{array}$$

Instantiation:

$$\begin{array}{c}
\frac{\Gamma_1 \vdash \tau}{\Gamma_1, \hat{\alpha}, \Gamma_2 \vdash \hat{\alpha} \simeq \tau \mid \Gamma_1, \hat{\alpha} = \tau, \Gamma_2} \\
\frac{\Gamma_1 \vdash \tau}{\Gamma_1, \hat{\alpha}, \Gamma_2 \vdash \tau \simeq \hat{\alpha} \mid \Gamma_1, \hat{\alpha} = \tau, \Gamma_2} \\
\frac{\Gamma_1, \hat{\alpha}_1, \Gamma_2, \hat{\alpha}_2, \Gamma_3 \vdash \hat{\alpha}_1 \simeq \hat{\alpha}_2 \mid \Gamma_1, \hat{\alpha}_1, \Gamma_2, \hat{\alpha}_2 = \hat{\alpha}_1, \Gamma_3}{\Gamma_1, \hat{\alpha}_3, \hat{\alpha}_2, \hat{\alpha}_1 = \hat{\alpha}_2 \rightarrow \hat{\alpha}_3, \Gamma_2 \vdash \sigma_2 \simeq \hat{\alpha}_2 \mid \Theta \quad \Theta \vdash \hat{\alpha}_3 \simeq [\Theta]\sigma_3 \mid \Delta} \\
\frac{\Gamma_1, \hat{\alpha}_1, \Gamma_2 \vdash \hat{\alpha}_1 \simeq \sigma_2 \rightarrow \sigma_3 \mid \Delta}{\Gamma_1, \hat{\alpha}_3, \hat{\alpha}_2, \hat{\alpha}_1 = \hat{\alpha}_2 \rightarrow \hat{\alpha}_3, \Gamma_2 \vdash \hat{\alpha}_2 \simeq \sigma_2 \mid \Theta \quad \Theta \vdash [\Theta]\sigma_3 \simeq \hat{\alpha}_3 \mid \Delta} \\
\frac{\Gamma_1, \hat{\alpha}_1, \Gamma_2 \vdash \sigma_2 \rightarrow \sigma_3 \simeq \hat{\alpha}_1 \mid \Delta}{}
\end{array}$$

$$\frac{\frac{\Gamma_1, \hat{\alpha}_1, \Gamma_2, \alpha_2 \vdash \hat{\alpha}_1 \simeq \sigma_2 \mid \Delta, \alpha_2, \Theta}{\Gamma_1, \hat{\alpha}_1, \Gamma_2 \vdash \hat{\alpha}_1 \simeq \forall \alpha_2. \sigma_2 \mid \Delta}}{\frac{\Gamma_1, \hat{\alpha}_2, \Gamma_2, \alpha_1 \mapsto \hat{\alpha}_1, \hat{\alpha}_1 \vdash \sigma_1[\alpha_1 \leftarrow \hat{\alpha}_1] \simeq \hat{\alpha}_2 \mid \Delta, \alpha_1 \mapsto \hat{\alpha}_1, \Theta}{\Gamma_1, \hat{\alpha}_2, \Gamma_2 \vdash \forall \alpha_1. \sigma_1 \simeq \hat{\alpha}_2 \mid \Delta}}$$

Subsumption:

$$\frac{\frac{\frac{\Gamma \vdash \hat{\alpha}_1 \leq \hat{\alpha}_2 \rightarrow \hat{\alpha}_3 \mid \Gamma, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_1 = \hat{\alpha}_2 \rightarrow \hat{\alpha}_3}{\Gamma \vdash \sigma_1 \rightarrow \sigma_2 \leq \sigma_1 \rightarrow \sigma_2 \mid \Gamma}}{\frac{\alpha \notin \text{fv}(\sigma_1)}{\Gamma \vdash \forall \alpha. \sigma_1 \rightarrow \sigma_2 \leq \sigma_1 \rightarrow \forall \alpha. \sigma_2 \mid \Gamma}}}{\frac{\nexists \sigma'_1 \rightarrow \sigma'_2 = \sigma_1. \alpha_1 \notin \text{fv}(\sigma'_1) \quad \Gamma, \hat{\alpha}_1 \vdash \sigma_1[\alpha_1 \leftarrow \hat{\alpha}_1] \leq \sigma_2 \mid \Delta}{\Gamma \vdash \forall \alpha_1. \sigma_1 \leq \sigma_2 \mid \Delta}}$$

