9.2.1) Given

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PHYS 513
HW # 9
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$$\tilde{V}_n(z) = \tilde{V}_n^+ e^{-j\beta z} + \tilde{V}_n^- e^{j\beta z}$$

where

$$V_n(z,t) = Re\left[\tilde{V}_n(z) e^{j\omega t}\right]$$

similarly

$$\tilde{E}_n(z) = \tilde{E}_n^+ e^{-jk_n z} + \tilde{E}^- e^{jk_n z}$$

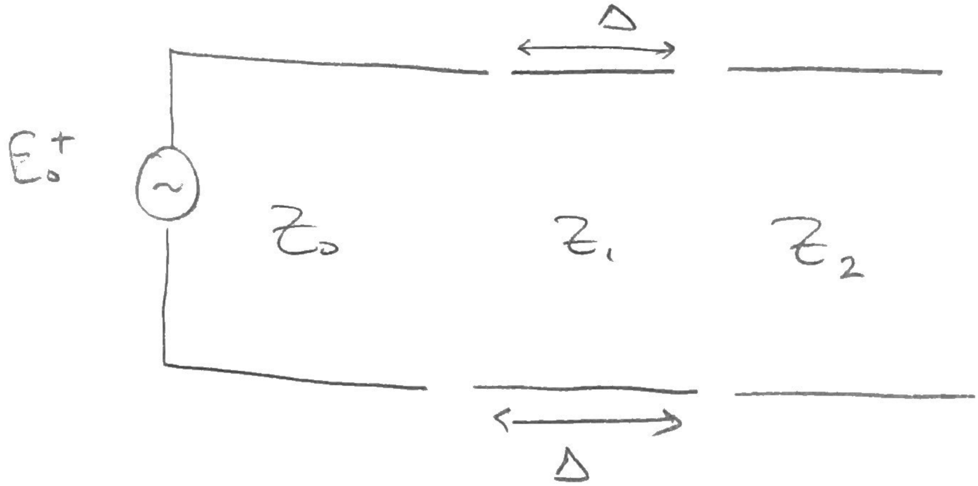
where

$$E_n(z,t) = Re\left[\tilde{E}_n(z) e^{j\omega t}\right]$$

Assume $\tilde{V}_0^+$ is known. At $z=0$ & $z=\Delta$, V & I are continous. Assume $V_2^- = 0$

find $\tilde{V}_0^-(z)$, $\tilde{V}_1^-(z)$, and $\tilde{V}_2^+(z)$. find $\tilde{V}_0^-(z)/\tilde{V}_0^+(z)$ and $\tilde{V}_1^-(z)/\tilde{V}_1^+(z)$

Step 1 break the problem down



$$\frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} = \frac{V_0^L}{Z_L} \quad (I) \qquad Z_L = Z_1 + Z_2 \quad (III)$$

$$V_0^+ + V_0^- = V_0^L \quad (II)$$

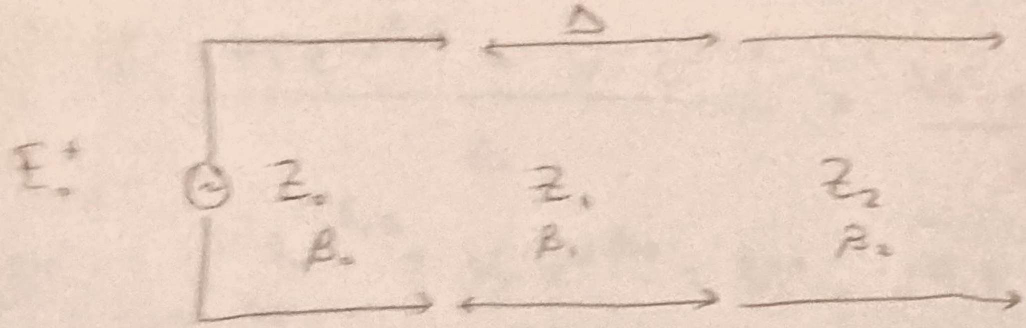
$$\frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} = \frac{V_0^+ + V_0^-}{Z_L}$$

$$\frac{V_0^+ - V_0}{Z_0} = \frac{V_0^+ + V_0^-}{Z_L}$$

$$Z_L(V_0^+ - V_0^-) = Z_0(V_0^+ + V_0^-)$$

$$V_0^+(Z_L - Z_0) = V_0^-(Z_0 + Z_L)$$

$$V_0^- = \frac{(Z_L - Z_0)}{(Z_0 + Z_L)} V_0^+$$



$$\left(\frac{V_0^+ e^{-j\beta_0 z}}{Z_0} - \frac{V_0^- e^{j\beta_0 z}}{Z_0}\right) = \frac{V_1^+ e^{-j\beta_1 z}}{Z_1}$$

