

# プログラミング言語周りノート

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2021 年 4 月 11 日

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## 第 1 章

## Preliminaries

## 1.1 基本的な表記

量子化 (quantifier) の束縛をコンマ (,) で続けて書く. 束縛の終わりをピリオド (.) で示す. 例えば,

$$\forall x_1 \in X_1, x_2 \in X_2. \exists y_1 \in Y_1, y_2 \in Y_2. x_1 = y_1 \wedge x_2 = y_2$$

は,

$$\forall x_1 \in X_1. \forall x_2 \in X_2. \exists y_1 \in Y_1. \exists y_2 \in Y_2. x_1 = y_1 \wedge x_2 = y_2$$

と等しい. また, 量子子の束縛において, *such that* を省略し, コンマ (,) で繋げて書く. 例えば,

$$\forall x \in \{0, 1\}, x \neq 0. x = 1$$

は,

$$\forall x \in \{0, 1\}. x \neq 0 \Rightarrow x = 1$$

と等しい. また,  $\Rightarrow$ ,  $\Leftrightarrow$  が他の記号と混同する場合, それぞれ *implies*, *iff* を使用する.

集合 (set) について, 以下の表記を用いる.

- 集合  $A$  について, その濃度 (cardinality) を  $|A|$  と表記する. なお,  $A$  が有限集合 (finite set) の時, 濃度とは要素の個数のことである.
- 集合  $A$  について,  $a \in A$  を  $a : A$  と表記する.
- 自然数 (natural number) の集合を  $\mathbb{N} = \{0, 1, \dots\}$  と表記する.
- 自然数  $n \in \mathbb{N}$  について,  $\{1, \dots, n\}$  を  $[n]$  と表記する.
- 集合  $A$  の冪集合を  $\mathcal{P}(A) = \{X \mid X \subseteq A\}$ , 有限冪集合を  $\mathcal{P}_{fin}(A) = \{X \in \mathcal{P}(A) \mid X \text{ は有限集合}\}$  と表記する.
- 集合  $A_1, \dots, A_n$  の直積 (cartesian product) を  $A_1 \times \dots \times A_n = \{(a_1, \dots, a_n) \mid a_1 \in A_1, \dots, a_n \in A_n\}$  と表記する. 集合  $A$  の  $n$  直積を  $A^n = \underbrace{A \times \dots \times A}_{n \text{ 項}}$  と表記する. 特に,  $A^0 = \{\epsilon\}$  である.
- 集合  $A_1, \dots, A_n$  の直和 (disjoin union) を  $A_1 \uplus \dots \uplus A_n = (A_1 \times \{1\}) \cup \dots \cup (A_n \times \{n\})$  と表記する. なお, 文脈から明らかな場合, 直和の添字を省略し,  $a \in A_i$  に対して,  $a \in A_1 \uplus \dots \uplus A_n$  と表記する.
- 集合  $A$  の  $B$  との差集合を  $A \setminus B = \{a \in A \mid a \notin B\}$  と表記する.

集合  $\Sigma$  について,  $\bigcup_{n \in \mathbb{N}} \Sigma^n$  を  $\Sigma^*$  と表記する. この時,  $\alpha \in \Sigma^*$  を  $\Sigma$  による列 (sequence) と呼ぶ. 列について, 以下の表記を用いる.

- $(\sigma_1, \dots, \sigma_n) \in \Sigma^n$  について,  $(\sigma_1, \dots, \sigma_n)$  を  $\sigma_1 \dots \sigma_n$  と表記する.
- 列  $\alpha = \sigma_1 \dots \sigma_n \in \Sigma^*$  について, その長さを  $|\alpha| = n$  と表記する.

集合  $A, B$  について,  $R \subseteq A \times B$  を関係 (relation) と呼ぶ. また,

$$A \rightarrow B \stackrel{\text{def}}{=} \{R \in \mathcal{P}(A \times B) \mid \forall x \in A, (x, y_1), (x, y_2) \in R. y_1 = y_2\}$$

という表記を導入し, 関係  $f : A \rightarrow B$  を  $A$  から  $B$  への部分関数 (partial function) と呼ぶ. さらに,

$$A \rightarrow B \stackrel{\text{def}}{=} \{f : A \rightarrow B \mid \forall x \in A. \exists y \in B. (x, y) \in f\}$$

という表記を導入し, 部分関数  $f : A \rightarrow B$  を (全) 関数 (function) と呼ぶ. 関係について, 以下の表記を用いる.

- 関係  $R \subseteq A \times B$  について,  $(a, b) \in R$  を  $a R b$  と表記する.
- 関係  $R \subseteq A \times B$  について, 定義域 (domain) を  $\text{dom}(R) = \{a \mid \exists b. (a, b) \in R\}$ , 値域 (range) を  $\text{cod}(R) = \{b \mid \exists a. (a, b) \in R\}$  と表記する.
- 部分関数  $f : A \rightarrow B$  について,  $(a, b) \in f$  を  $f(a) = b$  と表記する.

- 関係  $R_1 \subseteq A \times B$ ,  $R_2 \subseteq B \times C$  について, その**合成** (*composition*) を  $R_1; R_2 = R_2 \circ R_1 = \{(x, z) \in A \times C \mid \exists y \in B. (x, y) \in R_1, (y, z) \in R_2\}$  と表記する.
- 関係  $R \subseteq A \times B$ , 集合  $X \subseteq A$  について,  $R$  の  $X$  による**制限** (*restriction*) を  $R \upharpoonright_X = \{(a, b) \in R \mid a \in X\}$  と表記する. 特に関数  $f: A \rightarrow B$  の  $X \subseteq A$  による制限は, 関数  $f \upharpoonright_X: X \rightarrow B$  になる.
- $a \in A$ ,  $b \in B$  について, その組を  $a \mapsto b = (a, b)$ , 関数  $f: A \rightarrow B$  を  $f = x \mapsto f(x)$  と表記する.
- 2 項関係  $R \subseteq A^2$  について, その**推移閉包** (*transitive closure*), つまり以下を満たす最小の 2 項関係を  $R^+ \subseteq A^2$  と表記する.
  - 任意の  $(a, b) \in R$  について,  $(a, b) \in R^+$ .
  - 任意の  $(a, b) \in R^+$ ,  $(b, c) \in R^+$  について,  $(a, c) \in R^+$ .
- 2 項関係  $R \subseteq A^2$  について, その**反射推移閉包** (*reflexive transitive closure*) を  $R^* = R^+ \cup \{(a, a) \mid a \in A\}$  と表記する.

集合  $I$  について, その要素で添字付けられた対象の列  $\{a_i\}_{i \in I}$  を  $I$  で添字づけられた**族** (*indexed family*) と呼ぶ. 族について, 以下の表記を用いる.

- 族の集合を  $\prod_{i \in I} A_i = \{\{a_i\}_{i \in I} \mid \forall i \in I, a_i \in A_i\}$  と表記する.
- 集合の族  $A = \{A_i\}_{i \in I}$  について, 次の条件を満たす時,  $A$  は**互いに素** (*pairwise disjoint*) であるという.

$$\forall i_1, i_2 \in I, i_1 \neq i_2. A_{i_1} \cap A_{i_2} = \emptyset$$



## 第 2 章

## Basic Calculus



## 2.1 WIP: (Untyped) $\lambda$ -Calculus

## 2.2 Simply Typed $\lambda$ -Calculus

Alias: STLC,  $\lambda \rightarrow$  [GTL89]

### 2.2.1 Syntax

$e ::= x$	(variable)
$  e e$	(application)
$  \lambda x : \tau. e$	(abstraction)
$  c_A$	(constant)
$\tau ::= A$	(atomic type)
$  \tau \rightarrow \tau$	(function type)
$\Gamma ::= \cdot$	(empty)
$  \Gamma, x : \tau$	(cons)

Convention:

$$\tau_1 \rightarrow \tau_2 \rightarrow \cdots \rightarrow \tau_n \stackrel{\text{def}}{=} \tau_1 \rightarrow (\tau_2 \rightarrow (\cdots \rightarrow \tau_n) \cdots)$$

$$e_1 e_2 \cdots e_n \stackrel{\text{def}}{=} (\cdots (e_1 e_2) \cdots) e_n$$

Environment Reference:

$$\boxed{\Gamma(x) = \tau}$$

$$\frac{x = x'}{(\Gamma, x' : \tau)(x) = \tau} \quad \frac{x \neq x' \quad \Gamma(x) = \tau}{(\Gamma, x' : \tau')(x) = \tau}$$

Free Variable:

$$\boxed{fv(e) = \{\bar{x}'\}}$$

$$\frac{}{fv(x) = \{x\}} \quad \frac{fv(e_1) = X_1 \quad fv(e_2) = X_2}{fv(e_1 e_2) = X_1 \cup X_2} \quad \frac{fv(e) = X}{fv(\lambda x : \tau. e) = X \setminus \{x\}} \quad \frac{}{fv(c_A) = \emptyset}$$

Substitution:

部分関数  $\{x_1 \mapsto e_1, \dots, x_n \mapsto e_n\}$  を,  $[x_1 \leftarrow e_1, \dots, x_n \leftarrow e_n]$  または  $[x_1, \dots, x_n \leftarrow e_1, \dots, e_n]$  と表記する.

$$\boxed{e[x' \leftarrow e'] = e''}$$

$$\frac{[\bar{x}' \leftarrow \bar{e}'](x) = e}{x[\bar{x}' \leftarrow \bar{e}'] = e} \quad \frac{x \notin \text{dom}([\bar{x}' \leftarrow \bar{e}'])}{x[\bar{x}' \leftarrow \bar{e}'] = x}$$

$$\frac{e_1[\bar{x}' \leftarrow \bar{e}'] = e_1'' \quad e_2[\bar{x}' \leftarrow \bar{e}'] = e_2''}{(e_1 e_2)[\bar{x}' \leftarrow \bar{e}'] = e_1'' e_2''} \quad \frac{e([\bar{x}' \leftarrow \bar{e}'] \upharpoonright_{\text{dom}([\bar{x}' \leftarrow \bar{e}']) \setminus \{x\}}) = e''}{(\lambda x : \tau. e)[\bar{x}' \leftarrow \bar{e}'] = \lambda x : \tau. e''} \quad \frac{}{c_A[\bar{x}' \leftarrow \bar{e}'] = c_A}$$

$\alpha$ -Equality:

$$\boxed{e_1 \equiv_\alpha e_2}$$

$$\frac{x_1 = x_2}{x_1 \equiv_\alpha x_2} \quad \frac{x' \notin fv(e_1) \cup fv(e_2) \quad e_1[x_1 \leftarrow x'] \equiv_\alpha e_2[x_2 \leftarrow x']}{\lambda x_1 : \tau. e_1 \equiv_\alpha \lambda x_2 : \tau. e_2} \quad \frac{e_1 \equiv_\alpha e_2 \quad e_1' \equiv_\alpha e_2'}{e_1 e_1' \equiv_\alpha e_2 e_2'} \quad \frac{}{c_A \equiv_\alpha c_A}$$

**定理 1 (Correctness of Substitution).** 式  $e$ , 置換  $[\bar{x}' \leftarrow \bar{e}']$  について,  $X = \text{dom}([\bar{x}' \leftarrow \bar{e}'])$  とした時,

$$fv(e[\bar{x}' \leftarrow \bar{e}']) = (fv(e) \setminus X) \cup \bigcup_{x \in fv(e) \cap X} fv([\bar{x}' \leftarrow \bar{e}'](x)).$$

□

**定理 2 ( $\alpha$ -Equality Does Not Touch Free Variables).**  $e_1 \equiv_\alpha e_2$  ならば,  $fv(e_1) = fv(e_2)$ .

□

### 2.2.2 Typing Semantics

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c} \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{ T-Var} \\ \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \text{ T-Abs} \\ \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \text{ T-App} \\ \frac{}{\Gamma \vdash c_A : A} \text{ T-Const} \end{array}$$

特に,  $\cdot \vdash e : \tau$  の時,  $e : \tau$  と表記.

### 2.2.3 Evaluation Semantics (Call-By-Value)

$$\begin{array}{l} v ::= \lambda x : \tau. e \\ \quad | c_A \\ C ::= [] \\ \quad | C e \\ \quad | v C \end{array}$$

Small Step:

$$\boxed{e \Rightarrow e'}$$

$$\frac{}{(\lambda x : \tau. e) v \Rightarrow e[x \leftarrow v]} \quad \frac{e \Rightarrow e'}{C[e] \Rightarrow C[e']}$$

Big Step:

$$\boxed{e \Downarrow v}$$

$$\frac{e_1 \Downarrow \lambda x : \tau. e'_1 \quad e_2 \Downarrow v_2 \quad e'_1[x \leftarrow v_2] \Downarrow v}{e_1 e_2 \Downarrow v}$$

**定理 3 (Adequacy of Small Step and Big Step).**  $e \Rightarrow^* v$  iff  $e \Downarrow v$ .

□

**定理 4 (Type Soundness).**  $e : \tau$  の時,  $e \Rightarrow^* v$ ,  $e \Downarrow v$  となる  $v = \text{nf}(\Rightarrow, e)$  が存在し,

- $\tau = \tau_1 \rightarrow \tau_2$  の時,  $v \equiv_\alpha \lambda x' : \tau_1. e'$  となる  $\lambda x' : \tau_1. e'$  が存在する.
- $\tau = A$  の時,  $v \equiv_\alpha c_A$  となる  $c_A$  が存在する.

□

## 2.2.4 Equational Reasoning

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c}
\frac{\Gamma, x : \tau \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\lambda x : \tau. e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau} \text{Eq-}\beta\text{-Lam} \quad \frac{x \notin \text{fv}(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash (\lambda x : \tau_1. e) x \equiv e : \tau_1 \rightarrow \tau_2} \text{Eq-}\eta\text{-Lam} \\
\frac{e_1 \equiv_\alpha e_2 \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Refl} \\
\frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-Sym} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau \quad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau} \text{Eq-Trans} \\
\frac{\Gamma, x : \tau \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash \lambda x : \tau. e_1 \equiv \lambda x : \tau. e_2 : \tau \rightarrow \tau'} \text{Eq-Cong-Abs} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \rightarrow \tau \quad \Gamma \vdash e'_1 \equiv e'_2 : \tau'}{\Gamma \vdash e_1 e'_1 \equiv e_2 e'_2 : \tau} \text{Eq-Cong-App}
\end{array}$$

特に,  $\cdot \vdash e_1 \equiv e_2 : \tau$  の時,  $e_1 \equiv e_2 : \tau$  と表記.

**定理 5 (Respect Typing).**  $\Gamma \vdash e_1 \equiv e_2 : \tau$  ならば,  $\Gamma \vdash e_1 : \tau$  かつ  $\Gamma \vdash e_2 : \tau$ . □

**定理 6 (Respect Evaluation).**  $e_1 \equiv e_2 : \tau$  の時,  $e'_1 \Rightarrow^* e_1$ ,  $e_2 \Rightarrow^* e'_2$  ならば  $e'_1 \equiv e'_2 : \tau$ . □

**系 7.**  $e_1 \equiv e_2 : \tau$  の時,  $e_1 \Rightarrow^* e'_1$ ,  $e_2 \Rightarrow^* e'_2$  ならば  $e'_1 \equiv e'_2 : \tau$ . □

**証明.**  $e_1 \Rightarrow^* e_1$  より, 定理 6 から  $e_1 \equiv e'_2 : \tau$ . よって, T-Sym から  $e'_2 \equiv e_1 : \tau$  であり,  $e'_2 \Rightarrow^* e'_2$  より定理 6 から  $e'_2 \equiv e'_1 : \tau$ . 故に, T-Sym から  $e'_1 \equiv e'_2 : \tau$ . ■

## 2.3 WIP: System-T

## 2.4 WIP: PCF

## 2.5 System-F

Alias: F, Second Order Typed Lambda Calculus,  $\lambda 2$  [GTL89]

### 2.5.1 Syntax

$e ::= x$	(variable)
$  \lambda x : \tau. e$	(abstraction)
$  e e$	(application)
$  \Lambda t. e$	(universal abstraction)
$  e \tau$	(universal application)
$\tau ::= t$	(type variable)
$  \tau \rightarrow \tau$	(function type)
$  \forall t. \tau$	(polymorphic type)
$\Gamma ::= \cdot$	(empty)
$  \Gamma, x : \tau$	(variable cons)
$  \Gamma, t : \Omega$	(type variable cons)

Convention:

$$\tau_1 \rightarrow \tau_2 \rightarrow \cdots \rightarrow \tau_n \stackrel{\text{def}}{=} \tau_1 \rightarrow (\tau_2 \rightarrow (\cdots \rightarrow \tau_n) \cdots)$$

$$e_1 e_2 \cdots e_n \stackrel{\text{def}}{=} (\cdots (e_1 e_2) \cdots) e_n$$

Environment Reference:

$$\boxed{\Gamma(x) = \tau}$$

$$\frac{x = x'}{(\Gamma, x' : \tau)(x) = \tau} \quad \frac{x \neq x' \quad \Gamma(x) = \tau}{(\Gamma, x' : \tau')(x) = \tau} \quad \frac{\Gamma(x) = \tau}{(\Gamma, t : \Omega)(x) = \tau}$$

$$\frac{t = t'}{(\Gamma, t' : \Omega)(t) = \Omega} \quad \frac{t \neq t' \quad \Gamma(t) = \Omega}{(\Gamma, t' : \Omega')(t) = \Omega} \quad \frac{\Gamma(t) = \Omega}{(\Gamma, x : \tau)(t) = \Omega}$$

Free Variable:

$$\boxed{fv(e) = \{\bar{x}\}}$$

$$\frac{}{fv(x) = \{x\}} \quad \frac{fv(e_1) = X_1 \quad fv(e_2) = X_2}{fv(e_1 e_2) = X_1 \cup X_2} \quad \frac{fv(e) = X}{fv(\lambda x : \tau. e) = X \setminus \{x\}} \quad \frac{fv(e) = X}{fv(e \tau) = X} \quad \frac{fv(e) = X}{fv(\Lambda t. e) = X}$$

Substitution:

部分関数  $\{x_1 \mapsto e_1, \dots, x_n \mapsto e_n\}$  を,  $[x_1 \leftarrow e_1, \dots, x_n \leftarrow e_n]$  または  $[x_1, \dots, x_n \leftarrow e_1, \dots, e_n]$  と表記する.

$$\boxed{e[x' \leftarrow e'] = e''}$$

$$\frac{[\bar{x}' \leftarrow \bar{e}'](x) = e}{x[\bar{x}' \leftarrow \bar{e}'] = e} \quad \frac{x \notin \text{dom}([\bar{x}' \leftarrow \bar{e}'])}{x[\bar{x}' \leftarrow \bar{e}'] = x}$$

$$\frac{e_1[\bar{x}' \leftarrow \bar{e}'] = e''_1 \quad e_2[\bar{x}' \leftarrow \bar{e}'] = e''_2}{(e_1 e_2)[\bar{x}' \leftarrow \bar{e}'] = e''_1 e''_2} \quad \frac{e([\bar{x}' \leftarrow \bar{e}']) \upharpoonright_{\text{dom}([\bar{x}' \leftarrow \bar{e}']) \setminus \{x\}} = e''}{(\lambda x : \tau. e)[\bar{x}' \leftarrow \bar{e}'] = \lambda x : \tau. e''}$$

$$\frac{e[\bar{x}' \leftarrow \bar{e}'] = e''}{(e \tau)[\bar{x}' \leftarrow \bar{e}'] = e'' \tau} \quad \frac{e[\bar{x}' \leftarrow \bar{e}'] = e''}{(\Lambda t. e)[\bar{x}' \leftarrow \bar{e}'] = \Lambda t. e''}$$