第 9 章

Static Memory Management and Regions

第 10 章

Dynamic Memory Management and Gabage Collections

第 11 章

I/O Management and Concurrency

第 12 章

Code Generation and Virtual Machines

第 13 章

Program Stability and Compatibility

第 14 章

Program Separation and Linking

第 15 章

Syntax and Parsing

15.1 WIP: Parsing by LR Method

[Knu65]

15.2 Syntax and Semantics of PEG

[For02], [For04]

15.2.1 Syntax

e	::=	ϵ	(epsilon)
		σ	(terminal)
		A	(non-terminal)
		ee	(sequence)
		e / e	(alternative)
		e^*	(repetition)
		$!e$	(not predicate)
σ	\in	Σ	
A	\in	N	

定義 59. PEG 文法とは、以下による組 $G = (\Sigma, N, R, e_0)$ のことである。

Σ 終端記号の集合.

N 非終端記号の集合.

R $A \rightarrow e$ を満たす規則の集合. 規則は、非終端記号に対して必ず一つ.

e_0 初期式.

□

15.2.2 Structured Semantics

$$\begin{array}{c}
 \overline{\langle \epsilon, x \rangle \rightarrow s(\epsilon)} \\
 \overline{\langle \sigma, \sigma x \rangle \rightarrow s(\sigma)} \quad \frac{\sigma \neq \sigma'}{\overline{\langle \sigma, \sigma' x \rangle \rightarrow f}} \quad \overline{\langle \sigma, \epsilon \rangle \rightarrow f} \\
 \frac{A \leftarrow e \in R \quad \overline{\langle e, x \rangle \rightarrow o}}{\overline{\langle A, x \rangle \rightarrow o}} \\
 \frac{\overline{\langle e_1, x_1 x_2 y \rangle \rightarrow s(x_1)} \quad \overline{\langle e_2, x_2 y \rangle \rightarrow s(x_2)}}{\overline{\langle e_1 e_2, x_1 x_2 y \rangle \rightarrow s(x_1 x_2)}} \quad \frac{\overline{\langle e_1, x \rangle \rightarrow f}}{\overline{\langle e_1 e_2, x \rangle \rightarrow f}} \quad \frac{\overline{\langle e_1, x_1 y \rangle \rightarrow s(x_1)} \quad \overline{\langle e_2, y \rangle \rightarrow f}}{\overline{\langle e_1 e_2, x_1 y \rangle \rightarrow f}} \\
 \frac{\overline{\langle e_1, xy \rangle \rightarrow s(x)}}{\overline{\langle e_1 / e_2, xy \rangle \rightarrow s(x)}} \quad \frac{\overline{\langle e_1, x \rangle \rightarrow f} \quad \overline{\langle e_2, x \rangle \rightarrow o}}{\overline{\langle e_1 / e_2, x \rangle \rightarrow o}} \\
 \frac{\overline{\langle e, x_1 x_2 y \rangle \rightarrow s(x_1)} \quad \overline{\langle e^*, x_2 y \rangle \rightarrow s(x_2)}}{\overline{\langle e^*, x_1 x_2 y \rangle \rightarrow s(x_1 x_2)}} \quad \frac{\overline{\langle e, x \rangle \rightarrow f}}{\overline{\langle e^*, x \rangle \rightarrow s(\epsilon)}} \\
 \frac{\overline{\langle e, x \rangle \rightarrow f}}{\overline{\langle !e, x \rangle \rightarrow s(\epsilon)}} \quad \frac{\overline{\langle e, xy \rangle \rightarrow s(x)}}{\overline{\langle !e, xy \rangle \rightarrow f}}
 \end{array}$$

$$\llbracket (\Sigma, N, R, e_0) \rrbracket = \llbracket e_0 \rrbracket$$

$$\llbracket e \rrbracket = \{x \in \Sigma^* \mid \langle e, x \rangle \rightarrow s(x)\}$$

15.2.3 Equivalence

Abbreviations

$\& e = !(e)$	(and predicate)
$e^+ = ee^*$	(positive repetition)
$e^? = e/\epsilon$	(optional)

Associativity

$$\overline{\llbracket e_1/(e_2/e_3) \rrbracket} = \overline{\llbracket (e_1/e_2)/e_3 \rrbracket}$$

$$\overline{\llbracket e_1(e_2e_3) \rrbracket} = \overline{\llbracket (e_1e_2)e_3 \rrbracket}$$