and $V_0^+ e^{-j\beta_0 z} + V_0^- e^{j\beta_0 z} = V_1^+ e^{-j\beta_1 z}$

Start with these 2 equations

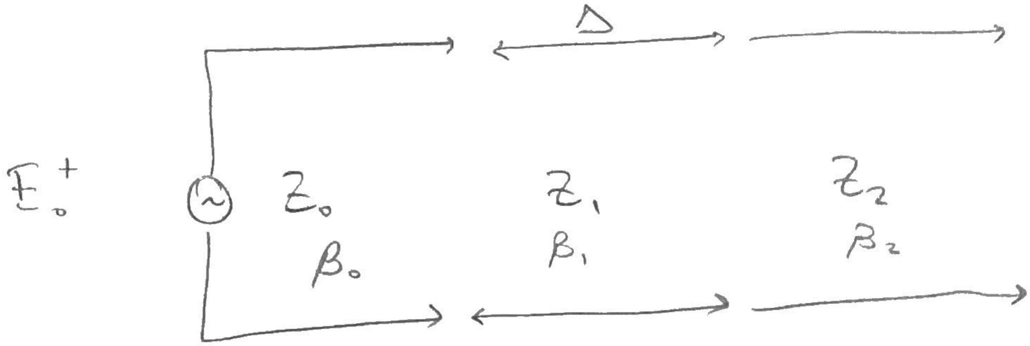
$$\frac{V_0^+ e^{-j\beta_0 z} - V_0^- e^{j\beta_0 z}}{Z_0} = \frac{V_0^+ e^{-j\beta_0 z} + V_0^- e^{j\beta_0 z}}{Z_1}$$

$Z_1 (V_0^+ e^{-j\beta_0 z} - V_0^- e^{j [illegible]} [illegible] Z_0 (V_0^+ e^{-j [illegible]} [illegible] \beta_0 z)$

$(Z_1 - Z_0) V_0^+ e^{-j\beta_0 z} = ([illegible] V_0^- e^{j\beta [illegible]}$

$$\boxed{V_0^- = \frac{(Z_1 - Z_0)}{(Z_1 + Z [illegible]} [illegible]}$$

$$\frac{V_0^-}{V_0^+} [illegible]$$



$$\left(\frac{V_0^+ e^{-j\beta_0 z}}{Z_0} - \frac{V_0^- e^{j\beta_0 z}}{Z_0}\right) = \frac{V_1^+ e^{-j\beta_1 z}}{Z_1}$$

and $$V_0^+ e^{-j\beta_0 z} + V_0^- e^{j\beta_0 z} = V_1^+ e^{-j\beta_1 z}$$

Start with these 2 equations

$$\frac{V_0^+ e^{-j\beta_0 z} - V_0^- e^{j\beta_0 z}}{Z_0} = \frac{V_0^+ e^{-j\beta_0 z} + V_0^- e^{j\beta_0 z}}{Z_1}$$

$$Z_1\left(V_0^+ e^{-j\beta_0 z} - V_0^- e^{j\beta_0 z}\right) = Z_0\left(V_0^+ e^{-j\beta_0 z} + V_0^- e^{j\beta_0 z}\right)$$

$$(Z_1 - Z_0)\, V_0^+ e^{-j\beta_0 z} = (Z_1 + Z_0)\, V_0^- e^{j\beta_0 z} \quad \checkmark$$

$$\boxed{V_0^- = \frac{(Z_1 - Z_0)}{(Z_1 + Z_0)}\, V_0^+ e^{-2j\beta_0 z}}$$

$$\boxed{\frac{V_0^-}{V_0^+} = \frac{(Z_1 - Z_0)}{(Z_1 + Z_0)}\, e^{-2j\beta_0 t}}$$

Using $$\frac{V_0^+ e^{-j\beta_0 z} - V_0^- e^{j\beta_0 z}}{Z_0} = \frac{V_1^+ e^{-j\beta_1 z}}{Z_1}$$

and $$V_0^+ e^{-j\beta_0 z} + V_0^- e^{j\beta_0 z} = V_1^+ e^{-j\beta_1 z}$$

$$V_0^- e^{j\beta_0 z} = V_1^+ e^{-j\beta_1 z} - V_0^+ e^{-j\beta_0 z}$$

$$\frac{V_0^+ e^{-j\beta_0 z} - \left(V_1^+ e^{-j\beta_1 z} - V_0^+ e^{-j\beta_0 z}\right)}{Z_0} = \frac{V_1^+ e^{-j\beta_1 z}}{Z_1}$$