Type Free Variable:

$$\boxed{tyfv(e) = \{\bar{x}\}}$$

$$\begin{array}{c} \frac{}{tyfv(x) = \emptyset} \quad \frac{tyfv(e_1) = T_1 \quad tyfv(e_2) = T_2}{tyfv(e_1 e_2) = T_1 \cup T_2} \quad \frac{tyfv(\tau) = T_1 \quad tyfv(e) = T_2}{tyfv(\lambda x : \tau. e) = T_1 \cup T_2} \\ \frac{tyfv(e) = T_1 \quad tyfv(\tau) = T_2}{tyfv(e \tau) = T_1 \cup T_2} \quad \frac{tyfv(e) = T}{tyfv(\Lambda t. e) = T \setminus \{t\}} \\ \frac{}{tyfv(t) = \{t\}} \quad \frac{tyfv(\tau_1) = T_1 \quad tyfv(\tau_2) = T_2}{tyfv(\tau_1 \rightarrow \tau_2) = T_1 \cup T_2} \quad \frac{tyfv(\tau) = T}{tyfv(\forall t. \tau) = T \setminus \{t\}} \end{array}$$

Type Substitution:

部分関数  $\{t_1 \mapsto \tau_1, \dots, t_n \mapsto \tau_n\}$  を,  $[t_1 \leftarrow \tau_1, \dots, t_n \leftarrow \tau_n]$  または  $[t_1, \dots, t_n \leftarrow t_1, \dots, t_n]$  と表記する.

$$\boxed{e[\bar{t} \leftarrow \bar{\tau}] = e'}$$

$$\begin{array}{c} \frac{}{x[\bar{t}' \leftarrow \bar{\tau}'] = x} \quad \frac{e_1[\bar{t}' \leftarrow \bar{\tau}'] = e_1'' \quad e_2[\bar{t}' \leftarrow \bar{\tau}'] = e_2''}{(e_1 e_2)[\bar{t}' \leftarrow \bar{\tau}'] = e_1'' e_2''} \quad \frac{\tau[\bar{t}' \leftarrow \bar{\tau}'] = \tau'' \quad e[\bar{t}' \leftarrow \bar{\tau}'] = e''}{(\lambda x : \tau. e)[\bar{t}' \leftarrow \bar{\tau}'] = \lambda x : \tau''. e''} \\ \frac{e[\bar{t}' \leftarrow \bar{\tau}'] = e'' \quad \tau[\bar{t}' \leftarrow \bar{\tau}'] = \tau''}{(e \tau)[\bar{t}' \leftarrow \bar{\tau}'] = e'' \tau''} \quad \frac{e([\bar{t}' \leftarrow \bar{\tau}'] \upharpoonright_{\text{dom}([\bar{t}' \leftarrow \bar{\tau}'] \setminus \{t\})}) = e''}{(\Lambda t. e)[\bar{t}' \leftarrow \bar{\tau}'] = \Lambda t. e''} \end{array}$$

$$\boxed{\tau[\bar{t}' \leftarrow \bar{\tau}'] = \tau''}$$

$$\begin{array}{c} \frac{[\bar{t}' \leftarrow \bar{\tau}'](t) = \tau}{t[\bar{t}' \leftarrow \bar{\tau}'] = \tau} \quad \frac{t \notin \text{dom}([\bar{t}' \leftarrow \bar{\tau}'])}{t[\bar{t}' \leftarrow \bar{\tau}'] = t} \quad \frac{\tau_1[\bar{t}' \leftarrow \bar{\tau}'] = \tau_1'' \quad \tau_2[\bar{t}' \leftarrow \bar{\tau}'] = \tau_2''}{(\tau_1 \rightarrow \tau_2)[\bar{t}' \leftarrow \bar{\tau}'] = \tau_1'' \rightarrow \tau_2''} \quad \frac{\tau([\bar{t}' \leftarrow \bar{\tau}'] \upharpoonright_{\text{dom}([\bar{t}' \leftarrow \bar{\tau}'] \setminus \{t\})}) = \tau''}{(\forall t. \tau)[\bar{t}' \leftarrow \bar{\tau}'] = \forall t. \tau''} \end{array}$$

$\alpha$ -Equality:

$$\boxed{e_1 \equiv_{\alpha} e_2}$$

$$\begin{array}{c} \frac{x_1 = x_2}{x_1 \equiv_{\alpha} x_2} \quad \frac{e_1 \equiv_{\alpha} e_2 \quad e_1' \equiv_{\alpha} e_2'}{e_1 e_1' \equiv_{\alpha} e_2 e_2'} \quad \frac{\tau_1 \equiv_{\alpha} \tau_2 \quad x' \notin fv(e_1) \cup fv(e_2) \quad e_1[x_1 \leftarrow x'] \equiv_{\alpha} e_2[x_2 \leftarrow x']}{\lambda x_1 : \tau_1. e_1 \equiv_{\alpha} \lambda x_2 : \tau_2. e_2} \\ \frac{e_1 \equiv_{\alpha} e_2 \quad \tau_1 \equiv_{\alpha} \tau_2}{e_1 \tau_1 \equiv_{\alpha} e_2 \tau_2} \quad \frac{t' \notin tyfv(e_1) \cup tyfv(e_2) \quad e_1[t_1 \leftarrow t'] \equiv_{\alpha} e_2[t_2 \leftarrow t']}{\Lambda t_1. e_1 \equiv_{\alpha} \Lambda t_2. e_2} \end{array}$$

$$\boxed{\tau_1 \equiv_{\alpha} \tau_2}$$

$$\frac{t_1 = t_2}{t_1 \equiv_{\alpha} t_2} \quad \frac{\tau_1 \equiv_{\alpha} \tau_2 \quad \tau_1' \equiv_{\alpha} \tau_2'}{\tau_1 \rightarrow \tau_1' \equiv_{\alpha} \tau_2 \rightarrow \tau_2'} \quad \frac{t' \notin tyfv(\tau_1) \cup tyfv(\tau_2) \quad \tau_1[t_1 \leftarrow t'] \equiv_{\alpha} \tau_2[t_2 \leftarrow t']}{\forall t_1. \tau_1 \equiv_{\alpha} \forall t_2. \tau_2}$$

**定理 8 (Correctness of Substitution).** 置換  $[\bar{x}' \leftarrow \bar{e}']$  について,  $X = \text{dom}([\bar{x}' \leftarrow \bar{e}'])$  とした時,

$$fv(e[\bar{x}' \leftarrow \bar{e}']) = (fv(e) \setminus X) \cup \bigcup_{x \in fv(e) \cap X} fv([\bar{x}' \leftarrow \bar{e}'](x)).$$

□

**定理 9 (Correctness of Type Substitution).** 式  $e$ , 型  $\tau$ , 型置換  $[\bar{t}' \leftarrow \bar{\tau}']$  について,  $T = \text{dom}([\bar{t}' \leftarrow \bar{\tau}'])$  とした時,

$$tyfv(e[\bar{t}' \leftarrow \bar{\tau}']) = (tyfv(e) \setminus T) \cup \bigcup_{t \in tyfv(e) \cap T} tyfv([\bar{t}' \leftarrow \bar{\tau}'](t))$$



$$tyfv(\tau[\bar{t}' \leftarrow \bar{\tau}']) = (tyfv(\tau) \setminus T) \cup \bigcup_{t \in tyfv(\tau) \cap T} tyfv([\bar{t}' \leftarrow \bar{\tau}'](t)).$$

□

定理 10 ( $\alpha$ -Equality Does Not Touch Free Variables).

- $\tau_1 \equiv_\alpha \tau_2$  ならば  $tyfv(\tau_1) = tyfv(\tau_2)$ .
- $e_1 \equiv_\alpha e_2$  ならば,  $fv(e_1) = fv(e_2)$ ,  $tyfv(e_1) = tyfv(e_2)$ .

□

## 2.5.2 Typing Semantics

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c} \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{ T-Var} \\ \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \text{ T-Abs} \\ \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \text{ T-App} \\ \frac{\Gamma, t : \Omega \vdash e : \tau}{\Gamma \vdash \Lambda t. e : \forall t. \tau} \text{ T-UnivAbs} \\ \frac{\Gamma \vdash e : \forall t. \tau_1}{\Gamma \vdash e \tau_2 : \tau_1[t \leftarrow \tau_2]} \text{ T-UnivApp} \\ \frac{\Gamma \vdash \tau \equiv_\alpha \tau' : \Omega \quad \Gamma \vdash e : \tau'}{\Gamma \vdash e : \tau} \text{ T-}\alpha\text{-Equiv} \end{array}$$

特に,  $\cdot \vdash e : \tau$  の時,  $e : \tau$  と表記.

## 2.5.3 Evaluation Semantics (Call-By-Value)

$$\begin{array}{l} v ::= \lambda x : \tau. e \\ \quad | \Lambda t. e \\ C ::= [] \\ \quad | C e \\ \quad | v C \\ \quad | C \tau \end{array}$$

Small Step:

$$\boxed{e \Rightarrow e'}$$

$$\begin{array}{c} \overline{(\lambda x : \tau. e) v \Rightarrow e[x \leftarrow v]} \\ \overline{(\Lambda t. e) \tau \Rightarrow e[t \leftarrow \tau]} \\ \frac{e \Rightarrow e'}{C[e] \Rightarrow C[e']} \end{array}$$

Big Step:

$$\boxed{e \Downarrow v}$$

$$\frac{e_1 \Downarrow \lambda x : \tau. e'_1 \quad e_2 \Downarrow v_2 \quad e'_1[x \leftarrow v_2] \Downarrow v}{e_1 e_2 \Downarrow v}$$

$$\frac{e \Downarrow \Lambda t. e'_1 \quad e'_1[t \leftarrow \tau] \Downarrow v}{e \tau \Downarrow v}$$

**定理 11 (Adequacy of Small Step and Big Step).**  $e \Rightarrow^* v$  iff  $e \Downarrow v$ . □

**定理 12 (Type Soundness).**  $e : \tau$  の時,  $e \Rightarrow^* v$ ,  $e \Downarrow v$  となる  $v = \text{nf}(\Rightarrow, e)$  が存在し,

- $\tau = \tau_1 \rightarrow \tau_2$  の時,  $v \equiv_\alpha \lambda x' : \tau_1. e'$  となる  $\lambda x' : \tau_1. e'$  が存在する.
- $\tau = \forall t. \tau_1$  の時,  $v \equiv_\alpha \Lambda t. e'$  となる  $\Lambda t. e'$  が存在する.

□

## 2.5.4 Equational Reasoning

$\Gamma \vdash e_1 \equiv e_2 : \tau$

$$\frac{\Gamma, x : \tau_2 \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\lambda x : \tau_2. e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau} \text{Eq-}\beta\text{-Lam} \quad \frac{x \notin \text{fv}(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash (\lambda x : \tau_1. e \ x) \equiv e : \tau_1 \rightarrow \tau_2} \text{Eq-}\eta\text{-Lam}$$

$$\frac{\Gamma, t : \Omega \vdash e : \tau}{\Gamma \vdash (\Lambda t. e) \tau_2 \equiv e[t \leftarrow \tau_2] : \tau[t \leftarrow \tau_2]} \text{Eq-}\beta\text{-UnivLam} \quad \frac{t \notin \text{tyfv}(e) \quad \Gamma \vdash e : \forall t'. \tau}{\Gamma \vdash (\Lambda t. e \ t) \equiv e : \forall t'. \tau} \text{Eq-}\eta\text{-UnivLam}$$

$$\frac{e_1 \equiv_\alpha e_2 \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Refl} \quad \frac{\tau \equiv_\alpha \tau' \quad \Gamma \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Type}$$

$$\frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-Sym} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau \quad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau} \text{Eq-Trans}$$

$$\frac{\Gamma, x : \tau \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash \lambda x : \tau. e_1 \equiv \lambda x : \tau. e_2 : \tau \rightarrow \tau'} \text{Eq-Cong-Abs} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \rightarrow \tau \quad \Gamma \vdash e'_1 \equiv e'_2 : \tau'}{\Gamma \vdash e_1 e'_1 \equiv e_2 e'_2 : \tau} \text{Eq-Cong-App}$$

$$\frac{\Gamma, t : \Omega \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \Lambda t. e_1 \equiv \Lambda t. e_2 : \forall (t. \tau)} \text{Eq-Cong-UnivAbs} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \forall t. \tau}{\Gamma \vdash e_1 \tau' \equiv e_2 \tau' : \tau[t \leftarrow \tau']} \text{Eq-Cong-UnivApp}$$

特に,  $\cdot \vdash e_1 \equiv e_2 : \tau$  の時,  $e_1 \equiv e_2 : \tau$  と表記.

**定理 13 (Respect Typing).**  $\Gamma \vdash e_1 \equiv e_2 : \tau$  ならば,  $\Gamma \vdash e_1 : \tau$  かつ  $\Gamma \vdash e_2 : \tau$ . □

**定理 14 (Respect Evaluation).**  $e_1 \equiv e_2 : \tau$  の時,  $e'_1 \Rightarrow^* e_1$ ,  $e_2 \Rightarrow^* e'_2$  ならば  $e'_1 \equiv e'_2 : \tau$ . □

**系 15.**  $e_1 \equiv e_2 : \tau$  の時,  $e_1 \Rightarrow^* e'_1$ ,  $e_2 \Rightarrow^* e'_2$  ならば  $e'_1 \equiv e'_2 : \tau$ . □

**証明.**  $e_1 \Rightarrow^* e_1$  より, 定理 14 から  $e_1 \equiv e'_2 : \tau$ . よって, T-Sym から  $e'_2 \equiv e_1 : \tau$  であり,  $e'_2 \Rightarrow^* e'_2$  より定理 14 から  $e'_2 \equiv e'_1 : \tau$ . 故に, T-Sym から  $e'_1 \equiv e'_2 : \tau$ . ■

## 2.5.5 Definability

Product

Product of  $\tau_1$  and  $\tau_2$ :

$$\tau_1 \times \tau_2 \stackrel{\text{def}}{=} \forall t. (\tau_1 \rightarrow \tau_2 \rightarrow t) \rightarrow t$$

$$\langle e_1, e_2 \rangle \stackrel{\text{def}}{=} \Lambda t. \lambda x : \tau_1 \rightarrow \tau_2 \rightarrow t. x \ e_1 \ e_2$$

$$\begin{aligned}\pi_1 e &\stackrel{\text{def}}{=} e \tau_1 \lambda x_1. \lambda x_2. x_1 \\ \pi_2 e &\stackrel{\text{def}}{=} e \tau_2 \lambda x_1. \lambda x_2. x_2\end{aligned}$$

Admissible typing rule:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2} \text{T-Product} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_1 e : \tau_1} \text{T-Proj-1} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_2 e : \tau_2} \text{T-Proj-2}$$

Admissible equality:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{aligned}\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \pi_1 \langle e_1, e_2 \rangle \equiv e_1 : \tau_1} \text{Eq-}\beta\text{-Product-1} \quad & \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \pi_2 \langle e_1, e_2 \rangle \equiv e_2 : \tau_2} \text{Eq-}\beta\text{-Product-2} \\ & \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \langle \pi_1 e, \pi_2 e \rangle \equiv e : \tau_1 \times \tau_2} \text{Eq-}\eta\text{-Product}\end{aligned}$$

### Existential Type

Existence of  $\exists t. \tau$ :

$$\begin{aligned}\exists t. \tau &\stackrel{\text{def}}{=} \forall t'. (\forall t. \tau \rightarrow t') \rightarrow t' \\ \text{pack} \langle \tau_t, e \rangle &\stackrel{\text{def}}{=} \Lambda t'. \lambda x : (\forall t. \tau \rightarrow t'). x \tau_t e \\ \text{unpack} \langle t, x \rangle &= e_1. \tau_2. e_2 \stackrel{\text{def}}{=} e_1 \tau_2 (\Lambda t. \lambda x : \tau. e_2)\end{aligned}$$

Admissible typing rule:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash e : \tau[t \leftarrow \tau_t]}{\Gamma \vdash \text{pack} \langle \tau_t, e \rangle : \exists t. \tau} \text{T-Pack} \quad \frac{\Gamma \vdash e_1 : \exists t. \tau \quad \Gamma, t : \Omega, x : \tau \vdash e_2 : \tau_2 \quad t \notin \text{tyfv}(\tau_2)}{\Gamma \vdash \text{unpack} \langle t, x \rangle = e_1. \tau_2. e_2 : \tau_2} \text{T-Unpack}$$

Admissible equality:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{aligned}\frac{\Gamma \vdash e_1 : \tau_1[t \leftarrow \tau_t] \quad \Gamma, t : \Omega, x : \tau_1 \vdash e_2 : \tau_2 \quad t \notin \text{tyfv}(\tau_2)}{\Gamma \vdash \text{unpack} \langle t, x \rangle = \text{pack} \langle \tau_t, e_1 \rangle. \tau_2. e_2 \equiv e_2[t \leftarrow \tau_t][x \leftarrow e_1] : \tau_2} \text{Eq-}\beta\text{-Exist} \\ \frac{\Gamma \vdash e : \exists t'. \tau \quad \tau' \equiv_\alpha \exists t'. \tau}{\Gamma \vdash \text{unpack} \langle t, x \rangle = e. \tau'. \text{pack} \langle t, x \rangle \equiv e : \exists t'. \tau} \text{Eq-}\eta\text{-Exist}\end{aligned}$$

## 2.6 System-F $\omega$

Alias: F  $\omega$ ,  $\lambda\omega$  [RRD14]

### 2.6.1 Syntax

$e ::= x$	(variable)
$\quad   \lambda x : \tau. e$	(abstraction)
$\quad   e e$	(application)
$\quad   \Lambda t : \kappa. e$	(universal abstraction)
$\quad   e \tau$	(universal application)
$\tau ::= t$	(type variable)
$\quad   \tau \rightarrow \tau$	(function type)
$\quad   \forall t : \kappa. \tau$	(polymorphic type)
$\quad   \lambda t : \kappa. \tau$	(type abstraction)
$\quad   \tau \tau$	(type application)
$\kappa ::= \Omega$	(type kind)
$\quad   \kappa \rightarrow \kappa$	(arrow kind)
$\Gamma ::= \cdot$	(empty)
$\quad   \Gamma, x : \tau$	(variable cons)
$\quad   \Gamma, t : \kappa$	(type variable cons)