Epsilon

$$\overline{\llbracket \epsilon/e \rrbracket} = \overline{\llbracket \epsilon \rrbracket}$$

$$\overline{\llbracket e\epsilon \rrbracket} = \overline{\llbracket e \rrbracket} \quad \overline{\llbracket \epsilon e \rrbracket} = \overline{\llbracket e \rrbracket}$$

Repetition

$$M ::= eM \mid \epsilon$$

$$\overline{\llbracket e^* \rrbracket} = \overline{\llbracket M \rrbracket}$$

15.2.4 Producing Analysis

$$s ::= 0 \mid 1, \quad o ::= s \mid f$$

- $\epsilon \rightarrow 0$
- $\sigma \rightarrow 1$
- $\sigma \rightarrow f$
- $A \leftarrow e \in R, \quad e \rightarrow o \text{ ならば } A \rightarrow o$
- $e_1 \rightarrow 0, \quad e_2 \rightarrow 0 \text{ ならば } e_1e_2 \rightarrow 0$
- $e_1 \rightarrow 1, \quad e_2 \rightarrow s \text{ ならば } e_1e_2 \rightarrow 1$
- $e_1 \rightarrow s, \quad e_2 \rightarrow 1 \text{ ならば } e_1e_2 \rightarrow 1$
- $e_1 \rightarrow f \text{ ならば } e_1e_2 \rightarrow f$
- $e_1 \rightarrow s, \quad e_2 \rightarrow f \text{ ならば } e_1e_2 \rightarrow f$
- $e_1 \rightarrow s \text{ ならば } e_1 / e_2 \rightarrow s$
- $e_1 \rightarrow f, \quad e_2 \rightarrow o \text{ ならば } e_1 / e_2 \rightarrow o$
- $e \rightarrow 1 \text{ ならば } e^* \rightarrow 1$
- $e \rightarrow f \text{ ならば } e^* \rightarrow f$
- $e \rightarrow s \text{ ならば } !e \rightarrow f$

- $e \rightarrow f$ ならば $e \rightarrow 0$

定理 60.

- $\langle e, x \rangle \rightarrow s(\epsilon)$ ならば, $e \rightarrow 0$
- $\langle e, xy \rangle \rightarrow s(x)$, $x \neq \epsilon$ ならば, $e \rightarrow 1$
- $\langle e, x \rangle \rightarrow f$ ならば, $e \rightarrow f$

□

系 61. $e \not\rightarrow 0$ ならば, $\langle e, xy \rangle \not\rightarrow s(x)$ かつ $\langle e, xy \rangle \not\rightarrow f$

□

15.3 Haskell Parsing with PEG

[Sim10]

15.3.1 Lexical Syntax

```

program ::= (lexeme | whitespace)*
lexeme  ::= qvarid
        | qconid
        | qvarsym
        | qconsym
        | literal
        | special
        | reservedop
        | reservedid
literal  ::= integer
        | float
        | char
        | string
special  ::= "(" | ")" | "," | ";" | "[" | "]" | "`" | "{" | "}"
whitespace ::= whitestuff+
whitestuff ::= whitechar | comment | ncomment

whitechar ::= newline | "\v" | " " | "\t" | (Unicode whitespace)
newline   ::= "\x\n" | "\r" | "\n" | "\f"
comment   ::= dashes (!symbol any*)? newline
dashes    ::= "-" ("-"+)
opencom   ::= "{-"
closecom  ::= "-}"
ncomment  ::= opencom ANYs (ncomment ANYs)* closecom
ANYs      ::= !(ANY* (opencom | closecom) ANY*) ANY*
ANY       ::= graphic | whitechar
any       ::= graphic | " " | "\t"
graphic   ::= small | large | symbol | digit | special | "\" | "'"
small     ::= "a" | "b" | ... | "z" | (Unicode lowercase letter) | "_"
large     ::= "A" | "B" | ... | "Z" | (Unicode uppercase letter) | (Unicode titlecase letter)
symbol    ::= "!" | "#" | "$" | "%" | "&" | "*" | "+" | "." | "/" | "<" | "=" | ">"
          | "?" | "@" | "\" | "^" | "|" | "-" | "~" | ":"
          | !(symbol | "_" | "\" | "'" | " ") uniSymbol
uniSymbol ::= (Unicode symbol) | (Unicode punctuation)
digit     ::= "0" | "1" | ... | "9" | (Unicode decimal digit)
octit     ::= "0" | "1" | ... | "7"
hexit     ::= digit | "A" | ... | "F" | "a" | ... | "f"
varid     ::= !(reservedid !other) small other*
conid     ::= large other*
other     ::= small | large | digit | "'"
reservedid ::= "case" | "class" | "data" | "default" | "deriving" | "do" | "else"
          | "foreign" | "if" | "import" | "in" | "infix" | "infixl" | "infixr"
          | "instance" | "let" | "module" | "newtype" | "of" | "then" | "type"
          | "where" | "_"
varsym    ::= !((reservedop | dashes) !symbol | ":" ) symbol+
consym    ::= !(reservedop !symbol) ":" symbol+
reservedop ::= "." | ":" | "::" | "=" | "\\" | "<-" | "->" | "@" | "~" | "=>"

