11.1.1) Show the following

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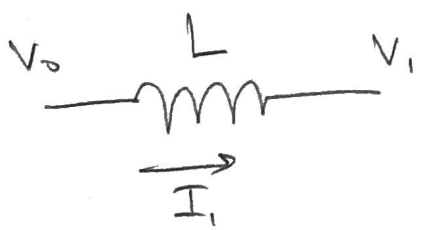
$$\tilde{V}_1 = \tilde{V}_0 - j\omega L \tilde{I}_1$$
$$\tilde{I}_2 = \tilde{I}_1 - j\omega C \tilde{V}_1$$
$$\tilde{V}_2 = \tilde{V}_1 - j\omega L \tilde{I}_2$$
$$\tilde{I}_2 = \frac{\tilde{V}_2}{Z_L}$$

Remember source is
$V_0 \cos(\omega t) = \mathrm{Re}[\tilde{V}_0]$

Circuit on page 2

Start with first inductor in loop 1



$$\Delta \tilde{V}_{01} = L \frac{d\tilde{I}_1}{dt}$$

$$\tilde{V}_0 - \tilde{V}_1 = L \frac{d\tilde{I}_1}{dt}$$

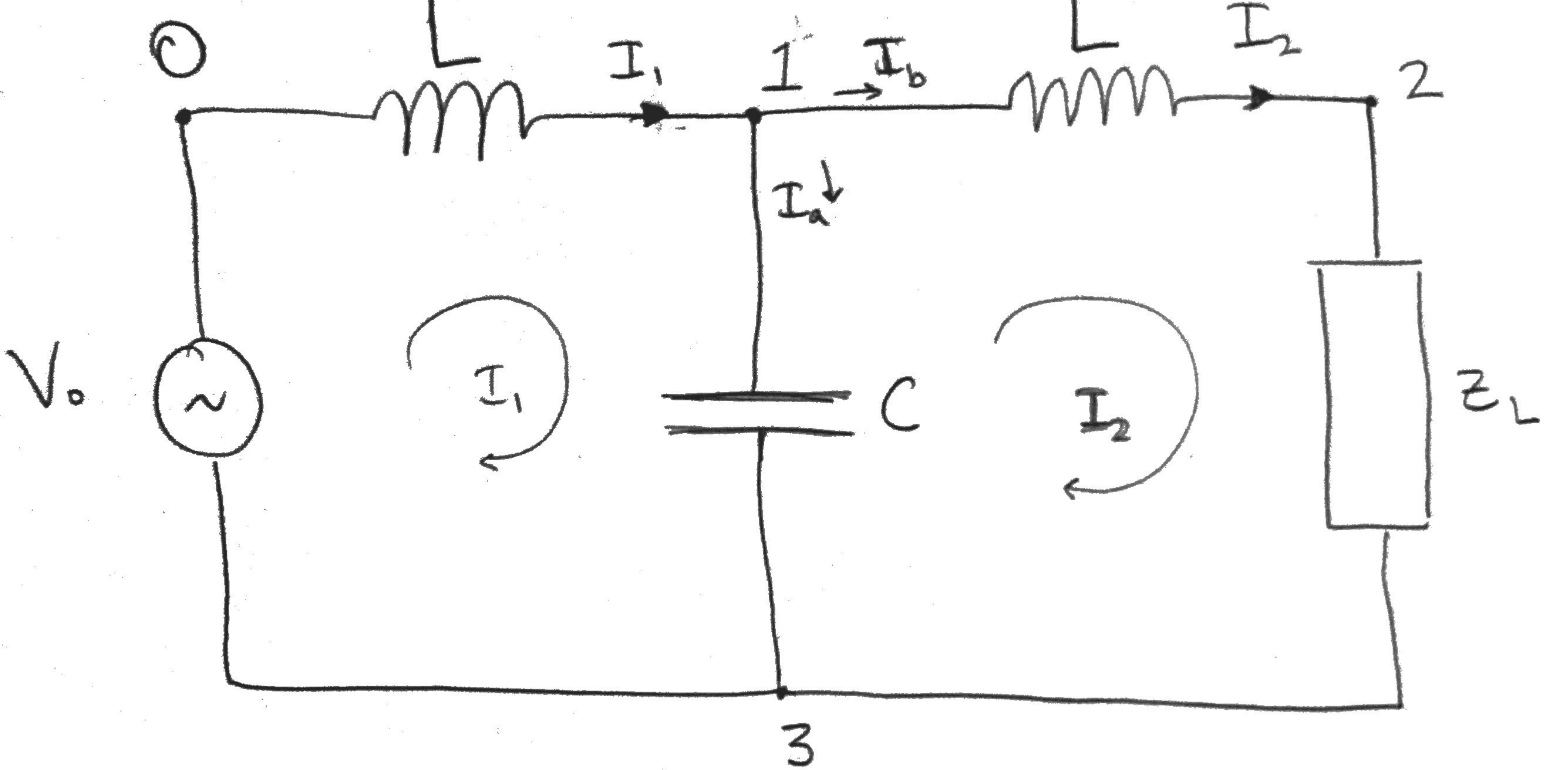
$$\frac{d\tilde{I}_1}{dt} = \frac{d}{dt}\left(\underbrace{\tilde{I}_1' e^{j\omega t}}\right)$$

