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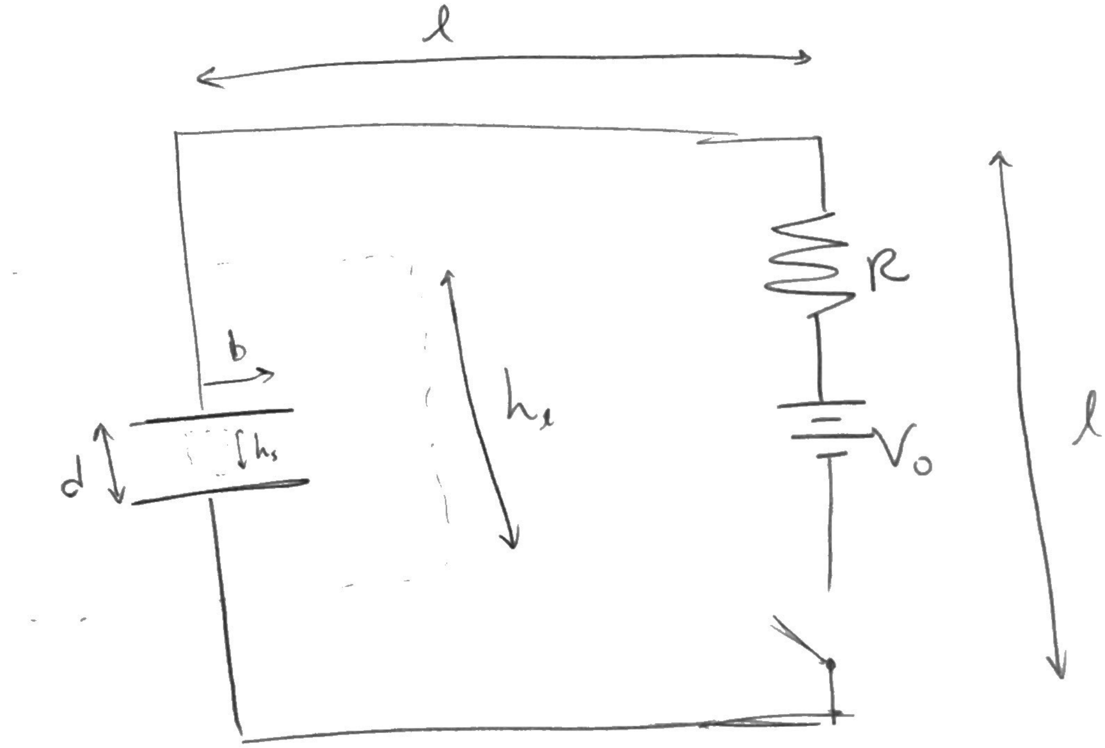
PHYS 513
September 30, 2020
HW #5

5.2) Show Poynting's theorem holds true for the areas below



Area 1 → r_s and h_s where $h_s < d$

Area 2 → r_l and h_l where $d < h_l < l$

Area 1

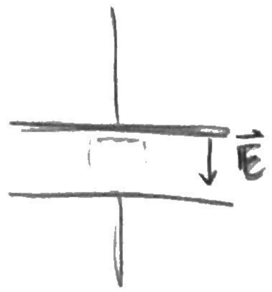
Assumptions:

- charge on plates are evenly distributed by σ. This allows for an E-field that is σ/ϵ_0 at the center of the plates
- $r_s << b$. This allows for fringe fields to go to 0, and allows E to be spatially constant

Area 2

Assumptions: (same as first" with below)

- $r_d >> d$ such that $B \to 0$ at r_d
- Assume that $\vec{E}$ in current carrying wire is 0 so $\vec{E} \cdot \vec{J} = 0$



E is time dependent, but I will leave it as $-E\hat{z}$ until the end

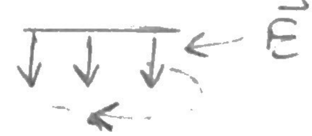
Poynting's Theorem

$$\int_V \left[\frac{\partial}{\partial t}\left(\frac{\vec{B}\cdot\vec{H}}{2}\right) + \frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right) + \vec{E}\cdot\vec{J} \right] dV =$$

$$-\oint_S (\vec{E}\times\vec{H})\cdot d\vec{S}$$

Find $\vec{B}$ field

$$\vec{\nabla}\times\vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$



$$\oint_L \vec{B}\cdot d\vec{l} = \int_S \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}\cdot d\vec{S}$$

