In [1]:
```
import numpy as np
from IPython import display
import matplotlib.pyplot as plt
from matplotlib.lines import Line2D
import copy
import scipy
from scipy.io import loadmat
from scipy.io import savemat
import seaborn as sns
sns.set_style('darkgrid')
import warnings
warnings.filterwarnings('ignore')
from IPython.display import Image
from utils import *
from envs.cart_pole_env import CartPoleEnv
from envs.hopper_env import HopperModEnv
from envs.cheetah_env import CheetahModEnv
```

Part 1 [30 pt] - Linear Environment

We start with the linear environment, similar to the one on the previous homework, and we consider optimizing for a sequence of actions, comparing shooting and collocation.

First, we define the environment:

In [2]:
```
class LinearEnv(object):
    def __init__(self, horizon=20, multiplier=1.):
        self.A = multiplier * 0.1 * np.array([[0.0481, -0.5049, 0.029
9, 2.6544, 1.0608],
                                             [2.3846, -0.2312, -0.1260, -0.7945,
0.5279],
                                             [1.4019, -0.6394, -0.1401, 0.5484,
0.1624],
                                             [-0.0254, 0.4595, -0.0862, 2.1750,
1.1012],
                                             [0.5172, 0.5060, 1.6579, -0.9407, -
1.4441]])
        self.B = np.array([[-0.7789, -1.2076],
                           [0.4299, -1.6041],
                           [0.2006, -1.7395],
                           [0.8302, 0.2295],
                           [-1.8465, 1.2780]])
        self.H = 20

        self.dx = self.A.shape[1]
        self.du = self.B.shape[1]
        self.Q = np.eye(self.dx)
        self.R = np.eye(self.du)
        self._init_state =  np.array([-1.9613, -1.3127, 0.0698, 0.093
5, 1.2494])
        self.reset()

    def step(self, act):
        cost = self._state.T @ self.Q @ self._state + act.T @ self.R
@ act
        state = self.A @ self._state + self.B @ act
        self._state = state.copy()
        return state, cost, False, {}

    def set_state(self, state):
        self._state = state.copy()

    def reset(self):
        self._state = self._init_state.copy()
        return self._init_state.copy()
```

In [3]:
```
env = LinearEnv()
```

Now, we implement the non-linear optimzation algorithms. A correct implementation should give an optimal cost of 7.461 for both methods, and a collocation error of 0.

[15 pt] Shooting

In the shooting method, we look for the sequences of actions that minimizes the total cost by directly substuting the constraints in the objective:

$$\min_{u_0,...,u_H} c(x_0, u_0) + c(f(x_0, u_0), u_1) + c(f(f(x_0, u_0), u_1) \cdots$$

In order to perform the optimization, we need to define the objective function to optimize. Fill in the code in `eval_shooting` which should return the cost of the trajectory with the specified sequences of actions.

In [4]:
```
def eval_shooting(env, actions):
    """
    Find the cumulative cost of the sequences of actions, which has s
hape [horizon, action dimension].
    Use the function step of the environment: env.step(action). It re
turns: next_state, cost, done,
    env_infos.
    """
    state = env.reset()
    actions = actions.reshape(env.H, env.du)
    horizon = env.H

    total_cost = 0


    """YOUR CODE HERE"""
    for action in actions:
        state, cost, _, _ = env.step(action)
        total_cost += cost

    """YOUR CODE ENDS HERE"""
    return total_cost
```

Once we have defined the objective function, we can use an off-the-shelf optimizer to find the optimal actions. In these case, we use [BFGS (https://docs.scipy.org/doc/scipy-0.16.0/reference/optimize.minimize-bfgs.html#optimize-minimize-bfgs)](https://docs.scipy.org/doc/scipy-0.16.0/reference/optimize.minimize-bfgs.html#optimize-minimize-bfgs), which is a quasi-Newton method.