const

$$\frac{d\tilde{I}_1}{dt} = \underbrace{\tilde{I}_1' e^{j\omega t}}\, j\omega$$

$$\tilde{V}_0 - \tilde{V}_1 = L(j\omega \tilde{I}_1)$$

$$\boxed{\tilde{V}_1 = \tilde{V}_0 - j\omega L \tilde{I}_1}$$



0
L
I1
1
Ib
L
I2
2
Ia
V0
I1
C
I2
ZL
3

Expand the two Loops into component voltages

$$V_0 + (V_1 - V_0) + (V_3 - V_1) = 0$$ first loop

$$(V_1 - V_3) + (V_2 - V_1) + (V_3 - V_2) = 0$$ second Loop

Choose $V_3 = 0$ since it feeds to the source

From the above

$$V_0 + V_1 - V_0 + V_3 - V_1 = 0$$

$$(\cancel{V_0 - V_0}) + (\cancel{V_1 - V_1}) + \cancelto{0}{V_3} = 0$$

Also

$$V_1 - V_3 + V_2 - V_1 + V_3 - V_2 = 0$$

$$(\overset{0}{\cancel{V_1 - V_1}}) + (\overset{0}{\cancel{V_2 - V_2}}) + (\overset{0}{\cancel{V_3 - V_3}}) = 0$$

from here use the relationship for

$$(V_1 - V_0) = -L\frac{dI_1}{dt}$$

$$V_1 = V_0 - L\frac{dI_1}{dt}$$

however these values are in the form of $\cos(\omega t + \phi_k)$

need a more effective way to represent $\cos(\omega t + \phi_k)$ where ϕ_k is a phase offset.

start

$$\cos(\omega t + \phi_k) = \frac{1}{2}\left(e^{j(\omega t + \phi_k)} + e^{-j(\omega t + \phi_k)}\right)$$

represent $e^{j(\omega t + \phi_k)}$ as ψ

$$\cos(\omega t + \varphi_k) = \frac{1}{2}(\psi + \psi^*)$$

Given that $\psi = a + bj$

$$\psi + \psi^* = 2a$$

Thus $\cos(\omega t + \varphi_k) = \frac{2}{2}\left(\mathrm{Re}\{e^{j(\omega t + \phi_k)}\}\right)$

$$\cos(\omega t+\phi_k) = Re\{e^{j\phi_k} e^{j\omega t}\}$$

where $e^{j\phi_k}$ is a phase offset which can be absorbed into a constant

Thus $V_o \cos(\omega t) = V_o Re\{e^{j\omega t}\}$

Given that V_o is a constant

$$V_o \cos(\omega t) = Re\{V_o e^{j\omega t}\}$$

Any V_k value can be represented thusly by a general solution of

$$V_k \cos(\omega t+\phi_k) = Re\{V_k e^{j\phi_k} e^{j\omega t}\}$$

where $V_k e^{j\phi_k}$ would become $\tilde{V}_k$

Similarly, since I is given by $\frac{V_k}{Z_k} = I_k$

I_k would be of the form $I_k \cos(\omega t+\theta_k) =$
$Re\{I_k e^{j\theta_k} e^{j\omega t}\}$