Convention:

$$\begin{aligned} \tau_1 \rightarrow \tau_2 \rightarrow \cdots \rightarrow \tau_n &\stackrel{\text{def}}{=} \tau_1 \rightarrow (\tau_2 \rightarrow (\cdots \rightarrow \tau_n) \cdots) \\ e_1 e_2 \cdots e_n &\stackrel{\text{def}}{=} (\cdots (e_1 e_2) \cdots) e_n \end{aligned}$$

Environment Reference:

$$\boxed{\Gamma(x) = \tau}$$

$$\begin{array}{ccc} \frac{x = x'}{(\Gamma, x' : \tau)(x) = \tau} & \frac{x \neq x' \quad \Gamma(x) = \tau}{(\Gamma, x' : \tau')(x) = \tau} & \frac{\Gamma(x) = \tau}{(\Gamma, t : \kappa)(x) = \tau} \\ \frac{t = t'}{(\Gamma, t' : \kappa)(t) = \kappa} & \frac{t \neq t' \quad \Gamma(t) = \kappa}{(\Gamma, t' : \kappa')(t) = \kappa} & \frac{\Gamma(t) = \kappa}{(\Gamma, x : \tau)(t) = \kappa} \end{array}$$

Free Variable:

$$\boxed{fv(e) = \{\bar{x}\}}$$

$$\begin{array}{ccccc} \frac{}{fv(x) = \{x\}} & \frac{fv(e) = X}{fv(\lambda x : \tau. e) = X \setminus \{x\}} & \frac{fv(e_1) = X_1 \quad fv(e_2) = X_2}{fv(e_1 e_2) = X_1 \cup X_2} & \frac{fv(e) = X}{fv(\Lambda t : \kappa. e) = X} & \frac{fv(e) = X}{fv(e \tau) = X} \end{array}$$

Substitution:

部分関数  $\{x_1 \mapsto e_1, \dots, x_n \mapsto e_n\}$  を,  $[x_1 \leftarrow e_1, \dots, x_n \leftarrow e_n]$  または  $[x_1, \dots, x_n \leftarrow e_1, \dots, e_n]$  と表記する.

$$\boxed{e[\overline{x'} \leftarrow \overline{e'}] = \overline{e''}}$$

$$\begin{array}{cc} \frac{[\overline{x'} \leftarrow \overline{e'}](x) = e}{x[\overline{x'} \leftarrow \overline{e'}] = e} & \frac{x \notin \text{dom}([\overline{x'} \leftarrow \overline{e'}])}{x[\overline{x'} \leftarrow \overline{e'}] = x} \end{array}$$

$$\begin{array}{c}
\frac{e(\overline{x'} \leftarrow \overline{e'}) \upharpoonright_{\text{dom}(\overline{x'} \leftarrow \overline{e'}) \setminus \{x\}} = e''}{(\lambda x : \tau. e)[\overline{x'} \leftarrow \overline{e'}] = \lambda x : \tau. e''} \quad \frac{e_1[\overline{x'} \leftarrow \overline{e'}] = e'_1 \quad e_2[\overline{x'} \leftarrow \overline{e'}] = e''_2}{(e_1 e_2)[\overline{x'} \leftarrow \overline{e'}] = e'_1 e''_2} \\
\frac{e[\overline{x'} \leftarrow \overline{e'}] = e''}{(\Lambda t : \kappa. e)[\overline{x'} \leftarrow \overline{e'}] = \Lambda t : \kappa. e''} \quad \frac{e[\overline{x'} \leftarrow \overline{e'}] = e''}{(e \tau)[\overline{x'} \leftarrow \overline{e'}] = e'' \tau}
\end{array}$$

Type Free Variable:

$$\boxed{\text{tyfv}(e) = \{\bar{t}\}}$$

$$\begin{array}{c}
\frac{}{\text{tyfv}(x) = \emptyset} \quad \frac{\text{tyfv}(\tau) = T_1 \quad \text{tyfv}(e) = T_2}{\text{tyfv}(\lambda x : \tau. e) = T_1 \cup T_2} \quad \frac{\text{tyfv}(e_1) = T_1 \quad \text{tyfv}(e_2) = T_2}{\text{tyfv}(e_1 e_2) = T_1 \cup T_2} \\
\frac{\text{tyfv}(e) = T}{\text{tyfv}(\Lambda t : \kappa. e) = T \setminus \{t\}} \quad \frac{\text{tyfv}(e) = T_1 \quad \text{tyfv}(\tau) = T_2}{\text{tyfv}(e \tau) = T_1 \cup T_2}
\end{array}$$

$$\boxed{\text{tyfv}(\tau) = \{\bar{t}\}}$$

$$\begin{array}{c}
\frac{}{\text{tyfv}(t) = \{t\}} \quad \frac{\text{tyfv}(\tau_1) = T_1 \quad \text{tyfv}(\tau_2) = T_2}{\text{tyfv}(\tau_1 \rightarrow \tau_2) = T_1 \cup T_2} \quad \frac{\text{tyfv}(\tau) = T}{\text{tyfv}(\forall t : \kappa. \tau) = T \setminus \{t\}} \\
\frac{\text{tyfv}(\tau) = T}{\text{tyfv}(\lambda t : \kappa. \tau) = T \setminus \{t\}} \quad \frac{\text{tyfv}(\tau_1) = T_1 \quad \text{tyfv}(\tau_2) = T_2}{\text{tyfv}(\tau_1 \tau_2) = T_1 \cup T_2}
\end{array}$$

Type Substitution:

部分関数  $\{t_1 \mapsto \tau_1, \dots, t_n \mapsto \tau_n\}$  を,  $[t_1 \leftarrow \tau_1, \dots, t_n \leftarrow \tau_n]$  または  $[t_1, \dots, t_n \leftarrow t_1, \dots, t_n]$  と表記する.

$$\boxed{e[\overline{t'} \leftarrow \overline{\tau'}] = e'}$$

$$\begin{array}{c}
\frac{}{x[\overline{t'} \leftarrow \overline{\tau'}] = x} \quad \frac{e_1[\overline{t'} \leftarrow \overline{\tau'}] = e'_1 \quad e_2[\overline{t'} \leftarrow \overline{\tau'}] = e''_2}{(e_1 e_2)[\overline{t'} \leftarrow \overline{\tau'}] = e'_1 e''_2} \quad \frac{\tau[\overline{t'} \leftarrow \overline{\tau'}] = \tau'' \quad e[\overline{t'} \leftarrow \overline{\tau'}] = e''}{(\lambda x : \tau. e)[\overline{t'} \leftarrow \overline{\tau'}] = \lambda x : \tau''. e''} \\
\frac{e[\overline{t'} \leftarrow \overline{\tau'}] = e'' \quad \tau[\overline{t'} \leftarrow \overline{\tau'}] = \tau''}{(e \tau)[\overline{t'} \leftarrow \overline{\tau'}] = e'' \tau''} \quad \frac{e([\overline{t'} \leftarrow \overline{\tau'}] \upharpoonright_{\text{dom}([\overline{t'} \leftarrow \overline{\tau'}] \setminus \{t\})}) = e''}{(\Lambda t : \kappa. e)[\overline{t'} \leftarrow \overline{\tau'}] = \Lambda t : \kappa. e''}
\end{array}$$

$$\boxed{\tau[\overline{t'} \leftarrow \overline{\tau'}] = \tau''}$$

$$\begin{array}{c}
\frac{}{t[\overline{t'} \leftarrow \overline{\tau'}] = \tau} \quad \frac{t \notin \text{dom}([\overline{t'} \leftarrow \overline{\tau'}])}{t[\overline{t'} \leftarrow \overline{\tau'}] = t} \\
\frac{\tau_1[\overline{t'} \leftarrow \overline{\tau'}] = \tau''_1 \quad \tau_2[\overline{t'} \leftarrow \overline{\tau'}] = \tau''_2}{(\tau_1 \rightarrow \tau_2)[\overline{t'} \leftarrow \overline{\tau'}] = \tau''_1 \rightarrow \tau''_2} \quad \frac{\tau([\overline{t'} \leftarrow \overline{\tau'}] \upharpoonright_{\text{dom}([\overline{t'} \leftarrow \overline{\tau'}] \setminus \{t\})}) = \tau''}{(\forall t : \kappa. \tau)[\overline{t'} \leftarrow \overline{\tau'}] = \forall t : \kappa. \tau''} \\
\frac{\tau([\overline{t'} \leftarrow \overline{\tau'}] \upharpoonright_{\text{dom}([\overline{t'} \leftarrow \overline{\tau'}] \setminus \{t\})}) = \tau''}{(\lambda t : \kappa. \tau)[\overline{t'} \leftarrow \overline{\tau'}] = \lambda t : \kappa. \tau''} \quad \frac{\tau_1[\overline{t'} \leftarrow \overline{\tau'}] = \tau''_1 \quad \tau_2[\overline{t'} \leftarrow \overline{\tau'}] = \tau''_2}{(\tau_1 \tau_2)[\overline{t'} \leftarrow \overline{\tau'}] = \tau''_1 \tau''_2}
\end{array}$$

$\alpha$ -Equality:

$$\boxed{e_1 \equiv_{\alpha} e_2}$$

$$\begin{array}{c}
\frac{x_1 = x_2}{x_1 \equiv_{\alpha} x_2} \quad \frac{e_1 \equiv_{\alpha} e_2 \quad e'_1 \equiv_{\alpha} e'_2}{e_1 e'_1 \equiv_{\alpha} e_2 e'_2} \quad \frac{\tau_1 \equiv_{\alpha} \tau_2 \quad x' \notin \text{fv}(e_1) \cup \text{fv}(e_2) \quad e_1[x_1 \leftarrow x'] \equiv_{\alpha} e_2[x_2 \leftarrow x']}{\lambda x_1 : \tau_1. e_1 \equiv_{\alpha} \lambda x_2 : \tau_2. e_2} \\
\frac{e_1 \equiv_{\alpha} e_2 \quad \tau_1 \equiv_{\alpha} \tau_2}{e_1 \tau_1 \equiv_{\alpha} e_2 \tau_2} \quad \frac{t' \notin \text{tyfv}(e_1) \cup \text{tyfv}(e_2) \quad e_1[t_1 \leftarrow t'] \equiv_{\alpha} e_2[t_2 \leftarrow t']}{\Lambda t_1 : \kappa. e_1 \equiv_{\alpha} \Lambda t_2 : \kappa. e_2}
\end{array}$$

$$\boxed{\tau_1 \equiv_{\alpha} \tau_2}$$

$$\frac{t_1 = t_2}{t_1 \equiv_\alpha t_2} \quad \frac{\tau_1 \equiv_\alpha \tau_2 \quad \tau'_1 \equiv_\alpha \tau'_2}{\tau_1 \rightarrow \tau'_1 \equiv_\alpha \tau_2 \rightarrow \tau'_2} \quad \frac{t' \notin \text{tyfv}(\tau_1) \cup \text{tyfv}(\tau_2) \quad \tau_1[t_1 \leftarrow t'] \equiv_\alpha \tau_2[t_2 \leftarrow t']}{\forall t_1 : \kappa. \tau_1 \equiv_\alpha \forall t_2 : \kappa. \tau_2}$$

$$\frac{t' \notin \text{tyfv}(\tau_1) \cup \text{tyfv}(\tau_2) \quad \tau_1[t_1 \leftarrow t'] \equiv_\alpha \tau_2[t_2 \leftarrow t']}{\lambda t_1 : \kappa. \tau_1 \equiv_\alpha \lambda t_2 : \kappa. \tau_2} \quad \frac{\tau_1 \equiv_\alpha \tau_2 \quad \tau'_1 \equiv_\alpha \tau'_2}{\tau_1 \tau'_1 \equiv_\alpha \tau_2 \tau'_2}$$

**定理 16 (Correctness of Substitution).** 置換  $[\bar{x}' \leftarrow \bar{e}']$  について,  $X = \text{dom}([\bar{x}' \leftarrow \bar{e}'])$  とした時,

$$fv(e[\bar{x}' \leftarrow \bar{e}']) = (fv(e) \setminus X) \cup \bigcup_{x \in fv(e) \cap X} fv([\bar{x}' \leftarrow \bar{e}'](x)).$$

□

**定理 17 (Correctness of Type Substitution).** 式  $e$ , 型  $\tau$ , 型置換  $[\bar{t}' \leftarrow \bar{\tau}']$  について,  $T = \text{dom}([\bar{t}' \leftarrow \bar{\tau}'])$  とした時,

$$\text{tyfv}(e[\bar{t}' \leftarrow \bar{\tau}']) = (\text{tyfv}(e) \setminus T) \cup \bigcup_{t \in \text{tyfv}(e) \cap T} \text{tyfv}([\bar{t}' \leftarrow \bar{\tau}'](t))$$

$$\text{tyfv}(\tau[\bar{t}' \leftarrow \bar{\tau}']) = (\text{tyfv}(\tau) \setminus T) \cup \bigcup_{t \in \text{tyfv}(\tau) \cap T} \text{tyfv}([\bar{t}' \leftarrow \bar{\tau}'](t)).$$

□

**定理 18 ( $\alpha$ -Equality Does Not Touch Free Variables).**

- $\tau_1 \equiv_\alpha \tau_2$  ならば  $\text{tyfv}(\tau_1) = \text{tyfv}(\tau_2)$ .
- $e_1 \equiv_\alpha e_2$  ならば,  $fv(e_1) = fv(e_2)$ ,  $\text{tyfv}(e_1) = \text{tyfv}(e_2)$ .

□

## 2.6.2 Typing Semantics

Kinding:

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\frac{\Gamma(t) = \kappa}{\Gamma \vdash t : \kappa} \text{ K-Var}$$

$$\frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma \vdash \tau_2 : \Omega}{\Gamma \vdash \tau_1 \rightarrow \tau_2 : \Omega} \text{ K-Arrow}$$

$$\frac{\Gamma, t : \kappa \vdash \tau : \Omega}{\Gamma \vdash \forall t : \kappa. \tau : \Omega} \text{ K-Forall}$$

$$\frac{\Gamma, t : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda t : \kappa_1. \tau : \kappa_1 \rightarrow \kappa_2} \text{ K-Abs}$$

$$\frac{\Gamma \vdash \tau_1 : \kappa_2 \rightarrow \kappa \quad \Gamma \vdash \tau_2 : \kappa_2}{\Gamma \vdash \tau_1 \tau_2 : \kappa} \text{ K-App}$$

Type equivalence:

$$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$$

$$\frac{\Gamma, t : \kappa_2 \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa_2}{\Gamma \vdash (\lambda t : \kappa_2. \tau_1) \tau_2 \equiv \tau_1[t \leftarrow \tau_2] : \kappa} \text{ T-Eq-}\beta\text{-Lam} \quad \frac{t \notin \text{tyfv}(\tau) \quad \Gamma \vdash \tau : \kappa_1 \rightarrow \kappa_2}{\Gamma \vdash (\lambda t : \kappa_1. \tau t) \equiv \tau : \kappa_1 \rightarrow \kappa_2} \text{ T-Eq-}\eta\text{-Lam}$$

$$\frac{\tau_1 \equiv_\alpha \tau_2 \quad \Gamma \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa} \text{ T-Eq-}\alpha\text{-Refl}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau_2 \equiv \tau_1 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa} \text{T-Eq-Sym} \quad \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa \quad \Gamma \vdash \tau_2 \equiv \tau_3 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_3 : \kappa} \text{T-Eq-Trans} \\
\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \Omega}{\Gamma \vdash \tau_1 \rightarrow \tau'_1 \equiv \tau_2 \rightarrow \tau'_2 : \Omega} \text{T-Eq-Cong-Arrow} \quad \frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash \forall t : \kappa. \tau_1 \equiv \forall t : \kappa. \tau_2 : \Omega} \text{Eq-Cong-Forall} \\
\frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \kappa'}{\Gamma \vdash \lambda t : \kappa. \tau_1 \equiv \lambda t : \kappa. \tau_2 : \kappa \rightarrow \kappa'} \text{T-Eq-Cong-Abs} \quad \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa' \rightarrow \kappa \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \kappa'}{\Gamma \vdash \tau_1 \tau'_1 \equiv \tau_2 \tau'_2 : \kappa} \text{Eq-Cong-App}
\end{array}$$

**定理 19 (Respect Kinding).**  $\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa$  ならば,  $\Gamma \vdash \tau_1 : \kappa$  かつ  $\Gamma \vdash \tau_2 : \kappa$ . □

Typing:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau : \Omega \quad \Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{T-Var} \\
\frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \text{T-Abs} \\
\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau} \text{T-App} \\
\frac{\Gamma, t : \kappa \vdash e : \tau}{\Gamma \vdash \Lambda t : \kappa. e : \forall t : \kappa. \tau} \text{T-UnivAbs} \\
\frac{\Gamma \vdash e : \forall t : \kappa. \tau_1 \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash e \tau_2 : \tau_1[t \leftarrow \tau_2]} \text{T-UnivApp} \\
\frac{\Gamma \vdash \tau \equiv \tau' : \Omega \quad \Gamma \vdash e : \tau'}{\Gamma \vdash e : \tau} \text{T-Equiv}
\end{array}$$

特に,  $\cdot \vdash e : \tau$  の時,  $e : \tau$  と表記.

**定理 20 (Respect Type Kind).**  $\Gamma \vdash e : \tau$  ならば,  $\Gamma \vdash \tau : \Omega$ . □

### 2.6.3 Evaluation Semantics (Call-By-Value)

$$\begin{array}{l}
v ::= \lambda x : \tau. e \\
\quad | \Lambda t : \kappa. e \\
C ::= [] \\
\quad | C e \\
\quad | v C \\
\quad | C \tau
\end{array}$$

Small Step:

$$\boxed{e \Rightarrow e'}$$

$$\begin{array}{c}
\overline{(\lambda x : \tau. e) v \Rightarrow e[x \leftarrow v]} \\
\overline{(\Lambda t : \kappa. e) \tau \Rightarrow e[t \leftarrow \tau]} \\
\frac{e \Rightarrow e'}{C[e] \Rightarrow C[e']}
\end{array}$$

Big Step:

$$\boxed{e \Downarrow v}$$

$$\frac{e_1 \Downarrow \lambda x : \tau. e'_1 \quad e_2 \Downarrow v_2 \quad e'_1[x \leftarrow v_2] \Downarrow v}{e_1 e_2 \Downarrow v}$$

$$\frac{e \Downarrow \Lambda t : \kappa.e'_1 \quad e'_1[t \leftarrow \tau] \Downarrow v}{e \tau \Downarrow v}$$

**定理 21 (Adequacy of Small Step and Big Step).**  $e \Rightarrow^* v$  iff  $e \Downarrow v$ . □

**定理 22 (Type Soundness).**  $e : \tau$  の時,  $e \Rightarrow^* v$ ,  $e \Downarrow v$  となる  $v = \text{nf}(\Rightarrow, e)$  が存在し,

- $\tau = \tau_1 \rightarrow \tau_2$  の時,  $v \equiv_\alpha \lambda x' : \tau_1.e'$  となる  $\lambda x' : \tau_1.e'$  が存在する.
- $\tau = \forall t : \kappa.\tau_1$  の時,  $v \equiv_\alpha \Lambda t : \kappa.e'$  となる  $\Lambda t : \kappa.e'$  が存在する.