In [5]:
```
def minimize_shooting(env, init_actions=None):
    if init_actions is None:
        init_actions = np.random.uniform(low=-.1, high=.1, size=(env.
H * env.du,))
    """YOUR CODE HERE"""
    res = minimize(fun=lambda actions: eval_shooting(env, actions),#
 Fill this with a function that returns the cumulative cost given the
states and actions,
                   x0=init_actions,# Fill this with the inital action
s
                   method='BFGS',
                   options={'xtol': 1e-6, 'disp': False, 'verbose': 2
})

    act_shooting = res.x
    print(res.message)
    print("The optimal cost is %.3f" % res.fun)
    policy_shooting = ActPolicy(env=env,
                                actions=act_shooting
                               )
    return policy_shooting
"""YOUR CODE ENDS HERE"""

policy_shooting = minimize_shooting(env)
```

```
Optimization terminated successfully.
The optimal cost is 7.461
```

[15 pt] Collocation

Now we will do the same, but for the collocation method. In addition to the objective function, we also have to formulate the equality constraints that capture the dynamics.

$$\begin{aligned} \min_{u_0, x_1, u_1, \ldots, x_H, u_H} \quad & c(x_0, u_0) + c(x_1, u_1) + \cdots + c(x_H, u_H) \\ \text{s.t.:} \quad & x_{t+1} = f(x_t, u_t) \quad \forall t \end{aligned}$$

Fill in the code in `eval_collocation` and `constraints` .

In [6]:
```
def eval_collocation(env, x):
    """
    Find the cost of the sequences of actions and state that have sha
pe [horizon, action dimension]
    and [horizon, state_dim], respectively.
    Use the function step of the environment: env.step(action). It re
turns: next_state, cost, done,
    env_infos.
    In order to set the environment at a specific state use the funct
ion env.set_state(state)
    """
    state = env.reset()
    total_cost = 0
    states, actions = x[:env.H * env.dx], x[env.H * env.dx:]
    states = states.reshape(env.H, env.dx)
    actions = actions.reshape(env.H, env.du)
    horizon = env.H
    """YOUR CODE HERE"""

    for i in range(horizon):
        state, cost, _, _ = env.step(actions[i])
        env.set_state(states[i])
        total_cost += cost

    """YOUR CODE ENDS HERE"""
    return total_cost

def constraints(env, x):
    """
    In optimization, the equality constraints are usually specified a
s h(x) = 0. In this case, we would have
    x_{t+1} - f(x_t, u_t) = 0. Here, you have to create a list that c
ontains the value of the different
    constraints, i.e., [x_1 - f(x_0, u_0), x_2 - f(x_1, u_1),..., x_H
- f(x_{H-1}, u_{H-1})].
    Use the function env.set_state(state) to set the state to the var
iable x_t.
    Use the function step of the environment: env.step(action), which
returns next_state, cost, done,
    env_infos; to obtain x_{t+1}.
    """
    state = env.reset()
    constraints = []
    states, actions = x[:env.H * env.dx], x[env.H * env.dx:]
    states = states.reshape(env.H, env.dx)
    actions = actions.reshape(env.H, env.du)
    horizon = env.H


    """YOUR CODE HERE"""
    for i in range(horizon):
        state, cost, _, _ = env.step(actions[i])
        constraints.append(states[i] - state)

    """YOUR CODE ENDS HERE"""
    return np.concatenate(constraints)
```

We can too use an off-the-shelf constraint optimzation algortihm, in thise case, we make use of the SLQP (https://docs.scipy.org/doc/scipy-0.16.0/reference/optimize.minimize-slsqp.html#optimize-minimize-slsqp) algorithm, which was seen in class.

In [7]:
```
def minimize_collocation(env, init_states_and_actions=None):
    if init_states_and_actions is None:
        init_states_and_actions = np.random.uniform(low=-.1, high=.1, size=(env.H * (env.du + env.dx),))

    """YOUR CODE HERE"""
    eq_cons = {'type': 'eq',
               'fun' : lambda x: constraints(env, x)
              }

# Fill this with a function that returns the cumulative cost given the states and actions,


    res = minimize(fun=lambda x: eval_collocation(env, x),
                   x0=init_states_and_actions,
                   method='SLSQP',
                   constraints=eq_cons,
                   options={'xtol': 1e-6, 'disp': False, 'verbose': 0, 'maxiter':201})
    print(res.message)
    print("The optimal cost is %.3f" % res.fun)
    states_collocation, act_collocation = res.x[:env.H * env.dx], res.x[env.H * env.dx:]
    states_collocation = states_collocation.reshape(env.H, env.dx)
    policy_collocation = ActPolicy(env,
                                   actions=act_collocation)
    """YOUR CODE ENDS HERE"""
    return policy_collocation, states_collocation

policy_collocation, states_collocation = minimize_collocation(env)
```

```
Optimization terminated successfully.
The optimal cost is 7.461
```

