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PHYS 513
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HW #6

6.3.1) Show

$$\mathbf{E} = E_{0x}\cos(k_z z - \omega t + \delta_x)\,\hat{x}$$

Satisfies

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

and

$$\mathbf{B} = B_{0x}\cos(k_z z - \omega t + \delta_x') + B_{0y}\cos(k_z z - \omega t + \delta_y')\,\hat{y}$$

satisfies

$$\nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Start with $\mathbf{E}$

$$\underline{\nabla^2 \mathbf{E}} = \frac{1}{c^2}\underline{\frac{\partial^2 \mathbf{E}}{\partial t^2}}$$

$$\nabla^2 \mathbf{E} = \nabla^2 E_x\,\hat{x} + \cancelto{0}{\nabla^2 E_y}\,\hat{y} + \cancelto{0}{\nabla^2 E_z}\,\hat{z}$$

$$= \left(\cancel{\frac{\partial^2}{\partial x^2}E_x} + \cancel{\frac{\partial^2}{\partial y^2}E_y} + \frac{\partial^2}{\partial z^2}E_x\right)\hat{x}$$

$$\nabla^2 \mathbf{E} = -k_z^2 E_{0x}\cos(k_z z - \omega t + \delta_x)\,\hat{x}$$

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left(E_{0x}\cos(k_z z - \omega t + \delta_x)\hat{x}\right)\frac{1}{c^2}$$

$$= \frac{-\omega^2}{c^2}E_0\cos(k_z z - \omega t + \delta_x)$$

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$-k_z^2 E_{0x}\cos(k_z z - \omega t + \delta_x') = -\frac{\omega^2}{c^2} E_{0x}\cos(k_z z - \omega t + \delta_x)$$

$$+k_z^2 \cancel{\mathbf{E}} = +\frac{\omega^2}{c^2}\cancel{\mathbf{E}}$$

$$\sqrt{k_z^2} = \sqrt{\frac{\omega^2}{c^2}} \rightarrow k_z = \pm\frac{\omega}{c}$$

$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$ is true when k_z is related to ω by c such $|k_z c| = |\omega|$

$$\nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$\nabla^2 B = \nabla^2 B_x \hat{x} + \nabla^2 B_y \hat{y} + \cancel{\nabla^2 B_z}^{0} \hat{z}$$

$$= \left(\cancel{\frac{\partial^2}{\partial x^2} B_x}^{0} + \cancel{\frac{\partial^2}{\partial y^2} B_x}^{0} + \frac{\partial^2}{\partial z^2} B_x\right)\hat{x} +$$

$$\left(\cancel{\frac{\partial^2}{\partial x^2} B_y}^{0} + \cancel{\frac{\partial^2}{\partial y^2} B_y}^{0} + \frac{\partial^2}{\partial z^2} B_y\right)\hat{y}$$

$$= \frac{\partial^2}{\partial z^2} B_x \hat{x} + \frac{\partial^2}{\partial z^2} B_y \hat{y}$$

$$= \frac{\partial^2}{\partial z^2} B_{0x}\cos(k_z z - \omega t + \delta_x')\hat{x} +$$

$$\frac{\partial^2}{\partial z^2} B_{0y}\cos(k_z z - \omega t + \delta_y')\hat{y}$$

$$\nabla^2 \mathbf{B} = -k_z^2\left(B_{0x}\cos(k_z z - \omega t + \delta_x')\hat{x} + B_{0y}\cos(k_z z - \omega t + \delta_y')\hat{y}\right)$$

$-k_z^2 \mathbf{B}$

$$\frac{1}{c^2}\frac{\partial^2 \mathbb{B}}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big(B_{0x}\cos(k_z z - \omega t + \delta_x')\hat{x} + B_{0y}\cos(k_z z - \omega t + \delta_y')\hat{y}\Big)$$

$$= \frac{-\omega^2}{c^2}\Big(B_{0x}\cos(k_z z - \omega t + \delta_x')\hat{x} + B_{0y}\cos(k_z z - \omega t + \delta_y')\hat{y}\Big) \quad \frac{-\omega^2}{c^2}\mathbb{B}$$

$$\nabla^2 \mathbb{B} = \frac{1}{c^2}\frac{\partial^2 \mathbb{B}}{\partial t^2}$$

$$-k_z^2 \cancel{\mathbb{B}} = \frac{-\omega^2}{c^2}\cancel{\mathbb{B}}$$

$$\sqrt{k_z^2} = \sqrt{\frac{\omega^2}{c^2}} \Rightarrow k_z = \pm\frac{\omega}{c}$$

