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PHYS 513
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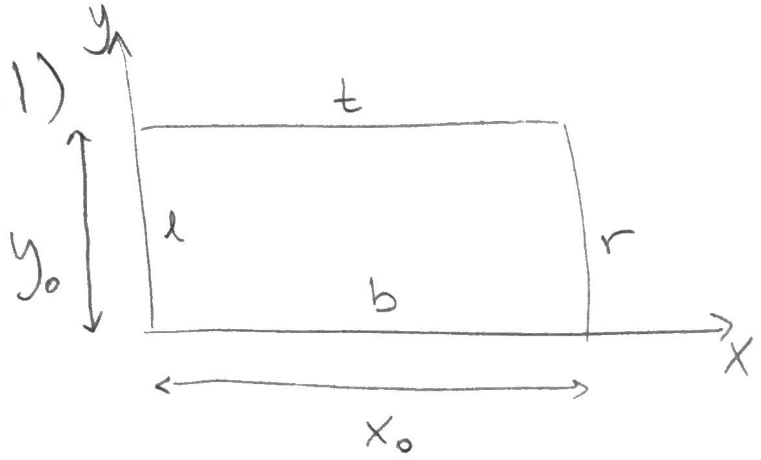
HW 2

2.1)



Calculate the potential with the following Boundary conditions

1.) $x=0 \rightarrow \Phi_l(0,y) = V_l$ (i)
$x = x_0 \rightarrow \Phi_l(x_0, y) = 0$ (ii)
$y=0 \rightarrow \Phi_l(x,0) = 0$ (iii)
$y=y_0 \rightarrow \Phi_l(x,y_0) = 0$ (iv)

$\Phi_l(x,y) = X(x)\,Y(y)$ & $\nabla^2 \Phi_l(x,y) = 0$

$$X(x)\,Y''(y) = X''(x)\,Y(y)$$

$$\frac{X(x)}{X''(x)} = \frac{Y(y)}{Y''(y)}$$

$$\frac{X(x)}{X''(x)} = k^2 \quad \& \quad \frac{Y(y)}{Y''(y)} = -k^2$$

$$X(x) = Ae^{kx} + Be^{-kx} \quad \& \quad Y(y) = C\sin(ky) + D\cos(ky)$$

Enter Boundary Condition (ii)

$$X(x_0) = 0 = Ae^{kx_0} + Be^{-kx_0}$$

$$Ae^{kx_0} = -Be^{-kx_0}$$

$$A = -A' e^{-kx_0} \quad \& \quad B = A' e^{kx_0}$$

$$X(x) = A'\left(e^{k(x_0-x)} - e^{k(x-x_0)}\right)$$ $A' \rightarrow A$ for simplicity

$$\underline{X(x) = 2A \sinh(k(x_0 - x))}$$ $\rightarrow$ when $x=0$, $X(0) > 0$ ✓

absorb 2 into A

Enter Boundary condition (iii)

$$Y(0) = 0 = C\sin(k \cdot 0) + D\cos(k0)$$

$$0 = 0 + D$$

$$\therefore \underline{D = 0}$$

Enter Boundary Condition (iv)

$$Y(y_0) = 0 = C\sin(ky_0)$$

$$\sin^{-1}(0) = ky_0$$

$$n\pi = ky_0 \quad \text{where } n = 1, 2, 3, \ldots$$

$$\therefore k = \frac{n\pi}{y_0} \quad \text{where } n = 1, 2, 3, \ldots$$

(I will leave this off until solutions are found)

Enter Boundary condition (i)

$$\Phi_\ell(x,y) = \underbrace{2CA}\sinh\left(\frac{n\pi}{y_0}(x_0-x)\right)\sin\left(\frac{n\pi y}{y_0}\right)$$

$$\Phi_\ell(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{y_0}(x_0-x)\right)\sin\left(\frac{n\pi y}{y_0}\right)$$

$$\Phi_\ell(0,y) = V_\ell = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi x_0}{y_0}\right)\sin\left(\frac{n\pi y}{y_0}\right)$$

<u>Fourier's Trick</u>

$$A_n \sinh\left(\frac{n\pi x_0}{y_0}\right) = \frac{2}{a}\int_0^a V_\ell \sin\left(\frac{n\pi y}{y_0}\right)dy$$

$$\therefore A_n = \frac{1}{\sinh\left(\frac{n\pi x_0}{y_0}\right)} \cdot \frac{2}{\cancel{a}} \cdot V_\ell \cdot \frac{\cancel{a} - \cancel{a}\cos(\pi n)}{\pi n}$$ ← only valid when n is odd, else 0

$$\therefore A_n = \frac{4V_\ell}{\pi n \sinh\left(\frac{n\pi x_0}{y_0}\right)} \quad \text{where } n = 1, 3, 5, \ldots$$

$$\boxed{\Phi_\ell(x,y) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_\ell}{\pi n} \frac{\sinh\left(\frac{n\pi}{y_0}(x_0-x)\right)\sin\left(\frac{n\pi y}{y_0}\right)}{\sinh\left(\frac{n\pi x_0}{y_0}\right)}}$$

2.) Calc with following boundary conditions

$\Phi_b(0, y) = 0$

$\Phi_b(x_0, y) = 0$

$\Phi_b(x, 0) = V_b$

$\Phi_b(x, y_0) = 0$

Same set up as part 1, but x and y are switched.