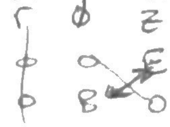
$$2\cancel{\pi}\cancel{r}\cdot B = \cancel{\pi} r^{\cancel{2}} \mu_0 \epsilon_0 \left(-\frac{\partial E}{\partial t}\right)$$

$$\underline{\vec{B} = -\mu_0 \epsilon_0 \frac{r}{2}\frac{\partial E}{\partial t}\hat{\phi}}$$

$$-\oint_S (\vec{E}\times\vec{H})\cdot d\vec{S} \longrightarrow -\frac{-1}{\mu_0}\oint(\vec{E}\times\vec{B})\cdot d\vec{S}$$

$$-\frac{1}{\mu_0}\oint_S(\vec{E}\times\vec{B})\cdot d\vec{S}$$

$$-\frac{1}{\mu_0}\oint_S\left(+E\hat{z}\times+\mu_0\epsilon_0\frac{r}{2}\frac{\partial E}{\partial t}\hat{\phi}\right)\cdot d\vec{S}$$



$$-\frac{\mu_0\epsilon_0}{\mu_0}\frac{\partial E}{\partial t}E\oint_S\frac{r}{2}(\hat{z}\times\hat{\phi})\cdot d\vec{S} \quad @\ r=r_s$$

$$+\epsilon_0\frac{\partial}{\partial t}\left(\frac{E^2}{2}\right)\frac{r_s}{2}\int+\hat{r}\cdot\hat{r}\ r\,d\phi\,dz$$

$$\epsilon_0\frac{\partial E}{\partial t}\left(\frac{E^2}{2}\right)\frac{r_s}{\cancel{2}}\cdot\cancel{2}\pi r_s h_s$$

$$-\oint_S(\vec{E}\times\vec{H})\cdot d\vec{S}=\epsilon_0\,\pi r_s^2 h_s\,\frac{\partial}{\partial t}\left(\frac{E^2}{2}\right)$$

$$\int_V \left[\cancel{\frac{\partial}{\partial t}\left(\frac{\vec{B}\cdot\vec{H}}{2}\right)} + \frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right) + \cancel{\vec{E}\cdot\vec{J}} \right] dV$$

$$\vec{B}\cdot\vec{H} = +\mu_0 \epsilon_0 \frac{r}{2}\frac{\partial E}{\partial t} \cdot +\epsilon_0 \frac{r}{2}\frac{\partial E}{\partial t}$$

$$= \mu_0 \epsilon^2 \frac{r^2}{4}\left(\frac{\partial E}{\partial t}\right)^2$$

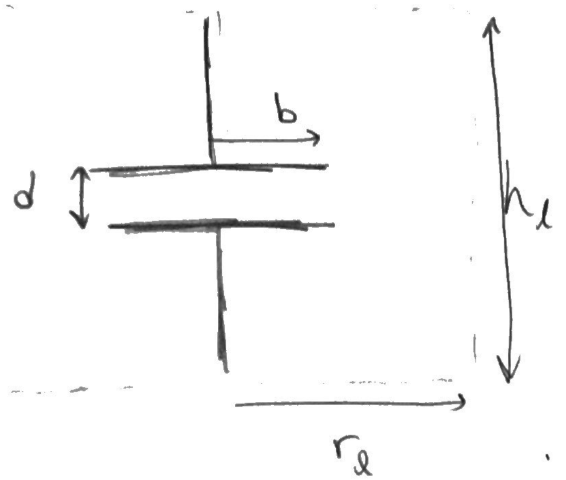
as $r_s \to 0$; $V \propto r_s^2 \to 0$ $\therefore$ $\vec{B}\cdot\vec{H} \to 0$

$\vec{E}\cdot\vec{J} = 0$ because $\vec{J} = 0$

$$\int_V \epsilon_0 \frac{\partial}{\partial t}\left(\frac{E^2}{2}\right) dV$$

$$\epsilon_0 \frac{\partial}{\partial t}\left(\frac{E^2}{2}\right) \int_V dV$$

$$\boxed{\epsilon_0 \frac{\partial}{\partial t}\left(\frac{E^2}{2}\right) \pi r_s^2 h_s = \epsilon_0 \pi r_s^2 h_s \frac{\partial}{\partial t}\left(\frac{E^2}{2}\right)} \checkmark$$