□

## 2.6.4 Equational Reasoning

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c} \frac{\Gamma, x : \tau_2 \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\lambda x : \tau_2.e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau} \text{Eq-}\beta\text{-Lam} \quad \frac{x \notin \text{fv}(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash (\lambda x : \tau_1.e x) \equiv e : \tau_1 \rightarrow \tau_2} \text{Eq-}\eta\text{-Lam} \\ \frac{\Gamma, t : \kappa \vdash e : \tau}{\Gamma \vdash (\Lambda t : \kappa.e) \tau_2 \equiv e[t \leftarrow \tau_2] : \tau[t \leftarrow \tau_2]} \text{Eq-}\beta\text{-UnivLam} \quad \frac{t \notin \text{tyfv}(e) \quad \Gamma \vdash e : \forall t : \kappa.\tau}{\Gamma \vdash (\Lambda t : \kappa.e t) \equiv e : \forall t : \kappa.\tau} \text{Eq-}\eta\text{-UnivLam} \\ \frac{e_1 \equiv_\alpha e_2 \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Refl} \quad \frac{\tau \equiv_\alpha \tau' \quad \Gamma \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-}\alpha\text{-Type} \\ \frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \text{Eq-Sym} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau \quad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau} \text{Eq-Trans} \\ \frac{\Gamma, x : \tau \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash \lambda x : \tau.e_1 \equiv \lambda x : \tau.e_2 : \tau \rightarrow \tau'} \text{Eq-Cong-Abs} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \rightarrow \tau \quad \Gamma \vdash e'_1 \equiv e'_2 : \tau'}{\Gamma \vdash e_1 e'_1 \equiv e_2 e'_2 : \tau} \text{Eq-Cong-App} \\ \frac{\Gamma, t : \kappa \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \Lambda t : \kappa.e_1 \equiv \Lambda t : \kappa.e_2 : (\forall t : \kappa.\tau)} \text{Eq-Cong-UnivAbs} \\ \frac{\Gamma \vdash e_1 \equiv e_2 : \forall t : \kappa.\tau \quad \Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash e_1 \tau_1 \equiv e_2 \tau_2 : \tau[t \leftarrow \tau_1]} \text{Eq-Cong-UnivApp} \end{array}$$

特に,  $\cdot \vdash e_1 \equiv e_2 : \tau$  の時,  $e_1 \equiv e_2 : \tau$  と表記.

**定理 23 (Respect Typing).**  $\Gamma \vdash e_1 \equiv e_2 : \tau$  ならば,  $\Gamma \vdash e_1 : \tau$  かつ  $\Gamma \vdash e_2 : \tau$ . □

**定理 24 (Respect Evaluation).**  $e_1 \equiv e_2 : \tau$  の時,  $e'_1 \Rightarrow^* e_1$ ,  $e_2 \Rightarrow^* e'_2$  ならば  $e'_1 \equiv e'_2 : \tau$ . □

**系 25.**  $e_1 \equiv e_2 : \tau$  の時,  $e_1 \Rightarrow^* e'_1$ ,  $e_2 \Rightarrow^* e'_2$  ならば  $e'_1 \equiv e'_2 : \tau$ . □

**証明.**  $e_1 \Rightarrow^* e_1$  より, 定理 14 から  $e_1 \equiv e'_2 : \tau$ . よって, T-Sym から  $e'_2 \equiv e_1 : \tau$  であり,  $e'_2 \Rightarrow^* e'_2$  より定理 14 から  $e'_2 \equiv e'_1 : \tau$ . 故に, T-Sym から  $e'_1 \equiv e'_2 : \tau$ . ■

## 2.6.5 Definability

Product

Product of  $\tau_1$  and  $\tau_2$ :

$$\begin{aligned} \tau_1 \times \tau_2 &\stackrel{\text{def}}{=} \forall t : \Omega. (\tau_1 \rightarrow \tau_2 \rightarrow t) \rightarrow t \\ \langle e_1, e_2 \rangle &\stackrel{\text{def}}{=} \Lambda t : \Omega. \lambda x : \tau_1 \rightarrow \tau_2 \rightarrow t. x e_1 e_2 \\ \pi_1 e &\stackrel{\text{def}}{=} e \tau_1 \lambda x_1. \lambda x_2. x_1 \\ \pi_2 e &\stackrel{\text{def}}{=} e \tau_2 \lambda x_1. \lambda x_2. x_2 \end{aligned}$$



Admissible kinding:

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma \vdash \tau_2 : \Omega}{\Gamma \vdash \tau_1 \times \tau_2 : \Omega} \text{ T-Product}$$

Admissible type equality:

$$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$$

$$\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \Omega}{\Gamma \vdash \tau_1 \times \tau'_1 \equiv \tau_2 \times \tau'_2 : \Omega} \text{ T-Eq-Product}$$

Admissible typing:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2} \text{ T-Product} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_1 e : \tau_1} \text{ T-Proj-1} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_2 e : \tau_2} \text{ T-Proj-2}$$

Admissible equality:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \pi_1 \langle e_1, e_2 \rangle \equiv e_1 : \tau_1} \text{ Eq-}\beta\text{-Product-1} \quad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \pi_2 \langle e_1, e_2 \rangle \equiv e_2 : \tau_2} \text{ Eq-}\beta\text{-Product-2}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \langle \pi_1 e, \pi_2 e \rangle \equiv e : \tau_1 \times \tau_2} \text{ Eq-}\eta\text{-Product}$$

### Existential Type

Existence of  $\exists t : \kappa. \tau$ :

$$\begin{aligned} \exists t : \kappa. \tau &\stackrel{\text{def}}{=} \forall t' : \Omega. (\forall t : \kappa. \tau \rightarrow t') \rightarrow t' \\ \text{pack}\langle \tau_t, e \rangle_{\exists t : \kappa. \tau} &\stackrel{\text{def}}{=} \Lambda t' : \Omega. \lambda x : (\forall t : \kappa. \tau \rightarrow t'). x \tau_t e \\ \text{unpack}\langle t : \kappa, x : \tau \rangle &= e_1. \tau_2. e_2 \stackrel{\text{def}}{=} e_1 \tau_2 (\Lambda t : \kappa. \lambda x : \tau. e_2) \end{aligned}$$

Admissible kinding:

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\frac{\Gamma, t : \kappa \vdash \tau : \Omega}{\Gamma \vdash \exists t : \kappa. \tau : \Omega} \text{ T-Exist}$$

Admissible type equality:

$$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$$

$$\frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash \exists t : \kappa. \tau_1 \equiv \exists t : \kappa. \tau_2 : \Omega} \text{ T-Eq-Cong-Exist}$$

Admissible typing rule:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\begin{array}{c}
\frac{\Gamma, t : \kappa \vdash \tau : \Omega \quad \Gamma \vdash \tau_t : \kappa \quad \Gamma \vdash e : \tau[t \leftarrow \tau_t]}{\Gamma \vdash \text{pack}\langle \tau_t, e \rangle_{\exists t : \kappa. \tau} : \exists t : \kappa. \tau} \text{ T-Pack} \\
\frac{\Gamma \vdash e_1 : \exists t : \kappa. \tau \quad \Gamma, t : \kappa, x : \tau \vdash e_2 : \tau_2 \quad t \notin \text{tyfv}(\tau_2)}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau \rangle = e_1. \tau_2. e_2 : \tau_2} \text{ T-Unpack}
\end{array}$$

Admissible equality:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau_t : \kappa \quad \Gamma \vdash e_1 : \tau_1[t \leftarrow \tau_t] \quad \Gamma, t : \kappa, x : \tau_1 \vdash e_2 : \tau_2 \quad t \notin \text{tyfv}(\tau_2)}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau_1 \rangle = \text{pack}\langle \tau_t, e_1 \rangle_{\exists t : \kappa. \tau_1}. \tau_2. e_2 \equiv e_2[t \leftarrow \tau_t][x \leftarrow e_1] : \tau_2} \text{ Eq-}\beta\text{-Exist} \\
\frac{\Gamma \vdash e : (\exists t : \kappa. \tau) \quad \tau' \equiv \exists t : \kappa. \tau}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau \rangle = e. \tau'. \text{pack}\langle t, x \rangle_{\exists t : \kappa. \tau} \equiv e : (\exists t : \kappa. \tau)} \text{ Eq-}\eta\text{-Exist}
\end{array}$$

## 2.7 $\lambda \mu$ -Calculus

Alias:  $\lambda \mu$  [Sel01][Roc05]

### 2.7.1 Syntax

$\tau ::= t$	(type variable)
$\top$	(top type)
$\tau \times \tau$	(product type)
$\tau \rightarrow \tau$	(function type)
$\perp$	(bottom type)
$e ::= x$	(variable)
$\langle \rangle$	(top value)
$\langle e, e \rangle$	(product)
$\pi_1 e$	(left projection)
$\pi_2 e$	(right projection)
$\lambda x : \tau. e$	(abstraction)
$e e$	(application)
$[\alpha]e$	(naming)
$\mu \alpha : \tau. e$	(un-naming)
$\Gamma ::= \cdot$	
$\Gamma, x : \tau$	
$\Delta ::= \cdot$	
$\alpha : \tau, \Delta$	

Environment Reference:

$$\boxed{\Gamma(x) = \tau}$$

$$\frac{x = x'}{(\Gamma, x' : \tau)(x) = \tau} \quad \frac{x \neq x' \quad \Gamma(x) = \tau}{(\Gamma, x' : \tau')(x) = \tau}$$

$$\boxed{\Delta(\alpha) = \tau}$$

$$\frac{\alpha = \alpha'}{(\alpha' : \tau, \Delta)(\alpha) = \tau} \quad \frac{\alpha \neq \alpha' \quad \Delta(\alpha) = \tau}{(\alpha' : \tau', \Delta)(\alpha) = \tau}$$

### 2.7.2 Typing Semantics

$$\boxed{\Gamma \vdash e : \tau \mid \Delta}$$

$$\begin{array}{c} \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau \mid \Delta} \text{ T-Var} \\ \frac{}{\Gamma \vdash \langle \rangle : \top \mid \Delta} \text{ T-Top} \\ \frac{\Gamma \vdash e_1 : \tau_1 \mid \Delta \quad \Gamma \vdash e_2 : \tau_2 \mid \Delta}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2 \mid \Delta} \text{ T-Product} \\ \frac{\Gamma \vdash e : \tau_1 \times \tau_2 \mid \Delta}{\Gamma \vdash \pi_1 e : \tau_1 \mid \Delta} \text{ T-Proj-1} \end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash e : \tau_1 \times \tau_2 \mid \Delta}{\Gamma \vdash \pi_2 e : \tau_2 \mid \Delta} \text{ T-Proj-2} \\
\frac{\Gamma, x : \tau_1 \vdash e : \tau_2 \mid \Delta}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2 \mid \Delta} \text{ T-Abs} \\
\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \mid \Delta \quad \Gamma \vdash e_2 : \tau_2 \mid \Delta}{\Gamma \vdash e_1 e_2 : \tau \mid \Delta} \text{ T-App} \\
\frac{\Delta(\alpha) = \tau \quad \Gamma \vdash e : \tau \mid \Delta}{\Gamma \vdash [\alpha]e : \perp \mid \Delta} \text{ T-Name} \\
\frac{\Gamma \vdash e : \perp \mid \alpha : \tau, \Delta}{\Gamma \vdash (\mu\alpha : \tau. e) : \tau \mid \Delta} \text{ T-Unname}
\end{array}$$

### 2.7.3 Equivalence

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau \mid \Delta}$$

$$\begin{array}{c}
\frac{\Gamma, x : \tau_2 \vdash e_1 : \tau \mid \Delta \quad \Gamma \vdash e_2 : \tau_2 \mid \Delta}{\Gamma \vdash (\lambda x : \tau_2. e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau \mid \Delta} \text{ Eq-}\beta\text{-Lam} \\
\frac{x \notin \text{fv}(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2 \mid \Delta}{\Gamma \vdash (\lambda x : \tau_1. e x) \equiv e : \tau_1 \rightarrow \tau_2 \mid \Delta} \text{ Eq-}\eta\text{-Lam} \\
\frac{\Gamma \vdash e : \top \mid \Delta}{\Gamma \vdash \langle \rangle \equiv e : \top \mid \Delta} \text{ Eq-}\eta\text{-Top} \\
\frac{\Gamma \vdash e_1 : \tau_1 \mid \Delta \quad \Gamma \vdash e_2 : \tau_2 \mid \Delta}{\Gamma \vdash \pi_1 \langle e_1, e_2 \rangle \equiv e_1 : \tau_1 \mid \Delta} \text{ Eq-}\beta\text{-Product-1} \\
\frac{\Gamma \vdash e_1 : \tau_1 \mid \Delta \quad \Gamma \vdash e_2 : \tau_2 \mid \Delta}{\Gamma \vdash \pi_2 \langle e_1, e_2 \rangle \equiv e_2 : \tau_2 \mid \Delta} \text{ Eq-}\beta\text{-Product-2} \\
\frac{\Gamma \vdash e : \tau_1 \times \tau_2 \mid \Delta}{\Gamma \vdash \langle \pi_1 e, \pi_2 e \rangle \equiv e : \tau_1 \times \tau_2 \mid \Delta} \text{ Eq-}\eta\text{-Product} \\
\frac{\alpha_1 \notin \text{fv}(e) \quad \Gamma \vdash e : \perp \mid \alpha : \tau_1 \times \tau_2, \Delta}{\Gamma \vdash \pi_1(\mu\alpha : \tau_1 \times \tau_2. e) \equiv \mu\alpha_1 : \tau_1. e[[\alpha](-) \leftarrow [\alpha_1](\pi_1(-))]] : \tau_1 \mid \Delta} \text{ Eq-}\zeta\text{-Product-1} \\
\frac{\alpha_2 \notin \text{fv}(e) \quad \Gamma \vdash e : \perp \mid \alpha : \tau_1 \times \tau_2, \Delta}{\Gamma \vdash \pi_2(\mu\alpha : \tau_1 \times \tau_2. e) \equiv \mu\alpha_2 : \tau_2. e[[\alpha](-) \leftarrow [\alpha_2](\pi_2(-))]] : \tau_2 \mid \Delta} \text{ Eq-}\zeta\text{-Product-2} \\
\frac{\Gamma \vdash e : \perp \mid \alpha_2 : \tau_\alpha, \Delta}{\Gamma \vdash [\alpha_1](\mu\alpha_2 : \tau_\alpha. e) \equiv e[\alpha_2 \leftarrow \alpha_1] : \perp \mid \Delta} \text{ Eq-}\beta\text{-Mu} \\
\frac{\Gamma \vdash e : \tau \mid \Delta}{\Gamma \vdash (\mu\alpha : \tau. [\alpha]e) \equiv e : \tau \mid \Delta} \text{ Eq-}\eta\text{-Mu} \\
\frac{\alpha_2 \notin \text{fv}(e_1) \cup \text{fv}(e_2) \quad \Gamma \vdash e_1 : \perp \mid \alpha : \tau_1 \rightarrow \tau_2, \Delta \quad \Gamma \vdash e_2 : \tau_2 \mid \Delta}{\Gamma \vdash (\mu\alpha : \tau_1 \rightarrow \tau_2. e_1) e_2 \equiv \mu\alpha_2 : \tau_2. e_1[[\alpha](-) \leftarrow [\alpha_2]((-) e_2)]] : \tau_2 \mid \Delta} \text{ Eq-}\zeta\text{-Mu}
\end{array}$$

### 2.7.4 Elaboration (Call-By-Value)

$$\boxed{\Gamma \vdash e : \tau \rightsquigarrow e'}$$

$$\begin{array}{c}
\frac{\Gamma(x_{x_0}) = V_\tau}{\Gamma \vdash x_0 : \tau \rightsquigarrow \lambda x_k : K_\tau. x_k x_{x_0}} \\
\frac{}{\Gamma \vdash \langle \rangle : \top \rightsquigarrow \lambda x_k : K_\top. x_k \langle \rangle} \\
\frac{\Gamma \vdash e_1 : \tau_1 \rightsquigarrow e'_1 \quad \Gamma \vdash e_2 : \tau_2 \rightsquigarrow e'_2}{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2 \rightsquigarrow \lambda x_k : K_{\tau_1 \times \tau_2}. e'_1 (\lambda x_1 : V_{\tau_1}. e'_2 (\lambda x_2 : V_{\tau_2}. x_k \langle x_1, x_2 \rangle))} \\
\frac{\Gamma \vdash e : \tau_1 \times \tau_2 \rightsquigarrow e'}{\Gamma \vdash \pi_1 e : \tau_1 \rightsquigarrow \lambda x_k : K_{\tau_1}. e' (\lambda x : V_{\tau_1} \times V_{\tau_2}. x_k (\pi_1 x))}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash e : \tau_1 \times \tau_2 \rightsquigarrow e'}{\Gamma \vdash \pi_2 e : \tau_2 \rightsquigarrow \lambda x_k : K_{\tau_2}.e' (\lambda x : V_{\tau_1} \times V_{\tau_2}.x_k (\pi_2 x))} \\
\frac{\Gamma, x_{x_0} : V_{\tau_1} \vdash e : \tau_2 \rightsquigarrow e'}{\Gamma \vdash (\lambda x_0 : \tau_1.e) : \tau_1 \rightarrow \tau_2 \rightsquigarrow \lambda x_k : K_{\tau_1 \rightarrow \tau_2}.x_k (\lambda x : V_{\tau_1} \times K_{\tau_2}.(\lambda x_{x_0} : V_{\tau_1}.e') (\pi_1 x) (\pi_2 x))} \\
\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \rightsquigarrow e'_1 \quad \Gamma \vdash e_2 : \tau_2 \rightsquigarrow e'_2}{\Gamma \vdash e_1 e_2 : \tau \rightsquigarrow \lambda x_k : K_{\tau}.e'_1 (\lambda x_1 : V_{\tau_2 \rightarrow \tau}.e'_2 (\lambda x_2 : V_{\tau_2}.x_1 \langle x_2, x_k \rangle))} \\
\frac{\Gamma, x_\alpha : K_\tau \vdash e : \perp \rightsquigarrow e'}{\Gamma \vdash (\mu \alpha : \tau.e) : \tau \rightsquigarrow \lambda x_\alpha : K_\tau.e' (\lambda x : \perp.\text{case } x \{ \})} \\
\frac{\Gamma(x_\alpha) = K_\tau \quad \Gamma \vdash e : \tau \rightsquigarrow e'}{\Gamma \vdash [\alpha]e : \tau \rightsquigarrow \lambda x_k : K_\perp.e' x_\alpha}
\end{array}$$

$$\boxed{V_\tau = \tau'}$$

$$\begin{array}{c}
\overline{V_\top = \top} \\
\frac{V_{\tau_1} = \tau'_1 \quad V_{\tau_2} = \tau'_2}{V_{\tau_1 \times \tau_2} = V_{\tau'_1} \times V_{\tau'_2}} \\
\frac{V_{\tau_1} = \tau'_1 \quad K_{\tau_2} = \tau'_2}{V_{\tau_1 \rightarrow \tau_2} = \tau'_1 \times \tau'_2 \rightarrow R} \\
\overline{V_\perp = \perp}
\end{array}$$