Thus the solution is

$$\boxed{\Phi_b(x,y) = \sum_{n=1,3,5\ldots}^{\infty} \frac{4V_b}{\pi n} \frac{\sinh\left(\frac{n\pi}{x_0}(y_0-y)\right)\sin\left(\frac{n\pi x}{x_0}\right)}{\sinh\left(\frac{n\pi y_0}{x_0}\right)}}$$

3.) Calc with following boundary conditions

$\Phi_t(0, y) = 0$ (i)

$\Phi_t(x_0, y) = 0$ (ii)

$\Phi_t(x, 0) = 0$ (iii)

$\Phi_t(x, y_0) = V_t$ (iv)

Given that $\Phi_t(x,y) = X(x)Y(y)$, then $X(x)$ is the same as in part 2

→ Start with Boundary Condition (iii)

$Y(0) = 0 = Ce^{k0} + De^{-k0}$ ← $k = \frac{n\pi}{x_0}$

$0 = C + D$

$C = -D$

$Y(y) = C'(e^{ky} - e^{-ky})$

$Y(y) = 2C' \sinh\left(\frac{n\pi y}{x_0}\right)$

Boundary condition at y_0 is the same for $y=0$ in part 2

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$\Phi_t(x, y_0) = V_t = C_n \sinh\left(\frac{n\pi y_0}{x_0}\right) \sin\left(\frac{n\pi x}{x_0}\right)$

Since Boundary condition leads to the same value for A_n (C_n)

The answer is

$$\boxed{\Phi_t(x,y) = \sum_{n=1,3,5\ldots}^{\infty} \frac{4V_0}{\pi n} \frac{\sinh\left(\frac{n\pi y}{x_0}\right)\sin\left(\frac{n\pi x}{x_0}\right)}{\sinh\left(\frac{n\pi y_0}{x_0}\right)}}$$

Note: the difference in this solution is that $\sinh\left(\frac{n\pi}{x_0}(y-y_0)\right)$ becomes $\sinh\left(\frac{n\pi}{x_0}y\right)$

4.) Calc with following boundary conditions

$\Phi_r(0,y) = 0$

$\Phi_r(x_0, y) = V_r$

$\Phi_r(x, 0) = 0$

$\Phi_r(x, y_0) = 0$

Same as geometry from 3, except switch x and y

$$\therefore \boxed{\Phi_r(x,y) = \sum_{n=1,3,5\ldots}^{\infty} \frac{4V_0}{\pi n} \frac{\sinh\left(\frac{n\pi x}{y_0}\right)\sin\left(\frac{n\pi y}{y_0}\right)}{\sinh\left(\frac{n\pi y_0}{x_0}\right)}}$$

5.) Justify $\Phi(x,y) = \Phi_l(x,y) + \Phi_r(x,y) + \Phi_t(x,y) + \Phi_b(x,y)$ is a general solution to this problem

Using the uniqueness theorem, any solution to Laplaces equation and satisfies the boundary conditions is the unique solution. Since Φ_l, Φ_r, Φ_t, & Φ_b satisfy each boundary independently, adding them all together gives a unique solution.

Technically should also mention superposition.

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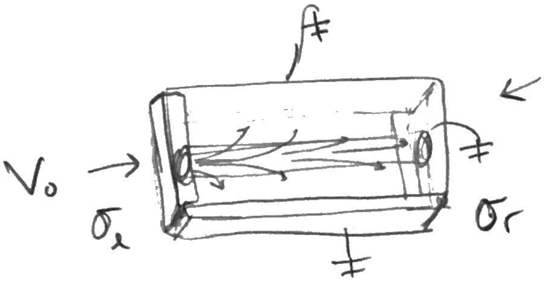
6.) Can Gauss's law be used in any way to check your solution?

Yes, this was discussed in class.



← given the same set up with exaggerated thickness create a gaussian cylinder

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \qquad \frac{Q_{enc}}{\epsilon_0} \rightarrow \frac{(\sigma_l + \sigma_r)\pi r^2}{\epsilon_0}$$

$$\oint \vec{E} \cdot d\vec{a} \rightarrow \underbrace{\cancel{\int \vec{E}_l \cdot d\vec{a}}}_{\perp} + \int \vec{E}_{in} \cdot d\vec{a} + \underbrace{\cancel{\int \vec{E}_r \cdot d\vec{a}}}_{\perp}$$

these are zero because they are perpendicular

$$\int \vec{E}_{in} \cdot d\vec{a} = \frac{(\sigma_l + \sigma_r)\pi r^2}{\epsilon_0}$$

↗

This E field could be used to validate the E_l solution.