$\mathcal{E}$ is only time dependent which I will leave as $-\mathcal{E}\hat{z}$

Poynting's theorem

$$\int_V \left[\frac{\partial}{\partial t}\left(\frac{\vec{B}\cdot\vec{H}}{2}\right) + \frac{\partial}{\partial t}\left(\frac{\vec{D}\cdot\vec{E}}{2}\right) + \vec{E}\cdot\vec{J}\right] dV =$$

$$-\oint_S (\vec{E}\times\vec{H})\cdot d\vec{S}$$

Find $\vec{B}$ field

$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0 \frac{\partial\vec{E}}{\partial t} \rightarrow \oint_L \vec{B}\cdot d\vec{l} = \int_S \mu_0\epsilon_0 \frac{\partial\vec{E}}{\partial t}\cdot d\vec{S}$$

$$B\cdot 2\cancel{\pi} r = -\mu_0\epsilon_0 \cancel{\pi} b^2 \frac{dE}{dt}$$

$$B = -\frac{\mu_0\epsilon_0 b^2}{2r}\frac{\partial E}{\partial t}\hat{\phi}$$

$$-\oint_S \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)\cdot d\vec{S}$$

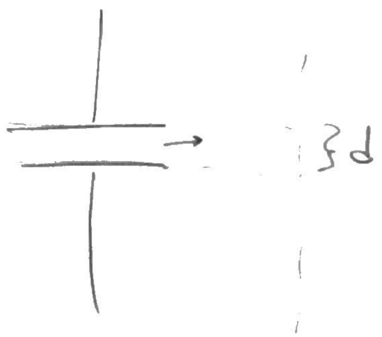
$$-\frac{1}{\cancel{\mu_0}}\oint_S \left(+E\,\hat{z}\times +\frac{\cancel{\mu_0}\epsilon_0\, b^2}{2r}\frac{dE}{dt}\,\hat{\phi}\right)\cdot d\vec{S}$$

$+\hat{r}$

$$+\,\epsilon_0\frac{dE}{dt}\,E\;\frac{d^2}{2}\oint_S \frac{1}{r}\left(\hat{z}\times\hat{\phi}\right)\cdot d\vec{S}$$

$$\frac{\epsilon_0\, b^2}{2}\,\frac{d}{dt}\left(\frac{E^2}{2}\right)\oint \frac{1}{\cancel{r}}\,\cancel{r}\,d\phi\,dz$$

$$\underline{\frac{\epsilon_0\, b^2}{\cancel{2}}\,\frac{d}{dt}\left(\frac{E^2}{2}\right)\,\cancel{2}\pi d}$$ ← this is the only section that has flux



$$\epsilon_0\, b^2\pi d\;\frac{d}{dt}\left(\frac{E^2}{2}\right)$$

$$\int_V \left[\frac{1}{\mu_0}\frac{d}{dt}\left(\frac{B^2}{2}\right) + \epsilon_0 \frac{d}{dt}\left(\frac{E^2}{2}\right) + \underbrace{\cancel{\vec{E}\cdot\vec{J}}}_{0}\right] dV$$

$$\int_V \epsilon_0 \frac{\partial}{\partial t}\left(\frac{E^2}{2}\right) dV = \underline{\underline{\epsilon_0 \frac{\partial}{\partial t}\left(\frac{E^2}{2}\right)\pi b^2 d}}$$

$$\int_V \left[\frac{1}{\mu_0}\frac{d}{dt}\left(\frac{B^2}{2}\right)\right] dV$$

$$\int_V \frac{1}{\mu_0}\frac{d}{dt}\left[\left(\cancel{\frac{\mu_0 \epsilon_0 b^2}{2r}\frac{\partial E}{dt}}\right)^2\right] dV$$

If $\quad r_1 >> d \; ; \; B \to 0$

$$\epsilon_0 \frac{d}{dt}\left(\frac{E^2}{2}\right)\pi b^2 d = \epsilon_0 b^2 \pi d \frac{d}{dt}\left(\frac{E^2}{2}\right) \checkmark$$