```

```

modid  ::= (conid "."*) conid
qvarid ::= (modid "."?) varid
qconid ::= (modid "."?) conid
qvarsym ::= (modid "."?) varsym
qconsym ::= (modid "."?) consym

decimal ::= digit+
octal   ::= octit+
hexadecimal ::= hexit+
integer ::= decimal
        | "0o" octal | "0O" octal
        | "0x" hexadecimal | "0X" hexadecimal
float   ::= decimal "." decimal exponent?
        | decimal exponent
exponent ::= ("e" | "E") ("+" | "-" ) decimal
char     ::= "'" (!("'" | "\\") graphic | " " | !"\\&" escape) "'"
string   ::= "\"" (!("\"" | "\\") graphic | " " | escape | gap)* "\""
escape   ::= "\\\"(charesc | ascii | decimal | "o" octal | "x" hexadecimal)
charesc  ::= "a" | "b" | "f" | "n" | "r" | "t" | "v" | "\\\" | "\\\" | "'" | "&"
ascii    ::= "^" cntrl | "NUL" | "SOH" | "STX" | "ETX" | "EOT" | "ENQ" | "ACK" | "BEL" | "BS"
        | "HT" | "LF" | "VT" | "FF" | "CR" | "SO" | "SI" | "DLE" | "DC1" | "DC2" | "DC3"
        | "DC4" | "NAK" | "SYN" | "ETB" | "CAN" | "EM" | "SUB" | "ESC" | "FS" | "GS" | "RS"
        | "US" | "SP" | "DEL"
cntrl    ::= "A" | "B" | ... | "Z" | "@" | "[" | "\\\" | "]" | "^" | "_"
gap      ::= "\\\" whitechar+ "\\\"

```

15.3.2 Preprocess for Layout

$$\begin{aligned}
L(s) &= \begin{cases} L_1(r', s) & (s = t : s', \text{pos}(t) = (r', c'), \text{islt}(t)) \\ \{c'\} : \langle c' \rangle : L_1(r', s) & (s = t : s', \text{pos}(t) = (r', c'), \text{islt}(t)) \\ \{1\} : \epsilon & (s = \epsilon) \end{cases} \\
L_1(r, s) &= \begin{cases} \langle c' \rangle : L_2(r', c', t, s') & (s = t : s', \text{pos}(t) = (r', c'), r \neq r') \\ L_2(r', c', t, s') & (s = t : s', \text{pos}(t) = (r', c'), r = r') \\ \epsilon & (s = \epsilon) \end{cases} \\
L_2(r_1, c_1, t_1, s) &= \begin{cases} t_1 : t_2 : L_1(r_2, s') & (\text{islt}(t_1), s = t_2 : s', \text{pos}(t_2) = (r_2, c_2), t_2 = "\{") \\ t_1 : \{c_2\} : \langle c_2 \rangle : t_2 : L_1(r_2, s') & (\text{islt}(t_1), s = t_2 : s', \text{pos}(t_2) = (r_2, c_2), t_2 \neq "\{") \\ t_1 : \{1\} : \epsilon & (\text{islt}(t_1), s = \epsilon) \\ t_1 : L_1(r_1, s) & (\text{islt}(t_1)) \end{cases} \\
\text{islt}(t) &= \begin{cases} \top & (t = \text{"module"}) \\ \perp & (\text{otherwise}) \end{cases} \\
\text{islt}(t) &= \begin{cases} \top & (t = \text{"let"}) \\ \top & (t = \text{"where"}) \\ \top & (t = \text{"do"}) \\ \top & (t = \text{"of"}) \\ \perp & (\text{otherwise}) \end{cases}
\end{aligned}$$

