Knowledge Representation & Reasoning
COMP9016

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First Order Logic

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Outline

>> Why FOL?

>> Syntax and semantics of FOL

>> Fun with sentences

>> Wumpus world in FOL

>> World[2,2]←Pit

> What is lacking in programming languages?

>> A second drawback: “There is a pit in [2,2] or [3,1]” or “If the wumpus is in [1,1] then he is not in [2,2].”

Pros and cons of propositional logic

>> Propositional logic is **declarative**: pieces of syntax correspond to facts.

>> Propositional logic allows partial/disjunctive/negated information (unlike most data structures and databases).

>> Propositional logic is **compositional**: meaning of $\boldsymbol{B_{1,1} \wedge P_{1,2}}$ is derived from meaning of $\mathbf{B_{1,1}}$ and of $P_{\mathbf{1,2}}$

>> Meaning in propositional logic is **context-independent** (unlike natural language, where meaning depends on context)

>> Propositional logic has very limited expressive power (unlike natural language) E.g., cannot say *"pits cause breezes in adjacent squares"* except by writing one sentence for each square

First-order logic

>> Whereas propositional logic assumes world contains **facts**, first-order logic (like natural language) assumes the world contains

> **Objects:** people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries . . .
> **Relations:** red, round, bogus, prime, multistoried . . .,
brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . .
> **Functions**: father of, best friend, third inning of, one more than, end of
. . .

"One plus two equals three."

"Squares neighboring the wumpus are smelly."

What are the Objects? Relation? Function?

Logics in general

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

>> **Ontological commitment** - what it assumes about the nature of reality.

>> **Epistemological commitments -** the possible states of knowledge that it allows with respect to each fact.

Syntax of FOL: Basic elements

Constants	$KingJohn, 2, UCB, \ldots$
Predicates	$Brother, >, \ldots$
Functions	$Sqrt, LeftLegOf, \ldots$
Variables	$x, y, a, b, \ldots$
Connectives	$\wedge \vee \neg \Rightarrow \ \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall \exists$

>> “All kings are persons”: $\forall$ x King(x) $\Rightarrow$ Person(x) .

>> “King John has a crown on his head”: $\exists$ x Crown(x) $\wedge$ OnHead(x, John)

Atomic sentences

Atomic sentence = $predicate(term_1, \ldots, term_n)$
or $term_1 = term_2$

Term = $function(term_1, \ldots, term_n)$
or $constant$ or $variable$

E.g., $Brother(KingJohn, RichardTheLionheart)$
$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

>> An **atomic sentence** (or **atom** for short) is formed from a predicate symbol optionally followed by a parenthesized list of terms, such as the above example.

Complex sentences

Complex sentences are made from atomic sentences using connectives

$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$

E.g. $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$

$>(1,2) \vee \leq(1,2)$

$>(1,2) \wedge \neg >(1,2)$

Truth in first-order logic

>> Sentences are true with respect to a **model** and an **interpretation**

>> Model contains ≥ 1 objects (**domain elements**) and relations among them

>> Interpretation specifies referents for:

constant symbols → objects predicate symbols → relations
function symbols → functional relations

An atomic sentence $predicate(term_1, \ldots, term_n)$ is true iff the objects referred to by $term_1, \ldots, term_n$ are in the relation referred to by $predicate$

