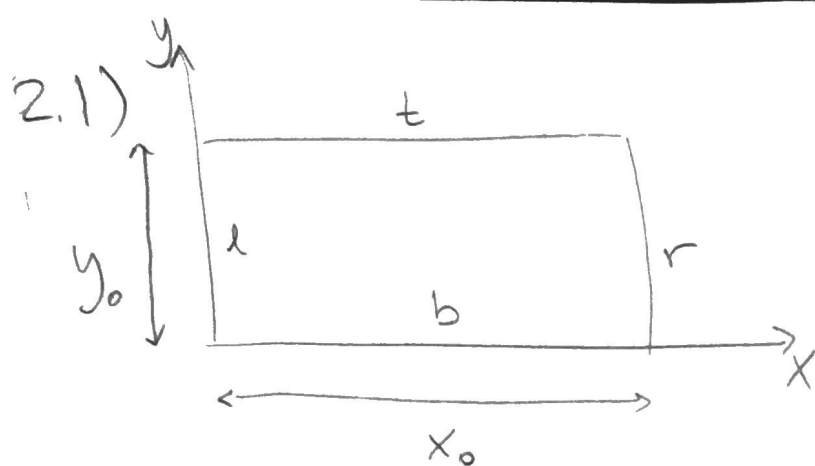


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 PHYS 513  
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 HW 2



Calculate the potential with the following  
 Boundary conditions

- 1.)  $x=0 \rightarrow \Phi_e(0, y) = V_e$  (i)  
 $x=x_0 \rightarrow \Phi_e(x_0, y) = 0$  (ii)  
 $y=0 \rightarrow \Phi_e(x, 0) = 0$  (iii)  
 $y=y_0 \rightarrow \Phi_e(x, y_0) = 0$  (iv)

$$\Phi_e(x, y) = X(x) Y(y) \quad \& \quad \nabla^2 \Phi_e(x, y) = 0$$

$$X(x) Y''(y) = X''(x) Y(y)$$

$$\frac{X(x)}{X''(x)} = \frac{Y(y)}{Y''(y)}$$

$$\frac{X(x)}{X''(x)} = k^2 \quad \& \quad \frac{Y(y)}{Y''(y)} = -k^2$$

$$X(x) = A e^{kx} + B e^{-kx} \quad \& \quad Y(y) = C \sin(ky) + D \cos(ky)$$

①

Enter Boundary Condition (ii)

$$X(x_0) = 0 = A e^{kx_0} + B e^{-kx_0}$$

$$A e^{kx_0} = -B e^{-kx_0}$$

$$A = -A' e^{-kx_0} \quad \& \quad B = A' e^{kx_0}$$

$$X(x) = A' (e^{k(x_0-x)} - e^{k(x-x_0)}) \quad A' \rightarrow A \text{ for simplicity}$$

$$\underline{X(x) = 2A \sinh(k(x_0 - x))} \rightarrow \text{when } x=0, X(0) > 0 \checkmark$$

absorb 2 into A

Enter Boundary condition (iii)

$$Y(0) = 0 = C \sin(k \cdot 0) + D \cos(k \cdot 0)$$

$$0 = 0 + D$$

$$\underline{\underline{\therefore D=0}}$$

Enter Boundary Condition (iv)

$$Y(y_0) = 0 = C \sin(k y_0)$$

$$\sin^{-1}(0) = k y_0$$

$$n\pi = k y_0 \quad \text{where } n = 1, 2, 3, \dots$$

$$\therefore k = \frac{n\pi}{y_0}$$

where  $n = 1, 2, 3, \dots$  (I will leave this off until solutions are found)

(2)

Enter Boundary condition (i)

$$\Phi_1(x, y) = \underbrace{2CA}_{\downarrow} \sinh\left(\frac{n\pi}{y_0}(x_0 - x)\right) \sin\left(\frac{n\pi y}{y_0}\right)$$

$$\Phi_1(x, y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{y_0}(x_0 - x)\right) \sin\left(\frac{n\pi y}{y_0}\right)$$

$$\Phi_1(0, y) = V_1 = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi x_0}{y_0}\right) \sin\left(\frac{n\pi y}{y_0}\right)$$

Fourier's Trick

$$A_n \sinh\left(\frac{n\pi x_0}{y_0}\right) = \frac{2}{a} \int_0^a V_1 \sin\left(\frac{n\pi y}{y_0}\right) dy$$

$$\therefore A_n = \frac{1}{\sinh\left(\frac{n\pi x_0}{y_0}\right)} \cdot \frac{2}{a} \cdot V_1 \cdot \frac{a - a \cos(\pi n)}{\pi n} \leftarrow \text{only valid when } n \text{ is odd, else } 0$$

$$\therefore A_n = \frac{4V_1}{\pi n \sinh\left(\frac{n\pi x_0}{y_0}\right)} \text{ where } n = 1, 3, 5, \dots$$

$$\Phi_1(x, y) = \sum_{n=1,3,5,\dots}^{\infty} \frac{4V_1}{\pi n} \frac{\sinh\left(\frac{n\pi}{y_0}(x_0 - x)\right) \sin\left(\frac{n\pi y}{y_0}\right)}{\sinh\left(\frac{n\pi x_0}{y_0}\right)}$$

③

2.) Calc with following boundary conditions

$$\Phi_b(0, y) = 0$$

$$\Phi_b(x_0, y) = 0$$

$$\Phi_b(x, 0) = V_b$$

$$\Phi_b(x, y_0) = 0$$

Same set up as part 1, but  $x$  and  $y$  are switched.