$\boxed{\nabla^2 \mathbb{B} = \frac{1}{c^2}\frac{\partial^2 \mathbb{B}}{\partial t^2} \text{ is true when } \underline{k_z} \text{ is related to } \omega \text{ by } c \text{ such that } |k_z c| = |\omega|}$

6.3.2) Show that $k_z, \omega, E_{ox}, B_{ox}, B_{oy}, \delta_x, \delta_x', $ or δ_y' must be related or 0 for $\mathbf{B}$ and $\mathbf{E}$ to be consistent with Maxwell's equations

Maxwell's Eq

$\sin \to \cos \to -\sin \to -\cos$

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 = 0 \quad (\rho = 0)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (\mathbf{J} = 0)$$

$$\mathbf{E} = E_{ox} \cos(k_z z - \omega t + \delta_x)\,\hat{x}$$

$$\mathbf{B} = B_{ox} \cos(k_z z - \omega t + \delta_x')\,\hat{x} + B_{oy} \cos(k_z z - \omega t + \delta_y')\,\hat{y}$$

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = \begin{matrix} 0\,\hat{x} \\ (\partial_z E_x - 0)\,\hat{y} \\ (0 - \underbrace{\cancel{\partial_y E_x}}_{0})\,\hat{z} \end{matrix}$$

$$\nabla \times \mathbf{E} = -k_z E_{ox} \sin(k_z z - \omega t + \delta_x)\,\hat{y}$$

$$\frac{\partial \mathbf{B}}{\partial t} = +\omega\left(+B_{ox} \sin(k_z z - \omega t + \delta_x')\,\hat{x} + B_{oy} \sin(k_z z - \omega t + \delta_y')\,\hat{y}\right)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$-k_z E_{ox} \sin(k_z z - \omega t + \delta_x)\hat{y} = -(\omega(B_{ox} \sin(k_z z - \omega t + \delta_x')\hat{x}$$
$$B_{oy} \sin(k_z z - \omega t + \delta_y')\hat{y})$$

$-\omega B_{ox} \sin(k_z z - \omega t + \delta_x')\hat{x}$ must equal 0

$B_{ox} = 0$ achieves this.

$$+k_z E_{ox} \sin(k_z z - \omega t + \delta_x)\cancel{\hat{y}} = +\omega B_{oy} \sin(k_z z - \omega t + \delta_y')\cancel{\hat{y}}$$

$$\phi = k_z z - \omega t$$

$$k_z E_{ox} \sin(\phi + \delta_x) = \omega B_{oy} \sin(\phi + \delta_y')$$

$$k_z E_{ox} (\sin(\phi)\cos(\delta_x) + \cos(\phi)\sin(\delta_x)) =$$
$$\omega B_{oy} (\sin(\phi)\cos(\delta_y') + \cos(\phi)\sin(\delta_y'))$$

If $\delta_x = \delta_y'$, then $k_z E_{ox} = \omega B_{oy}$ or

$$\frac{E_{ox}}{B_{oy}} = \frac{\omega}{k_z}$$

$$\nabla \times \mathbb{B} = \mu_0 \epsilon_0 \frac{\partial \mathbb{E}}{\partial t}$$

$$\nabla \times \mathbb{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & 0 \end{vmatrix} = \begin{matrix} (0 - \partial_z B_y)\hat{x} \\ (\partial_z B_x - 0)\hat{y} \\ (\underbrace{\cancel{\partial_x B_y}}_{0} - \underbrace{\cancel{\partial_y B_x}}_{0})\hat{z} \end{matrix}$$

$$\nabla \times \mathbb{B} = +k_z(+B_{0y}\sin(k_z z - \omega t + \delta_y')\hat{x}) + k_z(-B_{0x}\sin(k_z z - \omega t + \delta_x')\hat{y})$$

$$\frac{\partial \mathbb{E}}{\partial t} = +\omega(+E_{0x}\sin(k_z z - \omega t + \delta_x)\hat{x})$$

