11.1.1) Show the following

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Final (HW #11)
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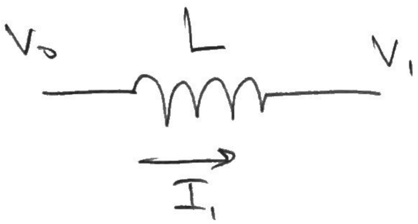
$$\tilde{V}_1 = \tilde{V}_0 - j\omega L \tilde{I}_1$$
$$\tilde{I}_2 = \tilde{I}_1 - j\omega C \tilde{V}_1$$
$$\tilde{V}_2 = \tilde{V}_1 - j\omega L \tilde{I}_2$$
$$\tilde{I}_2 = \frac{\tilde{V}_2}{Z_L}$$

Remember source is

$V_0 \cos(\omega t) = Re[\tilde{V}_0]$

Circuit on page 2

Start with first inductor in loop 1



$$\Delta \tilde{V}_{01} = L \frac{d\tilde{I}_1}{dt}$$

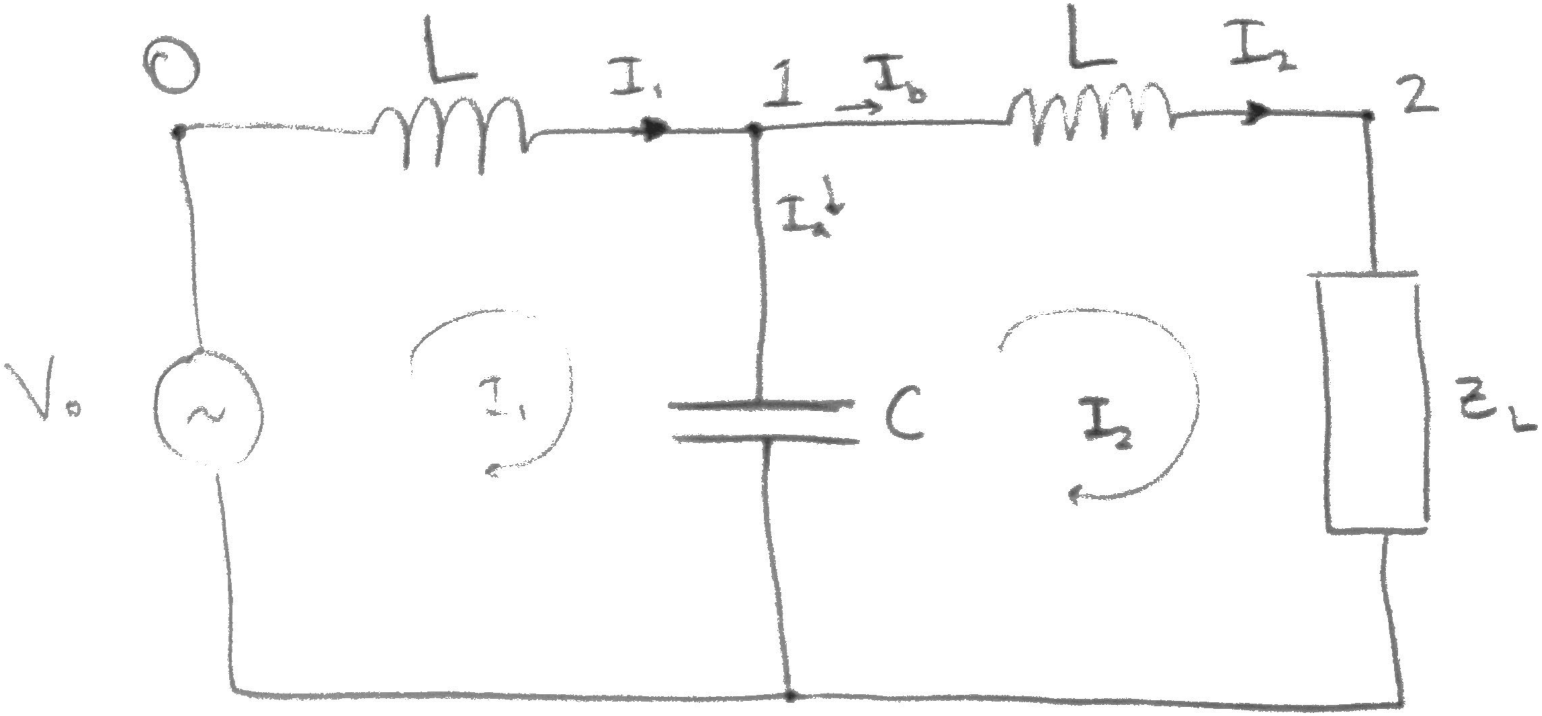
$$\tilde{V}_0 - \tilde{V}_1 = L \frac{d\tilde{I}_1}{dt}$$