Evaluation

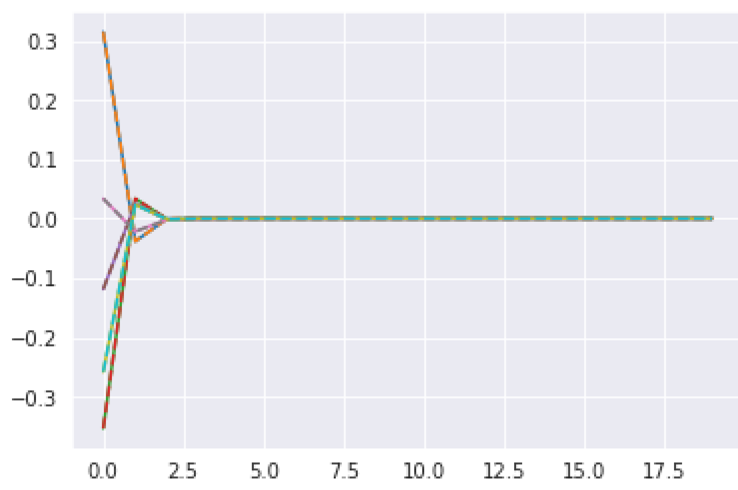
In [8]:
```
cost_shoot, states_shoot = rollout(env, policy_shooting)
cost_col, states_col = rollout(env, policy_collocation)
states_shoot, states_col = np.array(states_shoot), np.array(states_co
l)
error = np.linalg.norm(states_col - np.array(states_collocation))
ts = np.arange(states_shoot.shape[0])
print("---- Quantitative Metrics ---")
print("Shooting Cost %.3f" % cost_shoot)
print("Collocation Cost %.3f" % cost_col)
print("Collocation Error %.3f" % error)

print("\n\n---- Qualitative Metrics ---")
print("Evolution of the value of each dimension across 20 timesteps f
or the shooting methods.")
print("Both methods converge to the origin. Shooting: solid line(-);
Collocation: dashed line(--).")

for i in range(env.dx):
    plt.plot(ts, states_shoot[:, i], '-', ts, states_col[:, i], '--')
```

```
---- Quantitative Metrics ---
Shooting Cost 7.461
Collocation Cost 7.461
Collocation Error 0.000


---- Qualitative Metrics ---
Evolution of the value of each dimension across 20 timesteps for the
shooting methods.
Both methods converge to the origin. Shooting: solid line(-);  Colloc
ation: dashed line(--).
```



Part 2 [20 pt] - Stability

A discrete-time linear system is asymptotically stable if in the presence of no input the system converges towards the zero state. In practice, this means that the absolute value of the eigenvalues of the transition matrix must be smaller than 1. If that is not the case, the system is unstable.

For instance, the previous system is stable:

In [9]:
```
np.abs(np.linalg.eigvals(env.A))
```

Out[9]:
```
array([0.26052413, 0.14606684, 0.14606684, 0.09743496, 0.09743496])
```

[20 pt] Theoretical Question

Consider the linear system that we currently have, i.e.,

$$x_{t+1} = Ax_t + Bu_t$$

and we want to minimize the quadratic cost

$$\frac{1}{2}\sum_t x_t Q x_t$$

Hence, we have a linear quadratic regulator problem. Derive the gradient update for the action variables for both optimization methods: shooting and collocation. In the case of collocation, do not include the update due to the constraints.

Explain in a few lines why the shooting method might become unstable while the collocation method does not.

Refer to the pdf for reporting this question.

[0 pt] Empirical Behaviour

Now, we test the effect that you derived and see if the theory matches the empirical behavior. We use the same environment as in the previous part, but we just scale the transiton matrix so it has some eigenvalues larger than 1. Note this is the only change with respect to the previous part.

In [10]:
```
env = LinearEnv(multiplier=10.)
np.abs(np.linalg.eigvals(env.A))
```

Out[10]:
```
array([2.60524128, 1.46066843, 1.46066843, 0.97434961, 0.97434961])
```

Shooting

In [11]:
```
policy_shooting = minimize_shooting(env)
```

```
Desired error not necessarily achieved due to precision loss.
The optimal cost is 72284018840248.438
```

Collocation

In [12]:
```
policy_collocation, states_collocation = minimize_collocation(env)
```

```
Optimization terminated successfully.
The optimal cost is 201.812
```

