

11.1.1) Show the
following

$$\tilde{V}_1 = \tilde{V}_o - j\omega L \tilde{I}_1,$$

$$\tilde{I}_2 = \tilde{I}_1 - j\omega C \tilde{V}_1,$$

$$\tilde{V}_2 = \tilde{V}_1 - j\omega L \tilde{I}_2,$$

$$\tilde{I}_2 = \frac{\tilde{V}_2}{Z_L}$$

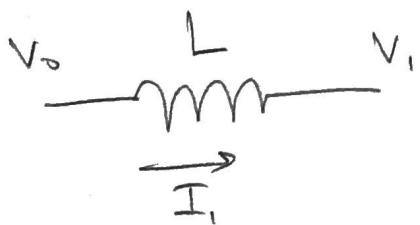
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Remember source is

$$\underline{V_o \cos(\omega t) = \text{Re}[\tilde{V}_o]}$$

Circuit on page 2

Start with first inductor in loop 1



$$\Delta \tilde{V}_o = L \frac{d \tilde{I}_1}{dt}$$

$$\tilde{V}_o - \tilde{V}_1 = L \frac{d \tilde{I}_1}{dt}$$

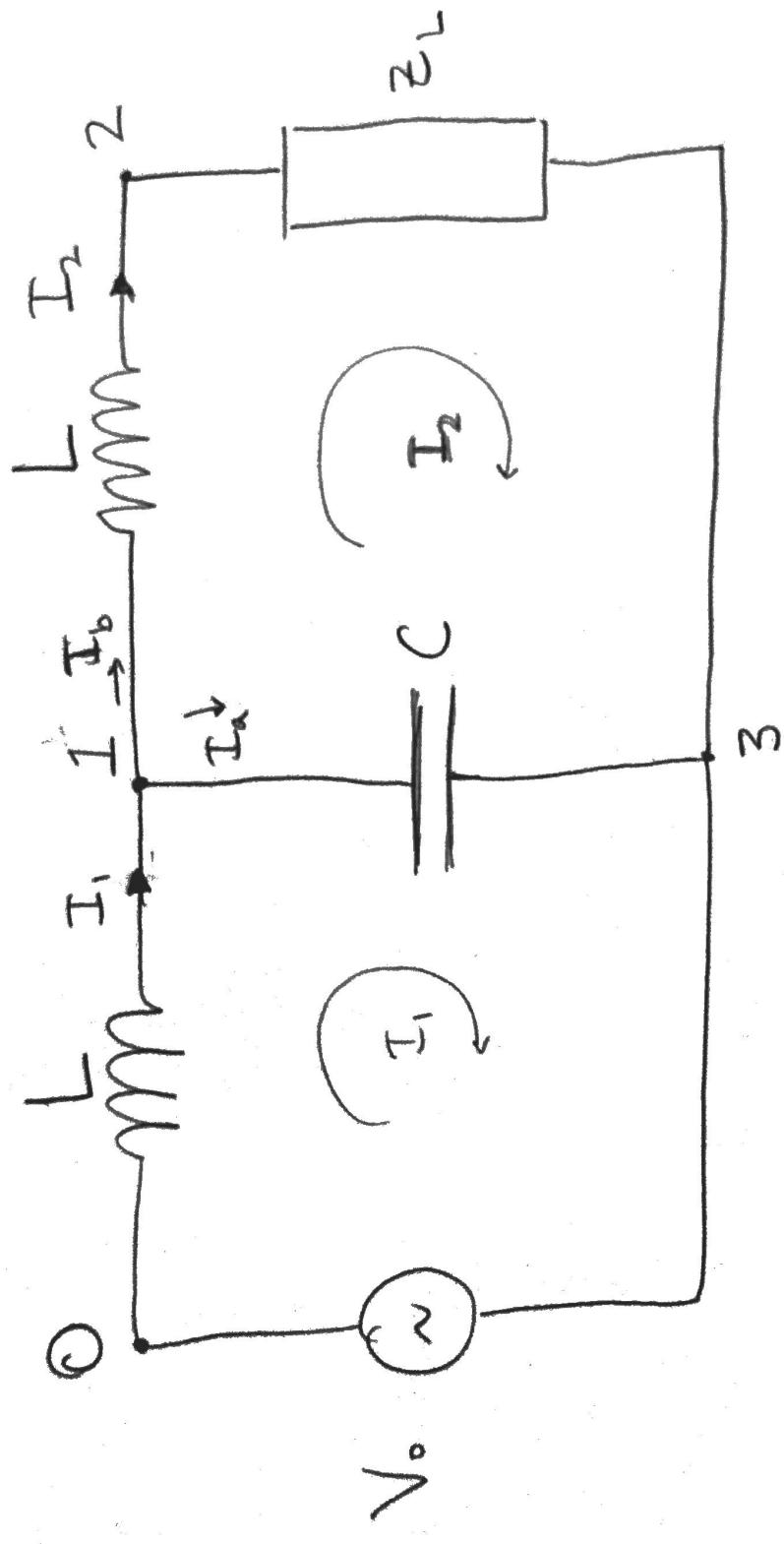
$$\tilde{V}_o - \tilde{V}_1 = L(j\omega \tilde{I}_1)$$