with this, the equation originally being solved would become

$$V_1 \cos(\omega t + \phi_1) = V_0 \cos(\omega t) - L \frac{d}{dt} I_1 \cos(\omega t + \theta_1)$$

$$Re\left\{ \tilde{V}_1 e^{j\omega t} \right\} = Re\left\{ \tilde{V}_0 e^{j\omega t} \right\} - Re\left\{ L \frac{d}{dt} \tilde{I}_1 e^{j\omega t} \right\}$$

Proof

$$Re\left\{ a_1 + j b_1 \right\} + Re\left\{ a_2 + j b_2 \right\} = a_1 + a_2$$

$$Re\left\{ a_1 + j b_1 + a_2 + j b_2 \right\} = a_1 + a_2 \quad \text{for } a_1, a_2 = \mathbb{R}$$

$$\rightarrow Re\left\{ \tilde{V}_1 e^{j\omega t} \right\} = Re\left\{ \tilde{V}_0 e^{j\omega t} - L \frac{d}{dt} \tilde{I}_1 e^{j\omega t} \right\}$$

$$\tilde{V}_1 e^{j\omega t} = \tilde{V}_0 e^{j\omega t} - L \frac{d}{dt} \tilde{I}_1 e^{j\omega t}$$

$$\tilde{V}_1 e^{j\omega t} = \tilde{V}_0 e^{j\omega t} - L j\omega \tilde{I}_1 e^{j\omega t}$$

$$\tilde{V}_1 \cancel{e^{j\omega t}} = (\tilde{V}_0 - j\omega L \tilde{I}_1) \cancel{e^{j\omega t}}$$

$$\boxed{\tilde{V}_1 = \tilde{V}_0 - j\omega L \tilde{I}_1}$$

Look at the inductor in loop 2

$$V_2 - V_1 = -L \frac{\partial I_2}{\partial t}$$

$$V_2 = V_1 - L \frac{\partial I_2}{\partial t}$$

Use the conversions shown before

$$\tilde{V}_2 e^{j\omega t} = \tilde{V}_1 e^{j\omega t} - L \frac{\partial}{\partial t} \tilde{I}_2 e^{j\omega t}$$

$$\hat{V}_2 \cancel{e^{j\omega t}} = (\tilde{V}_1 - j\omega L \tilde{I}_2) \cancel{e^{j\omega t}}$$

$$\boxed{\tilde{V}_2 = \tilde{V}_1 - j\omega L \tilde{I}_2}$$

Using the relationship through the load

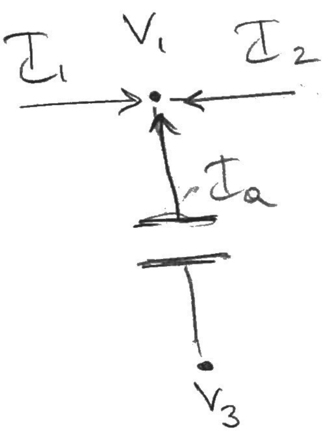
$$\cancel{V_3}^{0} - V_2 = -I_2 Z_L \rightarrow V_2 \cos(\omega t + \phi_2) = Z_L I_2 \cos(\omega t + \theta_2)$$

$$Re\{\tilde{V}_2 e^{j\omega t}\} = Re\{Z_L \tilde{I}_2 e^{j\omega t}\}$$

$$\boxed{\tilde{I}_2 = \frac{\tilde{V}_2}{Z_L}} \leftarrow \tilde{V}_2 = Z_L \tilde{I}_2$$

(sorry I didn't explicitly do this originally)