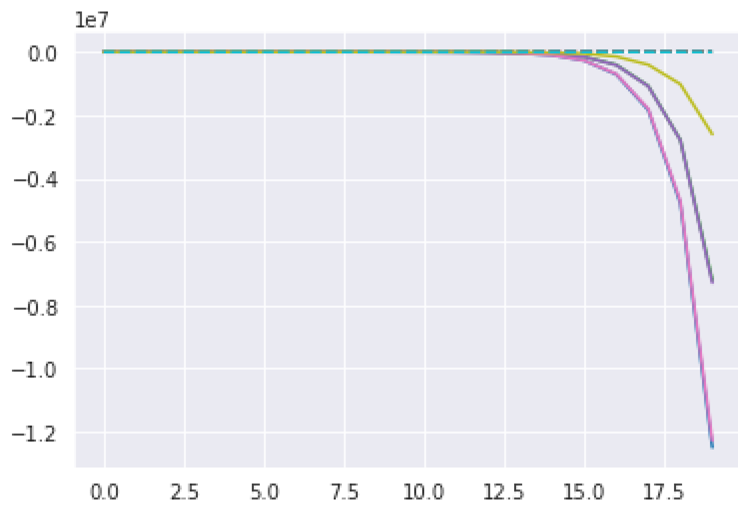
Evaluation

In [13]:
```
cost_shoot, states_shoot = rollout(env, policy_shooting)
cost_col, states_col = rollout(env, policy_collocation)
states_shoot, states_col = np.array(states_shoot), np.array(states_co
l)
error = np.linalg.norm(states_col - np.array(states_collocation))
print("---- Quantitative Metrics ---")
print("Shooting Cost %.3f" % cost_shoot)
print("Collocation Cost %.3f" % cost_col)
print("Collocation Error %.3f" % error)

print("\n\n---- Qualitative Metrics ---")
print("Evolution of the value of each dimension across 20 timesteps f
or the shooting methods.")
print("The shooting method diverges, while the collocation method ach
ieves the desired state. Shooting: solid line(-);  Collocation: dashe
d line(--).")
ts = np.arange(states_shoot.shape[0])
for i in range(env.dx):
    plt.plot(ts, states_shoot[:, i], '-', ts, states_col[:, i], '--')
```

```
---- Quantitative Metrics ---
Shooting Cost 72284018840248.438
Collocation Cost 201.812
Collocation Error 0.000


---- Qualitative Metrics ---
Evolution of the value of each dimension across 20 timesteps for the
shooting methods.
The shooting method diverges, while the collocation method achieves t
he desired state. Shooting: solid line(-);  Collocation: dashed line
(--).
```



Part 3 [0 pt] - Non-linear Environments

A nice thing of these algorithms is that they can be applied without any modification to non-linear environments such as the MuJoCo ones. For instance, here we learn a sequence of actions that leads to forward movement in the half-cheetah environment.

In [14]:
```
env = CheetahModEnv()
init_actions = np.random.uniform(low=-.25, high=.25, size=(env.H * env.du,))
action_shooting = minimize_shooting(env, init_actions)
cost_shooting, states_shooting = rollout(env, action_shooting)
```

```
Desired error not necessarily achieved due to precision loss.
The optimal cost is -20.400
```

SegmentLocal

Part 4 [30 pt] - Open-loop vs. Closed-loop

Until now, we have been optimizing directly the sequences of actions and then applying each of the actions in the sequences "blindly". While this suffices in deterministic environments, in the presence of noise it does not work out well usually. Because of the stochastic transitions, the state that you encounter at a specific time-step differs from the one predicted by the optimzation problem; as a result, the action found is no longer valid. In stochastic environments, we need close loop controllers in the form of either (i) parametric policies (e.g. linear feedback controllers or neural-networks), or (ii) non-parametric policies (e.g. model predictive control).

In the following, we will compare the different behaviour of open-loop and closed-loop control methods. Use the optimal cost for the action optimization methods to check the validity of your implementation.

In [15]:
```
env = CartPoleEnv()
```

Action Optimization

In [16]:
```
action_shooting = minimize_shooting(env)
```

```
Optimization terminated successfully.
The optimal cost is 0.008
```

[10 pt] Policy Optimization

We will start by learning a neural network policy using a shooting method. Fill in the code for `eval_policy`.

