6.2) $\mathbf{E} = E_{ox}(x,t)\hat{x} + E_{oy}(x,t)\hat{y} + E_{oz}(x,t)\hat{z}$

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HW # 6

and $\mathbf{B} = B_{ox}(x,t)\hat{x} + B_{oy}(x,t)\hat{y} + B_{oz}(x,t)\hat{z}$

6.2.1) Show $E_y(x,t)$, $E_z(x,t)$, $B_y(x,t)$ and $B_z(x,t)$ each individually obey a wave equation of the form

$$\frac{\partial^2 f}{\partial u^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

Where u is a place holder for one of the cartesian variables and f is one of the components stated above

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Start with Faraday's law for E_y

$$\nabla \times E_y = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & E_y & 0 \end{vmatrix} = \begin{matrix} (0 - \partial_z E_y)\hat{x} \\ 0\,\hat{y} \\ (\partial_x E_y - 0)\hat{z} \end{matrix}$$

$$\nabla \times E_y = \frac{\partial E_y}{\partial x}\hat{z} = -\frac{\partial B}{\partial t}$$

$$\frac{\partial B}{\partial t} = -\frac{\partial E_y}{\partial x}\hat{z}$$

Look at complete Ampere's Law when $J = 0$

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Apply time derivative to both sides

$$\frac{\partial}{\partial t}(\nabla \times B) = \frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\nabla \times \frac{\partial B}{\partial t} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

Plug $\frac{\partial B}{\partial t}$ in

$$\nabla \times \frac{-\partial E_y}{\partial x}\hat{z} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ 0 & 0 & \frac{\partial E_y}{\partial x} \end{array} \quad \begin{array}{l} \left(\cancel{\frac{\partial^2 E_y}{\partial x \partial y}}^{\,0} - 0\right)\hat{x} \\ = \left(0 - \frac{\partial^2 E_y}{\partial x^2}\right)\hat{y} \\ (0)\,\hat{z} \end{array}$$

$$+\left(+\frac{\partial^2 E_y}{\partial x^2}\right)\hat{y} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \qquad \mu_0 \epsilon_0 = \frac{1}{c^2}$$

$$\boxed{\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}}$$

✓ b/c $\frac{\partial^2 E}{\partial^2 t}$ is only dependent on E_y in $\hat{y}$ direction

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Do the same as E_y but with E_z

$$\nabla \times E_z = \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ 0 & 0 & E_z \end{array} \quad \begin{array}{l} \left(\cancel{\partial_y E_z}^{\,0} - 0\right)\hat{x} \\ = \left(0 - \partial_x E_z\right)\hat{y} \\ = 0\,\hat{z} \end{array}$$

$$-\nabla \times E_z = +\frac{\partial E_y}{\partial x}\hat{y} = +\frac{\partial B}{\partial t}$$

$$\frac{\partial B}{\partial t} = \frac{\partial E_y}{\partial x}\hat{y}$$

Look at complete Ampere's Law when $J=0$

$$\nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Apply time derivative to both sides

$$\frac{\partial}{\partial t}(\nabla \times B) = \frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

$$\nabla \times \frac{\partial B}{\partial t} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

Plug $\frac{\partial B}{\partial t}$ in

$$\nabla \times \frac{\partial E_z}{\partial x}\hat{y} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & \frac{\partial E_z}{\partial x} & 0 \end{vmatrix} = \begin{matrix} \left(0 - \cancel{\frac{\partial^2 E_z}{\partial x \partial z}}^{\,0}\right)\hat{x} \\ 0\,\hat{y} \\ \left(\frac{\partial^2 E_z}{\partial x^2} - 0\right)\hat{z} \end{matrix}$$

$$\boxed{\frac{\partial^2 E_z}{\partial x^2}\hat{z} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}}$$

b/c $\frac{\partial^2 E}{\partial t^2}$ is only dependent on E_z in $\hat{z}$

Start with B_y and Ampere's law

$$\nabla \times B_y = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & B_y & 0 \end{vmatrix} = \begin{matrix} \left(0 - \cancel{\partial_z B_y}^{\,0}\right)\hat{x} \\ 0\,\hat{y} \\ (\partial_x B_y - 0)\hat{z} \end{matrix}$$

$$\frac{\partial B_y}{\partial x}\hat{z} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Take time derivative of Faraday's law

$$\frac{\partial}{\partial t}(\nabla \times E) = \frac{\partial}{\partial t} - \left(\frac{\partial B}{\partial t}\right)$$

$$\nabla \times \frac{\partial E}{\partial t} = -\frac{\partial^2 B}{\partial t^2}$$

$$\frac{\partial E}{\partial t} = \frac{1}{\mu_0 \epsilon_0} \frac{\partial B_y}{\partial x}\hat{z}$$

$$\nabla \times \frac{1}{\mu_0 \epsilon_0} \frac{\partial B_y}{\partial x}\hat{z} = -\frac{\partial^2 B}{\partial t^2}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \frac{\partial B_y}{\partial x} \end{vmatrix} = \begin{matrix} \left(\cancel{\frac{\partial^2 B_y}{\partial x \partial y}}^{\,0} - 0\right)\hat{x} \\ \left(0 - \frac{\partial^2 B_y}{\partial x^2}\right)\hat{y} \\ 0\,\hat{z} \end{matrix}$$

$$+\frac{1}{\mu_0 \epsilon_0} \frac{\partial^2 B_y}{\partial x^2} \hat{y} = +\frac{\partial^2 B}{\partial t^2}$$

$$\boxed{\frac{\partial^2 B_y}{\partial x^2} \hat{y} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}}$$

