6.1.1) Find the magnetic field $\mathbb{B}$ that must exist using $\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t}$

$$\nabla \times \mathbb{E} = \nabla \times \left(E_{ox} \cos(k_z z - \omega t)\hat{x} \right)$$

$$\begin{matrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ E_x & 0 & 0 \end{matrix}$$

$$\langle 0,\ \partial_z E_x - 0,\ 0 - \cancelto{0}{\partial_y E_x} \rangle$$

$$\nabla \times \mathbb{E} = -k_z E_{ox} \sin(k_z z - \omega t)\,\hat{y} \quad \leftarrow \text{plug}$$

$$-\frac{\partial B}{\partial t} = -k_z E_{ox} \sin(k_z z - \omega t)\,\hat{y}$$

$$\int + dB = \int + k_z E_{ox} \sin(k_z z - \omega t)\,\hat{y}\, dt$$

$$\mathbb{B} = k_z E_{ox} \int \sin(k_z z - \omega t)\,\hat{y}\, dt$$

$$\phi = k_z z - \omega t$$

$$d\phi = -\omega\, dt \rightarrow dt = -\frac{d\phi}{\omega}$$

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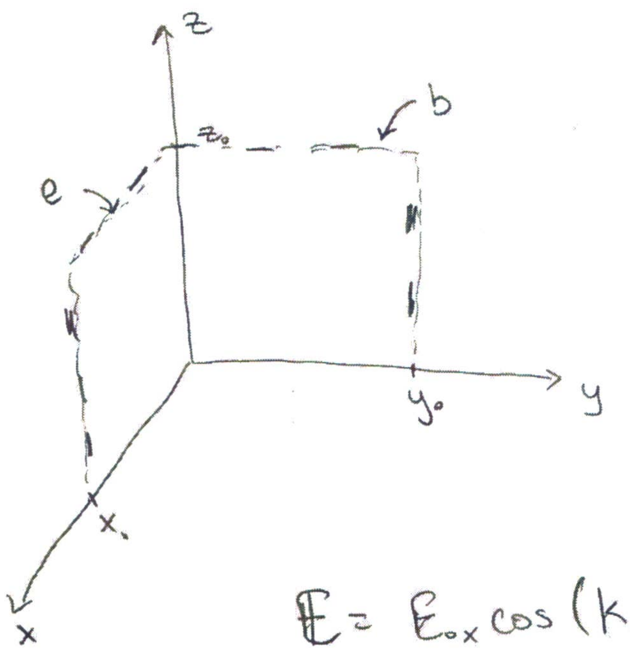
6.1) Faraday's law

$$\oint \mathbf{E}\cdot d\vec{l} = \frac{-\partial \Phi_B}{\partial t}$$

Generalized Ampere's law ($J=0$)

$$\oint \mathbf{B}\cdot d\vec{l} = \frac{1}{c^2}\frac{\partial \Phi_E}{\partial t}$$



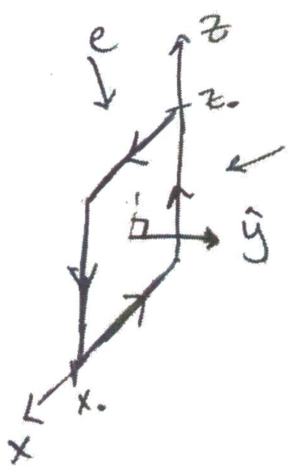
$$\mathbf{E} = E_{0x}\cos(k_z z - \omega t)\,\hat{x}$$

$$\mathbb{B} = \frac{k_z E_{ox}}{\omega} \int -\sin\phi \, d\phi \, \hat{y}$$

$$\mathbb{B} = \frac{k_z E_{ox}}{\omega} \cos\phi \, \hat{y}$$

$$\boxed{\mathbb{B} = \frac{k_z E_{ox}}{\omega} \cos(k_z z - \omega t) \, \hat{y}}$$

6,1,2) Show that $\vec{E}$ satisfies Faraday's Law in the form of $\oint \mathbb{E} \cdot d\vec{l} = -\frac{\partial \Phi_B}{dt}$ along the rectangle e



path of integration

call this $\hat{n}$ for clarity

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \mathbb{B} \cdot \hat{n} \, da$$

$$\oint \mathbf{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\int \frac{d\mathbf{B}}{dt}\, d\vec{a} \quad \leftarrow \text{area is static}$$

s → c → -s → -c

$$-\int \frac{d\mathbf{B}}{dt}\cdot d\vec{a} = -\int \frac{\partial}{\partial t}\left(E_{0x}\frac{k_z}{\omega}\cos(k_z z-\omega t)\right) dx\,dz\,\hat{y}\cdot\hat{y}$$

$$= -E_{0x}\frac{k_z}{\omega}\int \frac{\partial}{\partial t}\cos(k_z z-\omega t)\,dz\; x\Big|_0^{x_0}$$

$$=\left(-E_{0x}\frac{k_z}{\omega}\int_0^{z_0} +\omega(+\sin(k_z z-\omega t)\,dz)\, x_0\right)$$

$$\phi = k_z z-\omega t$$

$$d\phi = dz\,k_z \Rightarrow dz = \frac{d\phi}{k_z}$$

$$= -E_{0x}\frac{\cancel{k_z}}{\cancel{\omega}}\cdot\frac{\cancel{\omega}}{\cancel{k_z}}\,x_0\int_{-\omega t}^{k_z z_0-\omega t}\sin(\phi)\,d\phi$$

$$= -E_{0x}\,x_0\,-\cos\phi\Big|_{-\omega t}^{k_z z_0-\omega t}$$

✓

$$\boxed{-\frac{d\Phi_B}{dt} = E_{0x}\,x_0\left(\cos(k_z z_0-\omega t)-\cos(-\omega t)\right)}$$

① → $\int_0^{z_0}\mathbf{E}\cdot dz\Big|_{x=0}$

② → $\int_0^{x_0}\mathbf{E}\cdot d\vec{x}\Big|_{z=z_0}$

③ → $\int_{z_0}^{0}\mathbf{E}\cdot d\vec{z}\Big|_{x=x_0}$

④ → $\int_{x_0}^{0}\mathbf{E}\cdot d\vec{x}\Big|_{z=0}$

① $\displaystyle\int_0^{z_0} E_{0x}\cos(k_z z-\omega t)\,\overbrace{\hat{x}\cdot\hat{z}}^{0}\,dz = 0$

② $\displaystyle\int_0^{x_0} E_{0x}\cos(k_z z_0-\omega t)\,\overbrace{\hat{x}\cdot\hat{x}}^{1}\,dx = E_{0x}\cos(k_z z_0-\omega t)\,x_0$

③ $\displaystyle\int_{z_0}^{0} E_{0x}\cos(k_z z-\omega t)\,\overbrace{\hat{x}\cdot\hat{z}}^{0}\,dz = 0$

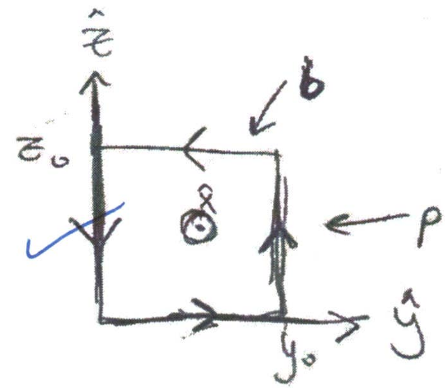
④ $\displaystyle\int_{x_0}^{0} E_{0x}\cos(0-\omega t)\,\hat{x}\cdot\hat{x}\,dx = -E_{0x}\cos(-\omega t)\,x_0$

$E_{0x}\,x_0\left(\cos(k_z z_0-\omega t)-\cos(-\omega t)\right)$

$=$

$E_{0x}\,x_0\left(\cos(k_z z_0-\omega t)-\cos(-\omega t)\right)$

6.1.3) Show that this $\mathbb{B}$ satisfies Ampere's Law along rectangle b. Draw the integral path.