Abbreviation:

$$\begin{array}{l}
K_\tau \stackrel{\text{def}}{=} V_\tau \rightarrow R \\
C_\tau \stackrel{\text{def}}{=} K_\tau \rightarrow R
\end{array}$$

定理 26.  $\Gamma \vdash e : \tau \rightsquigarrow e'$  ならば,  $\Gamma \vdash e' : C_\tau$ . □

定理 27.  $\Gamma \vdash e : \tau \mid \Delta \iff V(\Gamma), K(\Delta) \vdash e : \tau \rightsquigarrow e'$ . ただし,

$$\begin{array}{l}
V(\Gamma) \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} V(\Gamma'), x_{x'} : V_{\tau'} & (\Gamma = \Gamma', x' : \tau') \\ \cdot & (\Gamma = \cdot) \end{array} \right. \\
K(\Delta) \stackrel{\text{def}}{=} \left\{ \begin{array}{ll} x_\alpha : K_\tau, K(\Delta') & (\Delta = \alpha : \tau, \Delta') \\ \cdot & (\Delta = \cdot) \end{array} \right. .
\end{array}$$

□

### 2.7.5 Elaboration (Call-By-Name)

$$\boxed{\Gamma \vdash e : \tau \rightsquigarrow e'}$$

$$\begin{array}{c}
\frac{\Gamma(x_{x_0}) = C_\tau}{\Gamma \vdash x_0 : \tau \rightsquigarrow \lambda x_k : K_\tau.x_{x_0} x_k} \\
\frac{\Gamma \vdash \langle \rangle : \top \rightsquigarrow \lambda x_k : \perp.\text{case } x_k \{ \}}{\Gamma \vdash e_1 : \tau_1 \rightsquigarrow e'_1 \quad \Gamma \vdash e_2 : \tau_2 \rightsquigarrow e'_2} \\
\frac{\Gamma \vdash \langle e_1, e_2 \rangle : \tau_1 \times \tau_2 \rightsquigarrow \lambda x_k : K_{\tau_1} + K_{\tau_2}.\text{case } x_k \{ x_{k_1}.e'_1 x_{k_1} \mid x_{k_2}.e'_2 x_{k_2} \}}{\Gamma \vdash e : \tau_1 \times \tau_2 \rightsquigarrow e'} \\
\frac{\Gamma \vdash \pi_1 e : \tau_1 \rightsquigarrow \lambda x_k : K_{\tau_1}.e' (i_1 x_k)}{\Gamma, x_{x_1} : C_{\tau_1} \vdash e : \tau_2 \rightsquigarrow e'} \\
\frac{\Gamma \vdash (\lambda x_1 : \tau_1.e) : \tau_1 \rightarrow \tau_2 \rightsquigarrow \lambda x_k : C_{\tau_1} \times K_{\tau_2}.e'[x_{x_1} \leftarrow \pi_1 x_k] (\pi_2 x_k)}{\Gamma \vdash (\lambda x_1 : \tau_1.e) : \tau_1 \rightarrow \tau_2 \rightsquigarrow \lambda x_k : C_{\tau_1} \times K_{\tau_2}.e'[x_{x_1} \leftarrow \pi_1 x_k] (\pi_2 x_k)}
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \rightsquigarrow e'_1 \quad \Gamma \vdash e_2 : \tau_2 \rightsquigarrow e'_2}{\Gamma \vdash e_1 e_2 : \tau \rightsquigarrow \lambda x_k : K_\tau. e'_1 \langle e'_2, x_k \rangle} \\
\frac{\Gamma(x_\alpha) = K_\tau \quad \Gamma \vdash e : \tau \rightsquigarrow e'}{\Gamma \vdash [\alpha]e : \perp \rightsquigarrow \lambda x_k : K_\perp. e' x_\alpha} \\
\frac{\Gamma, x_\alpha : K_\tau \vdash e : \perp \rightsquigarrow e'}{\Gamma \vdash (\mu\alpha : \tau. e) : \tau \rightsquigarrow \lambda x_\alpha : K_\tau. e' \langle \rangle}
\end{array}$$

$$\boxed{K_\tau = \tau'}$$

$$\begin{array}{c}
\overline{K_\top = \perp} \\
\frac{K_{\tau_1} = \tau'_1 \quad K_{\tau_2} = \tau'_2}{K_{\tau_1 \times \tau_2} = \tau'_1 + \tau'_2} \\
\frac{C_{\tau_1} = \tau'_1 \quad K_{\tau_2} = \tau'_2}{K_{\tau_1 \rightarrow \tau_2} = \tau'_1 \times \tau'_2} \\
\overline{K_\perp = \top}
\end{array}$$

Abbreviation:

$$C_\tau \stackrel{\text{def}}{=} K_\tau \rightarrow R$$

定理 28.  $\Gamma \vdash e : \tau \rightsquigarrow e'$  ならば,  $\Gamma \vdash e' : C_\tau$ .

□

定理 29.  $\Gamma \vdash e : \tau \mid \Delta \iff C(\Gamma), K(\Delta) \vdash e : \tau \rightsquigarrow e'$ . ただし,

$$\begin{aligned}
C(\Gamma) &\stackrel{\text{def}}{=} \begin{cases} C(\Gamma'), x_{x'} : C_{\tau'} & (\Gamma = \Gamma', x' : \tau') \\ \cdot & (\Gamma = \cdot) \end{cases} \\
K(\Delta) &\stackrel{\text{def}}{=} \begin{cases} x_\alpha : K_\tau, K(\Delta') & (\Delta = \alpha : \tau, \Delta') \\ \cdot & (\Delta = \cdot) \end{cases} .
\end{aligned}$$

□

## 2.8 WIP: Lambda Bar Mu Mu Tilde Calculus

$\tilde{\lambda} \mu$ -Calculus

## 2.9 WIP: $\pi$ -Calculus





## 第 3 章

# Modules and Phase Distinction

### 3.1 Light-Weight F-ing modules

[RRD14]

#### 3.1.1 Internal Language

Having same power as System F  $\omega$

Syntax:

$$\begin{aligned}
 \kappa &::= \Omega \mid \kappa \rightarrow \kappa \\
 \tau &::= t \mid \tau \rightarrow \tau \mid \{\overline{l} : \tau\} \mid \forall t : \kappa. \tau \mid \exists t : \kappa. \tau \mid \lambda t : \kappa. \tau \mid \tau \tau \\
 e &::= x \mid \lambda x : \tau. e \mid e e \mid \{\overline{l} = \overline{e}\} \mid e.l \mid \Lambda t : \kappa. e \mid e \tau \mid \text{pack}\langle \tau, e \rangle_\tau \mid \text{unpack}\langle t : \kappa, x : \tau \rangle = e \text{ in } e \\
 \Gamma &::= \cdot \mid \Gamma, t : \kappa \mid \Gamma, x : \tau
 \end{aligned}$$

Abbreviation:

$$\begin{aligned}
 \Sigma.\overline{l} &\stackrel{\text{def}}{=} \begin{cases} (\Sigma.l).\overline{l'} & (\overline{l} = l \overline{l'}) \\ \Sigma & (\overline{l} = \epsilon) \end{cases} \\
 \overline{\tau_1} \rightarrow \tau_2 &\stackrel{\text{def}}{=} \begin{cases} \tau_1 \rightarrow (\overline{\tau_1} \rightarrow \tau_2) & (\overline{\tau_1} = \tau_1 \overline{\tau_1'}) \\ \tau_2 & (\overline{\tau_1} = \epsilon) \end{cases} \\
 \lambda \overline{x} : \overline{\tau}. e &\stackrel{\text{def}}{=} \begin{cases} \lambda x : \tau. \lambda \overline{x'} : \overline{\tau'}. e & (\overline{x} : \overline{\tau} = x : \tau \overline{x'} : \overline{\tau'}) \\ e & (\overline{x} : \overline{\tau} = \epsilon) \end{cases} \\
 e_0 \overline{e_1} &\stackrel{\text{def}}{=} \begin{cases} e_0 e_1 \overline{e_1'} & (\overline{e_1} = e_1 \overline{e_1'}) \\ e_0 & (\overline{e_1} = \epsilon) \end{cases} \\
 \forall \overline{t} : \overline{\kappa}. \tau &\stackrel{\text{def}}{=} \begin{cases} \forall t : \kappa. \forall \overline{t'} : \overline{\kappa'}. \tau & (\overline{t} : \overline{\kappa} = t : \kappa \overline{t'} : \overline{\kappa'}) \\ \tau & (\overline{t} : \overline{\kappa} = \epsilon) \end{cases} \\
 \Lambda \overline{t} : \overline{\kappa}. e &\stackrel{\text{def}}{=} \begin{cases} \Lambda t : \kappa. \Lambda \overline{t'} : \overline{\kappa'}. e & (\overline{t} : \overline{\kappa} = t : \kappa \overline{t'} : \overline{\kappa'}) \\ e & (\overline{t} : \overline{\kappa} = \epsilon) \end{cases} \\
 e \overline{\tau} &\stackrel{\text{def}}{=} \begin{cases} e \tau \overline{\tau'} & (\overline{\tau} = \tau \overline{\tau'}) \\ e & (\overline{\tau} = \epsilon) \end{cases} \\
 \text{let } \overline{x} : \overline{\tau} = \overline{e_1} \overline{t} : \overline{\kappa} = \overline{\tau} \text{ in } e_2 &\stackrel{\text{def}}{=} (\lambda \overline{x} : \overline{\tau}. \Lambda \overline{t} : \overline{\kappa}. e_2) \overline{e_1} \overline{\tau} \\
 \exists \overline{t} : \overline{\kappa}. \tau &\stackrel{\text{def}}{=} \begin{cases} \exists t : \kappa. \exists \overline{t'} : \overline{\kappa'}. \tau & (\overline{t} : \overline{\kappa} = t : \kappa \overline{t'} : \overline{\kappa'}) \\ \tau & (\overline{t} : \overline{\kappa} = \epsilon) \end{cases} \\
 \text{pack}\langle \overline{\tau}, e \rangle_{\exists \overline{t} : \overline{\kappa}. \tau_0} &\stackrel{\text{def}}{=} \begin{cases} \text{pack}\langle \tau, \text{pack}\langle \overline{\tau'}, e \rangle_{\exists \overline{t'} : \overline{\kappa'}. \tau_0} \rangle_{\exists \overline{t} : \overline{\kappa}. \tau_0} & (\overline{\tau} = \tau \overline{\tau'}, \overline{t} : \overline{\kappa} = t : \kappa \overline{t'} : \overline{\kappa'}) \\ e & (\overline{\tau} = \epsilon, \overline{t} : \overline{\kappa} = \epsilon) \end{cases} \\
 (\text{unpack}\langle \overline{t} : \overline{\kappa}, x : \tau \rangle = e_1 \text{ in } e_2) &\stackrel{\text{def}}{=} \begin{cases} \text{unpack}\langle t : \kappa, x_1 : \exists \overline{t'} : \overline{\kappa'}. \tau \rangle = e_1 \text{ in} & (\overline{t} : \overline{\kappa} = t : \kappa \overline{t'} : \overline{\kappa'}) \\ \text{unpack}\langle \overline{t'} : \overline{\kappa'}, x_2 : \tau \rangle = x_1 \text{ in } e_2 & \\ \text{let } x : \tau = e_1 \text{ in } e_2 & (\overline{t} : \overline{\kappa} = \epsilon) \end{cases}
 \end{aligned}$$

Kinding:

$$\boxed{\Gamma \vdash \tau : \kappa}$$

$$\frac{\Gamma(t) = \kappa}{\Gamma \vdash t : \kappa} \quad \frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma \vdash \tau_2 : \Omega}{\Gamma \vdash \tau_1 \rightarrow \tau_2 : \Omega} \quad \frac{\bigwedge_l \Gamma \vdash \tau_l : \Omega}{\Gamma \vdash \{\overline{l} : \tau_l\} : \Omega}$$

$$\frac{\Gamma, t : \kappa \vdash \tau : \Omega}{\Gamma \vdash \forall t : \kappa. \tau : \Omega} \quad \frac{\Gamma, t : \kappa \vdash \tau : \Omega}{\Gamma \vdash \exists t : \kappa. \tau : \Omega} \quad \frac{\Gamma, t : \kappa_1 \vdash \tau : \kappa_2}{\Gamma \vdash \lambda t : \kappa_1. \tau : \kappa_1 \rightarrow \kappa_2} \quad \frac{\Gamma \vdash \tau_1 : \kappa_2 \rightarrow \kappa \quad \Gamma \vdash \tau_2 : \kappa_2}{\Gamma \vdash \tau_1 \tau_2 : \kappa}$$

Type equivalence:

$$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$$

$$\frac{\Gamma, t : \kappa_2 \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa_2}{\Gamma \vdash (\lambda t : \kappa_2. \tau_1) \tau_2 \equiv \tau_1[t \leftarrow \tau_2] : \kappa} \quad \frac{t \notin \text{tyfv}(\tau) \quad \Gamma \vdash \tau : \kappa_1 \rightarrow \kappa_2}{\Gamma \vdash (\lambda t : \kappa_1. \tau t) \equiv \tau : \kappa_1 \rightarrow \kappa_2}$$

$$\frac{\tau_1 \equiv_{\alpha} \tau_2 \quad \Gamma \vdash \tau_1 : \kappa \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa} \quad \frac{\Gamma \vdash \tau_2 \equiv \tau_1 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa} \quad \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa \quad \Gamma \vdash \tau_2 \equiv \tau_3 : \kappa}{\Gamma \vdash \tau_1 \equiv \tau_3 : \kappa}$$

$$\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \Omega}{\Gamma \vdash \tau_1 \rightarrow \tau'_1 \equiv \tau_2 \rightarrow \tau'_2 : \Omega} \quad \frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash \forall t : \kappa. \tau_1 \equiv \forall t : \kappa. \tau_2 : \Omega}$$

$$\frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau_2 : \kappa'}{\Gamma \vdash \lambda t : \kappa. \tau_1 \equiv \lambda t : \kappa. \tau_2 : \kappa \rightarrow \kappa'} \quad \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa' \rightarrow \kappa \quad \Gamma \vdash \tau'_1 \equiv \tau'_2 : \kappa'}{\Gamma \vdash \tau_1 \tau'_1 \equiv \tau_2 \tau'_2 : \kappa}$$

Typing:

$$\boxed{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash \tau : \Omega \quad \Gamma(x) = \tau}{\Gamma \vdash x : \tau} \quad \frac{\Gamma \vdash \tau \equiv \tau' : \Omega \quad \Gamma \vdash e : \tau'}{\Gamma \vdash e : \tau}$$

$$\frac{\Gamma \vdash \tau_1 : \Omega \quad \Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x : \tau_1. e : \tau_1 \rightarrow \tau_2} \quad \frac{\Gamma \vdash e_1 : \tau_2 \rightarrow \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash e_1 e_2 : \tau}$$

$$\frac{\bigwedge_l \Gamma \vdash e_l : \tau_l}{\Gamma \vdash \{l = e_l\} : \{\overline{l} = \tau_l\}} \quad \frac{\Gamma \vdash e : \{\overline{l'} = \tau_{l'}\}}{\Gamma \vdash e.l : \tau_l}$$

$$\frac{\Gamma, t : \kappa \vdash e : \tau}{\Gamma \vdash \Lambda t : \kappa. e : (\forall t : \kappa. \tau)} \quad \frac{\Gamma \vdash e : (\forall t : \kappa. \tau_1) \quad \Gamma \vdash \tau_2 : \kappa}{\Gamma \vdash e \tau_2 : \tau_1[t \leftarrow \tau_2]}$$

$$\frac{\Gamma, t : \kappa \vdash \tau : \Omega \quad \Gamma \vdash \tau_t : \kappa \quad \Gamma \vdash e : \tau[t \leftarrow \tau_t]}{\Gamma \vdash \text{pack}\langle \tau_t, e \rangle_{\exists t : \kappa. \tau} : (\exists t : \kappa. \tau)} \quad \frac{\Gamma \vdash e_1 : (\exists t : \kappa. \tau_1) \quad \Gamma, t : \kappa, x : \tau_1 \vdash e_2 : \tau}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau_1 \rangle = e_1 \text{ in } e_2 : \tau}$$

Reduction:

$$v ::= \lambda x : \tau. e \mid \overline{\{l = e\}} \mid \Lambda t : \kappa. e \mid \text{pack}\langle \tau_t, e \rangle_{\exists t : \kappa. \tau}$$

$$C ::= [] \mid C e \mid v C \mid \overline{\{l = v, l = C, l = e\}} \mid C.l \mid C \tau \mid \text{pack}\langle \tau, C \rangle_{\tau} \mid \text{unpack}\langle t : \kappa, x : \tau \rangle = C \text{ in } e$$

$$\boxed{e \Rightarrow e'}$$

$$\overline{(\lambda x : \tau. e)v} \Rightarrow e[x \leftarrow v] \quad \overline{\{l' = v_{l'}\}.l} \Rightarrow v_l \quad \overline{(\Lambda t : \kappa. e)\tau} \Rightarrow e[t \leftarrow \tau]$$

$$\frac{}{\text{unpack}\langle t : \kappa, x : \tau \rangle = \text{pack}\langle \tau_t, v \rangle_{\tau_3} \text{ in } e \Rightarrow e[t \leftarrow \tau_t][x \leftarrow v]} \quad \frac{e \Rightarrow e'}{C[e] \Rightarrow C[e']}$$