15.3.3 PEG with Layout Tokens

```

module  ::= "module" modid exports? "where" body
        | body
body    ::= expbo bodyinl expbc
        | impbo bodyinl impbc
bodyinl ::= impdecls semi+ topdecls
        | impdecls
        | topdecls

```



```

impdecls ::= semi*(impdecl semi+)* impdecl
exports ::= "(" (export " , ")* export? ")"
export  ::= qvar
        | qtycon "(" (" (". ." | (cname " , ")* cname)? ")"?
        | "module" modid
impdecl  ::= "import" "qualified"? modid ("as" modid)? impspec?
impspec  ::= "(" (import " , ")* import? ")"
        | "hiding" "(" (import " , ")* import? ")"
import   ::= var
        | tycon "(" (" (". ." | (cname " , ")* cname)? ")"?
cname    ::= var | con

topdecls ::= (topdecl semi)* topdecl |
topdecl  ::= "type" simpletype "=" type
        | "data" (context "=>")? simpletype ("=" constrs)? deriving?
        | "newtype" (context "=>")? simpletype "=" newconstr deriving?
        | "class" (scontext "=>")? tycon tyvar ("where" cdecls)?
        | "instance" (scontext "=>")? qtycon inst ("where" idecls)?
        | "default" "(" ((type " , ")* type)? ")"
        | "foreign" fdecl
        | decl

decls    ::= expbo declsinl expbc
        | impbo declsinl impbc
declsinl ::= (decl semi)* decl |
decl     ::= (funlhs | pat) rhs
        | gendecl
cdecls   ::= expbo cdeclsinl expbc
        | impbo cdeclsinl impbc
cdeclsinl ::= (cdecl semi)* cdecl |
cdecl    ::= (funlhs | var) rhs
        | gendecl
idecls   ::= expbo ideclsinl expbc
        | impbo ideclsinl impbc
ideclsinl ::= (idecl semi)* idecl |
idecl    ::= (funlhs | var) rhs
        |
gendecl  ::= vars " : : " (context "=>")? type
        | fixity integer? ops
        |
ops      ::= (op " , ")* op
vars     ::= (var " , ")* var
fixity   ::= "infixl" | "infixr" | "infix"

type     ::= btype ("->" type)?
btype    ::= btype? atype
atype    ::= gtycon
        | tyvar
        | "(" (type " , ")+ type ")"
        | "[" type "]"
        | "(" type ")"
gtyson   ::= qtycon
        | "(" " "
        | "[" " "
        | "(" " -> " "
        | "(" " , "+" "

```

```

context ::= class
        | "(" ((class " ," )* class)? ")"
class   ::= qtycon tyvar
        | qtycon "(" tyvar atype+ ")"
scontext ::= simpleclass
        | "(" ((simpleclass " ," )* simpleclass)? ")"
simpleclass ::= qtycon tyvar

simpletype ::= tycon tyvar*
constrs   ::= (constr " | " )* constr
constr    ::= con expbo ((fielddecl " ," )* fielddecl)? expbc
            | (btype | " ! " atype) conop (btype | " ! " atype)
            | con (" ! " ? atype)*
newconstr ::= con expbo var " :: " type expbc
            | con atype
fielddecl ::= vars " :: " (type | " ! " atype)
deriving  ::= "deriving" dclass
            | "deriving" "(" ((dclass " ," )* dclass)? ")"
dclass    ::= qtycon
inst      ::= gtycon
            | "(" gtycon tyvar* ")"
            | "(" (tyvar " ," )+ tyvar ")"
            | "[" tyvar "]"
            | "(" tyvar "->" tyvar ")"

fdecl    ::= "import" callconv safety? impent var " :: " ftype
            | "export" callconv expent var " :: " ftype
callconv ::= "ccall" | "stdcall" | "cplusplus" | "jvm" | "dotnet"
            | (system-specific calling conventions)
impent   ::= string?
expent   ::= string?
safety   ::= "unsafe" | "safe"
ftype    ::= fatype "->" ftype
            | frtype
frtype   ::= fatype
            | "(" " " ")"
fatype   ::= qtycon atype*

funlhs   ::= var apat+
            | pat varop pat
            | "(" funlhs " )" apat+
rhs      ::= "=" exp ("where" decls)?
            | gdrhs ("where" decls)?
gdrhs    ::= guards "=" exp gdrhs?
guards   ::= " | " (guard " ," )* guard |
guard    ::= pat "<-" infixexp
            | "let" decls
            | infixexp