$$\left(2V_0^+ e^{-j\beta_0 z} - V_1^+ e^{-j\beta_1 z}\right) Z_1 = Z_0 V_1^+ e^{-j\beta_1 z}$$

$$2 Z_1 V_0^+ e^{-j\beta_0 z} = (Z_1 + Z_0) V_1^+ e^{-j\beta_1 z}$$

$$V_1^+ = \frac{2 Z_1 V_0^+ e^{-j\beta_0 z}}{Z_1 + Z_0} e^{j\beta_1 z}$$

$$\boxed{V_1^+ = \frac{2 Z_1}{(Z_1 + Z_0)} V_0^+ e^{-j(\beta_0 - \beta_1) z}}$$

$$\boxed{\frac{V_1^+}{V_0^+} = \frac{2 Z_1}{(Z_1 + Z_0)} e^{-j(\beta_0 - \beta_1) z}}$$

Do the same process for Z_2

$$\left(\frac{V_1^+ e^{-j\beta_1 z} - V_1^- e^{j\beta_1 z}}{Z_1} \right) = \frac{V_2^+ e^{-j\beta_2 z}}{Z_2}$$

and $V_1^+ e^{-j\beta_1 z} + V_1^- e^{j\beta_1 z} = V_2^+ e^{-j\beta_2 z}$

$$\left(\frac{V_1^+ e^{-j\beta_1 z} - V_1^- e^{j\beta_1 z}}{Z_1} \right) = \frac{\left(V_1^+ e^{-j\beta_1 z} + V_1^- e^{j\beta_1 z} \right)}{Z_2}$$

$$Z_2 \left(V_1^+ e^{-j\beta_1 z} - V_1^- e^{j\beta_1 z} \right) = Z_1 \left(V_1^+ e^{-j\beta_1 z} + V_1^- e^{j\beta_1 z} \right)$$

$$(Z_2 - Z_1) V_1^+ e^{-j\beta_1 z} = (Z_1 + Z_2) V_1^- e^{j\beta_1 z}$$

$$V_1^- = \frac{(Z_2 - Z_1)}{(Z_1 + Z_2)} e^{-2j\beta_1 z} V_1^+$$

$$V_1^- = \frac{(Z_2 - Z_1)}{(Z_1 + Z_2)} e^{-2j\beta_1 z} \left(\frac{2 Z_1}{Z_1 + Z_2} e^{-j(\beta_0 - \beta_1) z} \right) V_0^+$$

$$\boxed{V_1^- = \frac{2 Z_1 (Z_2 - Z_1)}{(Z_1 + Z_2)} V_0^+ e^{-j(\beta_1 + \beta_0) z}}$$

$$\boxed{\frac{V_1^-}{V_1^+} = \frac{(Z_2 - Z_1)}{(Z_1 + Z_2)} e^{-2j\beta_1 z}}$$

$$\frac{\left(V_1^+ e^{-j\beta_1 z} - V_1^- e^{j\beta_1 z}\right)}{Z_1} = \frac{V_2^+ e^{-j\beta_2 z}}{Z_2}$$

$$V_1^+ e^{-j\beta_1 z} + V_1^- e^{j\beta_1 z} = V_2^+ e^{-j\beta_2 z}$$

$$V_1^- e^{j\beta_1 z} = V_2^+ e^{-j\beta_2 z} - V_1^+ e^{-j\beta_1 z}$$

$$\frac{\left(V_1^+ e^{-j\beta_1 z} - \left(V_2^+ e^{-j\beta_2 z} - V_1^+ e^{-j\beta_1 z}\right)\right)}{Z_1} = \frac{V_2^+ e^{-j\beta_2 z}}{Z_2}$$

$$Z_2\left(2V_1^+ e^{-j\beta_1 z} - V_2^+ e^{-j\beta_2 z}\right) = Z_1 V_2^+ e^{-j\beta_2 z}$$

$$2V_1^+ Z_2 e^{-j\beta_1 z} = (Z_1 + Z_2) V_2^+ e^{-j\beta_2 z}$$

$$V_2^+ = \frac{2Z_2}{(Z_1 + Z_2)} e^{-j(\beta_1 - \beta_2)z} V_1^+$$

$$V_2^+ = \frac{2Z_2}{(Z_1 + Z_2)} e^{-j(\beta_1 - \beta_2)z} \frac{2Z_1}{(Z_1 + Z_0)} e^{-j(\beta_0 - \beta_1)z}$$

$$\boxed{V_2^+ = \frac{4Z_1 Z_2}{(Z_1 + Z_2)(Z_1 + Z_0)} e^{-j(\beta_0 - \beta_2)z} V_0^+}$$