Thus the solution is

$$\Phi_b(x, y) = \sum_{n=1,3,5,\dots} \frac{4V_b}{\pi n} \frac{\sinh\left(\frac{n\pi}{x_0}(y_0 - y)\right) \sin\left(\frac{n\pi x}{x_0}\right)}{\sinh\left(\frac{n\pi y_0}{x_0}\right)}$$

(4)

3.) Calc with following boundary conditions

$$\Phi_t(0, y) = 0 \quad (i)$$

$$\Phi_t(x_0, y) = 0 \quad (ii)$$

$$\Phi_t(x, 0) = 0 \quad (iii)$$

$$\Phi_t(x, y_0) = V_t \quad (iv)$$

Given that  $\Phi_t(x, y) = X(x)Y(y)$ , then

$X(x)$  is the same as in part 2

→ Start with Boundary Condition (iii)

$$Y(0) = 0 = C e^{k0} + D e^{-k0} \quad \leftarrow k = \frac{n\pi}{x_0}$$

$$0 = C + D$$

$$C = -D$$

$$Y(y) = C' (e^{ky} - e^{-ky})$$

$$Y(y) = 2C' \sinh\left(\frac{n\pi y}{x_0}\right)$$

Boundary condition at  $y_0$  is the same for  $y=0$  in part 2

⑤

$$\Phi_n(x, y) = V_t = C_n \sinh\left(\frac{n\pi y_0}{x_0}\right) \sin\left(\frac{n\pi x}{x_0}\right)$$

Since Boundary condition leads to the same value for  $A_n$  ( $C_n$ )

The answer is

$$\Phi_t(x, y) = \sum_{n=1,3,5,\dots}^{\infty} \frac{4V_0}{\pi n} \frac{\sinh\left(\frac{n\pi y}{x_0}\right) \sin\left(\frac{n\pi x}{x_0}\right)}{\sinh\left(\frac{n\pi y_0}{x_0}\right)}$$

Note: the difference in this solution is that  $\sinh\left(\frac{n\pi}{x_0}(y-y_0)\right)$  becomes

$$\sinh\left(\frac{n\pi y}{x_0}\right)$$

⑥

4.) Calc with following boundary conditions

$$\Phi_r(0, y) = 0$$

$$\Phi_r(x_0, y) = V_r$$

$$\Phi_r(x, 0) = 0$$

$$\Phi_r(x, y_0) = 0$$

Same as geometry from 3, except switch  $x$  and  $y$

$$\therefore \Phi_r(x, y) = \sum_{n=1,3,5,\dots}^{\infty} \frac{4V_0}{\pi n} \frac{\sinh\left(\frac{n\pi x}{y_0}\right) \sin\left(\frac{n\pi y}{y_0}\right)}{\sinh\left(\frac{n\pi y_0}{x_0}\right)}$$

(7)

5.) Just by  $\Phi(x,y) = \Phi_x(x,y) + \Phi_r(x,y) + \Phi_t(x,y) + \Phi_b(x,y)$   
is a general solution to this problem

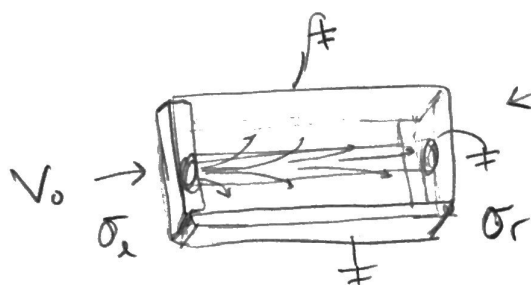
Using the uniqueness theorem, any solution to Laplace's equation and satisfies the boundary conditions is the unique solution. Since  $\Phi_x$ ,  $\Phi_r$ ,  $\Phi_t$ , &  $\Phi_b$  satisfy each boundary independently, adding them all together gives a unique solution.

⑧



6.) Can Gauss's law be used in any way to check your solution?

Yes, this was discussed in class.



← given the same set up with exaggerated thickness create a gaussian cylinder

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \quad \frac{Q_{enc}}{\epsilon_0} \rightarrow \frac{(\sigma_1 + \sigma_2) \pi r^2}{\epsilon_0}$$

$$\oint \vec{E} \cdot d\vec{a} \rightarrow \int \vec{E}_{out} \cdot d\vec{a} + \int \vec{E}_{in} \cdot d\vec{a} + \int \vec{E}_{side} \cdot d\vec{a}$$

these are zero because they are perpendicular

$$\int \vec{E}_{in} \cdot d\vec{a} = \frac{(\sigma_1 + \sigma_2) \pi r^2}{\epsilon_0}$$

→  
This E field could be used to validate the  $\Phi_1$  solution,

(9)