```

```

exp ::= infixexp ":" (context "=>")? type
    | infixexp
infixexp ::= "-" infixexp
    | lexp qop infixexp
    | lexp
lexp ::= "\\\" apat+ "->" exp
    | "let" decls "in" exp
    | "if" exp semi? "then" exp semi? "else" exp
    | "case" exp "of" casealts
    | "do" dostmts
    | fexp
fexp ::= aexp+
aexp ::= qcon expbo ((fbind " ,")* fbind)? expbc
    | aexp2 (expbo ((fbind " ,")* fbind)? expbc)*
    | qvar
aexp2 ::= literal
    | "(" exp ")"
    | "(" (exp " ,")+ exp ")"
    | "[" (exp " ,")* exp "]"
    | "[" exp "(" exp )? ". ." exp? "]"
    | "[" exp " | " (qual " ,")* qual "]"
    | "(" infixexp qop ")"
    | "(" !("-" infixexp) qop infixexp ")"
    | gcon

```

```

qual ::= pat "<-" exp
    | "let" decls
    | exp
casealts ::= expbo alts expbc
    | impbo alts impbc
alts ::= (alt semi)* alt
alt ::= pat "->" exp ("where" decls)?
    | pat gdpat ("where" decls)?
    |
gdpat ::= guards "->" exp gdpat?
dostmts ::= expbo stmts expbc
    | impbo stmts impbc
stmts ::= stmt* exp semi?
stmt ::= exp semi
    | pat "<-" exp semi
    | "let" decls semi
    | semi
fbind ::= qvar "=" exp

```

```

pat ::= lpat qconop pat
    | lpat
lpat ::= "-" (integer | float)
    | gcon apat+
    | apat
apat ::= var "@" apat)?
    | literal
    | "_"
    | "(" pat ")"
    | "(" (pat " ,")+ pat ")"
    | "[" (pat " ,")* pat "]"
    | "~" apat
    | qcon expbo ((fpat " ,")* fpat)? expbc
    | gcon
fpat ::= qvar "=" pat

```

```

gcon  ::= "(" ")"
      | "[" "]"
      | "(" " ", "+" ")"
      | qcon
var    ::= varid | "(" varsym ")"
qvar   ::= qvarid | "(" qvarsym ")"
con    ::= conid | "(" consym ")"
qcon   ::= qconid | "(" gconsym ")"
varop  ::= varsym | "^" varid "^"
qvarop ::= qvarsym | "^" qvarid "^"
conop  ::= consym | "^" conid "^"
qconop ::= gconsym | "^" qconid "^"
op     ::= varop | conop
qop    ::= qvarop | qconop
gconsym ::= ":" | qconsym
tyvar  ::= varid
tycon  ::= conid
qtycon ::= qconid

```

```

expbo  ::= [l]      "{" [0 : l]
expbc  ::= [0 : l]  "}" [l]
impbo  ::= [m : l]  {n} [n : m : l | n > m]
        ::= [m : l]  {n} [(n + 1) : m : l | n ≤ m]
        | [ε]       {n} [n : ε | n > 0]
impbc  ::= [m : l]  ε [l | m > 0]
semi   ::= ";"
        | [m : l]  ⟨n⟩ [m : l | m = n]

```

```

skip   ::= [m : l]  ⟨n⟩ [m : l | m < n]

```

15.4 WIP: A Memory Optimization for PEG with Cut Operations

[MMY08][MMY10]

15.5 WIP: SRB: An Abstract Machine of PEG

第 16 章

Analysis and Optimizations

第 17 章

Meta-Programming and Multi-Stage Programming

第 18 章

Generic Programming

第 19 章

Advanced Calculus

第 20 章

Quell: A Language for Functional Programming

20.1 WIP: Implementation Note of PEG Parser

Normalizing

$$\begin{aligned}
 e_{\text{RHS}} &::= e_1 / \dots / e_n / \epsilon & (n \in \mathbb{N}) \\
 &| e_1 / \dots / e_n & (n \in \mathbb{N}_{\geq 1}) \\
 e &::= !(u_1 \dots u_n) & (n \in \mathbb{N}_{\geq 1}) \\
 &| \&(u_1 \dots u_n) & (n \in \mathbb{N}_{\geq 1}) \\
 &| u_1 \dots u_n & (n \in \mathbb{N}_{\geq 1}) \\
 u &::= \sigma \\
 &| A
 \end{aligned}$$

