Understanding Sparse JL for Feature Hashing

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NeurIPS 2019

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- Document classification tasks (Weinberger et al. '09, etc)
- k-means/k-medians (Makarychev, Makarychev, Razenshteyn '18)
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Key question: What is the tradeoff between the dimension m, the projection time, and the performance in geometry preservation?

This paper: A theoretical analysis of this tradeoff for a state-of-the-art dimensionality reduction scheme on feature vectors. Could inform how to optimally set parameters in practice.

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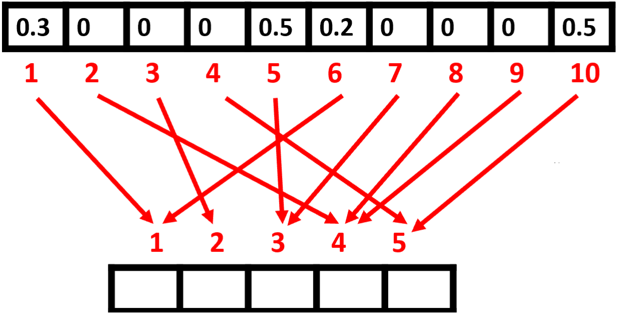

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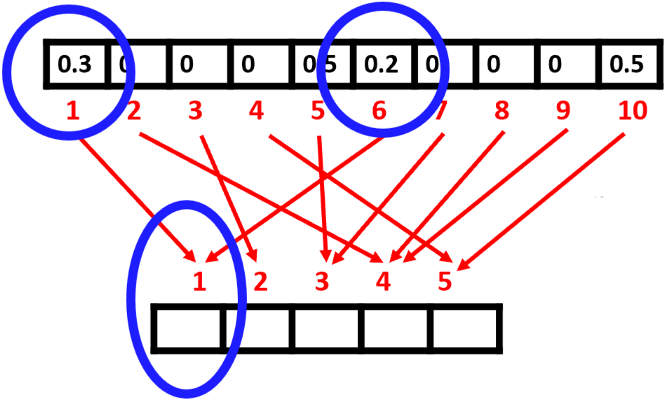

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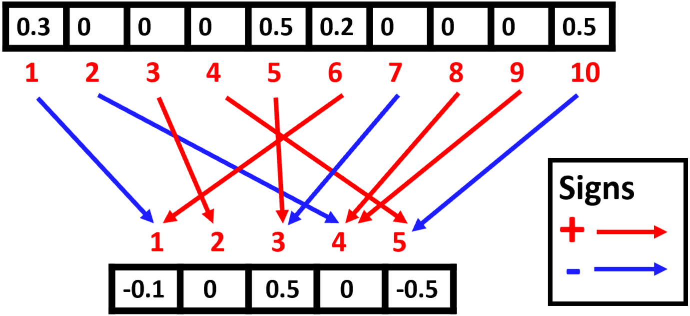

Use **random signs** to handle collisions (unbiased estimator of ℓ_2^2 norm):

$$f(x)_i = \sum_{j \in h^{-1}(i)} \sigma_j x_j.$$

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Use random signs to deal with collisions; scale the resulting vector by $\frac{1}{\sqrt{s}}$.

That is: $f(x)_i = \frac{1}{\sqrt{s}} \sum_{k=1}^{s} \left(\sum_{j \in h_k^{-1}(i)} \sigma_j^k x_j \right)$.

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This work

Analysis of tradeoff for sparse JL between # of hash functions s, dimension m, and performance in ℓ_2-norm preservation.

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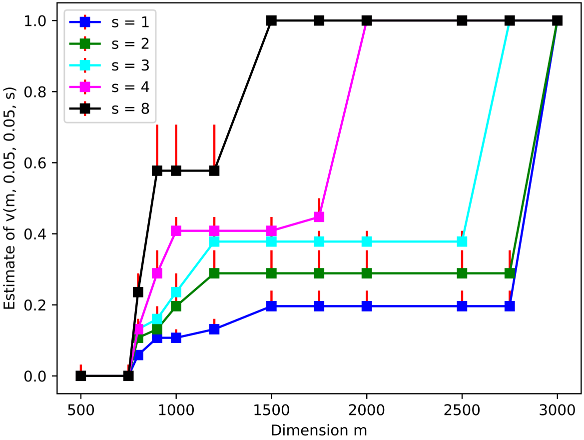
The function v captures the performance of sparse JL on feature vectors.