$$\tilde{V}_1 = \tilde{V}_o - j\omega L \tilde{I}_1$$

$$\frac{d \tilde{I}_1}{dt} = \frac{d}{dt} (\tilde{I}_1 e^{j\omega t})$$

const

$$\frac{d \tilde{I}_1}{dt} = \underline{\tilde{I}_1' e^{j\omega t}} - j\omega$$



Expand the two loops into component voltages

$$V_o + (V_1 - V_o) + (V_3 - V_1) = 0 \quad \text{first loop}$$

$$(V_1 - V_3) + (V_2 - V_1) + (V_3 - V_2) = 0 \quad \text{second loop}$$

Choose $V_3 = 0$ since it feeds to the source

from the above

$$V_o + V_1 - V_o + V_3 - V_1 = 0$$

$$\cancel{(V_o - V_o)} + \cancel{(V_1 - V_1)} + V_3 = 0$$

Also

$$V_1 - V_3 + V_2 - V_1 + V_3 - V_2 = 0$$

$$\cancel{(V_1^0 - V_1)} + \cancel{(V_2^0 - V_2)} + \cancel{(V_3^0 - V_3)} = 0$$

From here use the relationship for

$$(V_1 - V_0) = -L \frac{dI_1}{dt}$$

$$V_1 = V_0 - L \frac{dI_1}{dt}$$

however these values
are in the form
of $\cos(\omega t + \phi_k)$

need a more effective way to represent
 $\cos(\omega t + \phi_k)$ where ϕ_k is a phase offset.

start

$$\cos(\omega t + \phi_k) = \frac{1}{2} (e^{j(\omega t + \phi_k)} + e^{-j(\omega t + \phi_k)})$$

represent $e^{j(\omega t + \phi_k)}$ as ψ

$$\cos(\omega t + \phi_k) = \frac{1}{2} (\psi + \psi^*)$$

Given that $\psi = a + b j$

$$\psi + \psi^* = 2a$$

$$\text{Thus } \cos(\omega t + \phi_k) = \frac{1}{2} (\operatorname{Re}\{e^{j(\omega t + \phi_k)}\})$$

$$\cos(\omega t + \phi_k) = \operatorname{Re} \{ e^{j\phi_k} e^{j\omega t} \}$$

where $e^{j\phi_k}$ is a phase offset which can be absorbed into a constant

$$\text{Thus } V_o \cos(\omega t) = V_o \operatorname{Re} \{ e^{j\omega t} \}$$

Given that V_o is a constant

$$V_o \cos(\omega t) = \operatorname{Re} \{ V_o e^{j\omega t} \}$$

Any V_k value can be represented thusly by a general solution of

$$V_k \cos(\omega t + \phi_k) = \operatorname{Re} \{ V_k e^{j\phi_k} e^{j\omega t} \}$$