In [17]:
```
def eval_policy(env, policy, params):
    """
    Find the cost the policy with parameters params.
    Use the function step of the environment: env.step(action). It re
turns: next_state, cost, done,
    env_infos.

    You can set the parameters of the policy by policy.set_params(par
ams) and get the action for the current state
    with policy.get_action(state).
    """
    state = env.reset()
    total_cost = 0
    horizon = env.H
    policy.set_params(params)

    """YOUR CODE HERE"""
    for i in range(horizon):
        action = policy.get_action(state)
        state, cost, _, _ = env.step(action)
        total_cost += cost

    """YOUR CODE ENDS HERE"""
    return total_cost
```

In [18]:
```
def minimize_policy_shooting(env):
    policy_shooting = NNPolicy(env.dx, env.du, hidden_sizes=(10, 10))
    policy_shooting.init_params()
    params = policy_shooting.get_params()

    res = minimize(lambda x: eval_policy(env, policy_shooting, x),
                   params,
                   method='BFGS',
                   options={'xtol': 1e-6, 'disp': False, 'verbose': 2
})
    print(res.message)
    print("The optimal cost is %.3f" % res.fun)
    params_shooting = res.x
    policy_shooting.set_params(params_shooting)
    return policy_shooting

policy_shooting = minimize_policy_shooting(env)
```

```
Optimization terminated successfully.
The optimal cost is 0.008
```

[10 pt] Model Predictive Control

In [19]:
```
class MPCPolicy(object):
    def __init__(self, env, horizon):
        self.env = env
        self.H = horizon
        self.env = copy.deepcopy(env)
        np.random.seed(1)
        self.init_actions = np.random.uniform(low=-.1, high=.1, size=
(horizon * env.du,))

    def get_action(self, state, timestep):
        """
        Find the cost of the sequences of actions and state that have
shape [horizon, action dimension]
        and [horizon, state_dim], respectively.
        Use the function step of the environment: env.step(action). I
t returns, next_state, cost, done,
        env_infos.

        In order to set the environment at a specific state use the f
unction self.env.set_state(state)
        """
        env = self.env
        horizon = min(self.H, env.H - timestep)

        def eval_mpc(actions, state):
            actions = actions.reshape(horizon, env.du)
            total_cost = 0
            """YOUR CODE HERE"""
            self.env.set_state(state)
            for action in actions:
                state, cost, _, _ = env.step(action)
                total_cost += cost
            """YOUR CODE ENDS HERE"""
            return total_cost

        self.init_actions = np.random.uniform(low=-.1, high=.1, size=
(horizon * env.du,))
        res = minimize(lambda x: eval_mpc(x, state),
              self.init_actions,
              method='BFGS',
              options={'xtol': 1e-6, 'disp': False, 'verbose': 2}
             )
        act_shooting = res.x
        return act_shooting[:env.du]


    def reset(self):
        pass
```