Create the node analysis to look at the capacitor



$$I_1 - I_2 + \underline{I_a} = 0$$

$$\frac{1}{L}\int (V_0 - V_1)\,dt + \frac{1}{L}\int (V_2 - V_1)\,dt$$

$$\underline{+ C\left(-\frac{dV_2}{dt}\right)} = 0$$

$$I_a = -C \frac{dV_2}{dt}$$

$$I_1 - I_2 - C\frac{dV_2}{dt} = 0$$

$$I_1 \cos(\omega t + \theta_1) - I_2 \cos(\omega t + \theta_2) - C \frac{\partial}{\partial t} V_2 \cos(\omega t + \phi_2)$$
$$= 0$$

$$\mathrm{Re}\{\tilde{I}_1 e^{j\omega t}\} - \mathrm{Re}\{\tilde{I}_2 e^{j\omega t}\} - \mathrm{Re}\{C \frac{\partial}{\partial t}(\tilde{V}_2 e^{j\omega t})\} = 0$$

$$\mathrm{Re}\{\tilde{I}_2 e^{j\omega t}\} = \mathrm{Re}\{\tilde{I}_1 e^{j\omega t} - C \frac{\partial}{\partial t}(\tilde{V}_2 e^{j\omega t})\}$$

$$\tilde{I}_2 e^{j\omega t} = \tilde{I}_1 e^{j\omega t} - j\omega C \tilde{V}_2 e^{j\omega t}$$

$$\tilde{I}_2 \cancel{e^{j\omega t}} = (\tilde{I}_1 - j\omega C \tilde{V}_2) \cancel{e^{j\omega t}}$$

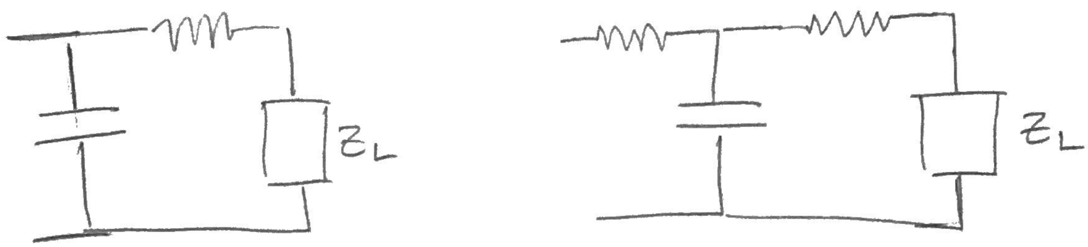
$$\boxed{\tilde{I}_2 = \tilde{I}_1 - j\omega C \tilde{V}_2}$$

$$\boxed{\begin{aligned} \tilde{V}_1 &= \tilde{V}_0 - j\omega L \tilde{I}_1 \\ \tilde{V}_2 &= \tilde{V}_1 - j\omega L \tilde{I}_2 \\ \tilde{I}_2 &= \tilde{I}_1 - j\omega C \tilde{V}_2 \\ \tilde{I}_2 &= \frac{\tilde{V}_2}{Z_L} \end{aligned}}$$

11.1.2) $\tilde{V}_o = V_o$ and previous equations,

Solve for Z_1 and Z_o where Z_1 & Z_o are

$\underline{Z_1}$ and Z_o



$Z_1 \rightarrow$ Capacitor $\overset{\text{Parallel}}{\downarrow}\,||$ (Inductor $\overset{\text{series}}{\downarrow}\,\&$ Z_L)