where $V_k e^{j\phi_k}$ would be come \tilde{V}_k

Similarly, since I is given by $\frac{V_k}{Z_k} = I_k$

I_k would be of the form $I_k \cos(\omega t + \phi_k) = \operatorname{Re} \{ I_k e^{j\phi_k} e^{j\omega t} \}$

with this, the equation originally being solved would become

$$V_1 \cos(\omega t + \phi_1) = V_0 \cos(\omega t) - L \frac{d}{dt} I_1 \cos(\omega t + \phi_1)$$

$$\operatorname{Re}\{\tilde{V}_1 e^{j\omega t}\} = \operatorname{Re}\{\tilde{V}_0 e^{j\omega t}\} - \operatorname{Re}\left\{L \frac{d}{dt} \tilde{I}_1 e^{j\omega t}\right\}$$

Proof

$$\operatorname{Re}\{a_1 + jb_1\} + \operatorname{Re}\{a_2 + jb_2\} = a_1 + a_2$$

$$\operatorname{Re}\{a_1 + jb_1 + a_2 + jb_2\} = a_1 + a_2 \quad \text{for } a_1, a_2 \in \mathbb{R}$$

$$\Rightarrow \operatorname{Re}\{\tilde{V}_1 e^{j\omega t}\} = \operatorname{Re}\{\tilde{V}_0 e^{j\omega t} - L \frac{d}{dt} \tilde{I}_1 e^{j\omega t}\}$$

$$\tilde{V}_1 e^{j\omega t} = \tilde{V}_0 e^{j\omega t} - L \frac{d}{dt} \tilde{I}_1 e^{j\omega t}$$

$$\tilde{V}_1 e^{j\omega t} = \tilde{V}_0 e^{j\omega t} - L j\omega \tilde{I}_1 e^{j\omega t}$$

$$\tilde{V}_1 e^{j\omega t} = (\tilde{V}_0 - j\omega L \tilde{I}_1) e^{j\omega t}$$

$$\boxed{\tilde{V}_1 = \tilde{V}_0 - j\omega L \tilde{I}_1}$$

Look at the inductor in loop 2

$$V_2 - V_1 = -L \frac{dI_2}{dt}$$

$$V_2 = V_1 - L \frac{dI_2}{dt}$$

Use the conversions shown before

$$\tilde{V}_2 e^{j\omega t} = \tilde{V}_1 e^{j\omega t} - L \frac{d}{dt} \tilde{I}_2 e^{j\omega t}$$

$$\tilde{V}_2 e^{j\omega t} = (\tilde{V}_1 - j\omega L \tilde{I}_2) e^{j\omega t}$$

$$\boxed{\tilde{V}_2 = \tilde{V}_1 - j\omega L \tilde{I}_2}$$

Using the relationship through the load

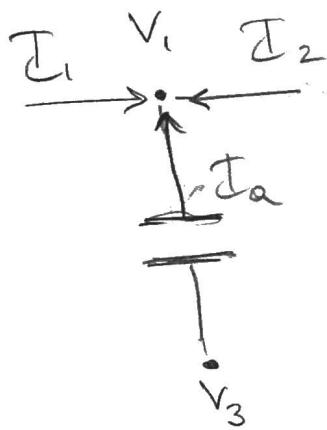
$$\underline{V_3}^0 \underline{V_2} = -\underline{I_2} Z_L \rightarrow V_2 \cos(\omega t + \phi_2) = Z_L I_2 \cos(\omega t + \phi_2)$$

$$\text{Re}\{\tilde{V}_2 e^{j\omega t}\} = \text{Re}\{Z_L \tilde{I}_2 e^{j\omega t}\}$$

$$\tilde{V}_2 = Z_L \tilde{I}_2$$

(sorry I didn't explicitly do this originally)