In [20]:
```
mpc_policy = MPCPolicy(env, env.H)
```

Evaluation

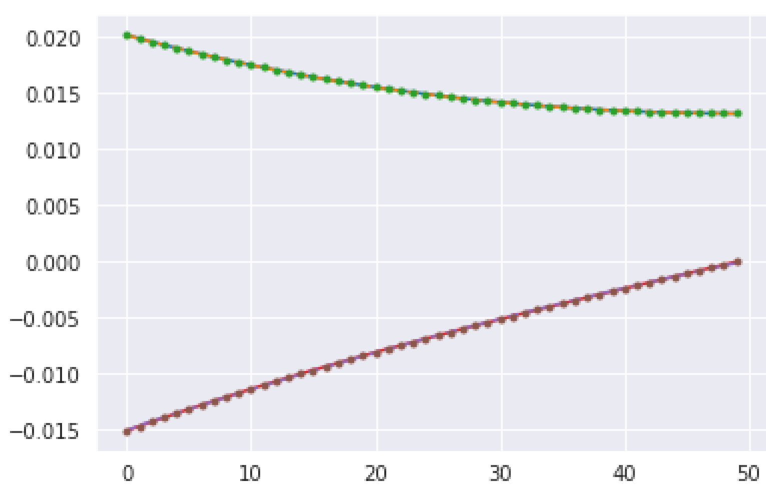
No noise

In [21]:
```
noise = 0.
cost_act, states_act = rollout(env, action_shooting, noise)
cost_pi, states_pi = rollout(env, policy_shooting, noise)
cost_mpc, states_mpc = rollout(env, mpc_policy, noise)
states_act, states_pi, states_mpc = np.array(states_act), np.array(st
ates_pi), np.array(states_mpc)
print("---- Quantitative Metrics ---")
print("Action Cost %.3f" % cost_act)
print("Policy Cost %.3f" % cost_pi)
print("MPC Cost %.3f" % cost_mpc)

print("\n\n---- Qualitative Metrics ---")
print("Evolution of the value of the angle and angular velocity of th
e cart-pole environment across 50 timesteps for the open-loop, policy
controller, and mpc controller.")
print("All the approaches achieve the same cost and follow the same t
rajectory. Open-loop: solid line(-);  Policy: dashed line(--). MPC: d
otted line(.)")
ts = np.arange(states_act.shape[0])
plt.plot(ts, states_act[:, 2], '-', ts, states_pi[:, 2], '--', states
_mpc[:, 2], '.')
plt.plot(ts, states_act[:, 3], '-', ts, states_pi[:, 3], '--', states
_mpc[:, 3], '.')
plt.show()
```

```
---- Quantitative Metrics ---
Action Cost 0.008
Policy Cost 0.008
MPC Cost 0.008


---- Qualitative Metrics ---
Evolution of the value of the angle and angular velocity of the cart-
pole environment across 50 timesteps for the open-loop, policy contro
ller, and mpc controller.
All the approaches achieve the same cost and follow the same trajecto
ry. Open-loop: solid line(-);  Policy: dashed line(--). MPC: dotted l
ine(.)
```



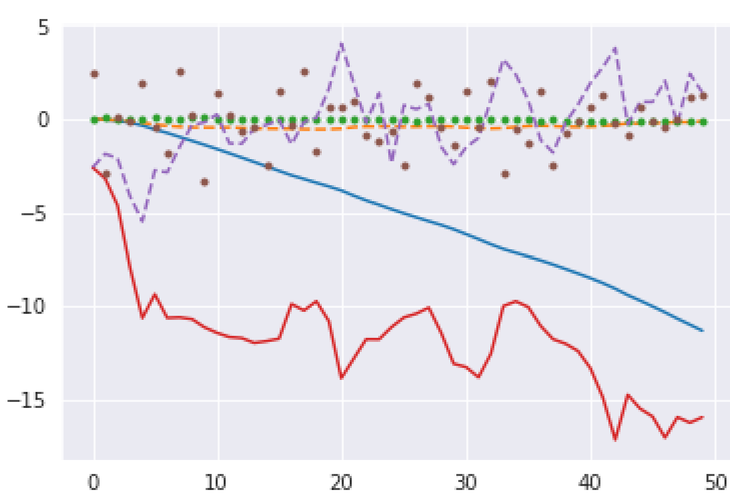
Noise

In [22]:
```
noise = 1.
cost_act, states_act = rollout(env, action_shooting, noise)
cost_pi, states_pi = rollout(env, policy_shooting, noise)
cost_mpc, states_mpc = rollout(env, mpc_policy, noise)
states_act, states_pi, states_mpc = np.array(states_act), np.array(st
ates_pi), np.array(states_mpc)
print("---- Quantitative Metrics ---")
print("Action Cost %.3f" % cost_act)
print("Policy Cost %.3f" % cost_pi)
print("MPC Cost %.3f" % cost_mpc)

print("\n\n---- Qualitative Metrics ---")
print("Evolution of the value of the angle and angular velocity of th
e cart-pole environment across 50 timesteps for the open-loop, policy
controller, and mpc controller.")
print("In the presence of noise, the open-loop controller fails to st
ablize the pole, while the policy and mpc controller succeed. The MPC
approach achieves the best performance. Open-loop: solid line(-);  Po
licy: dashed line(--). MPC: dotted line(.)")
ts = np.arange(states_act.shape[0])
plt.plot(ts, states_act[:, 2], '-', ts, states_pi[:, 2], '--', states
_mpc[:, 2], '.')
plt.plot(ts, states_act[:, 3], '-', ts, states_pi[:, 3], '--', states
_mpc[:, 3], '.')
plt.show()
```

```
---- Quantitative Metrics ---
Action Cost 4544.439
Policy Cost 99.745
MPC Cost 55.286


---- Qualitative Metrics ---
Evolution of the value of the angle and angular velocity of the cart-
pole environment across 50 timesteps for the open-loop, policy contro
ller, and mpc controller.
In the presence of noise, the open-loop controller fails to stablize
the pole, while the policy and mpc controller succeed. The MPC approa
ch achieves the best performance. Open-loop: solid line(-);  Policy:
dashed line(--). MPC: dotted line(.)
```



Why does the MPC method perform better than having a policy? Is there anyway we could make the performance of the policy better?

Reply in no more than 5 lines in the box below.