$$\frac{d\tilde{I}_1}{dt} = \frac{d}{dt}\Big(\underbrace{\overset{\text{const}}{\tilde{I}_1'}\, e^{j\omega t}}\Big)$$

$$\tilde{V}_0 - \tilde{V}_1 = L(j\omega \tilde{I}_1)$$

$$\frac{d\tilde{I}_1}{dt} = \underbrace{\tilde{I}_1' e^{j\omega t}}\, j\omega$$

$$\boxed{\tilde{V}_1 = \tilde{V}_0 - j\omega L \tilde{I}_1}$$


0
L
I1
1
Ia
Ib
L
I2
2
V0
I1
C
I2
ZL

Inductor in second loop (loop 2)

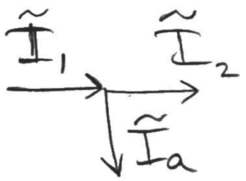
$$\Delta\tilde{V}_{12} = L\frac{\partial \tilde{I}_2}{\partial t} \qquad \frac{\partial \tilde{I}_2}{\partial t} = \frac{\partial}{\partial t}\left(\underbrace{\tilde{I}_2' e^{jwt}}\right)$$

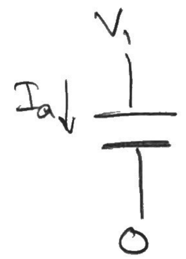
$$\tilde{V}_1 - \tilde{V}_2 = L\frac{\partial \tilde{I}_2}{\partial t} \qquad \frac{\partial \tilde{I}_2}{\partial t} = \underbrace{\tilde{I}_2' e^{jwt}}\, jw$$

$$\tilde{V}_1 - \tilde{V}_2 = L(jw\tilde{I}_2)$$

$$\boxed{\tilde{V}_2 = \tilde{V}_1 - jwL\tilde{I}_2}$$

Look at Node 1





$$\tilde{I}_a = C\frac{\partial \tilde{V}_1}{\partial t}$$

$$I_a = C\frac{\partial}{\partial t}\left(\underbrace{\tilde{V}_1' e^{jwt}}\right)$$

$$I_a = C\left(jw\,\underbrace{\tilde{V}_1' e^{jwt}}\right)$$

$$\tilde{I}_1 - \tilde{I}_2 - \tilde{I}_a = 0$$

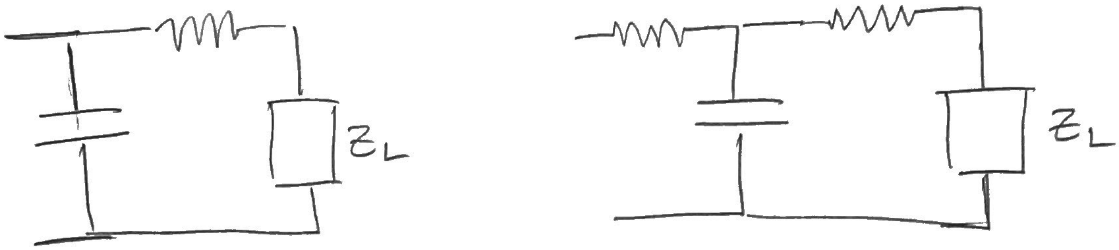
$$\tilde{I}_1 - C(jw\tilde{V}_1) = \tilde{I}_2$$

$$\boxed{\tilde{I}_2 = \tilde{I}_1 - jwC\tilde{V}_1}$$

11.1.2) $\tilde{V}_o = V_o$ and previous equations,

Solve for Z_1 and Z_o where Z_1 & Z_o are

$\underline{Z_1}$ and Z_o



$Z_1 \rightarrow$ Capacitor $\overset{\text{Parallel}}{\parallel}$ (Inductor $\overset{\text{series}}{\&}$ Z_L)