Create the node analysis to look at the capacitor



$$I_1 - I_2 + \underline{I_a} = 0$$

$$\frac{1}{L} \int (V_0 - V_1) dt + \frac{1}{L} \int (V_2 - V_1) dt$$

$$+ C \left(-\frac{\partial V_2}{\partial t} \right) = 0$$

$$I_a = -C \frac{\partial V_2}{\partial t}$$

$$I_1 - I_2 - C \frac{\partial V_2}{\partial t} = 0$$

$$\tilde{E}_1 \cos(\omega t + \phi_1) - \tilde{E}_2 \cos(\omega t + \phi_2) - C \frac{d}{dt} V_2 \cos(\omega t + \phi_2) = 0$$

$$\operatorname{Re}\{\tilde{I}_1 e^{j\omega t}\} - \operatorname{Re}\{\tilde{I}_2 e^{j\omega t}\} - \operatorname{Re}\left\{C \frac{d}{dt} (\tilde{V}_2 e^{j\omega t})\right\} = 0$$

$$\operatorname{Re}\{\tilde{I}_2 e^{j\omega t}\} = \operatorname{Re}\{\tilde{I}_1 e^{j\omega t} - C \frac{d}{dt} (\tilde{V}_2 e^{j\omega t})\}$$

$$\tilde{I}_2 e^{j\omega t} = \tilde{I}_1 e^{j\omega t} - j\omega C \tilde{V}_2 e^{j\omega t}$$

$$\tilde{I}_2 e^{j\omega t} = (\tilde{I}_1 - j\omega C \tilde{V}_2) e^{j\omega t}$$

$$\tilde{I}_2 = \tilde{I}_1 - j\omega C \tilde{V}_2$$

$$\tilde{V}_1 = \tilde{V}_o - j\omega L \tilde{I}_1$$

$$\tilde{V}_2 = \tilde{V}_1 - j\omega L \tilde{I}_2$$

$$\tilde{I}_2 = \tilde{I}_1 - j\omega C \tilde{V}_2$$

$$\tilde{I}_2 = \frac{\tilde{V}_2}{Z_L}$$

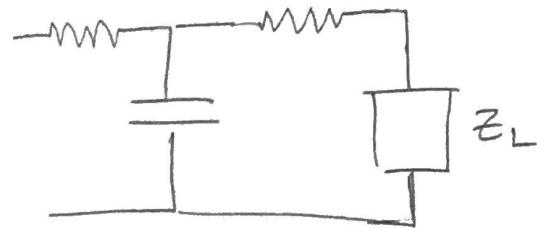
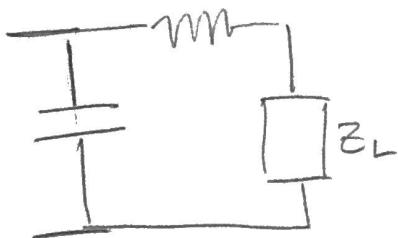
11.1.2) $\tilde{V}_o = V_o$ and previous equations,

Solve for Z_1 and Z_0 where $Z_1 \leq Z_0$ are

Z_1

and

Z_0



$Z_1 \rightarrow$ Capacitor \parallel (Inductor & Z_L)

\rightarrow Capacitor $\parallel Z_a$

$$Z_a = j\omega L + Z_L$$

$$(Z_1)^{-1} = \left(-\frac{j}{\omega C}\right)^{-1} + (j\omega L + Z_L)^{-1}$$

$$(Z_1)^{-1} = j\omega C + \frac{1}{j\omega L + Z_L}$$

$$(Z_1)^{-1} = \frac{j\omega C(j\omega L + Z_L) + 1}{(j\omega L + Z_L)}$$

$$Z_1 = \frac{(j\omega L + Z_L)}{j\omega C(j\omega L + Z_L) + 1}$$

This form is fine.