Equivalence:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\frac{\Gamma, x : \tau_2 \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\lambda x : \tau_2. e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau} \quad \frac{x \notin \text{fv}(e) \quad \Gamma \vdash e : \tau_1 \rightarrow \tau_2}{\Gamma \vdash (\lambda x : \tau_1. e x) \equiv e : \tau_1 \rightarrow \tau_2}$$

$$\frac{\bigwedge_{l'} \Gamma \vdash e_{l'} : \tau_{l'}}{\Gamma \vdash \{\overline{l'} = e_{l'}\}.l \equiv e_l : \tau_l} \quad \frac{\Gamma \vdash e : \{\overline{l} : \tau_l\}}{\Gamma \vdash \{\overline{l} = e.l\} \equiv e : \{\overline{l} : \tau_l\}}$$

$$\frac{\Gamma, t : \kappa \vdash e : \tau}{\Gamma \vdash (\Lambda t : \kappa. e) \tau_2 \equiv e[t \leftarrow \tau_2] : \tau[t \leftarrow \tau_2]} \quad \frac{t \notin \text{tyfv}(e) \quad \Gamma \vdash e : \forall t : \kappa. \tau}{\Gamma \vdash (\Lambda t : \kappa. e t) \equiv e : \forall t : \kappa. \tau}$$

$$\begin{array}{c}
\frac{\Gamma, t : \kappa \vdash \tau_1 \equiv \tau'_1 : \Omega \quad \Gamma \vdash \tau_t : \kappa \quad \Gamma \vdash e_1 : \tau_1[t \leftarrow \tau_t] \quad \Gamma, t : \kappa, x : \tau_1 \vdash e_2 : \tau}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau'_1 \rangle = \text{pack}\langle \tau_t, e_1 \rangle_{\exists t : \kappa. \tau_1} \text{ in } e_2 \equiv e_2[t \leftarrow \tau_t][x \leftarrow e_1] : \tau} \\
\frac{\Gamma \vdash e : \exists t : \kappa. \tau \quad \Gamma, t : \kappa \vdash \tau \equiv \tau' : \Omega}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau' \rangle = e \text{ in } \text{pack}\langle t, x \rangle_{\exists t : \kappa. \tau} \equiv e : (\exists t : \kappa. \tau)} \\
\frac{e_1 \equiv_{\alpha} e_2 \quad \Gamma \vdash e_1 : \tau \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \quad \frac{\Gamma \vdash \tau \equiv \tau' : \Omega \quad \Gamma \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash e_1 \equiv e_2 : \tau} \\
\frac{\Gamma \vdash e_2 \equiv e_1 : \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau \quad \Gamma \vdash e_2 \equiv e_3 : \tau}{\Gamma \vdash e_1 \equiv e_3 : \tau} \\
\frac{\Gamma, x : \tau \vdash e_1 \equiv e_2 : \tau'}{\Gamma \vdash \lambda x : \tau. e_1 \equiv \lambda x : \tau. e_2 : \tau \rightarrow \tau'} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau' \rightarrow \tau \quad \Gamma \vdash e'_1 \equiv e'_2 : \tau'}{\Gamma \vdash e_1 e'_1 \equiv e_2 e'_2 : \tau} \\
\frac{\bigwedge_l \Gamma \vdash e_{l,1} \equiv e_{l,2} : \tau_l}{\Gamma \vdash \{l = e_{l,1}\} \equiv \{l = e_{l,2}\} : \{l : \tau_l\}} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \{l : \tau_l, l' : \tau'_l\}}{\Gamma \vdash e_1.l \equiv e_2.l : \tau_l} \\
\frac{\Gamma, t : \kappa \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \Lambda t : \kappa. e_1 \equiv \Lambda t : \kappa. e_2 : (\forall t : \kappa. \tau)} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \forall t : \kappa. \tau \quad \Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash e_1 \tau_1 \equiv e_2 \tau_2 : \tau[t \leftarrow \tau_1]} \\
\frac{\Gamma \vdash \tau'_1 \equiv \tau'_2 : \kappa \quad \Gamma \vdash e_1 \equiv e_2 : \tau_1[t \leftarrow \tau'_1] \quad \Gamma, t : \kappa. \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash \text{pack}\langle \tau'_1, e_1 \rangle_{\exists t : \kappa. \tau_1} \equiv \text{pack}\langle \tau'_2, e_2 \rangle_{\exists t : \kappa. \tau_2} : (\exists t : \kappa. \tau_1)} \\
\frac{\Gamma, t : \kappa \vdash \tau'_1 \equiv \tau'_2 : \Omega \quad \Gamma \vdash e'_1 \equiv e'_2 : (\exists t : \kappa. \tau'_1) \quad \Gamma, t : \kappa, x : \tau'_1 \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash \text{unpack}\langle t : \kappa, x : \tau'_1 \rangle = e'_1 \text{ in } e_1 \equiv \text{unpack}\langle t : \kappa, x : \tau'_2 \rangle = e'_2 \text{ in } e_2 : \tau}
\end{array}$$

### 3.1.2 Syntax

$X ::= \dots$	(identifier)
$K ::= \dots$	(kind)
$T ::= \dots \mid P$	(type)
$E ::= \dots \mid P$	(expression)
$P ::= M$	(path)
$M ::= X$	(identifier)
$\mid \{B\}$	(bindings)
$\mid M.X$	(projection)
$B ::= \text{val } X = E$	(value binding)
$\mid \text{type } X = T$	(type binding)
$\mid \text{module } X = M$	(module binding)
$\mid \text{signature } X = S$	(signature binding)
$\mid \text{include } M$	(module including)
$\mid \epsilon$	(empty binding)
$\mid B; B$	(binding concatenation)
$S ::= P$	(signature path)
$\mid \{D\}$	(declarations)
$D ::= \text{val } X : T$	(value declaration)
$\mid \text{type } X = T$	(type binding)
$\mid \text{module } X : S$	(module declaration)
$\mid \text{signature } X = S$	(signature binding)
$\mid \text{include } S$	(signature including)
$\mid \epsilon$	(empty declaration)
$\mid D; D$	(declaration concatenation)

## 3.1.3 Signature

$\Sigma ::= [\tau]$	(anonymous value declaration)
$[= \tau : \kappa]$	(anonymous type declaration)
$[= \Sigma]$	(anonymous signature declaration)
$\overline{\{l_X : \Sigma\}}$	(structural signature)

Atomic Signature:

$$\begin{aligned}
[\tau] &\stackrel{\text{def}}{=} \{\text{val} : \tau\} \\
[e] &\stackrel{\text{def}}{=} \{\text{val} = e\} \\
[= \tau : \kappa] &\stackrel{\text{def}}{=} \{\text{type} : \forall t : (\kappa \rightarrow \Omega). t \tau \rightarrow t \tau\} \\
[\tau : \kappa] &\stackrel{\text{def}}{=} \{\text{type} = \Lambda t : (\kappa \rightarrow \Omega). \lambda x : (t \tau). x\} \\
[= \Sigma] &\stackrel{\text{def}}{=} \{\text{sig} : \Sigma \rightarrow \Sigma\} \\
[\Sigma] &\stackrel{\text{def}}{=} \{\text{sig} = \lambda x : \Sigma. x\}
\end{aligned}$$

$\boxed{\text{NotAtomic}(\Sigma)}$

$$\overline{\overline{\text{NotAtomic}(\{l_X : \Sigma\})}}$$

Admissible kinding:

$\boxed{\Gamma \vdash \tau : \kappa}$

$$\begin{aligned}
&\frac{\Gamma \vdash \tau : \Omega}{\Gamma \vdash [\tau] : \Omega} \text{ K-A-Val} \\
&\frac{\Gamma \vdash \tau : \kappa}{\Gamma \vdash [= \tau : \kappa] : \Omega} \text{ K-A-Typ} \\
&\frac{\Gamma \vdash \Sigma : \Omega}{\Gamma \vdash [= \Sigma] : \Omega} \text{ K-A-Sig}
\end{aligned}$$

Admissible type equivalence:

$\boxed{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}$

$$\begin{aligned}
&\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash [\tau_1] \equiv [\tau_2] : \Omega} \text{ T-Eq-Cong-A-Val} \\
&\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash [= \tau_1 : \kappa] \equiv [= \tau_2 : \kappa] : \Omega} \text{ T-Eq-Cong-A-Typ} \\
&\frac{\Gamma \vdash \Sigma_1 \equiv \Sigma_2 : \Omega}{\Gamma \vdash [= \Sigma_1] \equiv [= \Sigma_2] : \Omega} \text{ T-Eq-Cong-A-Sig}
\end{aligned}$$

Admissible typing:

$\boxed{\Gamma \vdash e : \tau}$

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash [e] : [\tau]} \text{ T-A-Val}$$

$$\frac{\Gamma \vdash \tau : \kappa}{\Gamma \vdash [\tau : \kappa] : [= \tau : \kappa]} \text{ T-A-Typ}$$

$$\frac{\Gamma \vdash \Sigma : \Omega}{\Gamma \vdash [\Sigma] : [= \Sigma]} \text{ T-A-Sig}$$

Admissible equivalence:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash [e].\text{val} \equiv e : \tau} \text{ Eq-}\beta\text{-A-Val} \quad \frac{\Gamma \vdash e : [\tau]}{\Gamma \vdash [e.\text{val}] \equiv e : [\tau]} \text{ Eq-}\eta\text{-A-Val} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash [e_1] \equiv [e_2] : [\tau]} \text{ Eq-Cong-A-Val}$$

$$\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash [\tau_1 : \kappa] \equiv [\tau_2 : \kappa] : [= \tau_1 : \kappa]} \text{ Eq-Cong-A-Typ}$$

$$\frac{\Gamma \vdash \Sigma_1 \equiv \Sigma_2 : \Omega}{\Gamma \vdash [\Sigma_1] \equiv [\Sigma_2] : [= \Sigma_1]} \text{ Eq-Cong-A-Sig}$$

### 3.1.4 Elaboration

Signature:

$$\boxed{\Gamma \vdash S \rightsquigarrow \Sigma}$$

$$\frac{\Gamma \vdash P : [= \Sigma] \rightsquigarrow e}{\Gamma \vdash P \rightsquigarrow \Sigma} \text{ S-Path}$$

$$\frac{\Gamma \vdash D \rightsquigarrow \Sigma}{\Gamma \vdash \{D\} \rightsquigarrow \Sigma} \text{ S-Struct}$$

Declarations:

$$\boxed{\Gamma \vdash D \rightsquigarrow \Xi}$$

$$\frac{\Gamma \vdash T : \Omega \rightsquigarrow \tau}{\Gamma \vdash \text{val } X : T \rightsquigarrow \{l_X : [\tau]\}} \text{ D-Val}$$

$$\frac{\Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash \text{type } X = T \rightsquigarrow \{l_X : [= \tau : \kappa]\}} \text{ D-Typ-Eq}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Sigma}{\Gamma \vdash \text{module } X : S \rightsquigarrow \{l_X : \Sigma\}} \text{ D-Mod}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Sigma}{\Gamma \vdash \text{signature } X = S \rightsquigarrow \{l_X : [= \Sigma]\}} \text{ D-Sig-Eq}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \{l_X : \Sigma\}}{\Gamma \vdash \text{include } S \rightsquigarrow \{l_X : \Sigma\}} \text{ D-Incl}$$

$$\frac{}{\Gamma \vdash \epsilon \rightsquigarrow \{\}} \text{ D-Emt}$$

$$\frac{\overline{\{l_{X_1}\}} \cap \overline{\{l_{X_2}\}} = \emptyset \quad \Gamma \vdash D_1 \rightsquigarrow \{l_{X_1} : \Sigma_1\} \quad \Gamma, x_{X_1} : \Sigma_1 \vdash D_2 \rightsquigarrow \{l_{X_2} : \Sigma_2\}}{\Gamma \vdash D_1; D_2 \rightsquigarrow \{l_{X_1} : \Sigma_1, l_{X_2} : \Sigma_2\}} \text{ D-Seq}$$

Module:

$$\boxed{\Gamma \vdash M : \Sigma \rightsquigarrow e}$$

$$\frac{\Gamma(x_X) = \Sigma}{\Gamma \vdash X : \Sigma \rightsquigarrow x_X} \text{ M-Var}$$

$$\frac{\Gamma \vdash B : \Sigma \rightsquigarrow e}{\Gamma \vdash \{B\} : \Sigma \rightsquigarrow e} \text{ M-Struct}$$

$$\frac{\Gamma \vdash M : \{l_X : \Sigma, \overline{l_{X'}} : \Sigma'\} \rightsquigarrow e}{\Gamma \vdash M.X : \Sigma \rightsquigarrow e.l_X} \text{ M-Dot}$$

Bindings:

$$\boxed{\Gamma \vdash B : \Sigma \rightsquigarrow e}$$

$$\begin{array}{c}
\frac{\Gamma \vdash E : \tau \rightsquigarrow e}{\Gamma \vdash \text{val } X = E : \{l_X : [\tau]\} \rightsquigarrow \{l_X = e\}} \text{ B-Val} \\
\frac{\Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash \text{type } X = T : \{l_X : [= \tau : \kappa]\} \rightsquigarrow \{l_X = [\tau : \kappa]\}} \text{ B-Typ} \\
\frac{\Gamma \vdash M : \Sigma \rightsquigarrow e \quad \text{NotAtomic}(\Sigma)}{\Gamma \vdash \text{module } X = M : \{l_X : \Sigma\} \rightsquigarrow \{l_X = e\}} \text{ B-Mod} \\
\frac{\Gamma \vdash S \rightsquigarrow \Sigma}{\Gamma \vdash \text{signature } X = S : \{l_X : [= \Sigma]\} \rightsquigarrow \{l_X = [\Sigma]\}} \text{ B-Sig} \\
\frac{\Gamma \vdash M : \{\overline{l_X : \Sigma}\} \rightsquigarrow e}{\Gamma \vdash \text{include } M : \{\overline{l_X : \Sigma}\} \rightsquigarrow e} \text{ B-Incl} \\
\frac{}{\Gamma \vdash \epsilon : \{\} \rightsquigarrow \{\}} \text{ B-Emt} \\
\frac{\overline{l'_{X_1}} = \overline{l_{X_1}} \setminus \overline{l_{X_2}} \quad \overline{l'_{X_1} : \Sigma'_1} \subseteq \overline{l_{X_1} : \Sigma_1} \quad \Gamma \vdash B_1 : \{\overline{l_{X_1} : \Sigma_1}\} \rightsquigarrow e_1}{\begin{array}{c} \overline{\Sigma = \{l'_{X_1} : \Sigma'_1, \overline{l_{X_2} : \Sigma_2}\}} \quad \Gamma, \overline{x_{X_1} : \Sigma_1} \vdash B_2 : \{\overline{l_{X_2} : \Sigma_2}\} \rightsquigarrow e_2 \\ \hline \text{let } x_1 = e_1 \text{ in} \\ \Gamma \vdash B_1; B_2 : \Sigma \rightsquigarrow \text{let } x_2 = (\text{let } x_{X_1} : \Sigma_1 = x_1.l_{X_1} \text{ in } e_2) \text{ in} \\ \{l'_{X_1} = x_1.l'_{X_1}, \overline{l_{X_2} = x_2.l_{X_2}}\} \end{array}} \text{ B-Seq}
\end{array}$$

Path:

$$\boxed{\Gamma \vdash P : \Sigma \rightsquigarrow e}$$

Use M-Dot.

$$\boxed{\Gamma \vdash T : \kappa \rightsquigarrow \tau}$$

$$\frac{\Gamma \vdash P : [= \tau : \kappa] \rightsquigarrow e}{\Gamma \vdash P : \kappa \rightsquigarrow \tau} \text{ T-Elab-Path}$$

$$\boxed{\Gamma \vdash E : \tau \rightsquigarrow e}$$

$$\frac{\Gamma \vdash P : [\tau] \rightsquigarrow e}{\Gamma \vdash P : \tau \rightsquigarrow e.\text{val}} \text{ E-Path}$$



## 3.2 F-ing modules

[RRD14]

### 3.2.1 Internal Language

See 3.1.1.

### 3.2.2 Syntax

$X ::= \dots$	(identifier)
$K ::= \dots$	(kind)
$T ::= \dots \mid P$	(type)
$E ::= \dots \mid P$	(expression)
$P ::= M$	(path)
$M ::= X$	(identifier)
$\mid \{B\}$	(bindings)
$\mid M.X$	(projection)
$\mid \text{fun } X : S \Rightarrow M$	(functor)
$\mid X X$	(functor application)
$\mid X :> S$	(sealing)
$B ::= \text{val } X = E$	(value binding)
$\mid \text{type } X = T$	(type binding)
$\mid \text{module } X = M$	(module binding)
$\mid \text{signature } X = S$	(signature binding)
$\mid \text{include } M$	(module including)
$\mid \epsilon$	(empty binding)
$\mid B; B$	(binding concatenation)
$S ::= P$	(signature path)
$\mid \{D\}$	(declarations)
$\mid (X : S) \rightarrow S$	((generative) functor signature)
$\mid S \text{ where type } \overline{X} = T$	(bounded signature)
$D ::= \text{val } X : T$	(value declaration)
$\mid \text{type } X = T$	(type binding)
$\mid \text{type } X : K$	(type declaration)
$\mid \text{module } X : S$	(module declaration)
$\mid \text{signature } X = S$	(signature binding)
$\mid \text{include } S$	(signature including)
$\mid \epsilon$	(empty declaration)
$\mid D; D$	(declaration concatenation)

### 3.2.3 Signature

$\Xi ::= \exists t : \kappa. \Sigma$	(abstract signature)
$\Sigma ::= [\tau]$	(atomic value declaration)

$[= \tau : \kappa]$	(atomic type declaration)
$[= \Xi]$	(atomic signature declaration)
$\{\overline{l_X : \Sigma}\}$	(structure signature)
$\forall t : \kappa. \Sigma \rightarrow \Xi$	(functor signature)

Atomic Signature:

$$\begin{aligned}
[\tau] &\stackrel{\text{def}}{=} \{\text{val} : \tau\} \\
[e] &\stackrel{\text{def}}{=} \{\text{val} = e\} \\
[= \tau : \kappa] &\stackrel{\text{def}}{=} \{\text{type} : \forall t : (\kappa \rightarrow \Omega). t \tau \rightarrow t \tau\} \\
[\tau : \kappa] &\stackrel{\text{def}}{=} \{\text{type} = \Lambda t : (\kappa \rightarrow \Omega). \lambda x : (t \tau). x\} \\
[= \Xi] &\stackrel{\text{def}}{=} \{\text{sig} : \Xi \rightarrow \Xi\} \\
[\Xi] &\stackrel{\text{def}}{=} \{\text{sig} = \lambda x : \Xi. x\}
\end{aligned}$$

$\text{NotAtomic}(\Sigma)$

$$\frac{}{\text{NotAtomic}(\{\overline{l_X : \Sigma}\})} \quad \frac{}{\text{NotAtomic}(\forall t : \kappa. \Sigma \rightarrow \Xi)}$$

Admissible kinding:

$\Gamma \vdash \tau : \kappa$

$$\begin{aligned}
&\frac{\Gamma \vdash \tau : \Omega}{\Gamma \vdash [\tau] : \Omega} \text{ K-A-Val} \\
&\frac{\Gamma \vdash \tau : \kappa}{\Gamma \vdash [= \tau : \kappa] : \Omega} \text{ K-A-Typ} \\
&\frac{\Gamma \vdash \Xi : \Omega}{\Gamma \vdash [= \Xi] : \Omega} \text{ K-A-Sig}
\end{aligned}$$

Admissible type equivalence:

$\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa$

$$\begin{aligned}
&\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \Omega}{\Gamma \vdash [\tau_1] \equiv [\tau_2] : \Omega} \text{ T-Eq-Cong-A-Val} \\
&\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash [= \tau_1 : \kappa] \equiv [= \tau_2 : \kappa] : \Omega} \text{ T-Eq-Cong-A-Typ} \\
&\frac{\Gamma \vdash \Xi_1 \equiv \Xi_2 : \Omega}{\Gamma \vdash [= \Xi_1] \equiv [= \Xi_2] : \Omega} \text{ T-Eq-Cong-A-Sig}
\end{aligned}$$

Admissible typing:

$\Gamma \vdash e : \tau$

$$\begin{aligned}
&\frac{\Gamma \vdash e : \tau}{\Gamma \vdash [e] : [\tau]} \text{ T-A-Val} \\
&\frac{\Gamma \vdash \tau : \kappa}{\Gamma \vdash [\tau : \kappa] : [= \tau : \kappa]} \text{ T-A-Typ} \\
&\frac{\Gamma \vdash \Xi : \Omega}{\Gamma \vdash [\Xi] : [= \Xi]} \text{ T-A-Sig}
\end{aligned}$$