$\hat{x}\times\hat{y}=\hat{z}$
$\hat{z}\times\hat{x}=\hat{y}$
$\hat{y}\times\hat{z}=\hat{x}$

← path of integration

$$\oint \mathbf{B}\cdot d\vec{l} = \frac{1}{c^2}\frac{\partial \Phi_E}{\partial t} \qquad s \to c \to -s \to -c$$

$$\frac{1}{c^2}\frac{d\Phi_E}{dt} = \int \frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{E})\, d\vec{a}$$

$$= \int \frac{1}{c^2}\frac{\partial}{\partial t}\left(E_{0x}\cos(k_z z-\omega t)\right)\overbrace{\hat{x}\cdot\hat{x}}^{1}\, dy\, dz$$

$$= \frac{E_{0x}}{c^2}\int_0^{y_0} dy \int_0^{z_0}\frac{\partial}{\partial t}\left(\cos(k_z z-\omega t)\right)dz$$

$$= \frac{E_{0x}\, y_0}{c^2}\int_0^{z_0} +\omega\cdot +\sin(k_z z-\omega t)\, dz$$

$$\phi = k_z z - \omega t \qquad d\phi = k_z z$$

$$dz = d\phi/k_z$$

$$= \frac{E_{0x}\, y_0}{c^2}\frac{\omega}{k_z}\int_{-\omega t}^{k_z z_0-\omega t}\sin\phi\, d\phi$$

$$\frac{1}{c^2}\frac{\partial\Phi_E}{dt} = \frac{E_{0x}\, y_0}{c^2}\frac{\omega}{k_z}\left(-\cos(k_z z_0-\omega t)+\cos(-\omega t)\right)$$

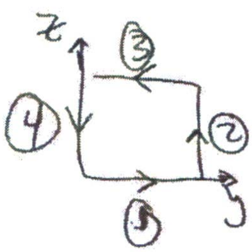
$$\oint \mathbf{B}\cdot d\vec{l} = $$

(1) → $\int_0^{y_0} B\cdot d\vec{y}\,\Big|_{z=0}$

(2) → $\int_0^{z_0} B\cdot d\vec{z}\,\Big|_{y=y_0}$

(3) $\int_{y_0}^{0} B\cdot d\hat{y}\,\Big|_{z=z_0}$

(4) $\int_{z_0}^{0} \mathbf{B}\cdot d\vec{z}\,\Big|_{y=0}$



① $\int_0^{y_0} \frac{k_z E_{ox}}{\omega} \cos(-\omega t)\ \overbrace{\hat{y}\cdot\hat{y}}^{1}\ dy = \frac{k_z E_{ox} y_0}{\omega} \cos(-\omega t)$

② $\int_0^{z_0} \frac{k_z E_{ox}}{\omega} \cos(k_z z-\omega t)\ \overbrace{\hat{y}\cdot\hat{z}}^{0}\ dz = 0$

③ $\int_{y_0}^{0} \frac{k_z E_{ox}}{\omega} \cos(k_z z_0 -\omega t)\ \hat{y}\cdot\hat{y}\ dy = \frac{-k_z E_{ox} y_0}{\omega} \cos(k_z z_0-\omega t)$

④ $\int_{z_0}^{0} \frac{k_z E_{ox}}{\omega} \cos(k_z z-\omega t)\ \overbrace{\hat{z}\cdot\hat{y}}^{0}\ dz = 0$

$$\frac{k_z E_{ox} y_0}{\omega}\left(\cos(-\omega t)-\cos(k_z z_0-\omega t)\right) = \oint \vec{B}\cdot d\vec{l}$$

$$\frac{k_z E_{ox} y_0}{\omega}\left(\cos(-\omega t)-\cos(k_z z_0-\omega t)\right) =$$

$$=$$

$$\frac{E_{ox} y_0}{c^2}\ \frac{\omega}{k_z}\left(\cos(-\omega t)-\cos(k_z z_0-\omega t)\right)$$ ✓

$$\frac{k_z E_{ox} y_0}{\omega} = \frac{E_{ox} y_0}{c^2}\frac{\omega}{k_z}$$

$$c^2 = \left(\frac{\omega}{k_z}\right)^2$$ ✓

6.2) $\mathbf{E} = E_{0x}(x,t)\hat{x} + E_{0y}(x,t)\hat{y} + E_{0z}(x,t)\hat{z}$

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and $B = B_{0x}(x,t)\hat{x} + B_{0y}(x,t)\hat{y} + B_{0z}(x,t)\hat{z}$

6.2.1) Show $E_y(x,t)$, $E_z(x,t)$, $B_y(x,t)$ and $B_z(x,t)$ each individually obey a wave equation of the form

$$\frac{\partial^2 f}{\partial u^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

Where u is a place holder for one of the cartesian variables and f is one of the components stated above

---

Start with Faraday's law for E_y

$$\nabla \times E_y = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & E_y & 0 \end{vmatrix} = \begin{matrix} (0 - \partial_z E_y)\hat{x} \\ 0\ \hat{y} \\ (\partial_x E_y - 0)\hat{z} \end{matrix}$$

$$\nabla \times E_y = \frac{\partial E_y}{\partial x}\hat{z} = -\frac{\partial B}{\partial t}$$

$$\frac{\partial B}{\partial t} = -\frac{\partial E_y}{\partial x}\hat{z}$$

Look at ~~complete~~ differential Ampere's Law when $J=0$

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Apply time derivative to both sides

$$\frac{\partial}{\partial t}(\nabla \times B) = \frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\nabla \times \frac{\partial B}{\partial t} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

Plug $\frac{\partial B}{\partial t}$ in

$$\nabla \times \frac{-\partial E_y}{\partial x}\hat{z} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \frac{\partial E_y}{\partial x} \end{vmatrix} \quad \left(\cancel{\frac{\partial^2 E_y}{\partial x \partial y}}^{\,0} - 0\right)\hat{x} \;\; -\left(0 - \frac{\partial^2 E_y}{\partial x^2}\right)\hat{y} \;\; (0)\,\hat{z}$$

$$+\left(+\frac{\partial^2 E_y}{\partial x^2}\right)\hat{y} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \qquad \mu_0 \epsilon_0 = \frac{1}{c^2}$$

$$\boxed{\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}}$$ ✓

b/c $\frac{\partial^2 E}{\partial^2 t}$ is only dependent on E_y in $\hat{y}$ direction ✓

Do the same as E_y but with E_z

$$\nabla \times E_z = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & E_z \end{vmatrix} \quad \left(\cancel{\partial_y E_z}^{\,0} - 0\right)\hat{x} \;\; = (0 - \partial_x E_z)\hat{y} \;\; = 0\,\hat{z}$$

$$-\nabla \times E_z = +\frac{\partial E_y}{\partial x}\hat{y} = +\frac{\partial B}{\partial t}$$ ✓

$$\frac{\partial B}{\partial t} = \frac{\partial E_y}{\partial x}\hat{y}$$

Look at complete Ampere's Law when $J=0$

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Apply time derivative to both sides

$$\frac{\partial}{\partial t}(\nabla \times B) = \frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

$$\nabla \times \frac{\partial B}{\partial t} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

Plug $\frac{\partial B}{\partial t}$ in

$$\nabla \times \frac{\partial E_z}{\partial x}\hat{y} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & \frac{\partial E_z}{\partial x} & 0 \end{vmatrix} = \begin{matrix} \left(0 - \cancelto{0}{\frac{\partial^2 E_z}{\partial x \partial z}}\right)\hat{x} \\ 0\,\hat{y} \\ \left(\frac{\partial^2 E_z}{\partial x^2} - 0\right)\hat{z} \end{matrix}$$

$$\boxed{\frac{\partial^2 E_z}{\partial x^2}\hat{z} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}}$$

b/c $\frac{\partial^2 E}{\partial t^2}$ is only dependent on E_z in $\hat{z}$

Start with B_y and Amperes law

$$\nabla \times B_y = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & B_y & 0 \end{vmatrix} = \begin{matrix} \left(0 - \cancelto{0}{\partial_z B_y}\right)\hat{x} \\ 0\,\hat{y} \\ (\partial_x B_y - 0)\hat{z} \end{matrix}$$

$$\frac{\partial B_y}{\partial x}\hat{z} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Take time derivative of Faraday's law

$$\frac{\partial}{\partial t}(\nabla \times E) = \frac{\partial}{\partial t} - \left(\frac{\partial B}{\partial t}\right)$$

$$\nabla \times \frac{\partial E}{\partial t} = -\frac{\partial^2 B}{\partial t^2}$$

$$\frac{\partial E}{\partial t} = \frac{1}{\mu_0 \epsilon_0}\frac{\partial B_y}{\partial x}\hat{z}$$

$$\nabla \times \frac{1}{\mu_0 \epsilon_0}\frac{\partial B_y}{\partial x}\hat{z} = -\frac{\partial^2 B}{\partial t^2}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \frac{\partial B_y}{\partial x} \end{vmatrix} = \begin{matrix} \left(\cancelto{0}{\frac{\partial^2 B_y}{\partial x \partial y}} - 0\right)\hat{x} \\ \left(0 - \frac{\partial^2 B_y}{\partial x^2}\right)\hat{y} \\ 0\,\hat{z} \end{matrix}$$

$$+\frac{1}{\mu_0 \epsilon_0} \frac{\partial^2 B_y}{\partial x^2}\hat{y} = +\frac{\partial^2 B}{\partial t^2}$$

$$\boxed{\frac{\partial^2 B_y}{\partial x^2}\hat{y} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}}$$

Start with B_z and Ampere's law

$$\nabla \times B_z = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \qquad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & B_z \end{vmatrix} = \begin{matrix} (\overset{0}{\cancel{\partial_y B_z}} - 0)\hat{x} \\ (0 - \partial_x B_z)\hat{y} \\ 0\,\hat{z} \end{matrix}$$

$$-\frac{\partial B_z}{\partial x}\hat{y} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \Rightarrow \frac{\partial E}{\partial t} = -\frac{1}{\mu_0 \epsilon_0} \frac{\partial B_z}{\partial x}\hat{y}$$

Take ~~time~~ derivative of Faraday's law

$$\nabla \times \frac{\partial E}{\partial t} = -\frac{\partial^2 B}{\partial t^2} \qquad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & \frac{\partial B_z}{\partial x} & 0 \end{vmatrix} = \begin{matrix} (0 - \overset{0}{\cancel{\frac{\partial^2 B_z}{\partial x \partial z}}})\hat{x} \\ 0\,\hat{y} \\ (\frac{\partial^2 B_z}{\partial x^2} - 0)\hat{z} \end{matrix}$$

$$\nabla \times \frac{+1}{\mu_0 \epsilon_0} \frac{\partial B_z}{\partial x}\hat{y} = +\frac{\partial^2 B}{\partial t^2}$$

$$\boxed{\frac{\partial^2 B_z}{\partial x^2}\hat{z} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}}$$

✓

6.2.2) Does it follow that

$E_x(x,t) = B_x(x,t) = 0$?