Start with B_z and Ampere's law

$$\nabla \times B_z = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & B_z \end{vmatrix} = \begin{matrix} (\cancel{\partial_y B_z}^{\,0} - 0)\hat{x} \\ (0 - \partial_x B_z)\hat{y} \\ 0\,\hat{z} \end{matrix}$$

$$-\frac{\partial B_z}{\partial x} \hat{y} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \Rightarrow \frac{\partial E}{\partial t} = -\frac{1}{\mu_0 \epsilon_0} \frac{\partial B_z}{\partial x} \hat{y}$$

Take time derivative of Faraday's law

$$\nabla \times \frac{\partial E}{\partial t} = -\frac{\partial^2 B}{\partial t^2}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & \frac{\partial B_z}{\partial x} & 0 \end{vmatrix} = \begin{matrix} \left(0 - \cancel{\frac{\partial^2 B_z}{\partial x \partial z}}^{\,0}\right)\hat{x} \\ 0\,\hat{y} \\ \left(\frac{\partial^2 B_z}{\partial x^2} - 0\right)\hat{z} \end{matrix}$$

$$\nabla \times \frac{+1}{\mu_0 \epsilon_0} \frac{\partial B_z}{\partial x} \hat{y} = +\frac{\partial^2 B}{\partial t^2}$$

$$\boxed{\frac{\partial^2 B_z}{\partial x^2} \hat{z} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2}}$$

6.2.2) Does it follow that

$E_x(x,t) = B_x(x,t) = 0$?

Given that $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$

and $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and $\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

$\nabla \cdot \mathbf{E} = 0$ use $\mathbf{E} = E_x(x,t)\,\hat{x}$

$$\frac{\partial E_x}{\partial x} = 0$$

$$\int \partial E_x = \int 0 \, dx$$

$B_x = g_2(t) + C_2$

$E_x = g_1(t) + C_1 \rightarrow$ same can be done for $\mathbf{B}$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = \begin{matrix} (0)\,\hat{x} \\ (\partial_z E_x - 0)\,\hat{y} \\ (0 - \partial_y E_x)\,\hat{z} \end{matrix}$$

$$\nabla \times \mathbf{E} = \cancel{\frac{\partial E_x}{\partial z}}^{\,0}\,\hat{y} - \cancel{\frac{\partial E_x}{\partial y}}^{\,0}\,\hat{z} = -\frac{\partial \mathbf{B}}{\partial t} = 0$$

$$-\frac{\partial B}{\partial t} = 0 \Rightarrow \int \partial B = -\int 0 \, dt$$

$B = C_2$ $\leftarrow$ same can be done with Ampere's law for $\mathbf{E}$

$$\boxed{E_x(x,t) = C_1 \;\&\; B_x(x,t) = C_2 \text{ not } 0}$$

6.2.3) Show how 1 is consistent with

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{B}}{\partial t^2}$$

from 6.2.1

$$\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} \qquad \frac{\partial^2 E_z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

$$\frac{\partial^2 B_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2} \qquad \frac{\partial^2 B_z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2}$$

Start : $\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$

$$\nabla^2 \mathbf{E} = \nabla^2 E_x \hat{x} + \nabla^2 E_y \hat{y} + \nabla^2 E_z \hat{z}$$ expand $\nabla^2 \mathbf{E}$

$$\nabla^2 E_x = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2}$$

$$\nabla^2 E_y = \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2}$$ compare to time derivative

$$\nabla^2 E_z = \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2}$$

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\partial^2 E_x}{\partial t^2}\hat{x} + \frac{\partial^2 E_y}{\partial t^2}\hat{y} + \frac{\partial^2 E_z}{\partial t^2}\hat{z}$$ expand time derivative

$$\nabla^2 E_x = \frac{\partial^2 E_x}{\partial t^2}\frac{1}{c^2} \leftarrow \text{skipping this one}$$

compare components

$$\nabla^2 E_y = \frac{\partial^2 E_y}{\partial t^2}\frac{1}{c^2}$$

$$\nabla^2 E_z = \frac{\partial^2 E_z}{\partial t^2}\frac{1}{c^2}$$

$$\frac{\partial^2 E_y}{\partial x^2} + \cancel{\frac{\partial^2 E_y}{\partial y^2}}^{0} + \cancel{\frac{\partial^2 E_y}{\partial z^2}}^{0} = \frac{\partial^2 E_y}{\partial t^2}\frac{1}{c^2}$$

If E_y depends only on x and t

$$\boxed{\frac{\partial^2 E_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2}}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \cancel{\frac{\partial^2 E_z}{\partial y^2}} + \cancel{\frac{\partial^2 E_z}{\partial z^2}} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

If E_z depends only on x and t

$$\boxed{\frac{\partial^2 E_z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}}$$

Do the same for $\mathbf{B}$

skipping to component step

$$\nabla^2 B_x = \frac{1}{c^2}\frac{\partial^2 B_x}{\partial t^2}$$ ← Skip this component

$$\nabla^2 B_y = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}$$

$$\nabla^2 B_z = \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2}$$

$$\frac{\partial^2 B_y}{\partial x^2} + \cancel{\frac{\partial^2 B_y}{\partial y^2}} + \cancel{\frac{\partial^2 B_y}{\partial z^2}} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}$$

If B_y depends only on x and t

$$\frac{\partial^2 B_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B_y}{\partial t^2}$$

$$\frac{\partial^2 B_z}{\partial x^2} + \cancel{\frac{\partial^2 B_z}{\partial y^2}} + \cancel{\frac{\partial^2 B_z}{\partial z^2}} = \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2}$$

If B_z depends only on x and t

$$\frac{\partial^2 B_z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2}$$