$$Z_1 = \frac{(j\omega L + Z_L)}{-\omega^2 CL + j\omega C Z_L + 1} \quad \leftarrow \text{this is also useful}$$

$$Z_1 = \frac{(j\omega L + Z_L)}{(1 - \omega^2 CL) + j\omega C Z_L} * \frac{(1 - \omega^2 CL) - j\omega C Z_L}{(1 - \omega^2 CL) - j\omega C Z_L}$$

wolfram alpha

$$Z_1 = \frac{-j(CL^2\omega^3 + CWZ_L^2 - LW)}{(1 - \omega^2 CL)^2 + \omega^2 C^2 Z_L^2} + Z_L$$

$$Z_o = j\omega L + Z_1$$

$$= j\omega L +$$

wolfram alpha again

$$Z_o = \frac{j(C^2L^3\omega^5 + C^2L\omega^3Z_L^2 - 3CL^2\omega^3 - CWZ^2 + 2L\omega)}{(1 - \omega^2 CL)^2 + \omega^2 C^2 Z_L^2} + Z$$

Alternative form straight from
Wolfram alpha (this makes section
3 easier)

This is what got plugged into
Wolfram alpha

$$Z_0 = jL\omega + \frac{1}{jC\omega + \frac{1}{Z_L + jL\omega}}$$

Parallel Series

Wolfram solution

$$Z_0 = \frac{-jCL^2\omega^3 - C\omega^2 Z_L + j2L\omega + Z_L}{-CL\omega^2 + jC\omega Z_L + 1}$$

this is the same
as the other solution

11.1.3) Using the previous solve for

1.) \tilde{I}_k for $k=1,2$ and \tilde{V}_k for $k=1,2$

Put everything in terms of $\tilde{V}_0 + Z_0$

$$\underline{\tilde{I}_1 = \frac{\tilde{V}_0}{Z_0}}$$

$$\tilde{V}_1 = \tilde{V}_0 - j\omega L \tilde{I}_1$$

$$\underline{\tilde{V}_1 = \tilde{V}_0 - j\omega L \frac{\tilde{V}_0}{Z_0}}$$

$$\tilde{I}_2 = \tilde{I}_1 - j\omega C \tilde{V}_1$$

$$\underline{\tilde{I}_2 = \frac{\tilde{V}_0}{Z_0} - j\omega C \left(\tilde{V}_0 - j\omega L \frac{\tilde{V}_0}{Z_0} \right)}$$

$$\tilde{V}_2 = \tilde{V}_1 - j\omega L \tilde{I}_2$$

$$\underline{\tilde{V}_2 = \tilde{V}_0 - j\omega L \frac{\tilde{V}_0}{Z_0} - j\omega L \left(\frac{\tilde{V}_0}{Z_0} - j\omega C \left(\tilde{V}_0 - j\omega L \frac{\tilde{V}_0}{Z_0} \right) \right)}$$

$$\tilde{I}_1 = \tilde{V}_0 \left(\frac{1}{Z_0} \right)$$

$$\tilde{V}_2 = \tilde{V}_0 \left(1 - \frac{j\omega L}{Z_0} - j\omega L \left(\frac{1}{Z_0} - j\omega C \left(1 - \frac{j\omega L}{Z_0} \right) \right) \right)$$

$$\tilde{V}_1 = \tilde{V}_0 \left(1 - \frac{j\omega L}{Z_0} \right)$$

$$\tilde{I}_2 = \tilde{V}_0 \left(\frac{1}{Z_0} - j\omega C \left(1 - \frac{j\omega L}{Z_0} \right) \right)$$

$$\tilde{I}_2 = \tilde{V}_o \left(\frac{1}{Z_0} \otimes j\omega C \left(1 \otimes \frac{j\omega L}{Z_0} \right) \right)$$

$$= \tilde{V}_o \left(\frac{1}{Z_0} - j\omega C - \frac{\omega^2 CL}{Z_0} \right)$$

$$\tilde{I}_2 = \tilde{V}_o \left(-j\omega C + \frac{1 - \omega^2 CL}{Z_0} \right)$$

$$\tilde{V}_2 = \tilde{V}_o \left(1 - \frac{j\omega L}{Z_0} - j\omega L \left(\frac{1}{Z_0} - j\omega C \left(1 - \frac{j\omega L}{Z_0} \right) \right) \right)$$

$$= \tilde{V}_o \left(1 - \frac{j\omega L}{Z_0} \otimes j\omega L \left(\otimes j\omega C + \frac{1 - \omega^2 CL}{Z_0} \right) \right)$$

$$= \tilde{V}_o \left(1 - \frac{j\omega L}{Z_0} - \omega^2 LC + \frac{-j\omega L + \omega^3 CL^2 j}{Z_0} \right)$$

$$\tilde{V}_2 = \tilde{V}_o \left(1 - \omega^2 LC + \frac{(j\omega CL^2 - 2j\omega L)}{Z_0} \right)$$