$\rightarrow$ Capacitor $\parallel$ Z_a

$$Z_a = j\omega L + Z_L$$

$$(Z_1)^{-1} = \left(\frac{-j}{\omega C}\right)^{-1} + (j\omega L + Z_L)^{-1}$$

$$(Z_1)^{-1} = j\omega C + \frac{1}{j\omega L + Z_L}$$

$$(Z_1)^{-1} = \frac{j\omega C(j\omega L + Z_L) + 1}{(j\omega L + Z_L)}$$

$$Z_1 = \frac{(j\omega L + Z_L)}{j\omega c(j\omega L + Z_L) + 1}$$

$$Z_1 = \frac{(j\omega L + Z_L)}{-\omega^2 cL + j\omega c Z_L + 1}$$ this is also useful

$$Z_1 = \frac{(j\omega L + Z_L)}{(1-\omega^2 cL) + j\omega c Z_L} * \frac{(1-\omega^2 cL) - j\omega c Z_L}{(1-\omega^2 cL) - j\omega c Z_L}$$

wolfram alpha

$$\boxed{Z_1 = \frac{-j(CL^2\omega^3 + C\omega Z^2 - L\omega) + Z_L}{(1-\omega^2 cL)^2 + \omega^2 c^2 Z_L^2}}$$

$$Z_0 = j\omega L + Z_1$$

$$= j\omega L +$$

wolfram alpha again

$$\boxed{Z_0 = \frac{j(c^2 L^3 \omega^5 + c^2 L \omega^3 Z_L^2 - 3cL^2\omega^3 - c\omega Z^2 + 2L\omega}{(1-\omega^2 cL)^2 + \omega^2 c^2 Z_L^2} + Z}$$

Alternative form straight from wolfram alpha (this makes section 3 easier)

This is what got plugged into wolfram alpha

series

$$Z_0 = jL\omega + \cfrac{1}{j c \omega + \cfrac{1}{Z_L + jL\omega}}$$

series

parallel

wolfram solution

$$Z_0 = \frac{-jCL^2\omega^3 - CL\omega^2 Z + j2L\omega + Z_L}{-CL\omega^2 + jC\omega Z_L + 1}$$

this is the same as the other solution

11.1.3) Using the previous solve for

1.) $\tilde{I}_k$ for $k=1,2$ and $\tilde{V}_k$ for $k=1,2$

Put everything in terms of $\tilde{V}_0$ + Z_0

$$\tilde{I}_1 = \frac{\tilde{V}_0}{Z_0}$$

$$\tilde{V}_1 = \tilde{V}_0 - j\omega L \tilde{I}_1$$

$$\tilde{V}_1 = \tilde{V}_0 - j\omega L \frac{\tilde{V}_0}{Z_0}$$

$$\tilde{I}_2 = \tilde{I}_1 - j\omega C \tilde{V}_1$$

$$\tilde{I}_2 = \frac{\tilde{V}_0}{Z_0} - j\omega C\left(\tilde{V}_0 - j\omega L \frac{\tilde{V}_0}{Z_0}\right)$$

$$\tilde{V}_2 = \tilde{V}_1 - j\omega L \tilde{I}_2$$

$$\tilde{V}_2 = \tilde{V}_0 - j\omega L \frac{\tilde{V}_0}{Z_0} - j\omega L\left(\frac{\tilde{V}_0}{Z_0} - j\omega C\left(\tilde{V}_0 - j\omega L \frac{\tilde{V}_0}{Z_0}\right)\right)$$

$$\tilde{I}_1 = \tilde{V}_0\left(\frac{1}{Z_0}\right)$$

$$\tilde{V}_1 = \tilde{V}_0\left(1 - \frac{j\omega L}{Z_0}\right)$$

$$\tilde{I}_2 = \tilde{V}_0\left(\frac{1}{Z_0} - j\omega C\left(1 - \frac{j\omega L}{Z_0}\right)\right)$$

$$\tilde{V}_2 = \tilde{V}_0\left(1 - \frac{j\omega L}{Z_0} - j\omega L\left(\frac{1}{Z_0} - j\omega C\left(1 - \frac{j\omega L}{Z_0}\right)\right)\right)$$