Given that $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$

and $\nabla \times \mathbf{E} = -\frac{dB}{dt}$ and $\nabla \times B = \mu_0 \epsilon_0 \frac{d\mathbf{E}}{dt}$.

$\nabla \cdot \mathbf{E} = 0$ use $\mathbf{E} = E_x(x,t)\hat{x}$

$\frac{\partial E_x}{dx} = 0$

$\int dE_x = \int 0\, dx$

$E_x = g_1(t) + C_1 \rightarrow$ same can be done for $\mathbf{B}$ $\rightarrow B_x = g_2(t) + C_2$

$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{dt}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ E_x & 0 & 0 \end{vmatrix} = \begin{matrix} (0)\hat{x} \\ (\partial z E_x - 0)\hat{y} \\ (0 - \partial y E_x)\hat{z} \end{matrix}$$

$\nabla \times E = \cancel{\frac{\partial E}{\partial z}}^{0}\hat{y} - \cancel{\frac{\partial E}{\partial y}}^{0}\hat{z} = -\frac{\partial \mathbf{B}}{dt} \cancel{\hat{x}} = 0$

$\int \frac{\partial B}{dt} = 0 \Rightarrow \int dB = -\int 0\, dt$

$B = C_2$ $\leftarrow$ same can be done with Ampere's law for E

$$\boxed{E_x(x,t) = C_1 \;\; \& \;\; B_x(x,t) = C_2 \text{ not } 0}$$

6.2.3) Show how 1 is consistent with

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{B}}{\partial t^2}$$

from 6.2.1

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} \qquad \frac{\partial^2 E_z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

$$\frac{\partial^2 B_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2} \qquad \frac{\partial^2 B_z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2}$$

Start : $\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$

$\nabla^2 \mathbf{E} = \nabla^2 E_x \hat{x} + \nabla^2 E_y \hat{y} + \nabla^2 E_z \hat{z}$ expand $\nabla^2 \mathbf{E}$

$$\nabla^2 E_x = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2}$$

$$\nabla^2 E_y = \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2}$$ compare to time derivative

$$\nabla^2 E_z = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2}$$

$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\partial^2 E_x}{\partial t^2}\hat{x} + \frac{\partial^2 E_y}{\partial t^2}\hat{y} + \frac{\partial^2 E_z}{\partial t^2}\hat{z}$ expand time derivative

$$\nabla^2 E_x = \frac{\partial^2 E_x}{\partial t^2}\frac{1}{c^2} \leftarrow \text{skipping this one}$$

$$\nabla^2 E_y = \frac{\partial^2 E_y}{\partial t^2}\frac{1}{c^2}$$

$$\nabla^2 E_z = \frac{\partial^2 E_z}{\partial t^2}\frac{1}{c^2}$$

compare components

$$\frac{\partial^2 E_y}{\partial x^2} + \cancel{\frac{\partial^2 E_y}{\partial y^2}}^{0} + \cancel{\frac{\partial^2 E_y}{\partial z^2}}^{0} = \frac{\partial^2 E_y}{\partial t^2}\frac{1}{c^2}$$

If E_y depends only on x and t

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \cancel{\frac{\partial^2 E_z}{\partial y^2}} + \cancel{\frac{\partial^2 E_z}{\partial z^2}} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

If E_z depends only on x and t

$$\frac{\partial^2 E_z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

Do the same for $\mathbf{B}$

skipping to component step

$$\nabla^2 B_x = \frac{1}{c^2}\frac{\partial^2 B_x}{\partial t^2}$$ ← Skip this component

$$\nabla^2 B_y = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}$$

$$\nabla^2 B_z = \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2}$$

$$\frac{\partial^2 B_y}{\partial x^2} + \cancel{\frac{\partial^2 B_y}{\partial y^2}} + \cancel{\frac{\partial^2 B_y}{\partial z^2}} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}$$

If B_y depends only on x and t

$$\frac{\partial^2 B_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}$$

$$\frac{\partial^2 B_z}{\partial x^2} + \cancel{\frac{\partial^2 B_z}{\partial y^2}} + \cancel{\frac{\partial^2 B_z}{\partial z^2}} = \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2}$$

If B_z depends only on x and t

$$\frac{\partial^2 B_z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2}$$

6.3.1) Show

$$\mathbb{E} = E_{0x}\cos(k_z z - \omega t + \delta_x)\hat{x}$$

Satisfies

$$\nabla^2 \mathbb{E} = \frac{1}{c^2}\frac{\partial^2 \mathbb{E}}{\partial t^2}$$

and

$$\mathbb{B} = B_{0x}\cos(k_z z - \omega t + \delta_x') + B_{0y}\cos(k_z z - \omega t + \delta_y')\hat{y}$$

satisfies

$$\nabla^2 \mathbb{B} = \frac{1}{c^2}\frac{\partial^2 \mathbb{E}}{\partial t^2}$$

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Start with $\mathbb{E}$

$$\underline{\nabla^2 \mathbb{E}} = \frac{1}{c^2}\underline{\frac{\partial^2 \mathbb{E}}{\partial t^2}}$$

$$\nabla^2 \mathbb{E} = \nabla^2 E_x \hat{x} + \cancel{\nabla^2 E_y}^{\,0}\hat{y} + \cancel{\nabla^2 E_z}^{\,0}\hat{z}$$

$$= \left(\cancel{\frac{\partial^2}{\partial x^2}E_x} + \cancel{\frac{\partial^2}{\partial y^2}E_y} + \frac{\partial^2}{\partial z^2}E_x\right)\hat{x}$$

$$\nabla^2 \mathbb{E} = -k_z^2 E_{0x}\cos(k_z z - \omega t + \delta_x)\hat{x}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbb{E}}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left(E_{0x}\cos(k_z z - \omega t + \delta_x)\hat{x}\right)\frac{1}{c^2}$$

$$= \frac{-\omega^2}{c^2}E_0\cos(k_z z - \omega t + \delta_x)$$

$$\nabla^2 \mathbb{E} = \frac{1}{c^2}\frac{\partial^2 \mathbb{E}}{\partial t^2}$$

$$-k_z^2 E_{0x}\cos(k_z z - \omega t + \delta_x) = -\frac{\omega^2}{c^2} E_{0x}\cos(k_z z - \omega t + \delta_x)$$

$$+k_z^2 \cancel{\mathbb{E}} = +\frac{\omega^2}{c^2}\cancel{\mathbb{E}}$$

$$\sqrt{k_z^2} = \sqrt{\frac{\omega^2}{c^2}} \Rightarrow k_z = \pm\frac{\omega}{c}$$

$\nabla^2 \mathbb{E} = \frac{1}{c^2}\frac{\partial^2 \mathbb{E}}{\partial t^2}$ is true when k_z is related to ω by c such $|k_z c| = |\omega|$ ✓

$$\nabla^2 \mathbb{B} = \frac{1}{c^2}\frac{\partial^2 \mathbb{B}}{\partial t^2}$$

$$\nabla^2 B = \nabla^2 B_x \hat{x} + \nabla^2 B_y \hat{y} + \cancel{\nabla^2 B_z}^{\,0} \hat{z}$$

$$= \left(\cancel{\frac{\partial^2}{\partial x^2} B_x}^{\,0} + \cancel{\frac{\partial^2}{\partial y^2} B_x}^{\,0} + \frac{\partial^2}{\partial z^2} B_x\right)\hat{x} +$$

$$\left(\cancel{\frac{\partial^2}{\partial x^2} B_y}^{\,0} + \cancel{\frac{\partial^2}{\partial y^2} B_y} + \frac{\partial^2}{\partial z^2} B_y\right)\hat{y}$$

$$= \frac{\partial^2}{\partial z^2} B_x \hat{x} + \frac{\partial^2}{\partial z^2} B_y \hat{y}$$

$$= \frac{\partial^2}{\partial z^2} B_{0x}\cos(k_z z - \omega t + \delta_x')\hat{x} +$$

$$\frac{\partial^2}{\partial z^2} B_{0y}\cos(k_z z - \omega t + \delta_y')\hat{y}$$

$$\nabla^2 \mathbb{B} = -k_z^2\left(B_{0x}\cos(k_z z - \omega t + \delta_x')\hat{x} + B_{0y}\cos(k_z z - \omega t + \delta_y')\hat{y}\right)$$