Plug in Z_0 to everything

$$\boxed{\tilde{I}_1 = \tilde{V}_o \left(\frac{-CL\omega^2 + jC\omega Z_L + 1}{-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L} \right)}$$

$$\tilde{V}_1 = V_0 \left(1 - \frac{j\omega L (CL\omega^2 + jC\omega Z_L + 1)}{-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L} \right)$$

$$\tilde{I}_2 = \tilde{V}_0 \left(-j\omega C + \frac{(1 - \omega^2 CL) (CL\omega^2 + j\omega C Z_L + 1)}{(-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L)} \right)$$

$$\tilde{V}_2 = V_0 \left(1 - \omega^2 LC + \frac{(j\omega CL^2 - 2j\omega L) (CL\omega^2 + j\omega C Z_L + 1)}{(-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L)} \right)$$

11.1.3.1) Find $I_k(t)$ & $V_k(t)$

$$I_1(t) = \operatorname{Re} \left\{ V_0 \left(\frac{-CL\omega^2 + jC\omega Z_L + 1}{-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L} \right) e^{j\omega t} \right\}$$

$$V_1(t) = \operatorname{Re} \left\{ V_0 \left(\frac{1 - j\omega L (CL\omega^2 + jC\omega Z_L + 1)}{-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L} \right) e^{j\omega t} \right\}$$

$$I_2(t) = \operatorname{Re} \left\{ V_0 \left(\frac{-j\omega C + (1 - \omega^2 CL) (CL\omega^2 + j\omega C Z_L + 1)}{-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L} \right) e^{j\omega t} \right\}$$

$$V_2(t) = \operatorname{Re} \left\{ V_0 \left(1 - \omega^2 LC + \frac{(j\omega CL^2 - 2j\omega L) (CL\omega^2 + j\omega C Z_L + 1)}{(-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L)} \right) e^{j\omega t} \right\}$$

(P knows these are subpar solutions, but P would genuinely take the real component of the complex impedance)

To further look at the solution,

The complex values for $\tilde{I}_1, \tilde{V}_1, \tilde{I}_2, \tilde{V}_2$

can be represented by as $a + jb$

with that in mind, the solution becomes

$$\tilde{I}_1(t) = \operatorname{Re} \{ (a_{\tilde{I}_1} + jb_{\tilde{I}_1}) e^{j\omega t} \}$$

$$= \operatorname{Re} \{ (a_{\tilde{I}_1} + jb_{\tilde{I}_1}) \cos \omega t + j \sin \omega t \}$$

$$= \operatorname{Re} \{ \underbrace{a_{\tilde{I}_1} \cos \omega t}_{\text{you actually have other parts}} + jb_{\tilde{I}_1} \cos \omega t + j \underbrace{a_{\tilde{I}_1} \sin \omega t}_{b_{\tilde{I}_1} \sin \omega t} - \underbrace{b_{\tilde{I}_1} \sin \omega t}_{\text{you actually have other parts}} \}$$

$$I_1(t) = \operatorname{Re} \{ \tilde{I}_1 \} \cos(\omega t) - \operatorname{Im} \{ \tilde{I}_1 \} \sin(\omega t)$$

This is true for the others as well

$$I_2(t) = \operatorname{Re} \{ \tilde{I}_2 \} \cos(\omega t) - \operatorname{Im} \{ \tilde{I}_2 \} \sin(\omega t)$$

$$V_1(t) = \operatorname{Re} \{ \tilde{V}_1 \} \cos(\omega t) - \operatorname{Im} \{ \tilde{V}_1 \} \sin(\omega t)$$

$$V_2(t) = \operatorname{Re} \{ \tilde{V}_2 \} \cos(\omega t) - \operatorname{Im} \{ \tilde{V}_2 \} \sin(\omega t)$$

