5.1) Describe the equations $\vec{\nabla}\cdot\vec{B}=0$, $\vec{\nabla}\times\vec{B}=\mu\vec{J}$ & $\vec{B}=-\vec{\nabla}\Phi_m$ using simple or familar problems. Also describe $B_\perp$ and $B_\parallel$ across a boundary

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HW # 5
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$$\vec{\nabla}\cdot\vec{B}=0$$

This equation can best be described initially in the integral form

$\int_V \vec{\nabla}\cdot\vec{B}=0$ for any volume V, also written as

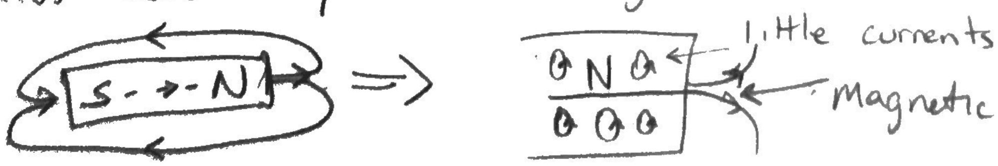
$\oint_S \vec{B}=0$ using divergence theorem, this equation shows that the magnetic flux over a closed surface always equals 0.

Working backwards, $\oint_S B=\int_V \vec{\nabla}\cdot\vec{B}$ shows a relationship between the magnetic flux and change in the magnetic field over the entire volume

Since $\vec{\nabla}\cdot\vec{B}$ is looking at the change in the local magnetic field, $\int_V \vec{\nabla}\cdot\vec{B}$ is the sum of the local changes. This means that $\vec{\nabla}\cdot\vec{B}=0$ states that ALL the small changes are 0. This can easily been seen with a simple magnetic field from a straight line current

$$B = \frac{\mu_0 I}{2\pi r}\hat{\phi} \qquad \vec{\nabla}\cdot\vec{B} = \frac{1}{r}\cancel{\frac{\partial}{\partial\phi}\frac{\mu_0 I}{2\pi r}} = 0$$

The power of $\vec{\nabla}\cdot\vec{B}=0$ really comes from $\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0$, where $\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0$ shows that electric fields start or end at charges (depending on the charge). $\vec{\nabla}\cdot\vec{B}=0$ shows that there are no magnetic monopoles because there are is no local gradient in the magnetic field. (I think this can be best illustrated by how magnetic fields are loops)



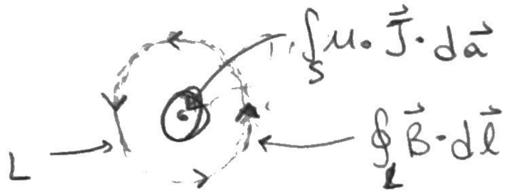
$\nabla \times B = \mu_0 J$

I am going to start with the integral form again

$\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \int_S \mu_0 \vec{J} \cdot d\vec{a}$ which can also be

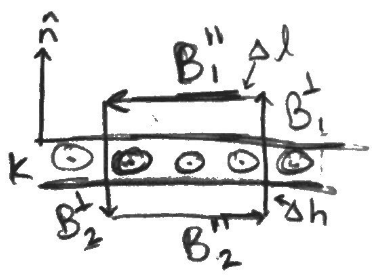
$\oint_L \vec{B} \cdot d\vec{l} = \int_S \mu_0 \vec{J} \cdot d\vec{a}$ using Stokes theorem.

This can visually be seen by



The area bounded by L can be reduced to the boundary of the current density, which is where $\vec{\nabla} \times \vec{B} = \mu_0 J$ holds true.

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ can best be shown with an infinite current sheet $\vec{K}$



$\vec{K}$

$\oint_L \vec{B} \cdot d\vec{l} = \int_S \mu_0 \vec{J} \cdot d\vec{a} \leftarrow (\hat{n} \times \Delta\vec{l})$

$\Delta h \to 0$

$B_1^{\parallel} \Delta l - B_2^{\parallel} \Delta l = \mu_0 \vec{K} \cdot (\hat{n} \times \Delta \vec{l})$

$(B_1^{\parallel} - B_2^{\parallel}) \cancel{\Delta l} = \mu_0 (\vec{K} \times \hat{n}) \cdot \cancel{\Delta \vec{l}}$

So B is perpendicular to $\vec{k}$

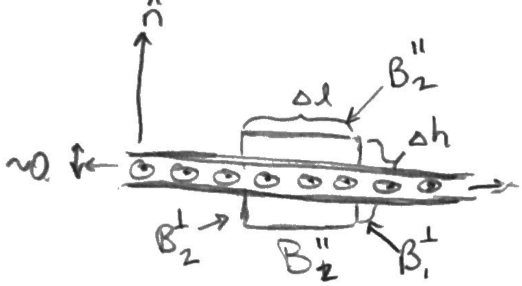
$$\vec{B} = -\vec{\nabla}\,\Psi_m$$

This approximation holds true when $\vec{\nabla} \times \vec{B} = 0$ because $\vec{\nabla} \times (\vec{\nabla}\,\Psi) = 0$ for all scalar potentials. This can hold true when there is a magnetization field $\vec{M}$ such that $\vec{H} + \vec{M} = 0$.

As explained in class, using a scalar potential can be very beneficial to simplify modeling efforts.

Boundary Conditions

Going back to the current sheet example



First $\nabla \cdot B = 0$

$$\int_V \vec{\nabla} \cdot \vec{B} \, dV = 0$$

$$\oint_S \vec{B} \cdot d\vec{a} = 0$$

$\hat{n}$

$$\oint_S \vec{B} \cdot \hat{n} = 0$$

$$\boxed{B^{\perp} = 0}$$

Second

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\int_S \vec{\nabla} \times \vec{B} \cdot d\vec{a} = \int_S \mu_0 \vec{J} \cdot d\vec{a}$$

$$\oint_L \vec{B} \cdot d\vec{l} = \int_S \mu_0 \vec{J} \cdot d\vec{a}$$

(because $B^{\perp} = 0$ we only need to look at $B^{\parallel}$)

$$\int \vec{B}_1^{\parallel} \cdot d\vec{l} - \int \vec{B}_2^{\parallel} \cdot d\vec{l} = \int \mu_0 \vec{K} \cdot (\hat{n} \times d\vec{l})$$

$$\int B_1^{\parallel} \cdot \cancel{d\vec{l}} - \int B_2^{\parallel} \cdot \cancel{d\vec{l}} = \int \mu_0 (\vec{K} \times \hat{n}) \cdot \cancel{d\vec{l}}$$

$$\boxed{B_1^{\parallel} - B_2^{\parallel} = \mu_0 (\vec{K} \times \hat{n})}$$