$\rightarrow$ Capacitor $||$ Z_a

$$Z_a = j\omega L + Z_L$$

$$(Z_1)^{-1} = \left(\frac{-j}{\omega C}\right)^{-1} + (j\omega L + Z_L)^{-1}$$

$$(Z_1)^{-1} = j\omega C + \frac{1}{j\omega L + Z_L}$$

$$(Z_1)^{-1} = \frac{j\omega C(j\omega L + Z_L) + 1}{(j\omega L + Z_L)}$$

$$Z_1 = \frac{(j\omega L + Z_L)}{j\omega c(j\omega L + Z_L) + 1}$$

$$Z_1 = \frac{(j\omega L + Z_L)}{-\omega^2 cL + j\omega c Z_L + 1}$$

this is also useful

$$Z_1 = \frac{(j\omega L + Z_L)}{(1-\omega^2 cL) + j\omega c Z_L} * \frac{(1-\omega^2 cL) - j\omega c Z_L}{(1-\omega^2 cL) - j\omega c Z_L}$$

wolfram alpha

$$\boxed{Z_1 = \frac{-j(CL^2\omega^3 + C\omega Z_L^2 - L\omega) + Z_L}{(1-\omega^2 cL)^2 + \omega^2 c^2 Z_L^2}}$$

$$Z_0 = j\omega L + Z_1$$

$$= j\omega L +$$

wolfram alpha again

$$\boxed{Z_0 = \frac{j(c^2L^3\omega^5 + c^2L\omega^3 Z_L^2 - 3cL^2\omega^3 - c\omega Z^2 + 2L\omega)}{(1-\omega^2 cL)^2 + \omega^2 c^2 Z_L^2} + Z}$$

Alternative form straight from wolfram alpha (this makes section 3 easier)

This is what got plugged into wolfram alpha

series

$$Z_0 = jL\omega + \cfrac{1}{j c\omega + \cfrac{1}{Z_L + jL\omega}}$$

parallel

series

wolfram solution

$$Z_0 = \frac{-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L}{-CL\omega^2 + jC\omega Z_L + 1}$$

this is the same as the other solution

11.1.3) Using the previous solve for

i.) $\tilde{I}_k$ for $k=1,2$ and $\tilde{V}_k$ for $k=1,2$

Put everything in terms of $\tilde{V}_o$ + Z_o

$$\tilde{I}_1 = \frac{\tilde{V}_o}{Z_o}$$

$$\tilde{V}_1 = \tilde{V}_o - j\omega L \tilde{I}_1$$

$$\tilde{V}_1 = \tilde{V}_o - j\omega L \frac{\tilde{V}_o}{Z_o}$$

$$\tilde{I}_2 = \tilde{I}_1 - j\omega C \tilde{V}_1$$

$$\tilde{I}_2 = \frac{\tilde{V}_o}{Z_o} - j\omega C\left(\tilde{V}_o - j\omega L \frac{\tilde{V}_o}{Z_o}\right)$$

$$\tilde{V}_2 = \tilde{V}_1 - j\omega L \tilde{I}_2$$

$$\tilde{V}_2 = \tilde{V}_o - j\omega L \frac{\tilde{V}_o}{Z_o} - j\omega L\left(\frac{\tilde{V}_o}{Z_o} - j\omega C\left(\tilde{V}_o - j\omega L \frac{\tilde{V}_o}{Z_o}\right)\right)$$

$$\tilde{I}_1 = \tilde{V}_o\left(\frac{1}{Z_o}\right)$$

$$\tilde{V}_1 = \tilde{V}_o\left(1 - \frac{j\omega L}{Z_o}\right)$$

$$\tilde{I}_2 = \tilde{V}_o\left(\frac{1}{Z_o} - j\omega C\left(1 - \frac{j\omega L}{Z_o}\right)\right)$$

$$\tilde{V}_2 = \tilde{V}_o\left(1 - \frac{j\omega L}{Z_o} - j\omega L\left(\frac{1}{Z_o} - j\omega C\left(1 - \frac{j\omega L}{Z_o}\right)\right)\right)$$

$$\tilde{I}_2 = \tilde{V}_0\left(\frac{1}{Z_0} - j\omega C\left(1 - \frac{j\omega L}{Z_0}\right)\right)$$

$$= \tilde{V}_0\left(\frac{1}{Z_0} - j\omega C - \frac{\omega^2 CL}{Z_0}\right)$$