Verify Answer by solving for Z_0

using $\tilde{I}_2 = \tilde{V}_o \left(\frac{1}{Z_0} - j\omega c \left(1 - \frac{j\omega L}{Z_0} \right) \right)$ and

$$\tilde{I}_2 = \frac{\tilde{V}_2}{Z_L} = \frac{\tilde{V}_o}{Z_L} \left(1 - \frac{j\omega L}{Z_0} - j\omega L \left(\frac{1}{Z_0} - j\omega c \left(1 - \frac{j\omega L}{Z_0} \right) \right) \right)$$

and solving for Z_0 (should be the same as original Z_0 in section 2)

$$\cancel{\left(\frac{1}{Z_0} - j\omega c \left(1 - \frac{j\omega L}{Z_0} \right) \right)} =$$

$$\cancel{\frac{1}{Z_L} \left(1 - \frac{j\omega L}{Z_0} - j\omega L \left(\frac{1}{Z_0} - j\omega c \left(1 - \frac{j\omega L}{Z_0} \right) \right) \right)}$$

$$\cancel{Z_L \left(\frac{1}{Z_0} - \left(j\omega c + \frac{\omega^2 c L}{Z_0} \right) \right)} =$$

$$1 - \frac{j\omega L}{Z_0} - j\omega L \left(\frac{1}{Z_0} - \cancel{j\omega c + \frac{\omega^2 c L}{Z_0}} \right)$$

$$Z_L \left(\frac{1}{Z_0} - j\omega c - \frac{\omega^2 c L}{Z_0} \right) =$$

$$1 - \frac{j\omega L}{Z_0} - \left(\frac{j\omega L}{Z_0} + \omega^2 c L - \frac{j\omega^3 c L^2}{Z_0} \right)$$

$$Z_L \left(\frac{1}{Z_0} - j\omega C - \frac{\omega^2 CL}{Z_0} \right) =$$

$$1 - \frac{j\omega L}{Z_0} - \frac{j\omega L}{Z_0} - \omega^2 CL + \frac{j\omega^3 CL^2}{Z_0}$$

$$\frac{Z_L}{Z_0} - \frac{\omega^2 CL Z_L}{Z_0} + \frac{2j\omega L}{Z_0} - \frac{j\omega^3 CL^2}{Z_0} =$$

$$j\omega C Z_L + 1 = \omega^2 c L$$

$$\frac{Z_L - \omega^2 CL Z_L + 2j\omega L - j\omega^3 CL^2}{Z_0} =$$

$$j\omega C Z_L + 1 = \omega^2 c L$$

$$Z_0 = \frac{Z_L - \omega^2 CL Z_L + 2j\omega L - j\omega^3 CL^2}{j\omega C Z_L + 1 - \omega^2 c L} \checkmark$$

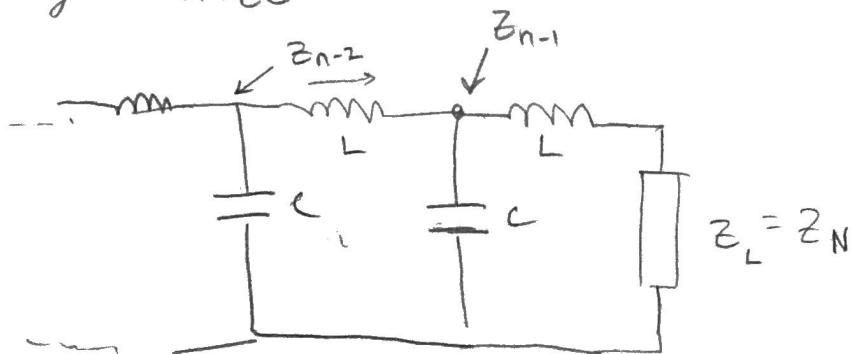
Same Z_0 from both methods

(this solution matches the wolfram solution)

