8.1.1) Calc L_i when J is uniform then when J is distributed δ deep

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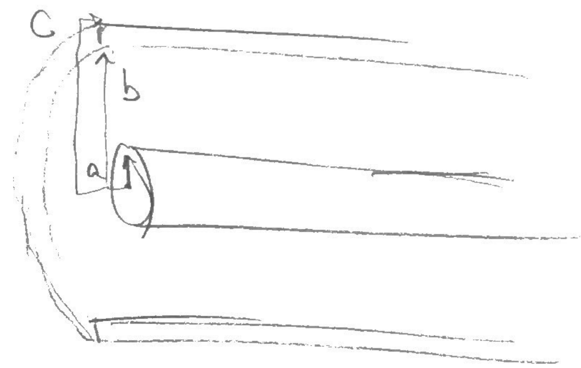
PHYS 513
HW 8
Nov 10, 2020



$J = \frac{I}{\pi a^2}\hat{z}$ ← uniform

Inner wire first

$$W = \frac{1}{2} L I^2 dV = \frac{\mu_0}{2}\int H^2 dV$$

$$\oint H \cdot dl = \int \vec{J} \cdot d\vec{A}$$

$$= \int_0^{2\pi}\int_0^r \frac{I}{\pi a^2} r\,dr\,d\phi\,(\hat{z}\cdot\hat{z}) I$$

$$H \cdot 2\cancel{\pi} \cancel{r} = I \frac{\cancel{\pi} r^{\cancel{2}}}{\cancel{\pi} a^2}$$

$$H = \frac{I r}{2\pi a^2}\hat{\phi}$$

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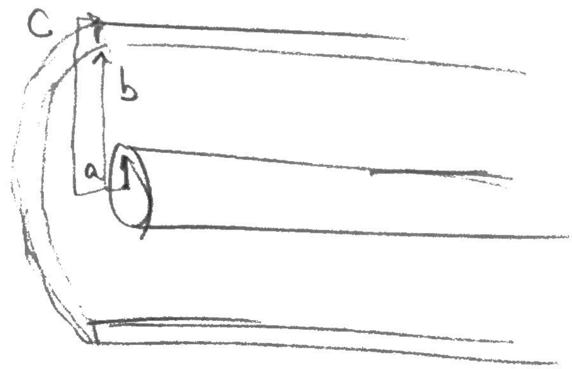


$J = \frac{I}{\pi a^2}\hat{z}$ ← uniform

Inner wire first

$W = \frac{1}{2} L I^2 dV = \frac{\mu_0}{2}\int \underline{H^2} dV$

$\oint H \cdot dl = \int \vec{J} \cdot d\vec{A}$

$= \int_0^{2\pi}\int_0^r \frac{I}{\pi a^2}\, r\,dr\,d\phi\,(\hat{z}\cdot\hat{z})\,1$

$H \cdot 2\cancel{\pi} \cancel{r} = I\frac{\cancel{\pi} r^{\cancel{2}}}{\cancel{\pi} a^2}$

$H = \frac{Ir}{2\pi a^2}\hat{\phi}$

$$\frac{\mu_0}{\cancel{2}} \int_0^{dz} \int_0^{2\pi} \int_0^{a} \left(\frac{Ir}{2\pi a^2}\right)^2 r\, d\phi\, dr\, dz = \cancel{\tfrac{1}{2}} L I^2$$

$$L \cancel{I^2} = \mu_0 \frac{\cancel{I^2}\, dz}{2\pi a^4} \int_0^{a} r^3\, dr$$

$$\boxed{L_{ia} = \frac{\mu_0}{8\pi}}$$

for outer shell

$$\oint H \cdot dl = \int \vec{J} \cdot d\vec{A} \qquad , \ b < r < c$$

Area $= \pi c^2 - \pi b^2$

$$H\, 2\pi r = \int_0^{2\pi}\int_b^{r} \frac{I}{\pi(c^2-b^2)} \cdot r\, dr\, d\phi \qquad J = \frac{I}{\pi(c^2-b^2)}\, \hat{z}$$

$$H \cdot \cancel{2\pi} r = \frac{\cancel{2\pi}\, I}{\pi(c^2-b^2)} \int_r^{c} r\, dr \Longrightarrow \frac{c^2 - r^2}{2}$$