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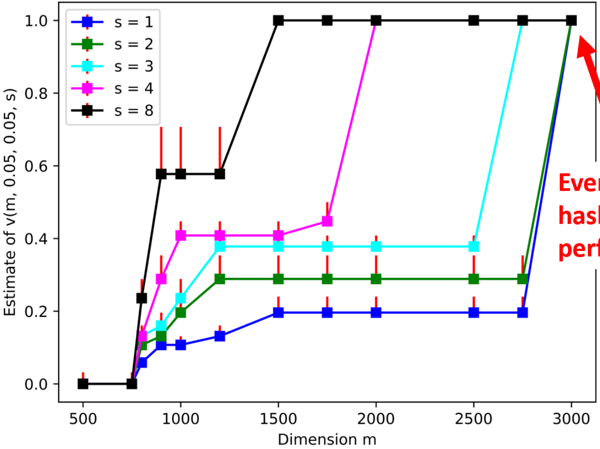

Even feature hashing has full performance

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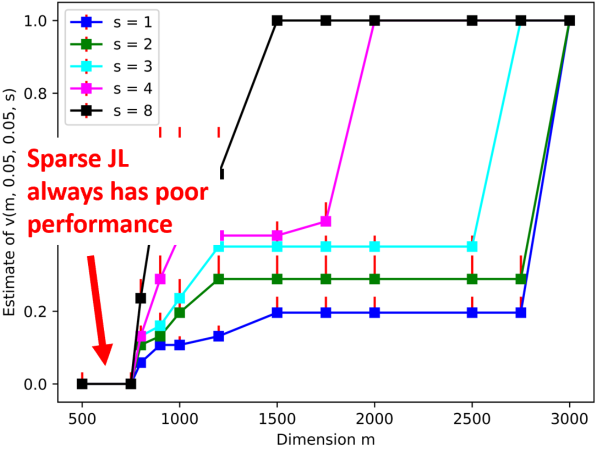

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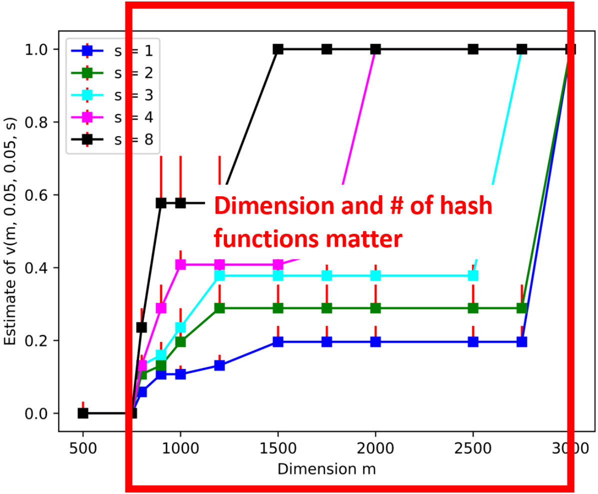

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for ϵ target error, δ target failure probability.

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Sparse JL can sometimes perform much better in practice on feature vectors than traditional theory on $\mathbb{R}^n$ suggests...

Performance on feature vectors (Weinberger et al. '09)

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$$S_v = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq v \|x\|_2\}.$$

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$v(m, \epsilon, \delta, s)$ is the supremum over $v \in [0, 1]$ such that:

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- $v(m, \epsilon, \delta, s) = 0 \implies$ poor performance
- $v(m, \epsilon, \delta, s) = 1 \implies$ full performance
- $v(m, \epsilon, \delta, s) \in (0, 1) \implies$ good performance on $x \in S_{v(m,\epsilon,\delta,s)}$

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We give a tight theoretical analysis of the function $v(m, \epsilon, \delta, s)$.

Informal statement of main result

Goal: $\mathbb{P}_{f \in \mathcal{F}}[\|f(x)\|_2 \in (1 \pm \epsilon)\|x\|_2] > 1 - \delta$.

$v(m, \epsilon, \delta, s) :=$ sup over $v \in [0, 1]$ s.t. sparse JL meets ℓ_2 goal on $x \in S_v$.