[10 pt] Response:

- This is because policy roll-out can be very unstable under noisy environments since small noises (numerical errors) can snowball as timestep increases. Therefore, having a policy can make the formulation too unstable to optimize.
- However, if we use the MPC method, we simply take the first actions and solve the optimization problem, so previous noise wouldn't affect the next optimization problem, and thus noises will not amplify as much.

Part 5 [20 pt] - Optimization Methods

In the previous parts, in order to optimize the collocation methods, we have used a built-in constrained optimization algorithm. However, creating our own contrained optimization solver is fairly easy given a general solver that minimizes unconstrained functions. Here, we implement two solvers by using the merit function.

[10 pt] Merit function

Given a standard constrained optimization problem:

$$\begin{aligned} &\min_x g_0(x) \\ \text{s.t.:} \quad & g_i(x) \leq 0 \quad \forall i \\ & h_j(x) = 0 \quad \forall j \end{aligned}$$

We can construct its *merit function* f_μ as

$$f_\mu(x) = g_0 + \mu \sum_i |g_i(x)|^+ + \mu \sum_j |h_j(x)|$$

The merit function allows us to transform a constrained optimization problem to an unconstrained one that has the same optimum as $\mu \to \infty$. Here, we will just solve collocation problems without any constrain on the state space. As a result, we will not have inequality constraints.

In [23]:
```
def merit_function(env, mu, x):
    """
    Implementation of the merit function. We use the previously defin
ed functions eval_collocation and constraints
    to obtain the cost and error of the variables.

    Note: code it in a way that mu can be either a scalar or a vector
    """
    cost = eval_collocation(env, x)
    cons = constraints(env, x)

    """YOUR CODE HERE"""
    merit_val = cost + np.sum(mu * np.abs(cons))# Fill this
    """YOUR CODE ENDS HERE"""
    return merit_val
```

[5 pt] Penalty Formulation

The easiest implementation is the penalty formulation. The penalty formulation iterates between finding the minimum of the merit function and increasing the scalar value of μ.

In [24]:
```
t = 1.5
mu = 1
init_states_and_actions = np.random.uniform(low=-.1, high=.1, size=(e
nv.H * (env.du + env.dx),))
num_iter = 5


for i in range(num_iter):
    """
    Otimization of the penalty function, which after finding the mini
mium for the merrit function we increase the
    value of mu. The value of mu should be increased as specified in
 the lecture.
    """

    """YOUR CODE HERE"""

    mu = t * mu # Fill this

    """YOUR CODE ENDS HERE"""


    res = minimize(lambda x: merit_function(env, mu, x),
                   init_states_and_actions,
                   method='BFGS',
                   options={'xtol': 1e-6, 'disp': False, 'verbose': 2, 'm
axiter':201}
                  )
    print("\nIteration %d:"% i)
    print("Value of mu %.3f" % mu)
    print("Inner optimization: %s" % res.message)
    print("Value of merit function %.3f" % res.fun)
    if np.linalg.norm(init_states_and_actions - res.x) < 1e-6: break
    init_states_and_actions = res.x



states_var_penalty, act_penalty = res.x[:env.H * env.dx], res.x[env.H
* env.dx:]
states_var_penalty = states_var_penalty.reshape(env.H, env.dx)
act_penalty = ActPolicy(env, act_penalty)
```

```
Iteration 0:
Value of mu 1.500
Inner optimization: Maximum number of iterations has been exceeded.
Value of merit function 0.488

Iteration 1:
Value of mu 2.250
Inner optimization: Maximum number of iterations has been exceeded.
Value of merit function 0.100

Iteration 2:
Value of mu 3.375
Inner optimization: Desired error not necessarily achieved due to pre
cision loss.
Value of merit function 0.059

Iteration 3:
Value of mu 5.062
Inner optimization: Desired error not necessarily achieved due to pre
cision loss.
Value of merit function 0.038

Iteration 4:
Value of mu 7.594
Inner optimization: Desired error not necessarily achieved due to pre
cision loss.
Value of merit function 0.028
```

[5 pt] Dual Descent

A better method is the dual descent formulation, which directly solves the Langrangian of the previous optimization problem:

$$\max_{\lambda_i, \nu_j} \min_x g_0 + \sum_i \lambda_i g_i(x) + \sum_j \nu_j h_j(x)$$

The dual descent method iterates between solving the inner minimization problem and taking a gradient step on the dual variables λ_i and ν_j. Here, again, we omit the g_i and λ_i terms since we do not have these constraints.