$$\tilde{I}_2 = \tilde{V}_0\left(-j\omega C + \frac{1-\omega^2 CL}{Z_0}\right)$$

$$\tilde{V}_2 = \tilde{V}_0\left(1 - \frac{j\omega L}{Z_0} - j\omega L\left(\frac{1}{Z_0} - j\omega C\left(1 - \frac{j\omega L}{Z_0}\right)\right)\right)$$

$$= \tilde{V}_0\left(1 - \frac{j\omega L}{Z_0} - j\omega L\left(-j\omega C + \frac{1-\omega^2 CL}{Z_0}\right)\right)$$

$$= \tilde{V}_0\left(1 - \frac{j\omega L}{Z_0} - \omega^2 LC + \frac{-j\omega L + \omega^3 CL^2 j}{Z_0}\right)$$

$$\tilde{V}_2 = \tilde{V}_0\left(1 - \omega^2 LC + \frac{j\omega CL^2 - 2j\omega L}{Z_0}\right)$$

plug in Z_0 to everything

$$\tilde{I}_1 = \tilde{V}_0\left(\frac{-CL\omega^2 + jC\omega Z_L + 1}{-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L}\right)$$

$$\boxed{\tilde{V}_1 = V_0\left(1 - \frac{j\omega L(CL\omega^2 + jC\omega Z_L + 1)}{-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L}\right)}$$

$$\boxed{\tilde{I}_2 = \tilde{V}_0\left(-j\omega C + \frac{(1-\omega^2 CL)(CL\omega^2 + j\omega C Z_L + 1)}{(-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L)}\right)}$$

$$\boxed{\tilde{V}_2 = V_0\left(1-\omega^2 LC + \frac{(j\omega CL^2 - 2j\omega L)(CL\omega^2 + j\omega C Z_L + 1)}{(-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L)}\right)}$$

11.1.3.1) Find $I_k(t)$ & $V_k(t)$

$$I_1(t) = \mathrm{Re}\left\{V_0\left(\frac{-CL\omega^2 + jC\omega Z_2 + 1}{-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L}\right)e^{j\omega t}\right\}$$

$$V_1(t) = \mathrm{Re}\left\{V_0\left(\frac{1 - j\omega L(CL\omega^2 + jC\omega Z_L + 1)}{-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L}\right)e^{j\omega t}\right\}$$

$$I_2(t) = \mathrm{Re}\left\{V_0\left(\frac{-j\omega C + (1-\omega^2 CL)(CL\omega^2 + j\omega C Z_L + 1)}{-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L}\right)e^{j\omega t}\right\}$$

$$V_2(t) = \mathrm{Re}\left\{V_0\left(1-\omega^2 LC + \frac{(j\omega CL^2 - 2j\omega L)(CL\omega^2 + j\omega C Z_L + 1)}{(-jCL^2\omega^3 - CL\omega^2 Z_L + j2L\omega + Z_L)}\right)e^{j\omega t}\right\}$$

(I know these are subpar solutions, but I would genuinely take the real component of the complex impedance

To further look at the solution,
The complex values for $\tilde{I}_1, \tilde{V}_1, \tilde{I}_2, \tilde{V}_2$ can be represented by as $a + jb$
with that in mind, the solution becomes

$$I_1(t) = Re\{(a_{I_1} + j\, b_{I_1})\, e^{j\omega t}\}$$

$$= Re\{(a_{I_1} + j b_{I_1}) \cos\omega t + j \sin\omega t\}$$

$$= Re\{\underline{a_{I_1} \cos\omega t} + j b_{I_1} \cos\omega t + j a_{I_1} \sin\omega t - \underline{b_{I_1} \sin\omega t}\}$$

you actually have other parts

$$\boxed{I_1(t) = Re\{\tilde{I}_1\}\cos(\omega t) - Im\{\tilde{I}_1\}\sin(\omega t)}$$