$-k_z^2 \mathbb{B}$

$$\frac{1}{c^2}\frac{\partial^2 \mathbb{B}}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big(B_{0x}\cos(k_z z - \omega t + \delta_x')\hat{x} + B_{0y}\cos(k_z z - \omega t + \delta_y')\hat{y}\Big)$$

$$= \frac{-\omega^2}{c^2}\Big(B_{0x}\cos(k_z z - \omega t + \delta_x')\hat{x} + B_{0y}\cos(k_z z - \omega t + \delta_y')\hat{y}\Big)$$

$$\frac{-\omega^2}{c^2}\mathbb{B}$$

$$\nabla^2 \mathbb{B} = \frac{1}{c^2}\frac{\partial^2 \mathbb{B}}{\partial t^2}$$

$$-k_z^2 \cancel{\mathbb{B}} = \frac{-\omega^2}{c^2}\cancel{\mathbb{B}}$$

$$\sqrt{k_z^2} = \sqrt{\frac{\omega^2}{c^2}} \Rightarrow k_z = \pm\frac{\omega}{c}$$

$\boxed{\nabla^2 \mathbb{B} = \frac{1}{c^2}\frac{\partial^2 \mathbb{B}}{\partial t^2}}$ is true when $\underline{k_z}$ is related to ω by c such that $|k_z c| = |\omega|$

6.3.2) Show that $k_z, \omega, E_{0x}, B_{0x}, B_{0y}, \delta_x, \delta_x', $ or δ_y' must be related or 0 for $\mathbb{B}$ and $\mathbf{E}$ to be consistent with Maxwell's equations

Maxwell's Eq

$\sin \to \cos \to -\sin \to -\cos$

$$\nabla \cdot \mathbb{E} = \rho/\epsilon_0 = 0 \quad (\rho = 0)$$

$$\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t}$$

$$\nabla \cdot \mathbb{B} = 0$$

$$\nabla \times \mathbb{B} = \mu_0 \mathbb{J} + \mu_0 \epsilon_0 \frac{\partial \mathbb{E}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial \mathbb{E}}{\partial t} \quad (\mathbb{J} = 0)$$

$$\mathbb{E} = E_{0x} \cos(k_z z - \omega t + \delta_x)\, \hat{x}$$

$$\mathbb{B} = B_{0x} \cos(k_z z - \omega t + \delta_x')\, \hat{x} + B_{0y} \cos(k_z z - \omega t + \delta_y')\, \hat{y}$$

$$\nabla \times \mathbb{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = \begin{matrix} 0\,\hat{x} \\ (\partial_z E_x - 0)\,\hat{y} \\ (0 - \cancel{\partial_y E_x}_{\,0})\,\hat{z} \end{matrix}$$

$$\nabla \times \mathbb{E} = -k_z E_{0x} \sin(k_z z - \omega t + \delta_x)\, \hat{y}$$

$$\frac{\partial \mathbb{B}}{\partial t} = +\omega\left(+B_{0x} \sin(k_z z - \omega t + \delta_x')\, \hat{x} + B_{0y} \sin(k_z z - \omega t + \delta_y')\, \hat{y}\right)$$

$$\nabla \times \mathbb{B} = \mu_0 \epsilon_0 \frac{\partial \mathbb{E}}{\partial t}$$

$$\nabla \times \mathbb{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & 0 \end{vmatrix} = \begin{matrix} (0 - \partial_z B_y)\hat{x} \\ (\partial_z B_x - 0)\hat{y} \\ (\underbrace{\cancel{\partial_x B_y}}_{0} - \underbrace{\cancel{\partial_y B_x}}_{0})\hat{z} \end{matrix}$$

$$\nabla \times \mathbb{B} = +k_z(+B_{oy}\sin(k_z z - \omega t + \delta_y')\hat{x}) + k_z(-B_{ox}\sin(k_z z - \omega t + \delta_x')\hat{y})$$

$$\frac{\partial \mathbb{E}}{\partial t} = +\omega(+E_{ox}\sin(k_z z - \omega t + \delta_x)\hat{x})$$

$$\nabla \times \mathbb{B} = \mu_0 \epsilon_0 \frac{\partial \mathbb{E}}{\partial t}$$

must be 0

$$k_z B_{oy}\sin(k_z z - \omega t + \delta_y')\hat{x} - B_{ox}\sin(k_z z - \omega t + \delta_x')\hat{y} =$$

$$\mu_0 \epsilon_0 \omega E_{ox}\sin(k_z z - \omega t + \delta_x)\hat{x}$$

If $B_{ox} = 0$, all components work out

$$k_z B_{oy}\sin(k_z z - \omega t + \delta_y')\hat{x} = \mu_0 \epsilon_0 \omega E_{ox}\sin(k_z z - \omega t + \delta_x)\hat{x}$$

If $\delta_y' = \delta_x$, then $k_z B_{oy} = \mu_0 \epsilon_0 \omega E_{ox}$. Using $\frac{E_{ox}}{B_{oy}} = \frac{\omega}{k_z}$ then $k_z = \mu_0 \epsilon_0 \omega (\omega / k_z)$

Thus $\left(\frac{\omega}{k_z}\right)^2 = \frac{1}{\mu_0 \epsilon_0}$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$-k_z E_{ox} \sin(k_z z - \omega t + \delta_x)\hat{y} = -\left(\omega\left(B_{ox}\sin(k_z z - \omega t + \delta_x')\hat{x}\right.\right.$$
$$\left.B_{oy}\sin(k_z z - \omega t + \delta_y')\hat{y}\right)$$

$-\omega B_{ox} \sin(k_z z - \omega t + \delta_x')\hat{x}$ must equal 0

$B_{ox} = 0$ achieves this.

$$+k_z E_{ox} \sin(k_z z - \omega t + \delta_x)\hat{y} = +\omega B_{oy} \sin(k_z z - \omega t + \delta_y')\hat{y}$$

$$\phi = k_z z - \omega t$$

$$k_z E_{ox} \sin(\phi + \delta_x) = \omega B_{oy} \sin(\phi + \delta_y')$$

$$k_z E_{ox}\left(\sin(\phi)\cos(\delta_x) + \cos(\phi)\sin(\delta_x)\right) =$$
$$\omega B_{oy}\left(\sin(\phi)\cos(\delta_y') + \cos(\phi)\sin(\delta_y')\right)$$

If $\delta_x = \delta_y'$, then $k_z E_{ox} = \omega B_{oy}$ or

$$\frac{E_{ox}}{B_{oy}} = \frac{\omega}{k_z}$$

technically,
one eqn + two unknowns B_{oy} + δ_y'
Integrate to show $B_{oy} = E_{ox}/c$. then
left with one eqn. + one unknown δ_y'.

Thus $B_{ox} = 0$

$$\frac{E_{ox}}{B_{oy}} = \frac{\omega}{k_z}, \quad \left(\frac{\omega}{k_z}\right)^2 = \frac{1}{\mu_o \epsilon_o} \quad \text{when } \delta_x = \delta_y'$$

Use this to see if other solutions to δ_x' and δ_y'

Resume from Ampere's law $\phi = k_z z - \omega t$

$$k_z B_{oy}(\sin\phi \cos\delta_y' + \cos\phi \sin\delta_y') = \mu_o \epsilon_o \omega E_{ox}(\sin\phi \cos\delta_x + \cos\phi \sin\delta_x)$$

$$(\sin\phi \cos\delta_y' + \cos\phi \sin\delta_y') = \overbrace{\mu_o \epsilon_o \frac{\omega}{k_z} \frac{E_{ox}}{B_{oy}}}^{1}(\sin\phi \cos\delta_x + \cos\phi \sin\delta_x)$$

$$(\sin\phi \cos\delta_y' + \cos\phi \sin\delta_y') = (\sin\phi \cos\delta_x + \cos\phi \sin\delta_x)$$

Pick an offset such that $\delta_y' = n\pi + \delta_x$

$$\sin\phi \cos(n\pi + \delta_x) + \cos\phi \sin(n\pi + \delta_x) = \sin\phi \cos\delta_x + \cos\phi \sin\delta_x$$

$$\sin\phi(\cos n\pi \cos\delta_x - \cancel{\sin n\pi \sin\delta_x}^{0}) + \cos\phi(\sin\delta_x \cos n\pi + \cancel{\cos\delta_x \sin n\pi}^{0}) = \sin\phi \cos\delta_x + \cos\phi \sin\delta_x$$

$$\pm(\sin\phi \cos\delta_x + \cos\phi \sin\delta_x) = \sin\phi \cos\delta_x + \cos\phi \sin\delta_x$$

This sign flip would imply that $E_{oy}/B_{oy} = \pm \frac{\omega}{k_z}$ Which can not be true. Therefore n must be an even integer.