Models for FOL: Example

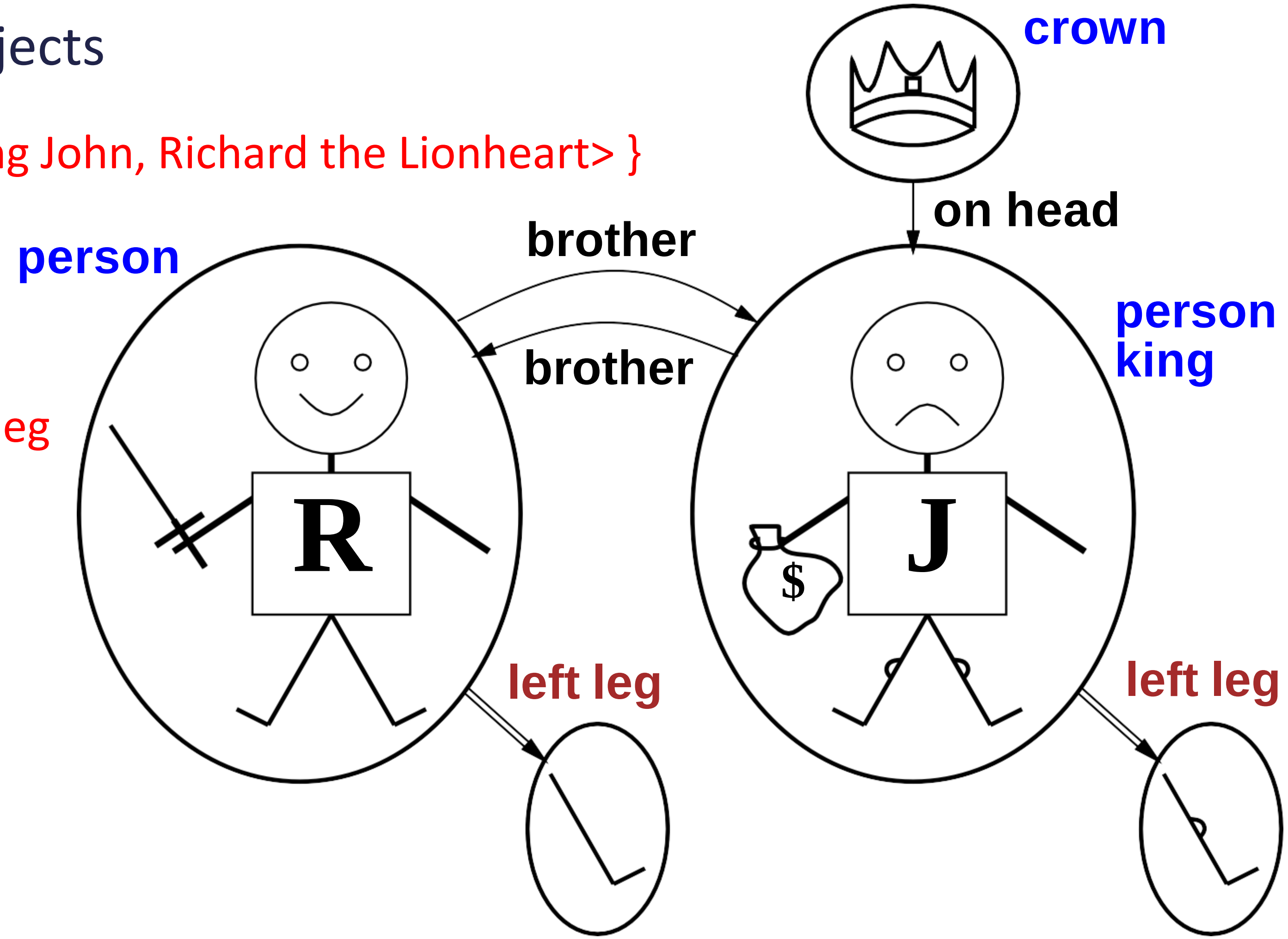
>> **Domain** of a model – set of objects

{ <Richard the Lionheart, King John>, <King John, Richard the Lionheart> }

>> **Functions**

<Richard the Lionheart> → Richard’s left leg

<King John> John’s left leg .



Truth example

Consider the interpretation in which

$Richard$ → Richard the Lionheart

$John$ → the evil King John

$Brother$ → the brotherhood relation

Under this interpretation, $Brother(Richard, John)$ is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

Models for FOL: Lots!

Entailment in propositional logic can be computed by enumerating models

We can enumerate the FOL models for a given KB vocabulary: For each number of domain elements n from 1 to ∞

For each k-ary predicate P_k in the vocabulary For each possible k-ary relation on n objects

For each constant symbol C in the vocabulary

For each choice of referent for C from n objects . . .

Computing entailment by enumerating FOL models is not easy!

Universal quantification

$\forall$ <*variables*> <*sentence*>

Everyone at Berkeley is smart:

$\forall x \quad At(x, Berkeley) \quad \Rightarrow \quad Smart(x)$

$\forall x\, P$ is true in a model m iff P is true with x being each possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of P

$$
\begin{aligned}
& (At(KingJohn, Berkeley) \Rightarrow Smart(KingJohn)) \\
\wedge\ & (At(Richard, Berkeley) \Rightarrow Smart(Richard)) \\
\wedge\ & (At(Berkeley, Berkeley) \Rightarrow Smart(Berkeley)) \\
\wedge\ & \ldots
\end{aligned}
$$

Universal quantification Example

"All Kings are persons"

$\forall\, x\; King(x) \Rightarrow Person(x)$

Extending the interpretation:

- x → Richard the Lionheart,
- x → King John,
- x → Richard's left leg,
- x → John's left leg,
- x → the crown.