Admissible equivalence:

$$\boxed{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

$$\begin{array}{c} \frac{\Gamma \vdash e : \tau}{\Gamma \vdash [e].\text{val} \equiv e : \tau} \text{Eq-}\beta\text{-A-Val} \quad \frac{\Gamma \vdash e : [\tau]}{\Gamma \vdash [e.\text{val}] \equiv e : [\tau]} \text{Eq-}\eta\text{-A-Val} \quad \frac{\Gamma \vdash e_1 \equiv e_2 : \tau}{\Gamma \vdash [e_1] \equiv [e_2] : [\tau]} \text{Eq-Cong-A-Val} \\[10pt] \frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash [\tau_1 : \kappa] \equiv [\tau_2 : \kappa] : [= \tau_1 : \kappa]} \text{Eq-Cong-A-Typ} \\[10pt] \frac{\Gamma \vdash \Xi_1 \equiv \Xi_2 : \Omega}{\Gamma \vdash [\Xi_1] \equiv [\Xi_2] : [= \Xi_1]} \text{Eq-Cong-A-Sig} \end{array}$$

### 3.2.4 (Generative) Elaboration

Signature:

$$\boxed{\Gamma \vdash S \rightsquigarrow \Xi}$$

$$\begin{array}{c} \frac{\Gamma \vdash P : [= \Xi] \rightsquigarrow e}{\Gamma \vdash P \rightsquigarrow \Xi} \text{S-Path} \\[10pt] \frac{\Gamma \vdash D \rightsquigarrow \Xi}{\Gamma \vdash \{D\} \rightsquigarrow \Xi} \text{S-Struct} \\[10pt] \frac{\Gamma \vdash S_1 \rightsquigarrow \overline{\exists t : \kappa. \Sigma} \quad \Gamma, t : \kappa, x_X : \Sigma \vdash S_2 \rightsquigarrow \Xi}{\Gamma \vdash (X : S_1) \rightarrow S_2 \rightsquigarrow \overline{\forall t : \kappa. \Sigma \rightarrow \Xi}} \text{S-Funct} \\[10pt] \frac{\Gamma \vdash S \rightsquigarrow \overline{\exists t_1 : \kappa_1} t : \kappa \overline{t_2 : \kappa_2. \Sigma} \quad \Sigma.l_X = [= t : \kappa] \quad \Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash S \text{ where type } \overline{X} = T \rightsquigarrow \overline{\exists t_1 : \kappa_1} t_2 : \kappa_2. \Sigma[t \leftarrow \tau]} \text{S-Where-Typ} \end{array}$$

Declarations:

$$\boxed{\Gamma \vdash D \rightsquigarrow \Xi}$$

$$\begin{array}{c} \frac{\Gamma \vdash T : \Omega \rightsquigarrow \tau}{\Gamma \vdash \text{val } X : T \rightsquigarrow \{l_X : [\tau]\}} \text{D-Val} \\[10pt] \frac{\Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash \text{type } X = T \rightsquigarrow \{l_X : [= \tau : \kappa]\}} \text{D-Typ-Eq} \\[10pt] \frac{\Gamma \vdash K \rightsquigarrow \kappa}{\Gamma \vdash \text{type } X : K \rightsquigarrow \overline{\exists t : \kappa. \{l_X : [= t : \kappa]\}}} \text{D-Typ} \\[10pt] \frac{\Gamma \vdash S \rightsquigarrow \overline{\exists t : \kappa. \Sigma}}{\Gamma \vdash \text{module } X : S \rightsquigarrow \overline{\exists t : \kappa. \{l_X : \Sigma\}}} \text{D-Mod} \\[10pt] \frac{\Gamma \vdash S \rightsquigarrow \Xi}{\Gamma \vdash \text{signature } X = S \rightsquigarrow \{l_X : [= \Xi]\}} \text{D-Sig-Eq} \\[10pt] \frac{\Gamma \vdash S \rightsquigarrow \overline{\exists t : \kappa. \{l_X : \Sigma\}}}{\Gamma \vdash \text{include } S \rightsquigarrow \overline{\exists t : \kappa. \{l_X : \Sigma\}}} \text{D-Incl} \\[10pt] \frac{}{\Gamma \vdash \epsilon \rightsquigarrow \{\}} \text{D-Emt} \\[10pt] \frac{\overline{\{l_{X_1}\}} \cap \overline{\{l_{X_2}\}} = \emptyset \quad \Gamma \vdash D_1 \rightsquigarrow \overline{\exists t_1 : \kappa_1. \{l_{X_1} : \Sigma_1\}} \quad \Gamma, t_1 : \kappa_1, x_{X_1} : \Sigma_1 \vdash D_2 \rightsquigarrow \overline{\exists t_2 : \kappa_2. \{l_{X_2} : \Sigma_2\}}}{\Gamma \vdash D_1; D_2 \rightsquigarrow \overline{\exists t_1 : \kappa_1} t_2 : \kappa_2. \overline{\{l_{X_1} : \Sigma_1} l_{X_2} : \Sigma_2\}}} \text{D-Seq} \end{array}$$

Matching:

$$\boxed{\Gamma \vdash \Sigma_1 \leq \overline{\exists t : \kappa. \Sigma_2} \uparrow \overline{\tau} \rightsquigarrow e}$$

$$\frac{\Gamma \vdash \Sigma_1 \leq \Sigma_2[t \leftarrow \tau_t] \rightsquigarrow e \quad \bigwedge_t \Gamma \vdash \tau_t : \kappa_t}{\Gamma \vdash \Sigma_1 \leq \overline{\exists t : \kappa_t. \Sigma_2} \uparrow \overline{\tau_t} \rightsquigarrow e} \text{U-Match}$$

Subtyping:

$$\boxed{\Gamma \vdash \Xi_1 \leq \Xi_2 \rightsquigarrow e}$$

$$\begin{array}{c}
\frac{\Gamma \vdash \tau_1 \leq \tau_2 \rightsquigarrow e}{\Gamma \vdash [\tau_1] \leq [\tau_2] \rightsquigarrow \lambda x : [\tau_1]. [e (x. \text{val})]} \text{U-Val} \\
\frac{\Gamma \vdash \tau_1 \equiv \tau_2 : \kappa}{\Gamma \vdash [= \tau_1 : \kappa] \leq [= \tau_2 : \kappa] \rightsquigarrow \lambda x : [= \tau_1 : \kappa]. x} \text{U-Typ} \\
\frac{\Gamma \vdash \Xi_1 \leq \Xi_2 \rightsquigarrow e_1 \quad \Gamma \vdash \Xi_2 \leq \Xi_1 \rightsquigarrow e_2}{\Gamma \vdash [= \Xi_1] \leq [= \Xi_2] \rightsquigarrow \lambda x : [= \Xi_1]. [\Xi_2]} \text{U-Sig} \\
\frac{\bigwedge_l \Gamma \vdash \Sigma_{l_1} \leq \Sigma_{l_2} \rightsquigarrow e_l}{\Gamma \vdash \{\overline{l} : \Sigma_{l_1}, \overline{l}' : \Sigma_{l_2}\} \leq \{\overline{l} : \Sigma_{l_2}\} \rightsquigarrow \lambda x : \{\overline{l} : \Sigma_{l_1}, \overline{l}' : \Sigma_{l_2}\}. \{\overline{l} = e_l (x.l)\}} \text{U-Struct} \\
\frac{\Gamma, \overline{t_2} : \overline{\kappa_2} \vdash \Sigma_2 \leq \exists \overline{t_1} : \overline{\kappa_1}. \Sigma_1 \uparrow \overline{\tau} \rightsquigarrow e_1 \quad \Gamma, \overline{t_2} : \overline{\kappa_2} \vdash \Xi_1[\overline{t_1} \leftarrow \overline{\tau}] \leq \Xi_2 \rightsquigarrow e_2}{\Gamma \vdash \forall \overline{t_1} : \overline{\kappa_1}. \Sigma_1 \rightarrow \Xi_1 \leq \forall \overline{t_2} : \overline{\kappa_2}. \Sigma_2 \rightarrow \Xi_2 \rightsquigarrow \lambda x_1 : (\forall \overline{t_1} : \overline{\kappa_1}. \Sigma_1 \rightarrow \Xi_1). \lambda x_2 : \Sigma_2. e_2 (x_1 \overline{\tau} (e_1 x_2))} \text{U-Funct} \\
\frac{\Gamma, \overline{t_1} : \overline{\kappa_1} \vdash \Sigma_1 \leq \exists \overline{t_2} : \overline{\kappa_2}. \Sigma_2 \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash \exists \overline{t_1} : \overline{\kappa_1}. \Sigma_1 \leq \exists \overline{t_2} : \overline{\kappa_2}. \Sigma_2 \rightsquigarrow \lambda x_1 : (\exists \overline{t_1} : \overline{\kappa_1}. \Sigma_1). \text{unpack}(\overline{t_1} : \overline{\kappa_1}, x'_1 : \Sigma_1) = x_1 \text{ in pack}(\overline{\tau}, e x'_1)_{\exists \overline{t_2} : \overline{\kappa_2}. \Sigma_2}} \text{U-Abs}
\end{array}$$

Module:

$$\boxed{\Gamma \vdash M : \Xi \rightsquigarrow e}$$

$$\begin{array}{c}
\frac{\Gamma(x_X) = \Sigma}{\Gamma \vdash X : \Sigma \rightsquigarrow x_X} \text{M-Var} \\
\frac{\Gamma \vdash B : \Xi \rightsquigarrow e}{\Gamma \vdash \{B\} : \Xi \rightsquigarrow e} \text{M-Struct} \\
\frac{\Gamma \vdash M : \exists \overline{t} : \overline{\kappa}. \{\overline{l_X} : \Sigma, \overline{l_{X'}} : \Sigma'\} \rightsquigarrow e}{\Gamma \vdash M.X : \exists \overline{t} : \overline{\kappa}. \Sigma \rightsquigarrow \text{unpack}(\overline{t} : \overline{\kappa}, x : \{\overline{l_X} : \Sigma, \overline{l_{X'}} : \Sigma'\}) = e \text{ in pack}(\overline{t}, x.l_X)_{\exists \overline{t} : \overline{\kappa}. \Sigma}} \text{M-Dot} \\
\frac{\Sigma \vdash S \rightsquigarrow \exists \overline{t} : \overline{\kappa}. \Sigma \quad \Gamma, \overline{t} : \overline{\kappa}, x_X : \Sigma \vdash M : \Xi \rightsquigarrow e}{\Gamma \vdash \text{fun } X : S \Rightarrow M : \forall \overline{t} : \overline{\kappa}. \Sigma \rightarrow \Xi \rightsquigarrow \Lambda \overline{t} : \overline{\kappa}. \lambda x_X : \Sigma. e} \text{M-Funct} \\
\frac{\Gamma(x_{X_1}) = \forall \overline{t} : \overline{\kappa}. \Sigma' \rightarrow \Xi \quad \Gamma(x_{X_2}) = \Sigma \quad \Gamma \vdash \Sigma \leq \exists \overline{t} : \overline{\kappa}. \Sigma' \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash X_1 X_2 : \Xi[\overline{t} \leftarrow \overline{\tau}] \rightsquigarrow x_{X_1} \overline{\tau} (e x_{X_2})} \text{M-App} \\
\frac{\Gamma(x_X) = \Sigma \quad \Gamma \vdash S \rightsquigarrow \exists \overline{t} : \overline{\kappa}. \Sigma' \quad \Gamma \vdash \Sigma \leq \exists \overline{t} : \overline{\kappa}. \Sigma' \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash X :> S : \exists \overline{t} : \overline{\kappa}. \Sigma' \rightsquigarrow \text{pack}(\overline{\tau}, e x_X)_{\exists \overline{t} : \overline{\kappa}. \Sigma'}} \text{M-Seal}
\end{array}$$

Bindings:

$$\boxed{\Gamma \vdash B : \Xi \rightsquigarrow e}$$

$$\begin{array}{c}
\frac{\Gamma \vdash E : \tau \rightsquigarrow e}{\Gamma \vdash \text{val } X = E : \{\overline{l_X} : [\tau]\} \rightsquigarrow \{\overline{l_X} = e\}} \text{B-Val} \\
\frac{\Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash \text{type } X = T : \{\overline{l_X} : [= \tau : \kappa]\} \rightsquigarrow \{\overline{l_X} = [\tau : \kappa]\}} \text{B-Typ} \\
\frac{\Gamma \vdash M : \exists \overline{t} : \overline{\kappa}. \Sigma \rightsquigarrow e \quad \text{NotAtomic}(\Sigma)}{\Gamma \vdash \text{module } X = M : \exists \overline{t} : \overline{\kappa}. \{\overline{l_X} : \Sigma\} \rightsquigarrow \text{unpack}(\overline{t} : \overline{\kappa}, x : \Sigma) = e \text{ in pack}(\overline{t}, \{\overline{l_X} = x\})_{\exists \overline{t} : \overline{\kappa}. \{\overline{l_X} : \Sigma\}}} \text{B-Mod} \\
\frac{\Gamma \vdash S \rightsquigarrow \Xi}{\Gamma \vdash \text{signature } X = S : \{\overline{l_X} : [= \Xi]\} \rightsquigarrow \{\overline{l_X} = [\Xi]\}} \text{B-Sig} \\
\frac{\Gamma \vdash M : \exists \overline{t} : \overline{\kappa}. \{\overline{l_X} : \Sigma\} \rightsquigarrow e}{\Gamma \vdash \text{include } M : \exists \overline{t} : \overline{\kappa}. \{\overline{l_X} : \Sigma\} \rightsquigarrow e} \text{B-Incl} \\
\frac{}{\Gamma \vdash \epsilon : \{\} \rightsquigarrow \{\}} \text{B-Emt}
\end{array}$$

$$\begin{array}{c}
\frac{\overline{l'_{X_1}} = \overline{l_{X_1}} \setminus \overline{l_{X_2}} \quad \overline{l'_{X_1}} : \overline{\Sigma'_1} \subseteq \overline{l_{X_1}} : \overline{\Sigma_1} \quad \Gamma \vdash B_1 : \exists \overline{t_1} : \overline{\kappa_1}. \{\overline{l_{X_1}} : \overline{\Sigma_1}\} \rightsquigarrow e_1}{\Sigma = \{\overline{l'_{X_1}} : \overline{\Sigma'_1}, \overline{l_{X_2}} : \overline{\Sigma_2}\} \quad \Gamma, \overline{t_1} : \overline{\kappa_1}, \overline{x_{X_1}} : \overline{\Sigma_1} \vdash B_2 : \exists \overline{t_2} : \overline{\kappa_2}. \{\overline{l_{X_2}} : \overline{\Sigma_2}\} \rightsquigarrow e_2} \text{ B-Seq} \\
\Gamma \vdash B_1; B_2 : \exists \overline{t_1} : \overline{\kappa_1} \overline{t_2} : \overline{\kappa_2}. \Sigma \rightsquigarrow \frac{\text{unpack}(\overline{t_1} : \overline{\kappa_1}, x_1) = e_1 \text{ in} \quad \text{unpack}(\overline{t_2} : \overline{\kappa_2}, x_2) = (\text{let } x_{X_1} : \overline{\Sigma_1} = x_1.l_{X_1} \text{ in } e_2) \text{ in}}{\text{pack}(\overline{t_1} \overline{t_2}, \{\overline{l'_{X_1}} = x_1.l'_{X_1}, \overline{l_{X_2}} = x_2.l_{X_2}\})_{\exists \overline{t_1} : \overline{\kappa_1} \overline{t_2} : \overline{\kappa_2}. \Sigma}}
\end{array}$$

Path:

$$\boxed{\Gamma \vdash P : \Sigma \rightsquigarrow e}$$

$$\frac{\Gamma \vdash P : \exists \overline{t} : \overline{\kappa}. \Sigma \quad \Gamma \vdash \Sigma : \Omega}{\Gamma \vdash P : \Sigma \rightsquigarrow \text{unpack}(\overline{t} : \overline{\kappa}, x) = e \text{ in } x} \text{ P-Mod}$$

$$\boxed{\Gamma \vdash T : \kappa \rightsquigarrow \tau}$$

$$\frac{\Gamma \vdash P : [\tau : \kappa] \rightsquigarrow e}{\Gamma \vdash P : \kappa \rightsquigarrow \tau} \text{ T-Elab-Path}$$

$$\boxed{\Gamma \vdash E : \tau \rightsquigarrow e}$$

$$\frac{\Gamma \vdash P : [\tau] \rightsquigarrow e}{\Gamma \vdash P : \tau \rightsquigarrow e.\text{val}} \text{ E-Path}$$

### 3.2.5 Modules as First-Class Values

$$\begin{aligned}
T &::= \dots \mid \text{pack } S \\
E &::= \dots \mid \text{pack } M : S \\
M &::= \dots \mid \text{unpack } E : S
\end{aligned}$$

Rootedness:

$$\boxed{t : \kappa \text{ rooted in } \Sigma \text{ at } \overline{l_X}}$$

$$\frac{t = \tau'}{t : \kappa \text{ rooted in } [\tau : \kappa] \text{ at } \epsilon} \quad \frac{t : \kappa \text{ rooted in } \{\overline{l_X} : \overline{\Sigma}\}.l \text{ at } \overline{l'}}{t : \kappa \text{ rooted in } \{\overline{l_X} : \overline{\Sigma}\} \text{ at } l \overline{l'}}$$

Rooted ordering:

$$t_1 : \kappa_1 \leq_{\Sigma} t_2 : \kappa_2 \iff \min\{\bar{l} \mid t_1 : \kappa_1 \text{ rooted in } \Sigma \text{ at } \bar{l}\} \leq \min\{\bar{l} \mid t_2 : \kappa_2 \text{ rooted in } \Sigma \text{ at } \bar{l}\}$$

Signature normalization:

$$\begin{aligned}
&\frac{\text{norm}_0(\tau) = \tau'}{\text{norm}([\tau]) = [\tau']} \\
&\frac{}{\text{norm}([\tau : \kappa]) = [\tau : \kappa]} \\
&\frac{}{\text{norm}([\Xi]) = [\Xi']} \\
&\frac{}{\text{norm}([\Xi]) = [\Xi']} \\
&\frac{\bigwedge_X \text{norm}(\Sigma_X) = \Sigma'_X}{\text{norm}(\{\overline{l_X} : \Sigma_X\}) = \{\overline{l_X} : \Sigma'_X\}}
\end{aligned}$$

$$\frac{\text{sort}_{\leq \Sigma'}(\overline{t : \kappa}) = \overline{t' : \kappa'} \quad \text{norm}(\Sigma) = \Sigma' \quad \text{norm}(\Xi) = \Xi'}{\text{norm}(\forall t : \kappa. \Sigma \rightarrow \Xi) = \forall t' : \kappa'. \Sigma' \rightarrow \Xi'}$$

$$\frac{\text{sort}_{\leq \Sigma'}(\overline{t : \kappa}) = \overline{t' : \kappa'} \quad \text{norm}(\Sigma) = \Sigma'}{\text{norm}(\exists t : \kappa. \Sigma) = \exists t' : \kappa'. \Sigma'}$$