Thus $\frac{1}{\sqrt{\mu_0 \epsilon_0}} = \left|\frac{\omega}{k_z}\right| = \frac{E_{0x}}{B_{0y}}$, $B_{0x} = 0$, and

$\delta_x = \delta_y' + n\pi$ when n is an even integer

6.3.3) If $\mathbb{E} = E_{0y} \cos(k_z z - \omega t + \delta_y) \hat{y}$

Faraday's law (I think this is for $\hat{y}$)

$$\nabla \times \mathbb{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ d_x & d_y & d_z \\ 0 & E_y & 0 \end{vmatrix} = \begin{matrix} 0 - d_z E_y \hat{x} \\ 0 \hat{y} \\ \partial_x E_y - 0 \hat{z} \end{matrix}$$

$$\nabla \times \mathbb{E} = -k_z(-E_{0y} \sin(k_z z - \omega t + \delta_y))\hat{x}$$

use $\frac{d\mathbb{B}}{dt}$ from 6.3.2 $\phi = k_z z - \omega t$

$$+k_z(+E_{0y} \sin(\phi + \delta_y))\hat{x} = -\omega(B_{0x} \sin(\phi + \delta_x')\hat{x} + B_{0y} \sin(\phi + \delta_y')\hat{y})$$

$B_{0y} = 0$ to cancel $\hat{y}$ component

$$k_z E_{0y} \sin(\phi + \delta_y)\hat{x} = -\omega B_{0x} \sin(\phi + \delta_x')\hat{x}$$

Assume trivial solution $\delta_y = \delta_x'$

$$k_z E_{0y} = -\omega B_{0x}$$

$$\frac{E_{0y}}{B_{0x}} = \frac{-\omega}{k_z}$$

Ampere's law

$$\frac{\partial \mathbf{E}}{\partial t} = +\omega\left(+E_{oy}\sin(k_z z - \omega t + \delta_y)\,\hat{y}\right)$$

use $\nabla \times \mathbf{B}$ from 6.3.2

$$\begin{matrix} \cancel{k_z B_{oy} \sin(k_z - \omega t + \delta_y')\hat{x}} \\ -k_z B_{ox} \sin(k_z z - \omega t + \delta_x')\hat{y} \end{matrix} = \mu_0 \epsilon_0 \omega E_{oy} \sin(k_z z - \omega t + \delta_y)\,\hat{y}$$

$B_{oy} = 0$

$$-k_z B_{ox} \sin(k_z z - \omega t + \delta_x') = \mu_0 \epsilon_0 \omega E_{oy} \sin(k_z z - \omega t + \delta_y)$$

if $\delta_x' = \delta_y$

$$-k_z B_{ox} = \mu_0 \epsilon_0 \omega E_{oy}$$

$$1 = \mu_0 \epsilon_0 \left(-\frac{\omega}{k_z}\right) \frac{E_{oy}}{B_{ox}} \leftarrow \left(-\frac{\omega}{k_z}\right)$$

$$\frac{\omega}{k_z} = \frac{\pm 1}{\sqrt{\mu_0 \epsilon_0}}$$

$$\boxed{\begin{matrix} \text{From } 6.3.2 \quad \delta_y = \delta_x' + n\pi \quad n = 0, 2, \ldots \\ \text{thus} \quad \dfrac{E_{oy}}{B_{ox}} = -\dfrac{\omega}{k}, \quad \dfrac{\omega}{k} = \dfrac{\pm 1}{\sqrt{\mu_0 \epsilon_0}}, \quad B_{oy} = 0 \end{matrix}}$$

6.3.4) Using $\mathbf{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$

show that 2 and 3 are consistent with

$$\mathbf{B} = \frac{1}{c}\hat{\mathbf{k}} \times \mathbf{E}$$

from 6.3.2 $\quad \dfrac{E_{ox}}{B_{oy}} = \dfrac{\omega}{k_z}$

from 6.3.3 $\quad \dfrac{E_{oy}}{B_{ox}} = -\dfrac{\omega}{k_z}$

using $\mathbf{k} = k_z\hat{z}$ and $\omega/k = c$

6.3.2 → $B_y\hat{y} = \frac{1}{c}\hat{k} \times E_x\hat{x}$ ✓

$$= \frac{k}{\omega}\hat{z} \times E_x\hat{x}$$

$$= \frac{E_x k}{\omega}(\hat{z} \times \hat{x})$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0\hat{x} + 1\hat{y} + 0\hat{z}$$

$$B_y\hat{y} = \frac{E_x k}{\omega}\hat{y}$$ ✓

From

6.3.3 $\dfrac{E_{oy}}{B_{ox}} = \dfrac{-\omega}{k_z}$

$B_x\hat{x} = \dfrac{-k_z}{\omega} E_y\hat{y} = \dfrac{1}{c}\hat{k} \times E_y\hat{y}$

$$\begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}$$

$0 - 1\,\hat{x}$
$0\,\hat{y}$
$0\,\hat{z}$

$B_x\hat{x} = \dfrac{E_y k}{\omega}\hat{z} \times \hat{y}$

$$\boxed{B_x\hat{x} = \dfrac{-E_y k}{\omega}\hat{x}}$$

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PHYS 313
October 13, 20
HW #6

6.4) Using $\mathbf{E} = \mathrm{Re}\left[\tilde{\mathbf{E}} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}\right]$

$$\mathbf{B} = \mathrm{Re}\left[\tilde{\mathbf{B}} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}\right]$$

$\mathbf{k} \equiv k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$

$\mathbf{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$

Show that $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{dt}$ leads to $\tilde{\mathbf{B}} = \frac{1}{c}\hat{\mathbf{k}} \times \tilde{\mathbf{E}}$ and explain how it relates to $\mathbf{B} = \frac{1}{c}\hat{\mathbf{k}} \times \mathbf{E}$

$$\nabla \times \mathbf{E} = \mathrm{Re}\left[e^{-i\omega t}\, \nabla \times \tilde{\mathbf{E}} e^{i\mathbf{k}\cdot\mathbf{r}}\right]$$

$$\frac{-\partial \mathbf{B}}{dt} = \mathrm{Re}\left[\mathbf{B}(-i\omega)\, e^{-i\omega t}\, e^{(i\mathbf{k}\cdot\mathbf{r})}\right]$$

Pick a direction such that $\tilde{\mathbf{B}}$ is only in the $\hat{x}$

$$\mathrm{Re}\left[e^{-i\omega t}\, \nabla \times \tilde{\mathbf{E}} e^{i\mathbf{k}\cdot\mathbf{r}}\right] \quad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{vmatrix} \quad \begin{array}{l} (\partial_y E_z - \partial_z E_y)_x \\ -(\partial_z E_x - \partial_x E_z)_y \\ (\partial_x E_y - \partial_y E_x)_z \end{array}$$

Since $\mathbf{B}$ was chosen to be in $\hat{x}$, E and k must be defined such that $\partial_y E_z - \partial_z E_y \neq 0$ and the other components are 0

$$\frac{\partial}{\partial y} e^{i\mathbf{k}\cdot\mathbf{r}} = ik_y\, e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$Re\left[e^{-i\omega t}\left(\frac{\partial}{\partial y}\tilde{E}_z e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{\partial}{\partial z}\tilde{E}_y e^{i\mathbf{k}\cdot\mathbf{r}}\right)\right]$$

$$Re\left[e^{-i\omega t}\left(\tilde{E}_z ik_y e^{i\mathbf{k}\cdot\mathbf{r}} - \tilde{E}_y ik_z e^{i\mathbf{k}\cdot\mathbf{r}}\right)\right]$$

Assuming $Im\left[\frac{\hat{k}\times\tilde{\mathbf{E}}}{c} = \mathbf{B}\right]$ such that the Re part can be dropped

Further more $e^{-i\omega t}$ $e^{i\mathbf{k}\cdot\mathbf{r}}$ is on both sides, so I will remove them

$$\begin{array}{ccc} x & y & z \\ k_x & k_y & k_z \\ E_x & E_y & E_z \end{array} = \begin{array}{c} k_y E_z - k_z E_y \\ \cancel{k_z E_x - k_x E_z} \\ \cancel{k_x E_y - k_y E_x} \end{array}$$

if $k_x = E_x = 0$

$$\tilde{E}_z \cancel{i} k_y - \tilde{E}_y \cancel{i} k_z = -\cancel{i}\omega \tilde{B}_x$$

$$\mathbf{k}\times\tilde{\mathbf{E}} = \omega\mathbf{B}$$

$$|\mathbf{k}|(\hat{k}\times\tilde{\mathbf{E}}) = \omega\mathbf{B} \qquad \frac{\omega}{k} = c$$

$$c^{-1}(\hat{k}\times\tilde{\mathbf{E}}) = \tilde{\mathbf{B}}$$ ✓

ok, but see Soln.