$$\begin{aligned}
 \text{norm}(N, []) &= (N, \emptyset) \\
 \text{norm}(N, [A \leftarrow e] + X) &= (N_2, \{A \leftarrow \text{alt}(a)\} \cup X_1 \cup X_2) \\
 &(\text{norm}(N, e) = (a, N_1, X_1), \text{norm}(N_1, X) = (N_2, X_2))
 \end{aligned}$$

$$\begin{aligned}
 \text{norm}(N, \epsilon) &= ([\epsilon], N, \emptyset) \\
 \text{norm}(N, \sigma) &= ([\sigma], N, \emptyset) \\
 \text{norm}(N, A) &= ([A], N, \emptyset) \\
 \text{norm}(N, e_1 e_2) &= (\text{seq}(a_1, a_2), N_2, X_1 \cup X_2) & (\text{norm}(N, e_1) = (a_1, N_1, X_1), \text{norm}(N_1, e_2) = (a_2, N_2, X_2)) \\
 \text{norm}(N, e_1 / e_2) &= (a_1 + a_2, N_2, X_1 \cup X_2) & (\text{norm}(N, e_1) = (a_1, N_1, X_1), \text{norm}(N_1, e_2) = (a_2, N_2, X_2)) \\
 \text{norm}(N, e^*) &= ([M], N' \uplus \{M\}, X \cup \{M \leftarrow AM / \epsilon\}) & (\text{norm}(N \uplus \{A\}, [A \leftarrow e]) = (N', X)) \\
 \text{norm}(N, \& e) &= ([M], N' \uplus \{M\}, X \cup \{M \leftarrow \&A\}) & (\text{norm}(N + \{A\}, [A \leftarrow e]) = (N', X)) \\
 \text{norm}(N, ! e) &= ([M], N' \uplus \{M\}, X \cup \{M \leftarrow !A\}) & (\text{norm}(N + \{A\}, [A \leftarrow e]) = (N', X))
 \end{aligned}$$

$$\begin{aligned}
 \text{seq}(a_1, a_2) &= [e_1 e_2 \mid e_1 \leftarrow a_1, e_2 \leftarrow a_2] \\
 \text{alt}([e_1, \dots, e_n]) &= e_1 / \dots / e_n & (\forall i < m. e_i \neq \epsilon, e_m = \epsilon) \\
 \text{alt}([e_1, \dots, e_n]) &= e_1 / \dots / e_n & (\forall i. e_i \neq \epsilon)
 \end{aligned}$$

$$\begin{aligned}
 \text{norm}((\Sigma, N, R, e_0)) &= (\Sigma, N', R', S) \\
 (R = \{A_1 \leftarrow e_1, \dots, A_n \leftarrow e_n\}, \text{norm}(N \uplus \{S\}, [S \leftarrow e_0, A_1 \leftarrow e_1, \dots, A_n \leftarrow e_n])) &= (N', R')
 \end{aligned}$$

Machine

State:

- a rule
- current position in rule

Transition:

- σ
- EOS
- otherwise

Output:

with backpoint バックポイントを設置し、バックポイントに戻った時の次の遷移を指定する。fail した場合一番直近の **backpoint** まで入力状態とスタックを戻す。reduce 時取り除かれる。
enter 非終端記号を参照する。メモ化されている場合その値を使う。それ以外の場合、reduce 時戻ってくる状態を記録し、次の状態に遷移する。
goto 次の状態に遷移する。
shift 入力を 1 つ消費し、次の状態に遷移する。
reduce 規則に沿ってスタックから要素を取り出してまとめ、メモし、スタックに新たに入れた後、enter 時に記録された状態に遷移する。

Optimization

1. unify transitions.
2. look ahead backpoints.

Example

$$\begin{array}{lcl}
 E & ::= & CA \\
 & | & \epsilon \\
 A & ::= & aB \\
 & | & a \\
 B & ::= & bA \\
 & | & b \\
 C & ::= & !abab \\
 & | & \& ab
 \end{array}$$

