5.1) Describe the equations $\vec{\nabla}\cdot\vec{B}=0$, $\vec{\nabla}\times\vec{B}=\mu\vec{J}$ & $\vec{B}=-\vec{\nabla}\Phi_m$ using simple or familar problems. Also describe $B_\perp$ and $B_\parallel$ across a boundary

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$\vec{\nabla}\cdot\vec{B}=0$

This equation can best be described initially in the integral form

$\int_V \vec{\nabla}\cdot\vec{B}=0$ for any volume V, also written as

$\oint_S \vec{B}=0$ using divergence theorem, this equation shows that the magnetic flux over a closed surface always equals 0.

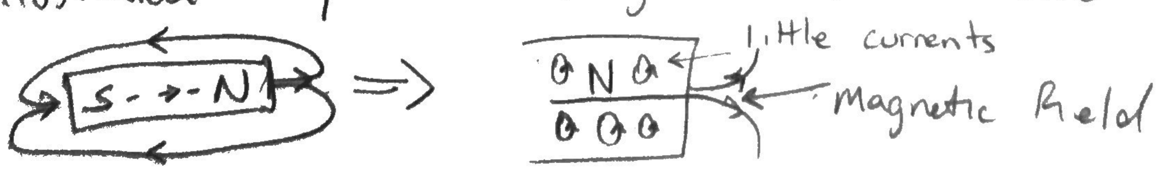
Working backwards, $\oint_S B=\int_V \vec{\nabla}\cdot\vec{B}$ shows a relationship between the magnetic flux and change in the magnetic field over the entire volume

Since $\vec{\nabla} \cdot \vec{B}$ is looking at the change in the local magnetic field, $\int_V \vec{\nabla} \cdot \vec{B}$ is the sum of the local changes. This means that $\vec{\nabla} \cdot \vec{B} = 0$ states that ALL the small changes are 0. This can easily been seen with a simple magnetic field from a straight line current

$$B = \frac{\mu_0 I}{2\pi r} \hat{\phi} \qquad \vec{\nabla} \cdot \vec{B} = \frac{1}{r} \cancel{\frac{\partial}{\partial \phi} \frac{\mu_0 I}{2\pi r}} = 0$$

I think this is the best explanation of the divergence B (in comparison to others and in terms of what I can think of). div(B) can be computed and shown to be zero for a wire (would have better if you considered eqn for finite segment and showed div(B)=0). Any current system can be created using small segments and so by superposition div(B)=0 for any current system.

The power of $\vec{\nabla} \cdot \vec{B} = 0$ really comes from $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$, where $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ shows that electric fields start or end at charges (depending on the charge). $\vec{\nabla} \cdot \vec{B} = 0$ shows that there are no magnetic monopoles because there ~~are~~ is no local gradient in the magnetic field. (I think this can be best illustrated by how magnetic fields are loops)



$\nabla \times B = \mu_0 J$

I am going to start with the integral form again

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \int_S \mu_0 \vec{J} \cdot d\vec{a}$$ which can also be

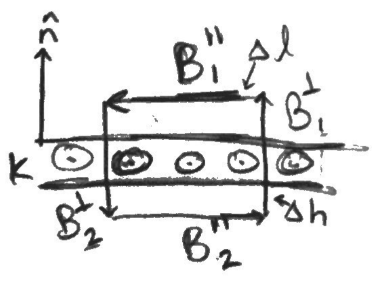
$$\oint_L \vec{B} \cdot d\vec{l} = \int_S \mu_0 \vec{J} \cdot d\vec{a}$$ using Stoke's theorem.

This can visually be seen by

Ideally you would have avoided appealing to the integral equation, which is easier to explain. One way of explaining the differential form is with this diagram. One can show the field is constant, equal, and opposite-signed, above and below using Biot Savart. dB_x/dz is not zero (where x is to the right and z is up in your diagram). So it follows that this derivative is not zero when there is a current in y. There are a total of six orientations of an infinite current sheet (counting + and - directions separately). With this, you get the six derivatives in the curl.

The area bounded by L can be reduced to the boundary of the current density, which is where $\vec{\nabla} \times \vec{B} = \mu_0 J$ holds true.

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ can best be shown with an infinite current sheet $\vec{K}$



$$\oint_L \vec{B} \cdot d\vec{l} = \int_S \mu_0 \underbrace{\vec{J}}_{\vec{K}} \cdot \underbrace{d\vec{a}}_{(\hat{n} \times \Delta \vec{l})}$$

$\Delta h \to 0$

$$B_1^{\parallel} \Delta l - B_2^{\parallel} \Delta l = \mu_0 \vec{K} \cdot (\hat{n} \times \Delta \vec{l})$$

$$(B_1^{\parallel} - B_2^{\parallel}) \cancel{\Delta l} = \mu_0 (\vec{K} \times \hat{n}) \cdot \cancel{\Delta \vec{l}}$$

So B is perpendicular to $\vec{k}$

$$\vec{B} = -\vec{\nabla}\, \Phi_m$$

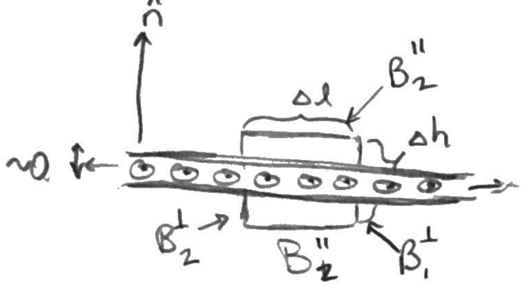
This approximation holds true when $\vec{\nabla} \times \vec{B} = 0$ because $\vec{\nabla} \times (\vec{\nabla}\Psi) = 0$ for all scalar potentials. This can hold true when there is a magnetization field $\vec{M}$ such that $\vec{H} + \vec{M} = 0$.

In this case, there isn't really anything to do - you know H so you can compute B. So knowing phi_m does not really help.

As explained in class, using a scalar potential can be very beneficial to simplify modeling efforts.

Boundary Conditions

Going back to the current sheet example



first $\nabla \cdot B = 0$

$$\int_V \vec{\nabla} \cdot \vec{B} \, dV = 0$$

$$\oint_S \vec{B} \cdot d\vec{a} = 0$$

$\hat{n}$

$$\oint_S \vec{B} \cdot \hat{n} = 0$$

$$\boxed{B^{\perp} = 0}$$

Second

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\int_S \vec{\nabla} \times \vec{B} \cdot d\vec{a} = \int_S \mu_0 \vec{J} \cdot d\vec{a}$$

$$\oint_L \vec{B} \cdot d\vec{l} = \int_S \mu_0 \vec{J} \cdot d\vec{a}$$

(because $B^{\perp} = 0$ we only need to look at $B^{\parallel}$)

$$\int \vec{B}_1^{\parallel} \cdot d\vec{l} - \int \vec{B}_2^{\parallel} \cdot d\vec{l} = \int \mu_0 \vec{k} \cdot (\hat{n} \times d\vec{l})$$

$$\int B_1^{\parallel} \cdot \cancel{d\vec{l}} - \int B_2^{\parallel} \cdot \cancel{d\vec{l}} = \int \mu_0 (\vec{k} \times \hat{n}) \cdot \cancel{d\vec{l}}$$

$$\boxed{B_1^{\parallel} - B_2^{\parallel} = \mu_0 (\vec{k} \times \hat{n})}$$