$\mathbf{B} = \frac{1}{c}\hat{k}\times\mathbf{E}$ is related to $\tilde{\mathbf{B}} = \frac{1}{c}\hat{k}\times\tilde{\mathbf{E}}$ by showing that there is an imaginary component to the $\mathbf{E}$ and $\mathbf{B}$ fields that follow the same equations. Furthermore the Real components of $\tilde{\mathbf{E}}$ & $\tilde{\mathbf{B}}$ are the real electric and magnetic fields.

7.2) 4.3b from book

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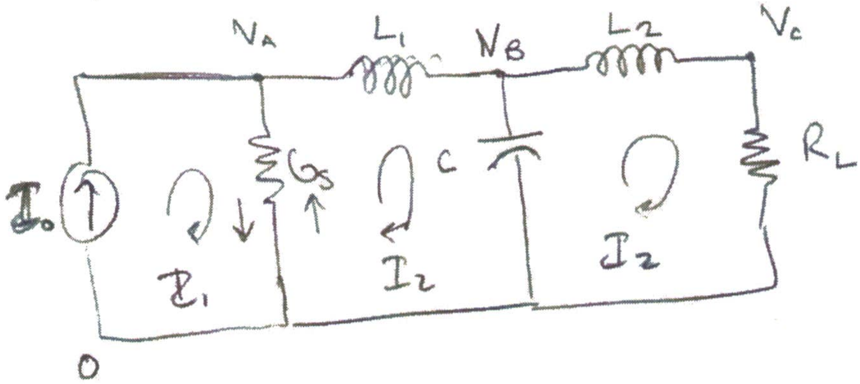
PHYS 513
HW #7
October 20, 2020



First loop

Not possible with current source ✓

Second loop

$$-\frac{(I_2-I_a)}{G_s} - L_1\frac{dI_2}{dt} - \frac{1}{C}\int(I_2-I_3)dt = 0$$ ✓

Third Loop

$$-\frac{1}{C}\int(I_3-I_2)dt - L_2\frac{dI_3}{dt} - R_L I_3 = 0$$

Nodes $\rightarrow \downarrow \rightarrow$

A: $I_o - V_A G_S - \frac{1}{L_1}\int (V_a - V_b)\,dt = 0$

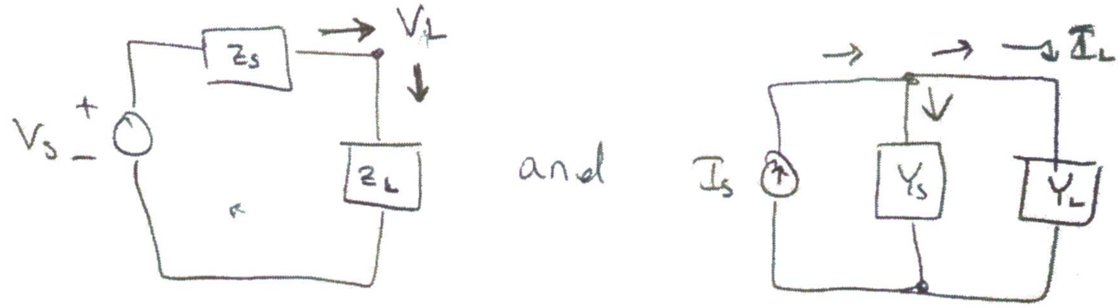
B: $\frac{1}{L_1}\int (V_b - V_a)\,dt + \frac{1}{L_2}\int (V_b - V_c)\,dt + C\frac{dV}{dt} = 0$

C: $\frac{1}{L_2}\int (V_c - V_b)\,dt + \frac{V_c}{R_L} = 0$ ✓

Should comment on relationship w/ other circuit

7.2) 4.3c

Show that



produce the same current to and Voltage across the Load.

$$\frac{V_s - V_L}{Z_s} - \frac{V_L}{Z_L} = 0$$

$$\frac{V_s}{Z_s} - V_L\left(\frac{1}{Z_s} + \frac{1}{Z_L}\right) = 0$$

$$\frac{V_s}{Z_s} = V_L\left(\frac{1}{Z_s} + \frac{1}{Z_L}\right)$$

$$\frac{V_s}{Z_s} = V_L\left(\frac{Z_L}{Z_s Z_L} + \frac{Z_s}{Z_L Z_s}\right)$$

$$\frac{V_s}{Z_s} = V_L\left(\frac{Z_L + Z_s}{Z_s Z_L}\right)$$

$$V_L = V_s\left(\frac{Z_L}{Z_s + Z_L}\right)$$

Check with Norton circuit

Check currents into node

$$I_s - V_L Y_s - V_L Y_L = 0$$

$$I_s - \frac{V_L}{Z_s} - \frac{V_L}{Z_L} = 0$$

$$I_s = \frac{V_L}{Z_s} + \frac{V_L}{Z_L}$$

$$I_s = \frac{Z_L V_L + Z_s V_L}{Z_s Z_L}$$

$$I_s Z_s Z_L = V_L (Z_L + Z_s)$$

$V_s \rightarrow$

$$\frac{I_s Z_s Z_L}{(Z_L + Z_s)} = V_L$$

$$V_L = \frac{V_s Z_L}{(Z_L + Z_s)}$$ $\leftarrow$ same voltage ✓

Given that Z_L and V_L are the same I_L should also be the same ✓

7.2) 4.3 d

V_s and Z_s are constant. Show that Power is maximized when $Z_L = Z_s^*$

$$P = VI \rightarrow Re[\tilde{V} e^{i\omega t}] \, Re[\tilde{I} e^{i\omega t}]$$

$$P = Re[(V_r + iV_i)(\cos(\omega t) + i\sin(\omega t))] \cdot$$
$$Re[(I_r + iI_i)(\cos(\omega t) + i\sin(\omega t))]$$

$$Re[(V_r + iV_i)(\cos(\omega t) + i\sin(\omega t))] =$$
$$Re[V_r\cos(\omega t) + iV_r\sin(\omega t) + iV_i\cos(\omega t) - V_i\sin(\omega t)]$$

same with I ↑

$$Re[(V_r + iV_i)(\cos(\omega t) + \cancel{i}\sin(\omega t))] = V_r\cos(\omega t) + V_i\sin(\omega t)$$

$$P = (V_r\cos(\omega t) - V_i\sin(\omega t))(I_r\cos(\omega t) - I_i\sin(\omega t))$$

$$P = V_r I_r \cos^2(\omega t) + V_i I_i \sin^2(\omega t) - (V_r I_i + V_i I_r)\cos(\omega t)\sin(\omega t)$$

$$\overline{P} = \frac{1}{2}(V_r I_r + V_i I_i) = \frac{1}{2} Re[\overline{VI^*}]$$

$$\overline{P} = \frac{1}{2} Re[\overline{VI^*}]$$

$$V = IZ$$

$$\overline{P} = \frac{1}{2} Re[\overline{VI^*}]$$

Maximize $\overline{P_L} = \frac{1}{2} Re[\overline{V_L I^*}]$

$$V_L = I Z_L$$

$$\overline{P_L} = \frac{1}{2} |\overline{I}|^2 \; Re[Z_L]$$

$$\overline{P_L} = \frac{1}{2} \left(\frac{|V_s|}{|Z_s + Z_L|}\right)^2 Re[Z_L]$$

$$\overline{P} = \frac{1}{2} \left(\frac{|V_s|}{|Z_{sR} + iZ_{si} + Z_{LR} + iZ_{Li}|}\right)^2 Re[Z_L]$$

$Re[Z_s + Z_L]^2 + Im[Z_s + Z_L]^2 \leftarrow$ get this to zero with $Im[Z_L] = -Im[Z_s]$

$$\overline{P} = \frac{1}{2} \left(\frac{|V_s|^2 Re[Z_L]}{(Re[Z_s] + Re[Z_L])^2}\right)$$ take ~~derivative~~ wrt $Re[Z_L]$

$u = Re[Z_L]$ $\qquad v = Re[Z_s]$

$$\frac{\partial}{\partial u} \overline{P} = \frac{\partial}{\partial u} \left(\frac{1}{2} \frac{|V_s|^2 u}{(u+v)^2} \right) = 0$$

$$\frac{\partial \overline{P}}{\partial u} = \left(\frac{1}{(u+v)^2} + \frac{-2u}{(u+v)^3} \right) \cancel{\frac{1}{2} |V_s|^2} = 0$$

$$\frac{(u+v) - 2u}{\cancel{(u+v)^3}} = 0$$ only care about numerator but require

