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PHYS 513
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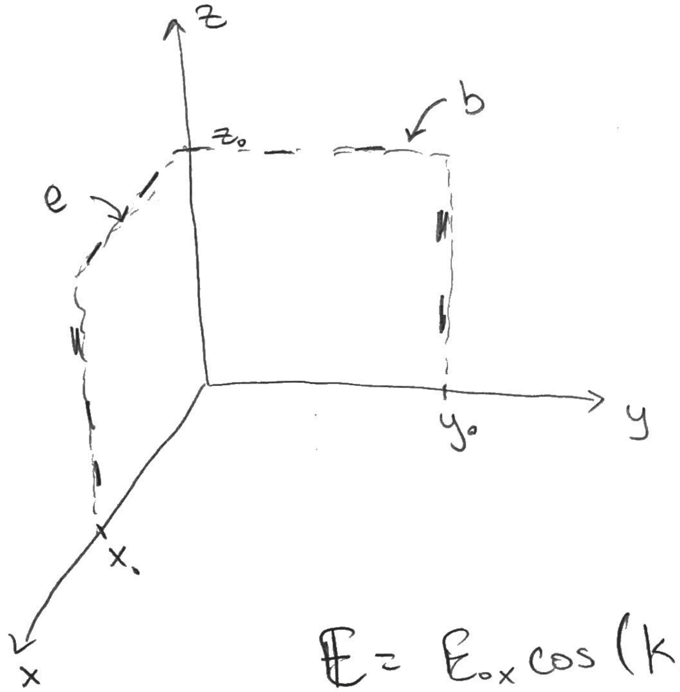
HW # 6

6.1) Faraday's law

$$\oint \mathbf{E} \cdot d\vec{l} = -\frac{\partial \Phi_B}{\partial t}$$

Generalized Ampere's law ($J=0$)

$$\oint \mathbf{B} \cdot d\vec{l} = \frac{1}{c^2} \frac{\partial \Phi_E}{\partial t}$$



$$\mathbf{E} = E_{0x} \cos(k_z z - \omega t)\,\hat{x}$$

6.1.1) Find the magnetic field $\mathbb{B}$ that must exist using $\nabla \times \mathbb{E} = -\frac{\partial \mathbb{B}}{\partial t}$

$$\nabla \times \mathbb{E} = \nabla \times \left(E_{ox} \cos(k_z z - \omega t)\hat{x} \right)$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial x & \partial y & \partial z \\ E_x & 0 & 0 \end{vmatrix}$$

$$\langle 0, \partial_z E_x - 0, 0 - \cancelto{0}{\partial_y E_x} \rangle$$

$$\nabla \times \mathbb{E} = -k_z E_{ox} \sin(k_z z - \omega t)\hat{y} \quad \leftarrow \text{plug in}$$

$$-\frac{\partial B}{\partial t} = -k_z E_{ox} \sin(k_z z - \omega t)\hat{y}$$

$$\int + dB = \int + k_z E_{ox} \sin(k_z z - \omega t)\hat{y}\, dt$$

$$\mathbb{B} = k_z E_{ox} \int \sin(k_z z - \omega t)\hat{y}\, dt$$

$$\phi = k_z z - \omega t$$

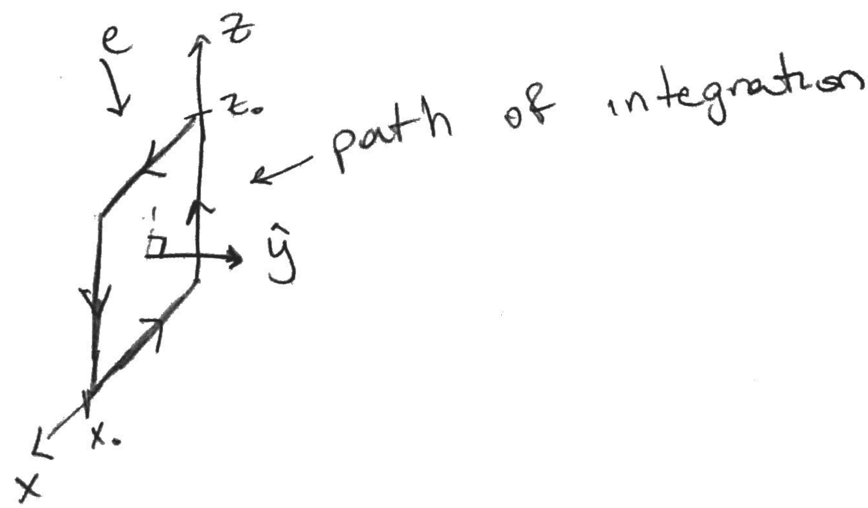
$$d\phi = -\omega\, dt \rightarrow dt = \frac{-d\phi}{\omega}$$

$$\mathbb{B} = \frac{k_z E_{0x}}{\omega} \int -\sin\phi \, d\phi \, \hat{y}$$

$$\mathbb{B} = \frac{k_z E_{0x}}{\omega} \cos\phi \, \hat{y}$$

$$\boxed{\mathbb{B} = \frac{k_z E_{0x}}{\omega} \cos(k_z z - \omega t) \, \hat{y}}$$