Type:

$$\boxed{\Gamma \vdash T : \kappa \rightsquigarrow \tau}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Xi}{\Gamma \vdash \text{pack } S : \Omega \rightsquigarrow \text{norm}(\Xi)} \text{ T-Pack}$$

Expression:

$$\boxed{\Gamma \vdash E : \tau \rightsquigarrow e}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Xi \quad \Gamma \vdash \Xi' \leq \text{norm}(\Xi) \rightsquigarrow e_1 \quad \Gamma \vdash M : \Xi' \rightsquigarrow e_2}{\Gamma \vdash (\text{pack } M : S) : \text{norm}(\Xi) \rightsquigarrow e_1 e_2} \text{ E-Pack}$$

Module:

$$\boxed{\Gamma \vdash M : \Xi \rightsquigarrow e}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Xi \quad \Gamma \vdash E : \text{norm}(\Xi) \rightsquigarrow e}{\Gamma \vdash (\text{unpack } E : S) : \text{norm}(\Xi) \rightsquigarrow e} \text{ M-Unpack}$$

### 3.2.6 Elaboration with Applicative Functor

$$S ::= \dots$$

$$| (X : S) \Rightarrow S \quad (\text{applicative functor signature})$$

$$\varphi ::= I \quad (\text{impure effect})$$

$$| P \quad (\text{pure effect})$$

$$\Sigma ::= \dots$$

$$| \{\overline{l_X : \Sigma}\}$$

$$| \forall t : \kappa. \Sigma \rightarrow_I \Xi \quad (\text{generative functor signature})$$

$$| \forall t : \kappa. \Sigma \rightarrow_P \Sigma \quad (\text{applicative functor signature})$$

Abbreviation:

$$\tau_1 \rightarrow_{\varphi} \tau_2 \stackrel{\text{def}}{=} \tau_1 \rightarrow \{l_{\varphi} : \tau_2\}$$

$$\lambda_{\varphi} x : \tau. e \stackrel{\text{def}}{=} \lambda x : \tau. \{l_{\varphi} = e\}$$

$$(e_1 e_2)_{\varphi} \stackrel{\text{def}}{=} (e_1 e_2).l_{\varphi}$$

$$\Gamma^{\varphi} \stackrel{\text{def}}{=} \begin{cases} \cdot & (\varphi = I) \\ \Gamma & (\varphi = P) \end{cases}$$

$$\text{tyenv}(\Gamma) \stackrel{\text{def}}{=} \begin{cases} \text{tyenv}(\Gamma') \, t : \kappa & (\Gamma = \Gamma', t : \kappa) \\ \text{tyenv}(\Gamma') & (\Gamma = \Gamma', x : \tau) \\ \epsilon & (\Gamma = \cdot) \end{cases}$$

$$\begin{aligned} \forall_P \Gamma. \tau_0 &\stackrel{\text{def}}{=} \begin{cases} \forall_P \Gamma'. \forall t : \kappa. \tau_0 & (\Gamma = \Gamma', t : \kappa) \\ \forall_P \Gamma'. \tau \rightarrow_P \tau_0 & (\Gamma = \Gamma', x : \tau) \\ \tau_0 & (\Gamma = \cdot) \end{cases} \\ \Lambda_P \Gamma. e &\stackrel{\text{def}}{=} \begin{cases} \Lambda_P \Gamma'. \Lambda t : \kappa. e & (\Gamma = \Gamma', t : \kappa) \\ \Lambda_P \Gamma'. \lambda_P x : \tau. e & (\Gamma = \Gamma', x : \tau) \\ e & (\Gamma = \cdot) \end{cases} \\ (e \ \Gamma)_P &\stackrel{\text{def}}{=} \begin{cases} (e \ \Gamma')_P t & (\Gamma = \Gamma', t : \kappa) \\ ((e \ \Gamma')_P x)_P & (\Gamma = \Gamma', x : \tau) \\ e & (\Gamma = \cdot) \end{cases} \end{aligned}$$

Effect combining:

$$\boxed{\varphi_1 \vee \varphi_2 = \varphi}$$

$$\overline{\varphi \vee \varphi} = \overline{\varphi} \quad \overline{I \vee P} = \overline{I} \quad \overline{P \vee I} = \overline{I}$$

Subeffects:

$$\boxed{\varphi_1 \leq \varphi_2}$$

$$\overline{\varphi \leq \varphi} \text{ F-Refl} \quad \overline{P \leq I} \text{ F-Sub}$$

Signature:

$$\boxed{\Gamma \vdash S \rightsquigarrow \Xi}$$

$$\begin{aligned} &\frac{\Gamma \vdash S_1 \rightsquigarrow \exists \overline{t_1 : \kappa_1}. \Sigma \quad \Gamma, \overline{t_1 : \kappa_1}, x_X : \Sigma \vdash S_2 \rightsquigarrow \Xi}{\Gamma \vdash (X : S_1) \rightarrow S_2 \rightsquigarrow \forall \overline{t_1 : \kappa_1}. \Sigma \rightarrow_I \Xi} \text{ S-Funct-I} \\ &\frac{\Gamma \vdash S_1 \rightsquigarrow \exists \overline{t_1 : \kappa_1}. \Sigma_1 \quad \Gamma, \overline{t_1 : \kappa_1}, x_X : \Sigma_1 \vdash S_2 \rightsquigarrow \exists \overline{t_2 : \kappa_2}. \Sigma_2}{\Gamma \vdash (X : S_1) \Rightarrow S_2 \rightsquigarrow \exists \overline{t'_2 : \kappa_1} \rightarrow \kappa_2. \forall \overline{t_1 : \kappa_1}. \Sigma_1 \rightarrow_P \Sigma_2[t_2 \leftarrow t'_2 \ t_1]} \text{ S-Funct-P} \end{aligned}$$

Subtyping:

$$\boxed{\Gamma \vdash \Xi_1 \leq \Xi_2 \rightsquigarrow e}$$

$$\frac{\Gamma, \overline{t_2 : \kappa_2} \vdash \Sigma_2 \leq \exists \overline{t_1 : \kappa_1}. \Sigma_1 \uparrow \bar{\tau} \rightsquigarrow e_1 \quad \Gamma, \overline{t_2 : \kappa_2} \vdash \Xi_1[\overline{t_1 \leftarrow \bar{\tau}}] \leq \Xi_2 \rightsquigarrow e_2 \quad \varphi_1 \leq \varphi_2}{\Gamma \vdash (\forall \overline{t_1 : \kappa_1}. \Sigma_1 \rightarrow_{\varphi_1} \Xi_1) \leq (\forall \overline{t_2 : \kappa_2}. \Sigma_2 \rightarrow_{\varphi_2} \Xi_2) \rightsquigarrow \frac{\lambda x_1 : (\forall \overline{t_1 : \kappa_1}. \Sigma_1 \rightarrow_{\varphi_1} \Xi_1). \Lambda \overline{t_2 : \kappa_2}. \lambda_{\varphi_2} x_2 : \Sigma_2. e_2 (x_1 \ \bar{\tau} (e_1 \ x_2))}{\varphi_1}} \text{ U-Funct}$$

Module:

$$\boxed{\Gamma \vdash M :_{\varphi} \Xi \rightsquigarrow e}$$

$$\begin{aligned} &\frac{\Gamma(x_X) = \Sigma}{\Gamma \vdash X :_P \Sigma \rightsquigarrow \Lambda_P \Gamma. x_X} \text{ M-Var} \\ &\frac{\Gamma \vdash B :_{\varphi} \Xi \rightsquigarrow e}{\Gamma \vdash \{B\} :_{\varphi} \Xi \rightsquigarrow e} \text{ M-Struct} \\ &\frac{\Gamma \vdash M :_{\varphi} \exists t : \kappa. \{\overline{l_X : \Sigma}, \overline{l_{X'} : \Sigma'}\} \rightsquigarrow e}{\Gamma \vdash M.X :_{\varphi} \exists \overline{t : \kappa}. \Sigma \rightsquigarrow \text{unpack}(\overline{t : \kappa}, x) = e \text{ in pack}(\overline{t}, \Lambda_P \Gamma. (x \ \Gamma^{\varphi})_P. l_X)} \text{ M-Dot} \\ &\frac{\Sigma \vdash S \rightsquigarrow \exists \overline{t : \kappa}. \Sigma \quad \Gamma, \overline{t : \kappa}, x_X : \Sigma \vdash M :_I \Xi \rightsquigarrow e}{\Gamma \vdash \text{fun } X : S \Rightarrow M :_P \forall \overline{t : \kappa}. \Sigma \rightarrow_I \Xi \rightsquigarrow \Lambda_P \Gamma. \Lambda \overline{t : \kappa}. \lambda_1 x_X : \Sigma. e} \text{ M-Funct-I} \\ &\frac{\Sigma \vdash S \rightsquigarrow \exists \overline{t : \kappa}. \Sigma \quad \Gamma, \overline{t : \kappa}, x_X : \Sigma \vdash M :_P \exists \overline{t_2 : \kappa_2}. \Sigma_2 \rightsquigarrow e}{\Gamma \vdash \text{fun } X : S \Rightarrow M :_P \exists \overline{t_2 : \kappa_2}. \forall \overline{t : \kappa}. \Sigma \rightarrow_P \Sigma_2 \rightsquigarrow e} \text{ M-Funct-P} \end{aligned}$$

$$\begin{array}{c}
\frac{\Gamma(x_{X_1}) = \forall \overline{t} : \overline{\kappa}. \Sigma' \rightarrow_{\varphi} \Xi \quad \Gamma(x_{X_2}) = \Sigma \quad \Gamma \vdash \Sigma \leq \exists \overline{t} : \overline{\kappa}. \Sigma' \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash X_1 X_2 :_{\varphi} \Xi[\overline{t} \leftarrow \overline{\tau}] \rightsquigarrow \Lambda_P \Gamma^{\varphi}. (x_{X_1} \overline{\tau} (e x_{X_2}))_{\varphi}} \text{M-App} \\
\frac{\overline{t}_{\Gamma} : \overline{\kappa}_{\Gamma} = \text{tyenv}(\Gamma) \quad \Gamma(x_X) = \Sigma \quad \Gamma \vdash S \rightsquigarrow \exists \overline{t} : \overline{\kappa}. \Sigma' \quad \Gamma \vdash \Sigma \leq \exists \overline{t} : \overline{\kappa}. \Sigma' \uparrow \overline{\tau} \rightsquigarrow e}{\Gamma \vdash X :> S :_P \exists \overline{t}' : \overline{t}_{\Gamma} : \overline{\kappa}_{\Gamma} \rightarrow \kappa. \Sigma' [t \leftarrow t' \overline{t}_{\Gamma}] \rightsquigarrow \text{pack}(\lambda \overline{t}_{\Gamma} : \overline{\kappa}_{\Gamma}. \tau, \Lambda_P \Gamma. e x_X)} \text{M-Seal} \\
\frac{\Gamma \vdash S \rightsquigarrow \Xi \quad \Gamma \vdash E : \text{norm}(\Xi) \rightsquigarrow e}{\Gamma \vdash (\text{unpack } E : S) :_I \text{norm}(\Xi) \rightsquigarrow e} \text{M-Unpack}
\end{array}$$

定理 30 (Typing for module elaboration).

- $\Gamma \vdash M :_I \Xi \rightsquigarrow e$  ならば,  $\Gamma \vdash e : \Xi$ .
- $\Gamma \vdash M :_P \exists \overline{t} : \overline{\kappa}. \Sigma \rightsquigarrow e$  ならば,  $\cdot \vdash e : \exists \overline{t} : \overline{\kappa}. \forall_P \Gamma. \Sigma$ .

□

Bindings:

$$\boxed{\Gamma \vdash B :_{\varphi} \Xi \rightsquigarrow e}$$

$$\begin{array}{c}
\frac{\Gamma \vdash E : \tau \rightsquigarrow e}{\Gamma \vdash \text{val } X = E :_P \{l_X : [\tau]\} \rightsquigarrow \Lambda_P \Gamma. \{l_X = e\}} \text{B-Val} \\
\frac{\Gamma \vdash T : \kappa \rightsquigarrow \tau}{\Gamma \vdash \text{type } X = T :_P \{l_X : [\tau : \kappa]\} \rightsquigarrow \Lambda_P \Gamma. \{l_X = [\tau : \kappa]\}} \text{B-Typ} \\
\frac{\Gamma \vdash M :_{\varphi} \exists \overline{t} : \overline{\kappa}. \Sigma \rightsquigarrow e \quad \text{NotAtomic}(\Sigma)}{\Gamma \vdash \text{module } X = M :_{\varphi} \exists \overline{t} : \overline{\kappa}. \{l_X : \Sigma\} \rightsquigarrow \text{unpack}(\overline{t} : \overline{\kappa}, x) = e \text{ in } \text{pack}(\overline{t}, \Lambda_P \Gamma^{\varphi}. \{l_X = x \Gamma^{\varphi}\})} \text{B-Mod} \\
\frac{\Gamma \vdash S \rightsquigarrow \Xi}{\Gamma \vdash \text{signature } X = S :_P \{l_X : [= \Xi]\} \rightsquigarrow \Lambda_P \Gamma. \{l_X = [\Xi]\}} \text{B-Sig} \\
\frac{\Gamma \vdash M :_{\varphi} \exists \overline{t} : \overline{\kappa}. \{l_X : \Sigma\} \rightsquigarrow e}{\Gamma \vdash \text{include } M :_{\varphi} \exists \overline{t} : \overline{\kappa}. \{l_X : \Sigma\} \rightsquigarrow e} \text{B-Incl} \\
\frac{}{\Gamma \vdash \epsilon :_P \{\} \rightsquigarrow \Lambda_P \Gamma. \{\}} \text{B-Emt} \\
\frac{\overline{l}'_{X_1} = \overline{l}_{X_1} \setminus \overline{l}_{X_2} \quad \overline{l}'_{X_1} : \overline{\Sigma}'_1 \subseteq \overline{l}_{X_1} : \overline{\Sigma}_1 \quad \Gamma \vdash B_1 :_{\varphi_1} \exists \overline{t}_1 : \overline{\kappa}_1. \{\overline{l}_{X_1} : \overline{\Sigma}_1\} \rightsquigarrow e_1 \quad \Sigma = \{\overline{l}'_{X_1} : \overline{\Sigma}'_1, \overline{l}_{X_2} : \overline{\Sigma}_2\} \quad \Gamma, \overline{t}_1 : \overline{\kappa}_1, \overline{x}_{X_1} : \overline{\Sigma}_1 \vdash B_2 :_{\varphi_2} \exists \overline{t}_2 : \overline{\kappa}_2. \{\overline{l}_{X_2} : \overline{\Sigma}_2\} \rightsquigarrow e_2}{\Gamma \vdash B_1; B_2 :_{\varphi_1 \vee \varphi_2} \exists \overline{t}_1 : \overline{\kappa}_1 \overline{t}_2 : \overline{\kappa}_2. \Sigma \rightsquigarrow \text{unpack}(\overline{t}_1 : \overline{\kappa}_1, x_1) = e_1 \text{ in } \text{unpack}(\overline{t}_2 : \overline{\kappa}_2, x_2) = (\text{let } \overline{x}_{X_1} = \Lambda_P \Gamma^{\varphi_1 \vee \varphi_2}. (x_1 \Gamma^{\varphi_1})_{P}. l_{X_1} \text{ in } e_2) \text{ in } \text{pack}(\overline{t}_1 \overline{t}_2, \Lambda_P \Gamma^{\varphi_1 \vee \varphi_2}. \text{let } \overline{x}_{X_1} = (x_1 \Gamma^{\varphi_1})_{P}. l_{X_1} \text{ in } \{\overline{l}'_{X_1} = (x_1 \Gamma^{\varphi_1})_{P}. l'_{X_1}, l_{X_2} = (x_2 (\Gamma, \overline{t}_1 : \overline{\kappa}_1, \overline{x}_{X_1} : \overline{\Sigma}_1)^{\varphi_2})_{P}. l_{X_2}\})} \text{B-Seq}
\end{array}$$

Path:

$$\boxed{\Gamma \vdash P : \Sigma \rightsquigarrow e}$$

$$\frac{\Gamma \vdash P :_{\varphi} \exists \overline{t} : \overline{\kappa}. \Sigma \quad \Gamma \vdash \Sigma : \Omega}{\Gamma \vdash P : \Sigma \rightsquigarrow \text{unpack}(\overline{t} : \overline{\kappa}, x) = e \text{ in } (x \Gamma^{\varphi})_P} \text{P-Mod}$$

Expression:

$$\boxed{\Gamma \vdash E : \tau \rightsquigarrow e}$$

$$\frac{\Gamma \vdash S \rightsquigarrow \Xi \quad \Gamma \vdash \exists \overline{t} : \overline{\kappa}. \Sigma \leq \text{norm}(\Xi) \rightsquigarrow e_1 \quad \Gamma \vdash M :_{\varphi} \exists \overline{t} : \overline{\kappa}. \Sigma \rightsquigarrow e_2}{\Gamma \vdash (\text{pack } M : S) : \text{norm}(\Xi) \rightsquigarrow e_1 \text{ (unpack}(\overline{t} : \overline{\kappa}, x) = e_2 \text{ in pack}(\overline{t} : \overline{\kappa}, (x \Gamma^{\varphi})_P))} \text{E-Unpack}$$





## 第 4 章

# Control Operators



## 第 5 章

# Coherent Implicit Parameter



## 第 6 章

# Polymorphic Record Type



## 第 7 章

# Type Checking and Inference





## 第 8 章

# Static Memory Management and Regions



## 第 9 章

# Dynamic Memory Management and Gabage Collection



## 第 10 章

# I/O Management and Concurrency



## 第 11 章

# Code Generation and Virtual Machines





## 第 12 章

# Program Stability and Compatibility



## 第 13 章

# Program Separation and Linking



## 第 14 章

# Syntax and Parsing



## 第 15 章

# Analysis and Optimizations





## 第 16 章

# Meta-Programming and Multi-Stage Programming



## 第 17 章

# Generic Programming



## 第 18 章

# Advanced Calculus



## 第 19 章

### Some Notes of Quell Ideas



## 19.1 Quell Syntax and Identations

### 19.1.1 Syntax

## 19.2 Quell Modules

### 19.2.1 Syntax

$$\begin{aligned}
 e &::= \dots \\
 &\quad | \text{ letrec}\{B\} \text{ in } e \\
 &\quad | P \\
 \tau &::= \dots \\
 &\quad | P \\
 P &::= M \\
 M &::= x \\
 &\quad | \{B\} \\
 &\quad | M.x \\
 &\quad | \text{ fun } x : S. M \\
 &\quad | x \ x \\
 &\quad | x : S \\
 B &::= x = e \\
 &\quad | \text{ type } t = T \\
 &\quad | \text{ module } x = M \\
 &\quad | \text{ use } B \\
 &\quad | \epsilon \\
 &\quad | B; B \\
 T &::= \lambda x. T \\
 &\quad | \tau \\
 S &::= P \\
 &\quad | \{D\} \\
 &\quad | (x : S) \rightarrow S \\
 &\quad |
 \end{aligned}$$



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