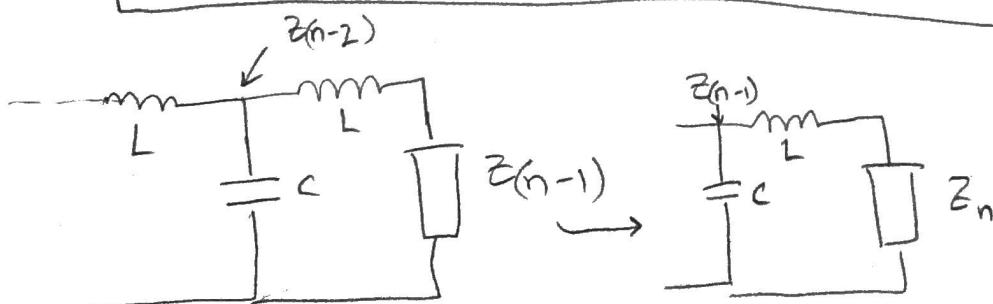
No. 1.4) Using the N ladder circuit, solve the following

- 1.) Write an iterative equation for Z_{n-1} in terms of Z_n, w, L , and C

Look at a single node then generalize



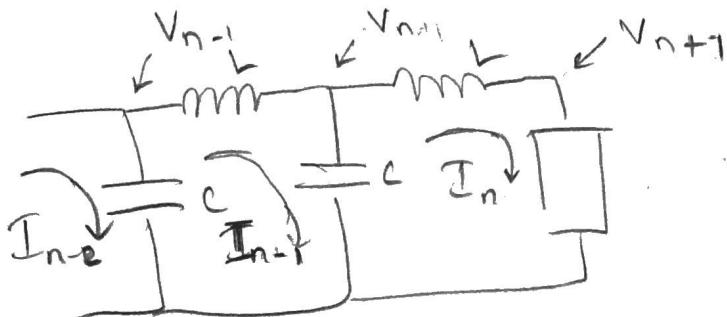
$$Z_{n-1} = \left(j\omega C + (Z_n + j\omega L)^{-1} \right)^{-1}$$



2.) Write an equation that relates

\tilde{V}_{n+1} to \tilde{V}_n and \tilde{I}_n

use the same as last section



$$V_n - V_{n+1} = L \frac{dI_n}{dt}$$

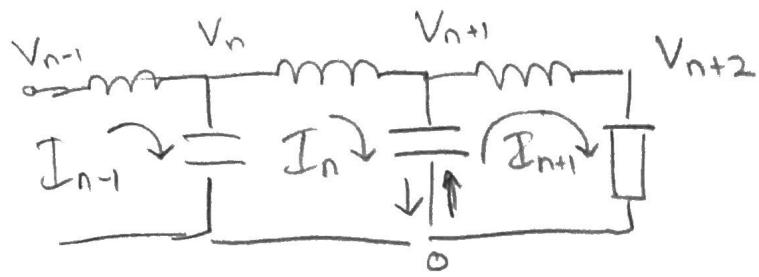
$$\frac{dI}{dt} = j\omega I$$

$$V_{n-1} - V_n = L \frac{dI_{n-1}}{dt}$$

ii) $\tilde{V}_{n+1} = \tilde{V}_n - j\omega L \tilde{I}_n$

Should note that you are using a different indexing convention. In my diagram, current I_2 flowed into node 2. Here you would have current I_1 flowing into node 2.

3.) Write an equation that relates \tilde{I}_{n+1} to \tilde{I}_n and \tilde{V}_{n+1}



$$\tilde{I}_n - \tilde{I}_{n+1} = C \frac{d\tilde{V}_{n+1}}{dt}$$

$$\frac{d\tilde{V}}{dt} = j\omega \tilde{V}$$

$$\boxed{\tilde{I}_{n+1} = \tilde{I}_n - j\omega C \tilde{V}_{n+1}}$$