$$\nabla \times \mathbb{B} = \mu_0 \epsilon_0 \frac{\partial \mathbb{E}}{\partial t}$$

must be 0

$$k_z B_{0y}\sin(k_z z - \omega t + \delta_y')\hat{x} - B_{0x}\sin(k_z z - \omega t + \delta_x')\hat{y} =$$

$$\mu_0 \epsilon_0 \omega E_{0x}\sin(k_z z - \omega t + \delta_x)\hat{x}$$

If $B_{0x} = 0$, all components work out

$$k_z B_{0y}\sin(k_z z - \omega t + \delta_y')\cancel{\hat{x}} = \mu_0 \epsilon_0 \omega E_{0x}\sin(k_z z - \omega t + \delta_x)\cancel{\hat{x}}$$

If $\delta_y' = \delta_x$, then $k_z B_{0y} = \mu_0 \epsilon_0 \omega E_{0x}$. Using $\frac{E_{0x}}{B_{0y}} = \frac{\omega}{k_z}$ then $k_z = \mu_0 \epsilon_0 \omega (\omega / k_z)$

Thus $\left(\frac{\omega}{k_z}\right)^2 = \frac{1}{\mu_0 \epsilon_0}$

Thus $B_{ox_y} = 0$

$$\frac{E_{ox}}{B_{oy}} = \frac{\omega}{k_z}, \quad \left(\frac{\omega}{k_z}\right)^2 = \frac{1}{\mu_0 \epsilon_0} \quad \text{when } \delta_x = \delta_y'$$

Use this to see if other solutions to δ_x' and δ_y'

Resume from Ampere's law $\phi = k_z z - \omega t$

$$k_z B_{oy}(\sin\phi \cos\delta_y' + \cos\phi \sin\delta_y') = \mu_0 \epsilon_0 \omega E_{ox}(\sin\phi \cos\delta_x + \cos\phi \sin\delta_x)$$

$$(\sin\phi \cos\delta_y' + \cos\phi \sin\delta_y') = \overset{1}{\mu_0 \epsilon_0 \frac{\omega}{k_z} \frac{E_{ox}}{B_{oy}}}(\sin\phi \cos\delta_x + \cos\phi \sin\delta_x)$$

$$(\sin\phi \cos\delta_y' + \cos\phi \sin\delta_y') = (\sin\phi \cos\delta_x + \cos\phi \sin\delta_x)$$

Pick an offset such that $\delta_y' = n\pi + \delta_x$

$$\sin\phi \cos(n\pi + \delta_x) + \cos\phi \sin(n\pi + \delta_x) = \sin\phi \cos\delta_x + \cos\phi \sin\delta_x$$

$$\sin\phi(\cos n\pi \cos\delta_x - \overset{0}{\cancel{\sin n\pi \sin\delta_x}}) + \cos\phi(\sin\delta_x \cos n\pi + \overset{0}{\cancel{\cos\delta_x \sin n\pi}}) = \sin\phi \cos\delta_x + \cos\phi \sin\delta_x$$

$$\pm(\sin\phi \cos\delta_x + \cos\phi \sin\delta_x) = \sin\phi \cos\delta_x + \cos\phi \sin\delta_x$$

This sign flip would imply that $E_{oy}/B_{oy} = \pm\frac{\omega}{k_z}$ Which can not be true. Therefore n must be an even integer.

Thus $\frac{1}{\sqrt{\mu_0 \epsilon_0}} = \left|\frac{\omega}{k_z}\right| = \frac{E_{0x}}{B_{0y}}$, $B_{0x} = 0$, and

$\delta_x = \delta_y' + n\pi$ when n is an even integer

6.3.3) If $\mathbb{E} = E_{0y}\cos(k_z z - \omega t + \delta_y)\hat{y}$

Faraday's law (I think this is for $\hat{y}$)

$$\nabla \times \mathbb{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ d_x & d_y & d_z \\ 0 & E_y & 0 \end{vmatrix} = \begin{matrix} 0 - d_z E_y \ \hat{x} \\ 0\ \hat{y} \\ \cancel{\partial_x E_y} - 0 \ \hat{z} \end{matrix}$$

$$\nabla \times \mathbb{E} = -k_z(-E_{0y}\sin(k_z z - \omega t + \delta_y))\hat{x}$$

use $\frac{\partial \mathbb{B}}{\partial t}$ from 6.3.2 $\phi = k_z z - \omega t$

$$+k_z(+E_{0y}\sin(\phi + \delta_y))\hat{x} = -\omega\left(B_{0x}\sin(\phi + \delta_x')\hat{x} + B_{0y}\sin(\phi + \delta_y')\hat{y}\right)$$