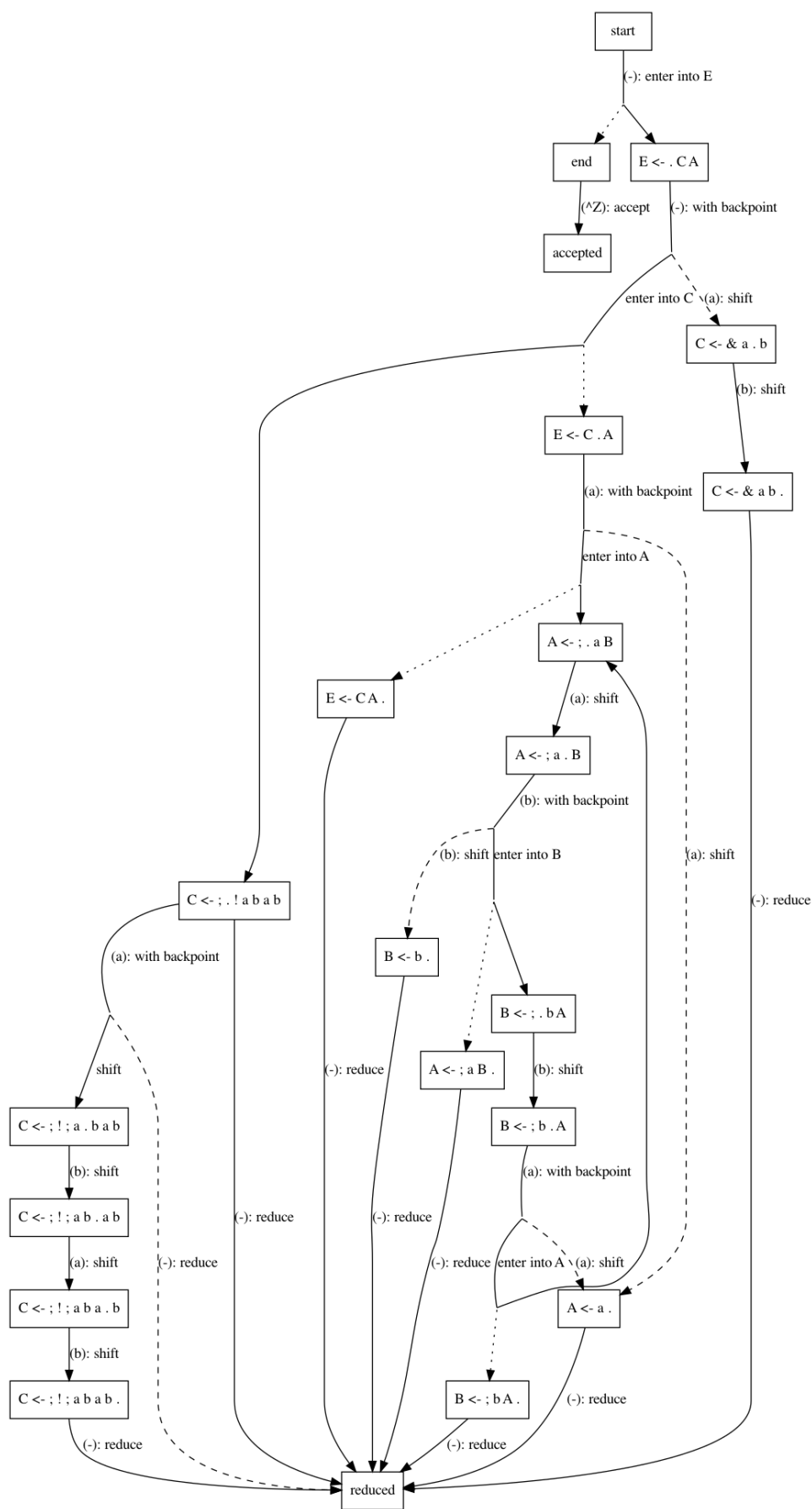


図 20.2 最適化された状態遷移図

20.2 Quell Syntax and Identations

20.2.1 Syntax

20.3 Quell Modules

20.3.1 Syntax

$$\begin{array}{ll}
 e & ::= \dots \\
 & \quad | \text{ letrec } \{B\} \text{ in } e \\
 & \quad | P \\
 \tau & ::= \dots \\
 & \quad | P \\
 P & ::= M \\
 M & ::= x \\
 & \quad | \{B\} \\
 & \quad | M.x \\
 & \quad | \text{ fun } x : S. M \\
 & \quad | x \ x \\
 & \quad | x : S \\
 B & ::= x = e \\
 & \quad | \text{ type } t = T \\
 & \quad | \text{ module } x = M \\
 & \quad | \text{ use } B \\
 & \quad | \epsilon \\
 & \quad | B; B \\
 T & ::= \lambda x. T \\
 & \quad | \tau \\
 S & ::= P \\
 & \quad | \{D\} \\
 & \quad | (x : S) \rightarrow S \\
 & \quad |
 \end{array}$$

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