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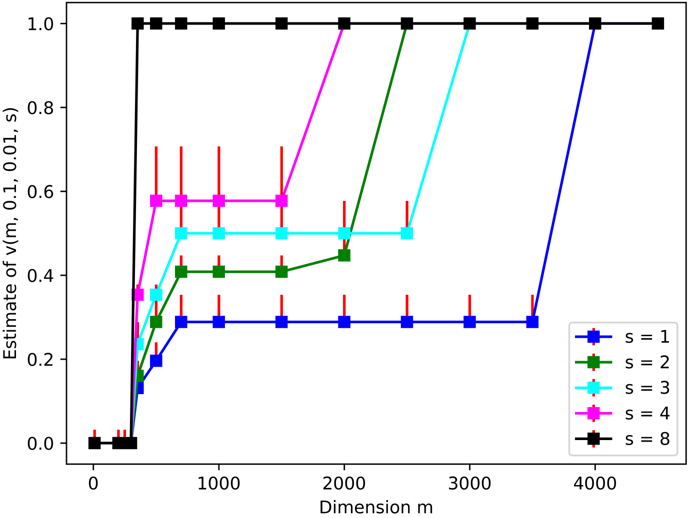
Theorem (Informal)

Sparse JL has **four regimes** *in terms of how it performs on norm preservation. For error ϵ and failure probability δ, sparse JL with projected dimension m and s hash functions has performance $v(m, \epsilon, \delta, s)$ equal to:*

$$\begin{cases} 1 \textit{ (full performance)} & \textit{High } m \\ \sqrt{s} B_1 \textit{ (partial performance)} & \textit{Middle } m \\ \sqrt{s} \min(B_1, B_2) \textit{ (partial performance)} & \textit{Middle } m \\ 0 \textit{ (poor performance)} & \textit{Small } m, \end{cases}$$

where B_1, B_2 are functions of m, ϵ, δ.

$v(m, \epsilon, \delta, s)$ on more synthetic data



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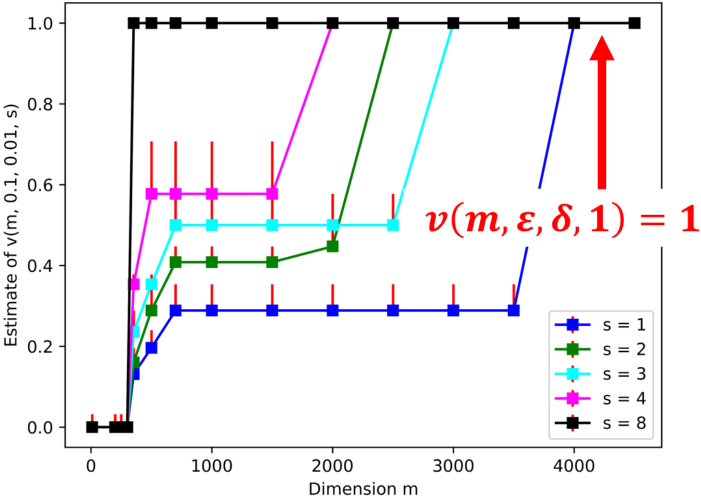

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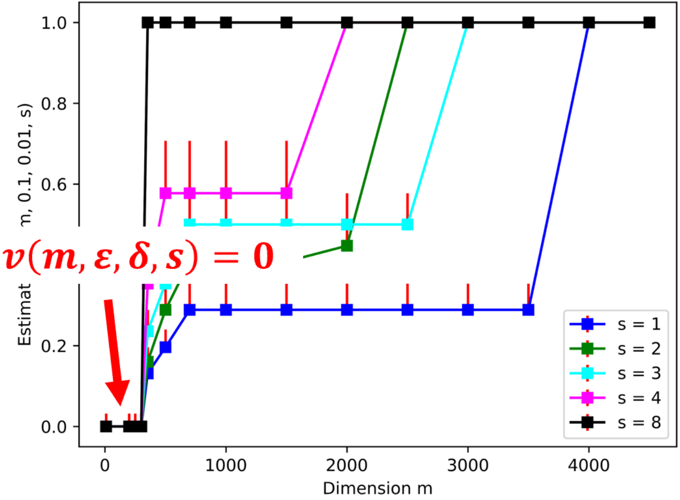