$$H = \frac{I\,(c^2 - r^2)}{2\pi(c^2-b^2)\, r}\, \hat{\phi}$$

$$\frac{1}{2}LI^2 = \mu_0 \int_0^{dz}\int_0^{2\pi}\int_b^c \left(\frac{I(c^2-r^2)}{2\pi(c^2-b^2)r} \right)^2 r\,dr\,d\phi\,dz$$

$$\frac{1}{2}L\cancel{I^2} = \frac{\cancel{I^2}\,dz\,\mu_0}{\cancel{2}\pi(c^2-b^2)^2} \int_b^c \frac{(c^2-r^2)^2}{r}\,dr$$

$$\int_b^c \frac{c^4}{r} - \frac{2c^2 r^{\cancel{2}}}{\cancel{r}} + \frac{r^4}{r}\,dr$$

$$\frac{1}{2}L = \frac{dz\,\mu_0}{2\pi(c^2-b^2)^2}\left(c^4 \ln(c/b) - c^2(c^2-b^2) + \frac{1}{4}(c^4-b^4) \right)$$

$$L = \frac{\mu_0\,dz}{\pi}\left(\frac{c^4 \ln(c/b)}{(c^2-b^2)^2} - \frac{c^2}{(c^2-b^2)} + \frac{1}{4}\frac{c^2+b^2}{(c^2-b^2)} \right)$$

$$\boxed{L = \frac{\mu_0\,dz}{\pi}\left(\frac{c^4 \ln(c/b)}{(c^2-b^2)^2} - \frac{b^2-3c^2}{(c^2-b^2)} \right)}$$

text book has an extra $\frac{1}{2}$

This page is where

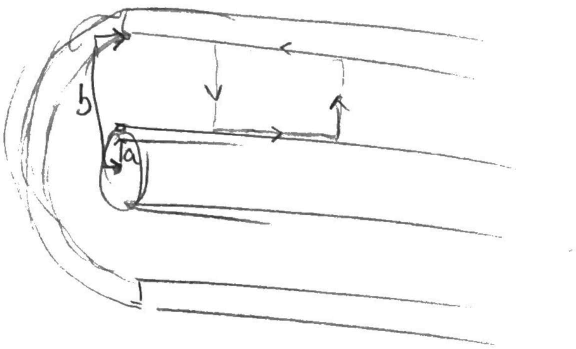
I add the δ calculations

I am not sure why

I am not reducing $\frac{L i_a}{2\pi}$

or $\frac{L i_b}{2\pi}$

8.1.2) Derive L_{ext}



$$L = \frac{\int B \cdot dA}{I}$$

$$\oint B \cdot dl = I_{enc} \quad \leftarrow \text{no } \hat{r} \text{ dependence}$$

$$B = \frac{I \mu_0}{2\pi r} \hat{\phi}$$

$$L = \frac{\int_0^{dz} \int_a^b \frac{I \mu_0}{2\pi r} \, dr \, dz}{I}$$

$$= \frac{\mu_0 \cancel{I}}{2\pi \cancel{I}} dz \int_a^b \frac{1}{r} dr$$

$$\boxed{L_{ext} = \frac{\mu_0 \, dz}{2\pi} \ln\left(\frac{b}{a}\right)}$$

8.1.3) Gap is small and defined as $\delta = b-a$ and $\delta/a \ll 1$, show that this external inductance is similar to parallel plate

Parallel plate inductance

$$L \equiv \mu_0 \frac{d}{w} dz \quad H$$ ← from book 2.5 (3)

$$L = \frac{\mu_0 \, dz}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$L = \frac{\mu_0 \, dz}{2\pi} \ln\left(\frac{a+\delta}{a}\right)$$

Using above for parallel transmission lines

$$L = \mu_0 \frac{d}{w} dz$$ where $w = 2\pi a$ (width) and $d = \delta$

$$L = \mu_0 \frac{\delta}{2\pi a} dz$$

Taylor expand $\ln\left(\frac{a+\delta}{a}\right)$ for $\frac{\delta}{a} \ll 1$

$$L_{ext} = \frac{\mu_0 \, dz}{2\pi} \ln\left(\frac{a+\delta}{a}\right) \rightarrow \frac{\mu_0 \, dz}{2\pi}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right)$$

if $\delta^2 \ll \delta$ then

$$\boxed{\frac{\mu_0 \, dz}{2\pi} \ln\left(\frac{a+\delta}{a}\right) \approx \frac{\mu_0 \, dz}{2\pi} \frac{\delta}{a}}$$