This is true for the others as well

$$I_2(t) = Re\{\tilde{I}_2\}\cos(\omega t) - Im\{\tilde{I}_1\}\sin(\omega t)$$

$$V_1(t) = Re\{\tilde{V}_1\}\cos(\omega t) - Im\{\tilde{V}_1\}\sin(\omega t)$$

$$V_2(t) = Re\{\tilde{V}_2\}\cos(\omega t) - Im\{\tilde{V}_2\}\sin(\omega t)$$

Verify Answer by solving for Z_0 using $\tilde{I}_2 = \tilde{V}_0\left(\frac{1}{Z_0} - j\omega c\left(1 - \frac{j\omega L}{Z_0}\right)\right)$ and

$$\tilde{I}_2 = \frac{\tilde{V}_2}{Z_L} = \frac{\tilde{V}_0}{Z_L}\left(1 - \frac{j\omega L}{Z_0} - j\omega L\left(\frac{1}{Z_0} - j\omega c\left(1 - \frac{j\omega L}{Z_0}\right)\right)\right)$$

and solving for Z_0 (should be the same as original Z_0 in section 2

$$\cancel{\tilde{V}_0}\left(\frac{1}{Z_0} - j\omega c\left(1 - \frac{j\omega L}{Z_0}\right)\right) =$$

$$\frac{\cancel{\tilde{V}_0}}{\cancel{Z_L}}\left(1 - \frac{j\omega L}{Z_0} - j\omega L\left(\frac{1}{Z_0} - j\omega c\left(1 - \frac{j\omega L}{Z_0}\right)\right)\right)$$

$$Z_L\left(\frac{1}{Z_0} - \left(j\omega c + \frac{\omega^2 cL}{Z_0}\right)\right) =$$

$$1 - \frac{j\omega L}{Z_0} - j\omega L\left(\frac{1}{Z_0} - \left(j\omega c + \frac{\omega^2 cL}{Z_0}\right)\right)$$

$$Z_L\left(\frac{1}{Z_0} - j\omega c - \frac{\omega^2 cL}{Z_0}\right) =$$

$$1 - \frac{j\omega L}{Z_0} - \left(\frac{j\omega L}{Z_0} + \omega^2 cL - \frac{j\omega^3 cL^2}{Z_0}\right)$$

$$Z_L\left(\frac{1}{Z_0} - j\omega C - \frac{\omega^2 CL}{Z_0}\right) =$$

$$1 - \frac{j\omega L}{Z_0} - \frac{j\omega L}{Z_0} - \omega^2 CL + \frac{j\omega^3 CL^2}{Z_0}$$

$$\frac{Z_L}{Z_0} - \frac{\omega^2 CLZ_L}{Z_0} + \frac{2j\omega L}{Z_0} - \frac{j\omega^3 CL^2}{Z_0} =$$

$$j\omega CZ_L + 1 - \omega^2 CL$$

$$\frac{Z_L - \omega^2 CLZ_L + 2j\omega L - j\omega^3 CL^2}{Z_0} =$$

$$j\omega CZ_L + 1 - \omega^2 CL$$

$$Z_0 = \frac{Z_L - \omega^2 CLZ_L + 2j\omega L - j\omega^3 CL^2}{j\omega CZ_L + 1 - \omega^2 CL} \checkmark$$

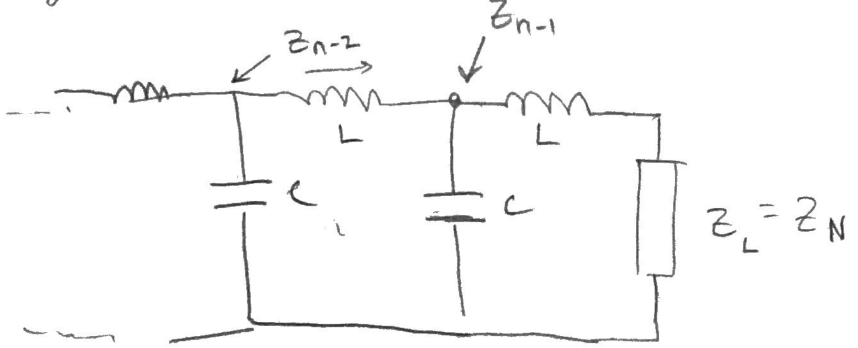
same Z_0 from both methods

(this solution matches the wolfram solution)