$B_{0y} = 0$ to cancel $\hat{y}$ component

$$k_z E_{0y}\sin(\phi + \delta_y)\hat{x} = -\omega B_{0x}\sin(\phi + \delta_x')\hat{x}$$

Assume trivial solution $\delta_y = \delta_x'$

$$k_z E_{0y} = -\omega B_{0x}$$

$$\frac{E_{0y}}{B_{0x}} = \frac{-\omega}{k_z}$$

Ampere's law

$$\frac{\partial \mathbf{E}}{dt} = +\omega\left(+E_{oy}\sin(k_z z-\omega t+\delta_y)\,\hat{y}\right)$$

use $\nabla \times \mathbf{B}$ from 6.3.2

$$\underline{k_z B_{oy}\sin(k_z-\omega t+\delta_y')\hat{x}} - k_z B_{ox}\sin(k_z z-\omega t+\delta_x')\hat{y} = \mu_0\epsilon_0\omega E_{oy}\sin(k_z z-\omega t+\delta_y)\,\hat{y}$$

$B_{oy}=0$

$$-k_z B_{ox}\sin(k_z z-\omega t+\delta_x') = \mu_0\epsilon_0\omega E_{oy}\sin(k_z z-\omega t+\delta_y)$$

if $\delta_x' = \delta_y$

$$-k_z B_{ox} = \mu_0\epsilon_0\,\omega\,E_{oy}$$

$$1 = \mu_0\epsilon_0\left(-\frac{\omega}{k_z}\right)\frac{E_{oy}}{B_{ox}} \leftarrow \left(\frac{-\omega}{k_z}\right)$$

$$\frac{\omega}{k_z} = \frac{\pm 1}{\sqrt{\mu_0\epsilon_0}}$$

$$\boxed{\begin{array}{l}\text{From } 6.3.2 \quad \delta_y = \delta_x' + n\pi \\ \text{thus} \quad \dfrac{E_{oy}}{B_{ox}} = -\dfrac{\omega}{k}, \quad \dfrac{\omega}{k} = \dfrac{\pm 1}{\sqrt{\mu_0\epsilon_0}}, \quad B_{oy}=0\end{array}}$$

6.3.4) Using $\mathbb{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$

show that 2 and 3 are consistent with

$$\mathbb{B} = \frac{1}{c} \hat{\mathbb{k}} \times \mathbb{E}$$

from 6.3.2 $\qquad \dfrac{E_{ox}}{B_{oy}} = \dfrac{\omega}{k_z}$

from 6.3.3 $\qquad \dfrac{E_{oy}}{B_{ox}} = -\dfrac{\omega}{k_z}$

Using $\mathbb{k} = k_z \hat{z}$ and $\omega / k = c$

6.3.2 → $\displaystyle B_y \hat{y} = \frac{1}{c} \hat{k} \times E_x \hat{x}$

$$= \frac{k}{\omega} \hat{z} \times E_x \hat{x}$$

$$= \frac{E_x k}{\omega} (\hat{z} \times \hat{x})$$

$$\begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} = \begin{array}{c} 0 \hat{x} \\ 1 \hat{y} \\ 0 \hat{z} \end{array}$$

$$B_y \hat{y} = \frac{E_x k}{\omega} \hat{y} \checkmark$$

from

6.3.3 $\frac{E_{0y}}{B_{0x}} = \frac{-\omega}{k_z}$

$B_x \hat{x} = -\frac{k_z}{\omega} E_y \hat{y} = \frac{1}{c} \hat{k} \times E_y \hat{y}$

$$\begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}$$

$B_x \hat{x} = \frac{E_y k}{\omega} \hat{z} \times \hat{y}$

$0 - 1 \hat{x}$
$0 \hat{y}$
$0 \hat{z}$

$$\boxed{B_x \hat{x} = \frac{-E_y k}{\omega} \hat{x}}$$