However, using the merit function instead of the Lagrangian results in a more stable behavior. For this excersice, we use the merit function. In such case, the function $h_j(x)$ is $|x_{j+1} - f(x_j, u_j)|$.

In [25]:
```
init_states_and_actions = np.random.uniform(low=-.1, high=.1, size=(e
nv.H * (env.du + env.dx),))
nu = 1.5 * np.ones_like(constraints(env, init_states_and_actions))
alpha = 1
num_iter = 5

for i in range(num_iter):
    """
    Otimization using dual descent, at each iteration we find the opt
imal for the merrit function, and then take
    a gradient step for nu.
    """
    res = minimize(lambda x: merit_function(env, nu, x),
                   init_states_and_actions,
                   method='BFGS',
                   options={'xtol': 1e-6, 'disp': False, 'verbose': 0, 'm
axiter':201}
                  )
    print("\nIteration %d:"% i)
    print("Norm of nu %.3f" % np.linalg.norm(nu))
    print("Inner optimization: %s" % res.message)
    print("Value of lagrangian %.3f" % res.fun)
    if np.linalg.norm(init_states_and_actions - res.x) < 1e-6: break

    init_states_and_actions = res.x

    """
    Use the function constraints(env, init_state_and_actions) and the
learning rate alpha to update the
    value of mu.
    """
    """YOUR CODE HERE """
    nu = nu + alpha * constraints(env, init_states_and_actions) # Fil
l this
    """YOUR CODE ENDS HERE"""

states_var_dual_descent, act_dual_descent = res.x[:env.H * env.dx], r
es.x[env.H * env.dx:]
states_var_dual_descent = states_var_dual_descent.reshape(env.H, env.
dx)
act_dual_descent = ActPolicy(env, act_dual_descent)
```

```
Iteration 0:
Norm of nu 21.213
Inner optimization: Maximum number of iterations has been exceeded.
Value of lagrangian 0.511

Iteration 1:
Norm of nu 21.220
Inner optimization: Maximum number of iterations has been exceeded.
Value of lagrangian 0.081

Iteration 2:
Norm of nu 21.220
Inner optimization: Desired error not necessarily achieved due to pre
cision loss.
Value of lagrangian 0.038

Iteration 3:
Norm of nu 21.220
Inner optimization: Desired error not necessarily achieved due to pre
cision loss.
Value of lagrangian 0.023

Iteration 4:
Norm of nu 21.220
Inner optimization: Desired error not necessarily achieved due to pre
cision loss.
Value of lagrangian 0.018
```

Evaluation

In [26]:
```
cost_penalty, states_penalty = rollout(env, act_penalty)
cost_dual_descent, states_dual_descent = rollout(env, act_dual_descen
t)
states_penalty, states_dual_descent = np.array(states_penalty), np.ar
ray(states_dual_descent)
error_penalty = np.linalg.norm(states_penalty - np.array(states_var_p
enalty))
error_dual_descent = np.linalg.norm(states_dual_descent - np.array(st
ates_var_dual_descent))

print("---- Quantitative Metrics ---")
print("Cost Penalty %.3f" % cost_penalty)
print("Cost Dual Descent %.3f" % cost_dual_descent)

print("Error Penalty %.3f" % error_penalty)
print("Error Dual Descent %.3f" % error_dual_descent)

print("\n\n---- Qualitative Metrics ---")
print("Evolution of the value of the angle and angular velocity of th
e cart-pole environment across 50 timesteps for the penalty and dual
 descent methods.")
print("Dual descent yields to slighlthly better results. Both present
non-zero error on the constraints and fail to stabilize the cart-pol
e.  Penalty: solid line(-);  Dual descent: dashed line(--).")
ts = np.arange(states_penalty.shape[0])
plt.plot(ts, states_penalty[:, 2], '-', ts, states_dual_descent[:, 2
], '--')
plt.plot(ts, states_penalty[:, 3], '-', ts, states_dual_descent[:, 3
], '--')
plt.show()
```

```
---- Quantitative Metrics ---
Cost Penalty 0.017
Cost Dual Descent 0.017
Error Penalty 0.000
Error Dual Descent 0.000


---- Qualitative Metrics ---
Evolution of the value of the angle and angular velocity of the cart-
pole environment across 50 timesteps for the penalty and dual descent
methods.
Dual descent yields to slighlthly better results. Both present non-ze
ro error on the constraints and fail to stabilize the cart-pole.  Pen
alty: solid line(-);  Dual descent: dashed line(--).
```