$u + v \neq 0$

$\cancel{u} + v - \cancel{2}u = 0$

$v = u$ ← sub back in

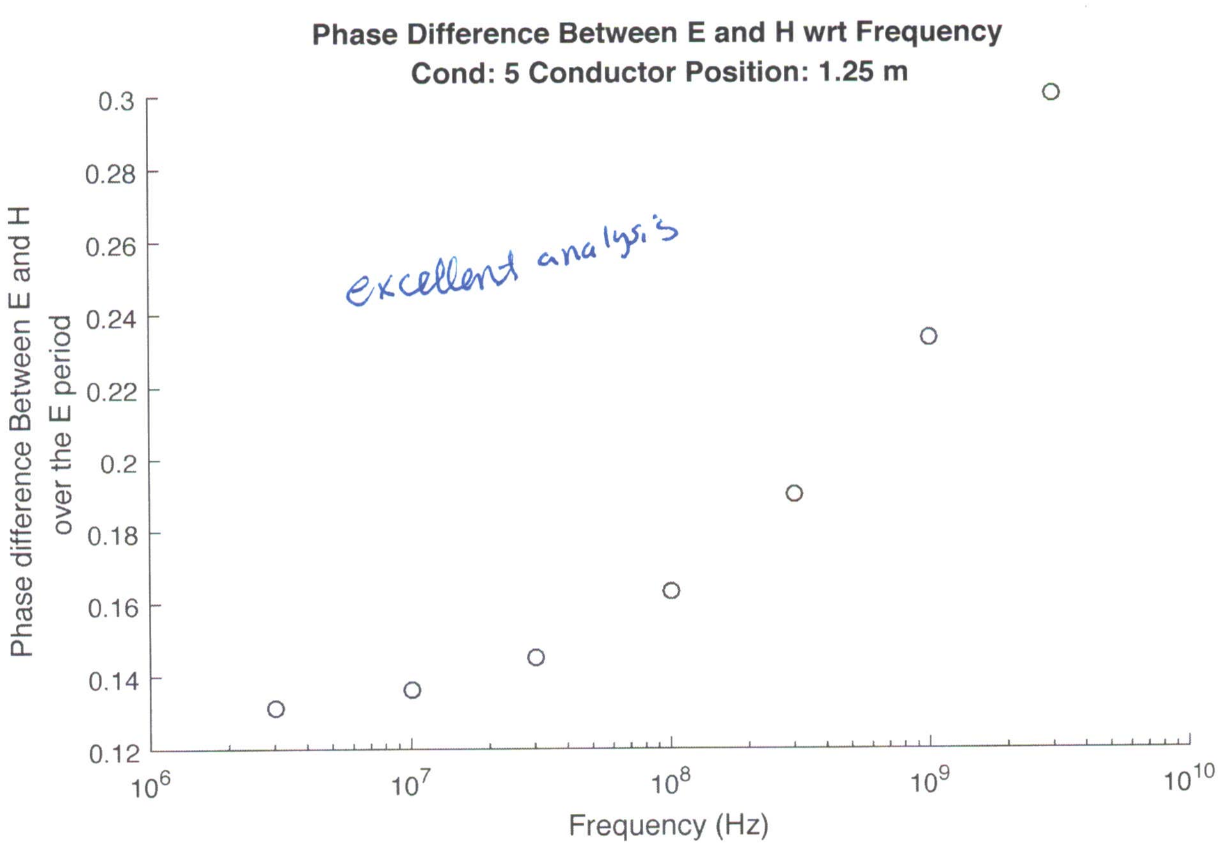
$Re[Z_s] = Re[Z_L]$

$(Im[Z_L] = -Im[Z_s])$

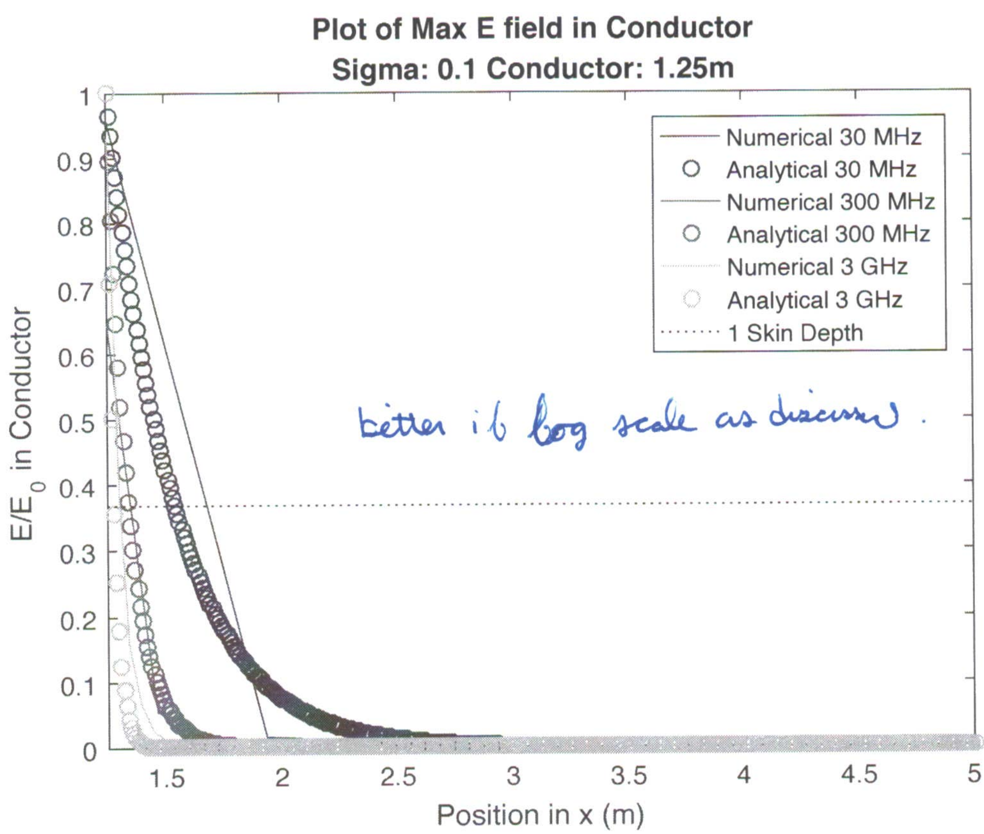
Plug in $Z_L = Z_s^*$

This should be the local max because if $Z_L \ll Z_s$ then $\bar{P} \to 0$ and if $Z_L \gg Z_s$ $\bar{P} \propto \frac{1}{Z_L}$ which goes to 0.

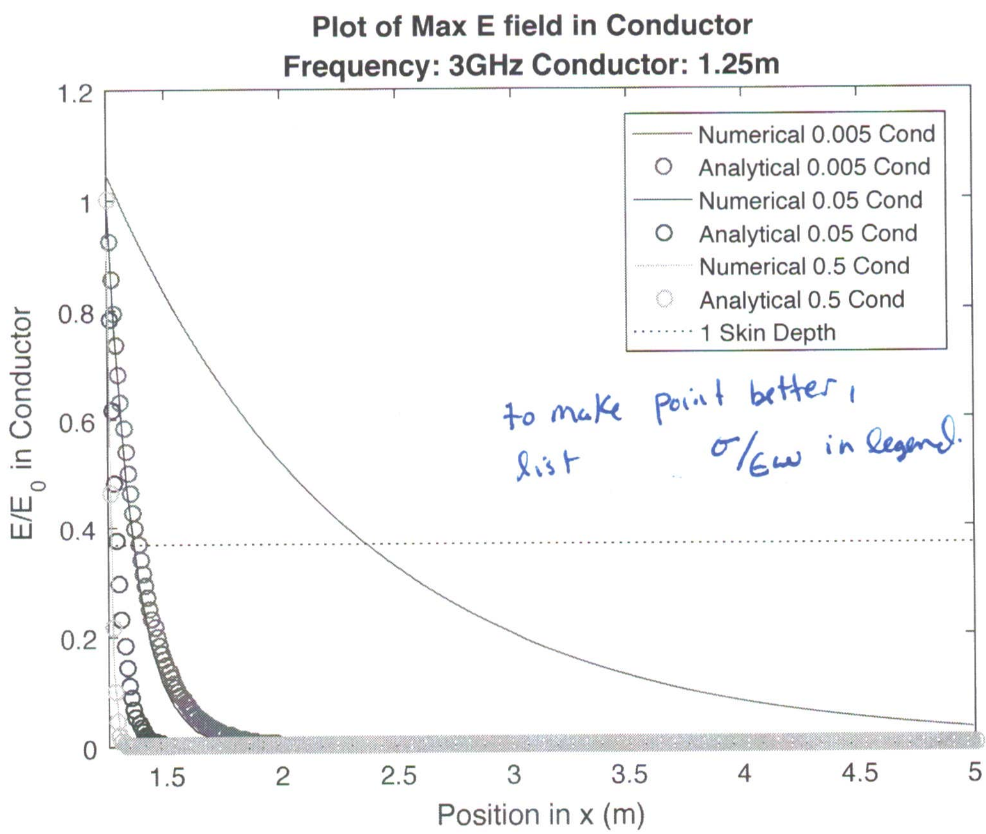
$\boxed{\therefore P \text{ is maximized when } Z_L = Z_s^*}$



