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PHYS 313
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HW #6

6.4) Using $\mathbf{E} = \mathrm{Re}\left[\tilde{\mathbf{E}} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}\right]$

$$\mathbf{B} = \mathrm{Re}\left[\tilde{\mathbf{B}} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}\right]$$

$$\mathbf{k} \equiv k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$$

$$\mathbf{r} \equiv x\hat{x} + y\hat{y} + z\hat{z}$$

Show that $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{dt}$ leads to

$\tilde{\mathbf{B}} = \frac{1}{c}\hat{\mathbf{k}} \times \tilde{\mathbf{E}}$ and explain how it relates to

$\mathbf{B} = \frac{1}{c}\hat{\mathbf{k}} \times \mathbf{E}$

$$\nabla \times \mathbf{E} = \mathrm{Re}\left[e^{-i\omega t}\, \nabla \times \tilde{\mathbf{E}} e^{i\mathbf{k}\cdot\mathbf{r}}\right]$$

$$-\frac{\partial \mathbf{B}}{dt} = \mathrm{Re}\left[\mathbf{B}(-i\omega)\, e^{-i\omega t}\, e^{(i\mathbf{k}\cdot\mathbf{r})}\right]$$

Pick a direction such that $\tilde{\mathbf{B}}$ is only in the $\hat{x}$

$$\mathrm{Re}\left[e^{-i\omega t}\, \nabla \times \tilde{\mathbf{E}} e^{i\mathbf{k}\cdot\mathbf{r}}\right] \quad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{vmatrix} \quad \begin{array}{l} (\partial_y E_z - \partial_z E_y)_x \\ -(\partial_z E_x - \partial_x E_z)_y \\ (\partial_x E_y - \partial_y E_x)_z \end{array}$$

Since $\mathbf{B}$ was chosen to be in $\hat{x}$ E and k must be defined such that $\partial_y E_z - \partial_z E_y \neq 0$ and the other components are 0

$$\frac{\partial}{\partial y} e^{i\mathbf{k}\cdot\mathbf{r}} = ik_y e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$\mathrm{Re}\left[e^{-i\omega t}\left(\frac{\partial}{\partial y}\tilde{E}_z e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{\partial}{\partial z}\tilde{E}_y e^{i\mathbf{k}\cdot\mathbf{r}}\right)\right]$$

$$\mathrm{Re}\left[e^{-i\omega t}\left(\tilde{E}_z\, ik_y e^{i\mathbf{k}\cdot\mathbf{r}} - \tilde{E}_y\, ik_z e^{i\mathbf{k}\cdot\mathbf{r}}\right)\right]$$

Assuming $\mathrm{Im}\left[\frac{\hat{k}\times\vec{E}}{c} = \mathbf{B}\right]$ such that the Re part can be dropped

Further more $e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{r}}$ is on both sides, so I will remove them

$$\begin{array}{ccc|c} x & y & z & k_y E_z - k_z E_y \\ k_x & k_y & k_z & = \cancel{k_z E_x} - \cancel{k_x E_z} \\ E_x & E_y & E_z & \cancel{k_x E_y} - \cancel{k_y E_x} \end{array}$$

If $k_x = E_x = 0$

$$\tilde{E}_z \cancel{i} k_y - \tilde{E}_y \cancel{i} k_z = -\cancel{i}\omega \tilde{B}_x$$

$$\mathbf{k}\times\tilde{\mathbf{E}} = \omega\mathbf{B}$$

$$|\mathbf{k}|(\hat{k}\times\tilde{\mathbf{E}}) = \omega\mathbf{B} \qquad \frac{\omega}{k} = c$$

$$c^{-1}(\hat{k}\times\tilde{\mathbf{E}}) = \tilde{\mathbf{B}} \quad \checkmark$$

$\mathbf{B} = \frac{1}{c}\hat{k}\times\mathbf{E}$ is related to $\tilde{\mathbf{B}} = \frac{1}{c}\hat{k}\times\tilde{\mathbf{E}}$ by showing that there is an imaginary component to the $\mathbf{E}$ and $\mathbf{B}$ fields that follow the same equations. Furthermore the Real components of $\tilde{\mathbf{E}}$ & $\tilde{\mathbf{B}}$ are the real electric and magnetic fields.