6.1.2) Show that $\vec{E}$ satisfies Faraday's Law in the form of $\oint \mathbb{E} \cdot d\vec{l} = -\frac{\partial \Phi_B}{dt}$ along the rectangle e



$$\oint \mathbb{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\int \frac{d\mathbb{B}}{dt}\, d\vec{a} \quad \leftarrow \text{area is static}$$

$s \to c \to -s \to c$

$$-\int \frac{d\mathbb{B}}{dt}\cdot d\vec{a} = -\int \frac{\partial}{\partial t}\left(E_{0x}\frac{k_z}{\omega}\cos(k_z z-\omega t)\right) dx\, dz\, \hat{y}\cdot\hat{y}$$

$$= -E_{0x}\frac{k_z}{\omega}\int \frac{\partial}{\partial t}\cos(k_z z-\omega t)\, dz \; x\Big|_0^{x_0}$$

$$= \left(-E_{0x}\frac{k_z}{\omega}\int_0^{z_0} +\omega(+\sin(k_z z-\omega t)\, dz)\, x_0\right)$$

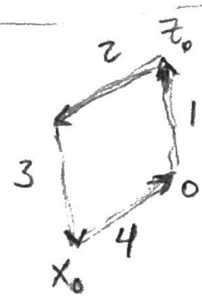
$$\phi = k_z z-\omega t$$

$$d\phi = dz\, k_z \Rightarrow dz = \frac{d\phi}{k_z}$$

$$= -E_{0x}\frac{\cancel{k_z}}{\cancel{\omega}}\cdot\frac{\cancel{\omega}}{\cancel{k_z}}x_0\int_{-\omega t}^{k_z z_0-\omega t}\sin(\phi)\, d\phi$$

$$= -E_{0x}\, x_0 \, -\cos\phi\Big|_{-\omega t}^{k_z z_0-\omega t}$$

$$-\frac{d\Phi_B}{dt} = E_{0x}x_0\left(\cos(k_z z_0-\omega t) - \cos(-\omega t)\right)$$



1 $\rightarrow \int_0^{z_0} \mathbb{E}\cdot dz \Big|_{x=0}$

2 $\rightarrow \int_0^{x_0} \mathbb{E}\cdot d\vec{x} \Big|_{z=z_0}$

3 $\rightarrow \int_{z_0}^{0} \mathbb{E}\cdot d\vec{z} \Big|_{x=x_0}$

4 $\rightarrow \int_{x_0}^{0} \mathbb{E}\cdot d\vec{x} \Big|_{z=0}$

① $\int_0^{z_0} E_{0x}\cos(k_z z-\omega t)\,\overbrace{\hat{x}\cdot\hat{z}}^{0}\,dz = 0$

② $\int_0^{x_0} E_{0x}\cos(k_z z_0-\omega t)\,\overbrace{\hat{x}\cdot\hat{x}}\,dx = E_{0x}\cos(k_z z_0-\omega t)\,x_0$

③ $\int_{z_0}^{0} E_{0x}\cos(k_z z-\omega t)\,\overbrace{\hat{x}\cdot\hat{z}}^{0}\,dz = 0$

④ $\int_{x_0}^{0} E_{0x}\cos(10-\omega t)\,\hat{x}\cdot\hat{x}\,dx = -E_{0x}\cos(-\omega t)\,x_0$

$E_{0x}\,x_0\left(\cos(k_z z_0-\omega t)-\cos(-\omega t)\right)$

$=$

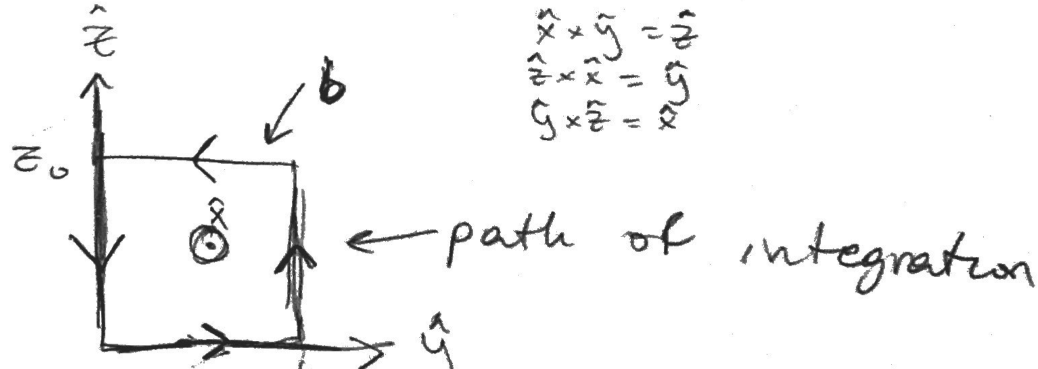
$E_{0x}\,x_0\left(\cos(k_z z_0-\omega t)-\cos(-\omega t)\right)$ ✓

6.1.3) Show that this $\mathbb{B}$ satisfies Ampere's Law along rectangle b. Draw the integral path.