$$\frac{\left(V_1^+ e^{-j\beta_1 z} - V_1^- e^{j\beta_1 z}\right)}{Z_1} = \frac{V_2^+ e^{-j\beta_2 z}}{Z_2}$$

$$V_1^+ e^{-j\beta_1 z} + V_1^- e^{j\beta_1 z} = V_2^+ e^{-j\beta_2 z}$$

$$V_1^- e^{j\beta_1 z} = V_2^+ e^{-j\beta_2 z} - V_1^+ e^{-j\beta_1 z}$$

$$\frac{\left(V_1^+ e^{-j\beta_1 z} - \left(V_2^+ e^{-j\beta_2 z} - V_1^+ e^{-j\beta_1 z}\right)\right)}{Z_1} = \frac{V_2^+ e^{-j\beta_2 z}}{Z_2}$$

$$Z_2\left(2V_1^+ e^{-j\beta_1 z} - V_2^+ e^{-j\beta_2 z}\right) = Z_1 V_2^+ e^{-j\beta_2 z}$$

$$2V_1^+ Z_2 e^{-j\beta_1 z} = (Z_1 + Z_2) V_2^+ e^{-j\beta_2 z}$$

$$V_2^+ = \frac{2Z_2}{(Z_1+Z_2)} e^{-j(\beta_1 - \beta_2)z} V_1^+$$

$$V_2^+ = \frac{2Z_2}{(Z_1+Z_2)} e^{-j(\beta_1 - \beta_2)z} \frac{2Z_1}{(Z_1+Z_0)} e^{-j(\beta_0 - \beta_1)z}$$

$$\boxed{V_2^+ = \frac{4Z_1 Z_2}{(Z_1+Z_2)(Z_1+Z_0)} e^{-j(\beta_0 - \beta_2)z} V_0^+}$$

9.2.21) Show

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HW # 9
November 16, 2020

$$V(z,t) = V^+[\cos(\omega t - \beta z) + \rho \cos(\omega t + \beta z)]$$

can be written as two standing waves

$$V(z,t) = A\cos(\omega t)\cos(\beta z) + B\sin(\omega t)\sin(\beta z)$$

and find A + B

$$\cos(\omega t - \beta z) \to \cos(u - v)$$
$$\cos(\omega t + \beta z) \to \cos(u + v)$$

$$\cos(u - v) = \cos u \cos v + \sin u \sin v \quad (I)$$
$$\cos(u + v) = \cos u \cos v - \sin u \sin v \quad (II)$$

$$(I) + (II) = 2\cos u \cos v = \underline{\cos(u - v) + \cos(u + v)}$$
$$(I) - (II) = 2\sin u \sin v = \underline{\cos(u - v) - \cos(u + v)}$$

define γ such that

$$\gamma + (\gamma - \rho) = 1$$

$$V(z,t) = V^+\Big[\big(\gamma + (\gamma-\rho)\big)\cos(\mu - \nu) + \big(\gamma + (\rho - \gamma)\big)\cos(\mu+\nu)\Big]$$

$$= V^+\Big[\gamma\{\cos(\mu-\nu) + \cos(\mu+\nu)\} + \qquad \leftarrow (\mathrm{I}) + (\mathrm{II})$$

$$(\gamma-\rho)\{\cos(\mu-\nu) - \cos(\mu+\nu)\}\Big] \qquad \leftarrow (\mathrm{I}) - (\mathrm{II})$$

$$= V^+\Big[2\gamma \cos\mu \cos\nu + 2(\gamma-\rho)\sin\mu \sin\nu\Big]$$

Solve for γ

$$2\gamma - \rho = 1 \Rightarrow \gamma = \frac{\rho+1}{2}$$

$$= V^+\Big[(\rho+1)\cos\mu\cos\nu + (1-\rho)\sin\mu\sin\nu\Big]$$

$$\boxed{V(z,t) = V^+\Big[(\rho+1)\cos\omega t\cos\beta z + (1-\rho)\sin\omega t\sin\beta z\Big]}$$

9.2.2.3) The plot generated from the code creates a standing wave with V_{max} and V_{min}. These V_{max} and V_{min} relate to the two standing waves found in part 9.2.1 by the relationship of the coefficients $(1-\rho)$ and $(1+\rho)$. If $\rho < 0$ $|(1-\rho)| \rightarrow (1+|\rho|)$ and $|(1+\rho)| \rightarrow (1-|\rho|)$. Therefore, the standing wave ratio is given by

$$S = \frac{1+|\rho|}{1-|\rho|}$$

$|\rho|$ can be determined from the plot of the ratios, and if the reflected wave is given, so can the sign.