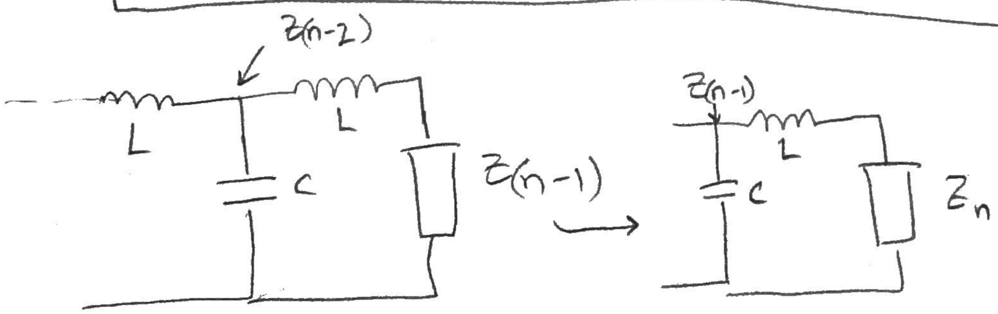
III.1.4) Using the N ladder circuit, solve the following

1.) Write an iterative equation for Z_{n-1} in terms of $Z_n, \omega, L,$ and C

Look at a single node then generalize

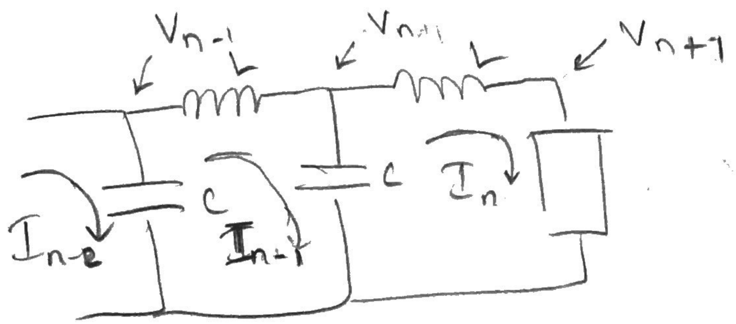


$$\boxed{Z_{n-1} = \left(j\omega C + \left(Z_n + j\omega L\right)^{-1}\right)^{-1}}$$



2.) Write an equation that relates $\tilde{V}_{n+1}$ to $\tilde{V}_n$ and $\tilde{I}_n$

use the same as last section



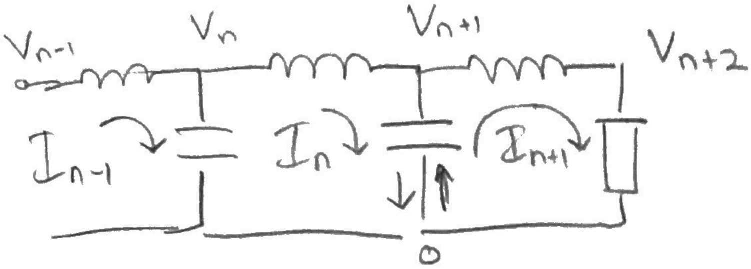
$$V_n - V_{n+1} = L \frac{\partial I_n}{\partial t}$$

$$\frac{\partial I}{\partial t} = j\omega I$$

$$V_{n-1} - V_n = L \frac{\partial I_{n-1}}{\partial t}$$

$$\boxed{\tilde{V}_{n+1} = \tilde{V}_n - j\omega L \tilde{I}_n}$$

3.) Write an equation that relates $\tilde{I}_{n+1}$ to $\tilde{I}_n$ and $\tilde{V}_{n+1}$



$$\tilde{I}_n - \tilde{I}_{n+1} = C \frac{\partial \tilde{V}_{n+1}}{dt}$$

$$\frac{\partial \tilde{V}}{dt} = j\omega \tilde{V}$$

$$\boxed{I_{n+1} = \tilde{I}_n - j\omega C \tilde{V}_{n+1}}$$