$$\tilde{I}_1 = V_0 \left(\frac{1}{Z_0} \right)$$

$$= V_0$$

Verify Answer by solving for Z_o

using $\tilde{I}_2 = \tilde{V}_o\left(\frac{1}{Z_o} - j\omega c\left(1 - \frac{j\omega L}{Z_o}\right)\right)$ and

$$\tilde{I}_2 = \frac{\tilde{V}_2}{Z_L} = \frac{\tilde{V}_o}{Z_L}\left(1 - \frac{j\omega L}{Z_o} - j\omega L\left(\frac{1}{Z_o} - j\omega c\left(1 - \frac{j\omega L}{Z_o}\right)\right)\right)$$

and solving for Z_o (should be the same as original Z_o in section 2

$$\cancel{\tilde{V}_o}\left(\frac{1}{Z_o} - j\omega c\left(1 - \frac{j\omega L}{Z_o}\right)\right) =$$

$$\frac{\cancel{\tilde{V}_o}}{\cancel{Z_L}}\left(1 - \frac{j\omega L}{Z_o} - j\omega L\left(\frac{1}{Z_o} - j\omega c\left(1 - \frac{j\omega L}{Z_o}\right)\right)\right)$$

$$Z_L\left(\frac{1}{Z_o} - \left(j\omega c + \frac{\omega^2 cL}{Z_o}\right)\right) =$$

$$1 - \frac{j\omega L}{Z_o} - j\omega L\left(\frac{1}{Z_o} - \left(j\omega c + \frac{\omega^2 cL}{Z_o}\right)\right)$$

$$Z_L\left(\frac{1}{Z_o} - j\omega c - \frac{\omega^2 cL}{Z_o}\right) =$$

$$1 - \frac{j\omega L}{Z_o} - \left(\frac{j\omega L}{Z_o} + \omega^2 cL - \frac{j\omega^3 cL^2}{Z_o}\right)$$

$$Z_L\left(\frac{1}{Z_0} - j\omega C - \frac{\omega^2 CL}{Z_0}\right) =$$

$$1 - \frac{j\omega L}{Z_0} - \frac{j\omega L}{Z_0} - \omega^2 CL + \frac{j\omega^3 CL^2}{Z_0}$$

$$\frac{Z_L}{Z_0} - \frac{\omega^2 CLZ_L}{Z_0} + \frac{2j\omega L}{Z_0} - \frac{j\omega^3 CL^2}{Z_0} =$$

$$j\omega CZ_L + 1 - \omega^2 CL$$

$$\frac{Z_L - \omega^2 CLZ_L + 2j\omega L - j\omega^3 CL^2}{Z_0} =$$

$$j\omega CZ_L + 1 - \omega^2 CL$$

$$Z_0 = \frac{Z_L - \omega^2 CLZ_L + 2j\omega L - j\omega^3 CL^2}{j\omega CZ_L + 1 - \omega^2 CL}$$ ✓ ✓

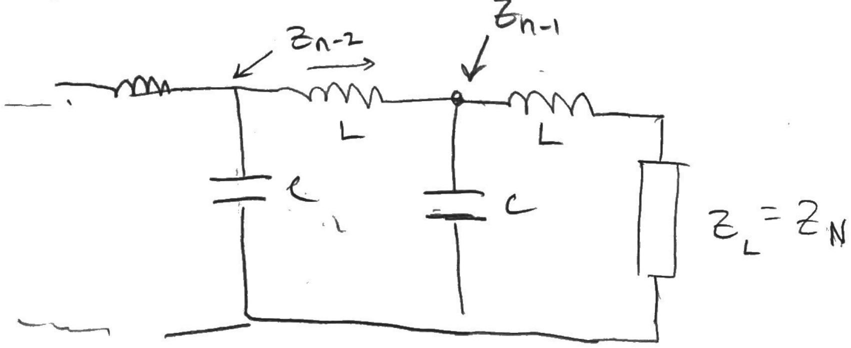
same Z_0 from both methods

(this solution matches the wolfram solution)