Equivalent to asserting the following five sentences

Rich. the Lionheart is a king ⇒ Rich. the Lionheart is a person.

King John is a king ⇒ King John is a person.

Richard's left leg is a king ⇒ Richard's left leg is a person.

John's left leg is a king ⇒ John's left leg is a person.

The crown is a king ⇒ the crown is a person.

P	Q	$P \Rightarrow Q$
false	*false*	*true*
false	*true*	*true*
true	*false*	*false*
true	*true*	*true*

A common mistake to avoid

Typically, $\Rightarrow$ is the main connective with $\forall$

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x \; At(x, \; Berkeley) \wedge Smart(x)$$

means "Everyone is at Berkeley and everyone is smart"

Existential quantification

$\exists \langle variables \rangle \quad \langle sentences \rangle$

Someone at Stanford is smart:

$\exists x \quad At(x, Stanford) \wedge Smart(x)$

$\exists x \quad P$ is true in a model m iff P is true with x being some possible object in the model

Roughly speaking, equivalent to the disjunction of instantiations of P

$$
\begin{aligned}
& (At(KingJohn, Stanford) \wedge Smart(KingJohn)) \\
\vee\ & (At(Richard, Stanford) \wedge Smart(Richard)) \\
\vee\ & (At(Stanford, Stanford) \wedge Smart(Stanford)) \\
\vee\ & \ldots
\end{aligned}
$$

Existential quantification

"All Kings are persons"

$\exists x \; Crown(x) \; \wedge \; OnHead(x,John)$

More precisely, ***∃x P*** is true in a given model if P is true in at least one extended interpretation that assigns x to a domain element.

That is, at least one of the following is true:

Richard the Lionheart is a crown ∧ Richard the Lionheart is on John's head;
King John is a crown ∧ King John is on John's head;
Richard's left leg is a crown ∧ Richard's left leg is on John's head;
John's left leg is a crown ∧ John's left leg is on John's head;
The crown is a crown ∧ the crown is on John's head.

Another common mistake to avoid

Typically, $\wedge$ is the main connective with $\exists$

Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists\, x\ At(x, Stanford) \Rightarrow Smart(x)$$

is true if there is anyone who is not at Stanford!

Properties of quantifiers

$\forall x \quad \forall y$ is the same as $\forall y \quad \forall x$ (why??)

$\exists x \quad \exists y$ is the same as $\exists y \quad \exists x$ (why??)

$\exists x \quad \forall y$ is **not** the same as $\forall y \; \exists x$

$\exists x \; \forall y \; Loves(x, y)$

"There is a person who loves everyone in the world"

$\forall y \; \exists x \; Loves(x, y)$

"Everyone in the world is loved by at least one person"

Quantifier duality: *each can be expressed using the other*

$\forall x \, Likes(x, IceCream) \qquad \neg \exists x \neg Likes(x, IceCream)$

$\exists x \, Likes(x, Broccoli) \qquad \neg \forall x \neg Likes(x, Broccoli)$

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Fun with sentences

Brothers are siblings

$$\forall x, y\, Brother(x, y) \Rightarrow Sibling(x, y)$$

"Sibling" is symmetric

$$\forall x, y\, Sibling(x, y) \Leftrightarrow Sibling(y, x).$$

One's mother is one's female parent

$$\forall x, y\, Mother(x, y) \Leftrightarrow (Female(x) \wedge Parent(x, y)).$$

A first cousin is a child of a parent's sibling

$$\forall x, y\ FirstCousin(x, y) \Leftrightarrow \exists p, ps\ Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)$$

Equality

$term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object

E.g., $1 = 2$ *and* $\forall_x \; \times(Sqrt(x), Sqrt(x)) = x$ *are satisfiable*
$2 = 2$ *is valid*

E.g., definition of (full) $Sibling$ in terms of $Parent$:

$$\forall_{x,y} \; Sibling(x, y) \Leftrightarrow [\neg(x = y) \wedge \exists_{m,f} \neg(m = f) \wedge Parent(m, x) \wedge Parent(f, x) \wedge Parent(m, y) \wedge Parent(f, y)]$$

Interacting with FOL KBs

>> Suppose a wumpus-world agent is using an FOL KB and perceives a smell and a breeze (but no glitter) at $t = 5$:

$$Tell(KB, Percept([Smell, Breeze, None], 5))$$
$$Ask(KB, \exists a\ Action(a, 5))$$

I.e., does KB entail any actions at $t = 5$?