Phase Difference Between E and H wrt Frequency
Cond: 5 Conductor Position: 1.25 m
Phase difference Between E and H over the E period
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
10^6
10^7
10^8
10^9
10^{10}
Frequency (Hz)
excellent analysis

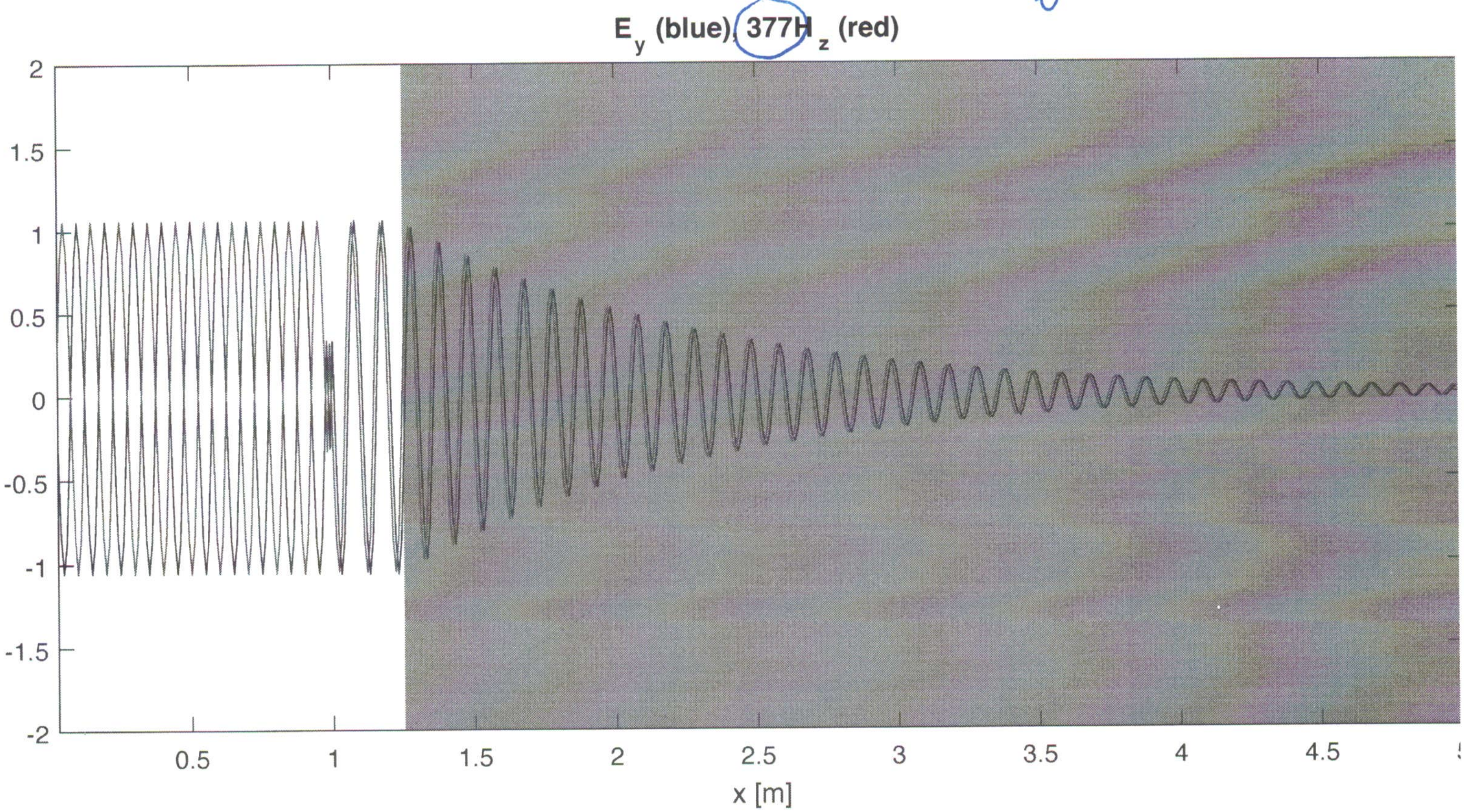


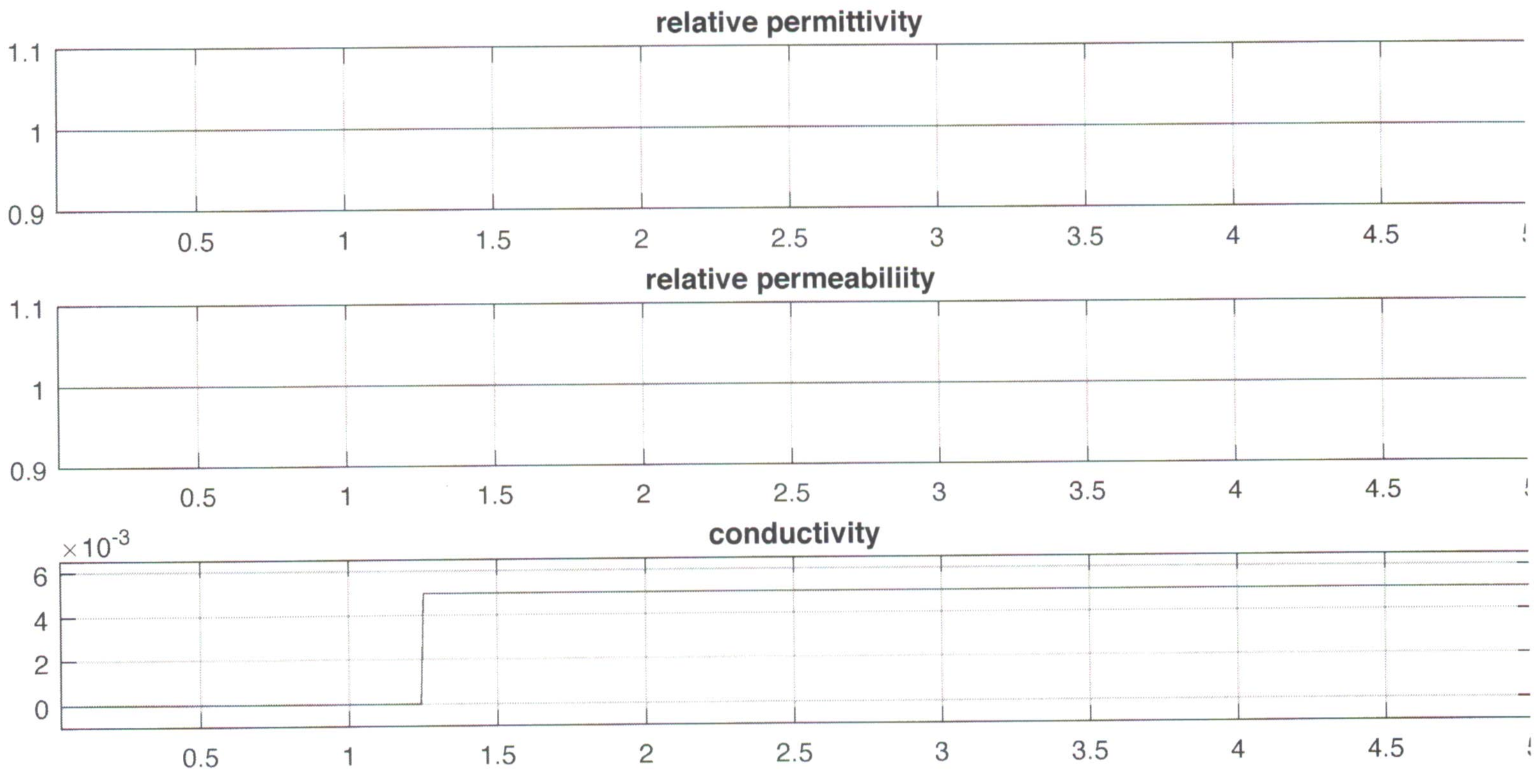
better if log scale as discussed.



to make point better, list $\sigma/\epsilon\omega$ in legend.

do you know where this comes from?





relative permittivity
1.1
1
0.9
0.5
1
1.5
2
2.5
3
3.5
4
4.5
relative permeabiliity
1.1
1
0.9
0.5
1
1.5
2
2.5
3
3.5
4
4.5
conductivity
×10-3
6
4
2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5