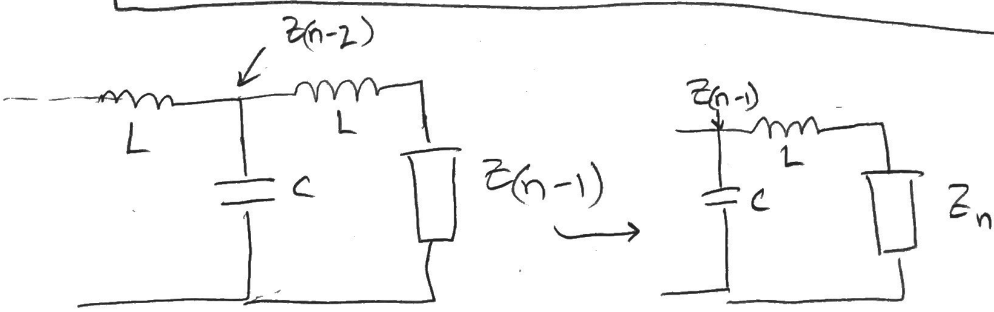
III.1.4) Using the N ladder circuit, solve the following

1.) Write an iterative equation for Z_{n-1} in terms of Z_n, ω, L, and C

Look at a single node then generalize

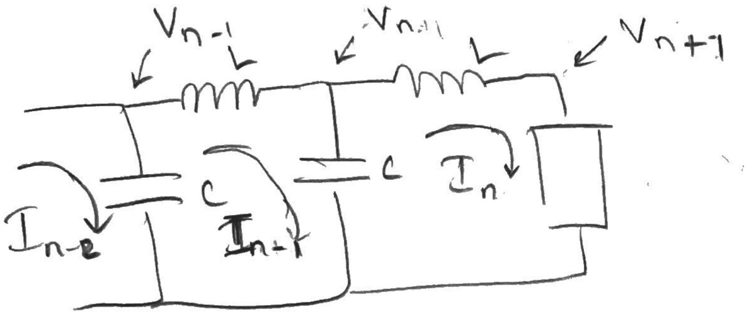


$$\boxed{Z_{n-1} = \left(j\omega C + (Z_n + j\omega L)^{-1}\right)^{-1}}$$



2.) Write an equation that relates $\tilde{V}_{n+1}$ to $\tilde{V}_n$ and $\tilde{I}_n$

Use the same as last section

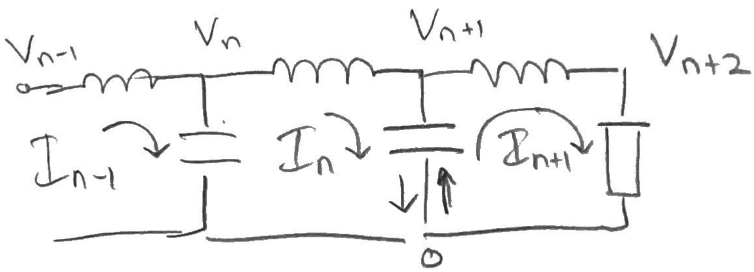


$$V_n - V_{n+1} = L \frac{\partial I_n}{\partial t} \qquad \frac{\partial I}{\partial t} = j\omega I$$

$$V_{n-1} - V_n = L \frac{\partial I_{n-1}}{\partial t}$$

$$\boxed{\tilde{V}_{n+1} = \tilde{V}_n - j\omega L \tilde{I}_n}$$

3.) Write an equation that relates $\tilde{I}_{n+1}$ to $\tilde{I}_n$ and $\tilde{V}_{n+1}$

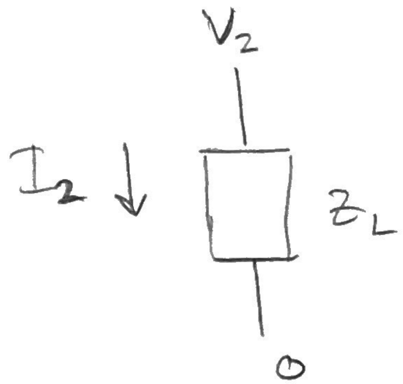


$$\tilde{I}_n - \tilde{I}_{n+1} = C \frac{\partial \tilde{V}_{n+1}}{\partial t}$$

$$\frac{\partial \tilde{V}}{\partial t} = j\omega \tilde{V}$$

$$\boxed{I_{n+1} = \tilde{I}_n - j\omega C \tilde{V}_{n+1}}$$

Voltage across Z_L



$$V_2 = I_2 Z_L$$

$$\boxed{I_2 = \frac{V_2}{Z_L}}$$