>> Answer: $Yes, \{a/Shoot\}$ ← substitution (binding list)

Given a sentence S and a substitution σ,

$S\sigma$ denotes the result of plugging σ into S; e.g.,
$S = Smarter(x, y)$
$\sigma = \{x/Biden, y/Trump\}$
$S\sigma = Smarter(Biden, Trump)$

$Ask(KB, S)$ returns some/all σ such that $KB \models S\sigma$

The kinship domain

>> The domain of family relationships, or kinship. This domain includes facts such as "Elizabeth is the mother of Charles" and "Charles is the father of William" and rules such as "One's grandmother is the mother of one's parent."

>> The objects in our domain are people.

>> We have two unary predicates, Male and Female.

>> Kinship **relations**—parenthood, brotherhood, marriage, and so on—are represented by **binary predicates**: *Parent, Sibling, Brother , Sister , Child , Daughter, Son, Spouse, Wife, Husband, Grandparent , Grandchild , Cousin, Aunt, and Uncle.*

>> We use functions for *Mother* and *Father* , because every person has exactly one of each of these (at least according to nature's design).

The kinship domain

>> We can go through each function and predicate, writing down what we know in terms of the other symbols. For example, one's mother is one's female parent:

$\forall_{m,c}\ Mother(c) = m \Leftrightarrow Female(m) \wedge Parent(m,c)$.

One's husband is one's male spouse:

$\forall_{w,\ h}\ Husband(h,w) \Leftrightarrow Male(h) \wedge Spouse(h,w)$.

Male and female are disjoint categories:

$\forall_{x}\ \mathrm{Male}(x) \Leftrightarrow \neg\mathrm{Female}(x)$.

Parent and child are inverse relations:

$\forall_{p,\ c}\ \mathrm{Parent}(p, c) \Leftrightarrow \mathrm{Child}\ (c, p)$.

A grandparent is a parent of one's parent:

$\forall_{g,\ c}\ Grandparent\ (g, c) \Leftrightarrow \exists p\ Parent(g, p) \wedge Parent(p, c)$.

A sibling is another child of one's parents:

$\forall_{x,\ y}\ Sibling(x, y) \Leftrightarrow x\ = y \wedge \exists p\ Parent(p, x) \wedge Parent(p, y)$

Theorems

>> Not all logical sentences about a domain are axioms. Some are theorems—that is, they are entailed by the axioms. For example, consider the assertion that siblinghood is symmetric:

$\forall_{x,\,y}$ *Sibling(x, y)* $\Leftrightarrow$ *Sibling(y, x) - Is this an axiom or a theorem?*

>> If we ASK the KB what will be returned?

>> Not all axioms are definitions.

$\forall_{x}$ *Person(x)* $\Leftrightarrow$ *. . .*

>> Partial specifications of properties that every person has and properties that make something a person can be written

Knowledge base for the wumpus world

>> Percept ([Stench, Breeze, Glitter , None, None], 5)

>> Turn(Right), Turn(Left), Forward , Shoot , Grab, Climb

>> *ASKVARS*($\exists_a$ *BestAction(a, 5))* ,

"Perception"

$$\forall b, g, t \quad Percept([Stench, b, g], t) \quad \Rightarrow \quad Stench(t)$$

$$\forall s, b, t \quad Percept([s, b, Glitter], t) \quad \Rightarrow \quad AtGold(t)$$

Reflex: $\forall t \quad AtGold(t) \Rightarrow Action(Grab, t)$

Reflex with internal state: do we have the gold already?

$$\forall t \; AtGold(t) \wedge \neg Holding(Gold, t) \Rightarrow Action(Grab, t)$$

$Holding(Gold, \; t)$ cannot be observed

$\Rightarrow$ keeping track of change is essential

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Deducing hidden properties

Properties of locations:

$$\forall x, t\, At(Agent, x, t) \wedge Smelt(t) \Rightarrow Smelly(x)$$
$$\forall x, t\, At(Agent, x, t) \wedge Breeze(t) \Rightarrow Breezy(x)$$

Squares are breezy near a pit:

Diagnostic rule—infer cause from effect

$$\forall y\, Breezy(y) \Rightarrow \exists x\, Pit(x) \wedge Adjacent(x, y)$$

Causal rule—infer effect from cause

$$\forall x, y\, Pit(x) \wedge Adjacent(x, y) \Rightarrow Breezy(y)$$

Neither of these is complete—e.g., the causalrule doesn't say whether squares far away from pits can be breezy

Definition for the $Breezy$ predicate:

$$\forall y\; Breezy(y) \Leftrightarrow [\exists x\; Pit(x) \wedge Adjacent(x, y)]$$

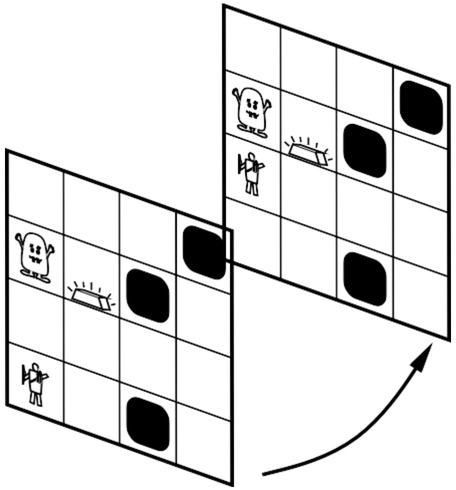
Keeping track of change

Facts hold in **situations**, rather than eternally
E.g., $Holding(Gold, Now)$ rather than just $Holding(Gold)$

Situation calculus is one way to represent change in FOL:

Adds a situation argument to each non-eternal predicate E.g.,
Now in $Holding(Gold, Now)$ denotes a situation

Situations are connected by the $Result$ function
$Result(a, s)$ is the situation that results from doing a in s



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Describing actions I

"Effect" axiom—describe changes due to action

$$\forall s\, AtGold(s) \Rightarrow Holding(Gold, Result(Grab, s))$$

"Frame" axiom—describe non-changes due to action

$$\forall s\, HaveArrow(s) \Rightarrow HaveArrow(Result(Grab, s))$$

Frame problem: find an elegant way to handle non-change

(a) representation—avoid frame axioms

(b) inference—avoid repeated "copy-overs" to keep track of state

Qualification problem: true descriptions of real actions require endless caveats—what if gold is slippery or nailed down or . . .

Ramification problem: real actions have many secondary consequences— what about the dust on the gold, wear and tear on gloves, . . .

Describing actions II

Successor-state axioms solve the representational frame problem

Each axiom is "about" a predicate (not an action per se):

P true afterwards $\Leftrightarrow$ [an action made P true
$\vee$ P true already and no action made P false]

For holding the gold:

$$\forall a, s\, Holding(Gold, Result(a, s)) \Leftrightarrow$$
$$[(a = Grab \wedge AtGold(s))$$
$$\vee\ (Holding(Gold, s) \wedge a \neq Release)]$$

Making plans

Initial condition in KB:

$At(Agent, [1, 1], S_0)$
$At(Gold, [1, 2], S_0)$

Query: $Ask(KB, \exists s\, Holding(Gold, s))$

i.e., in what situation will I be holding the gold?

Answer: $\{s/Result(Grab, Result(Forward, S_0))\}$

i.e., go forward and then grab the gold

This assumes that the agent is interested in plans starting at S_0 and that S_0 is the only situation described in the KB

Making plans: A better way

Represent plans as action sequences $[a_1, a_2, \ldots, a_n]$

$PlanResult(p, s)$ is the result of executing p in s

Then the query $Ask(KB, \exists p \quad Holding(Gold, PlanResult(p, S_0)))$
has the solution $\{p/[Forward, Grab]\}$

Definition of $PlanResult$ in terms of $Result$:

$$\forall s \; PlanResult([\,], s) = s$$
$$\forall a, p, s \quad PlanResult([a|p], s) = PlanResult(p, Result(a, s))$$

Planning systems are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

Summary

>> First-order logic:
- objects and relations are semantic primitives
- syntax: constants, functions, predicates, equality, quantifiers

>> Increased expressive power: sufficient to define Wumpus world

Situation calculus:

> Conventions for describing actions and change in FOL

> Can formulate planning as inference on a situation calculus KB

References

[1] Russell, S. and Norvig, P., 2002. Artificial intelligence: a modern approach Logical Agents, Chapter 8.

[2] – Based on Lecture slides, chapter08.pdf, by Russell, S.