$\hat{x}\times\hat{y}=\hat{z}$

$\hat{z}\times\hat{x}=\hat{y}$

$\hat{y}\times\hat{z}=\hat{x}$



$$\oint \mathbf{B}\cdot d\vec{l} = \frac{1}{c^2}\frac{\partial \Phi_E}{\partial t}$$

$s \to c \to -s \to -c$

$$\frac{1}{c^2}\frac{d\Phi_E}{dt} = \int \frac{1}{c^2}\frac{\partial}{\partial t}(\mathbf{E})\, d\vec{a}$$

$$= \int \frac{1}{c^2}\frac{\partial}{\partial t}\left(E_{0x}\cos(k_z z - \omega t)\right)\overbrace{\hat{x}\cdot\hat{x}}^{1}\, dy\, dz$$

$$= \frac{E_{0x}}{c^2}\int_0^{y_0} dy \int_0^{z_0}\frac{\partial}{\partial t}\left(\cos(k_z z - \omega t)\right) dz$$

$$= \frac{E_{0x}\, y_0}{c^2}\int_0^{z_0} +\omega \cdot +\sin(k_z z - \omega t)\, dz$$

$$\phi = k_z z - \omega t \qquad d\phi = k_z z$$

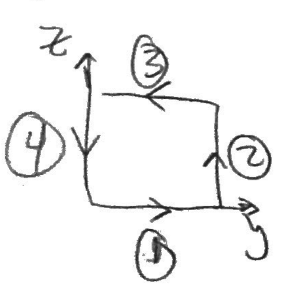
$$dz = d\phi / k_z$$

$$= \frac{E_{0x}\, y_0}{c^2}\frac{\omega}{k_z}\int_{-\omega t}^{k_z z_0 - \omega t} \sin\phi\, d\phi$$

$$\frac{1}{c^2}\frac{\partial \Phi_E}{dt} = \frac{E_{0x}\, y_0}{c^2}\frac{\omega}{k_z}\left(-\cos(k_z z_0 - \omega t) + \cos(-\omega t)\right)$$

$$\oint \mathbf{B}\cdot d\vec{l} = \quad (1) \to \int_0^{y_0} \mathbf{B}\cdot d\vec{y}\,\Big|_{z=0}$$

$$(2) \to \int_0^{z_0} \mathbf{B}\cdot d\vec{z}\,\Big|_{y=y_0}$$



$$(3) \quad \int_{y_0}^{0} \mathbf{B}\cdot d\hat{y}\,\Big|_{z=z_0}$$

$$(4) \quad \int_{z_0}^{0} \mathbf{B}\cdot d\vec{z}\,\Big|_{y=0}$$

① $\int_0^{y_0} \frac{k_z E_{0x}}{\omega} \cos(-\omega t) \overbrace{\hat{y}\cdot\hat{y}}^{1} dy = \frac{k_z E_{0x} y_0}{\omega} \cos(-\omega t)$

② $\int_0^{z_0} \frac{k_z E_{0x}}{\omega} \cos(k_z z - \omega t) \overbrace{\hat{y}\cdot\hat{z}}^{0} dz = 0$

③ $\int_{y_0}^{0} \frac{k_z E_{0x}}{\omega} \cos(k_z z_0 - \omega t) \hat{y}\cdot\hat{y}\, dy = \frac{-k_z E_{0x} y_0}{\omega} \cos(k_z z_0 - \omega t)$

④ $\int_{z_0}^{0} \frac{k_z E_{0x}}{\omega} \cos(k_z z - \omega t) \overbrace{\hat{z}\cdot\hat{y}}^{0} dz = 0$

$$\frac{k_z E_{0x} y_0}{\omega}\left(\cos(-\omega t) - \cos(k_z z_0 - \omega t)\right) = \oint \vec{B} \cdot d\vec{l}$$

$$\frac{k_z E_{0x} y_0}{\omega}\left(\cos(-\omega t) - \cos(k_z z_0 - \omega t)\right) =$$

$$= \frac{E_{0x} y_0}{c^2} \frac{\omega}{k_z}\left(\cos(-\omega t) - \cos(k_z z_0 - \omega t)\right)$$

$$\frac{k_z E_{0x} y_0}{\omega} = \frac{E_{0x} y_0}{c^2} \frac{\omega}{k_z}$$

$$c^2 = \left(\frac{\omega}{k_z}\right)^2 \checkmark$$