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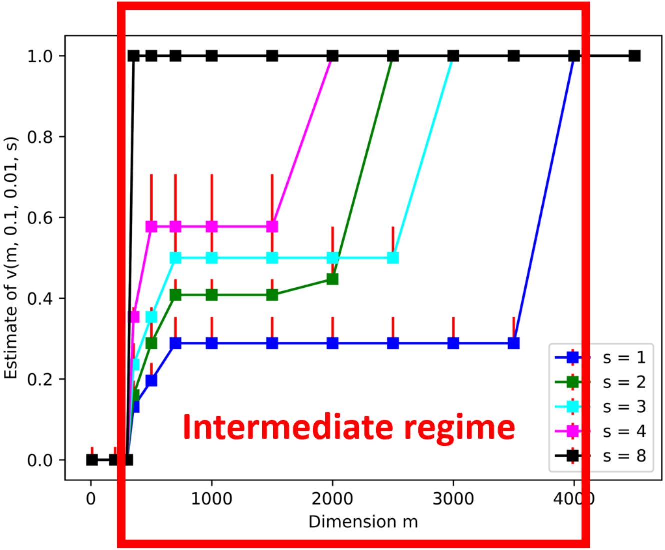
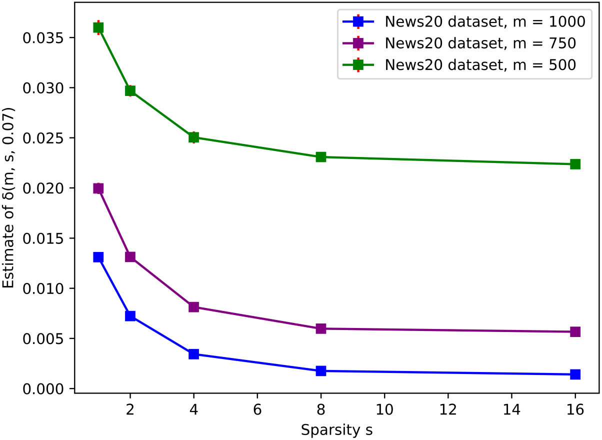


Sparse JL on News20 dataset



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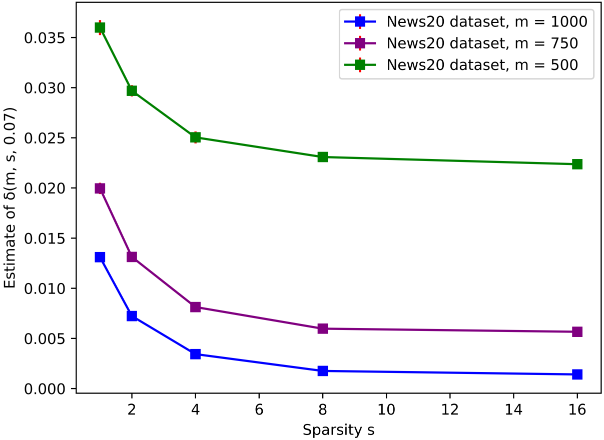

Sparse JL with ≥ 4 hash functions can perform much better than feature hashing in practice.

Comparison to previous work

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Bounds on v (Weinberger et al '09,..., Freksen et al. '18):

- $v(m, \epsilon, \delta, \mathbf{1})$ understood
- $v(m, \epsilon, \delta, s)$ bound for *multiple hashing* (a suboptimal construction)

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Bounds for sparse JL on full space $\mathbb{R}^n$:

- Can set $m \approx \epsilon^{-2} \log(1/\delta)$, $s \approx \epsilon^{-1} \log(1/\delta)$ (Kane and Nelson '12)
- Can set $m \approx \min(2\epsilon^{-2}/\delta, \epsilon^{-2} \log(1/\delta) e^{\Theta(\epsilon^{-1} \log(1/\delta)/s)})$ (Cohen '16)

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Tight bounds on $v(m,\epsilon,\delta,s)$ **for a general** $s > 1$ *for sparse JL.*

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Tight bounds on $v(m, \epsilon, \delta, s)$ **for a general** $s > 1$ *for sparse JL.*

$\implies$ *Characterization of sparse JL performance in terms of* ϵ, δ, *and* ℓ_∞*-to-*ℓ_2 *norm ratio for a general # of hash functions* s

Conclusion

Tight analysis of $v(m, \epsilon, \delta, s)$ for uniform sparse JL for a general s. Could inform how to optimally set s and m in practice.

Characterization of sparse JL performance in terms of ϵ, δ, and ℓ_∞-to-ℓ_2 norm ratio for a general # of hash functions s.

Evaluation on real-world and synthetic data (sparse JL can perform much better than feature hashing).

Proof technique involves a new perspective on analyzing JL distributions.

Thank you!

EXTRA MATERIAL: MAIN RESULT AND PROOF

Main result

Theorem

Under mild conditions, $v(m, \epsilon, \delta, s)$ is equal to $f'(m, \epsilon, \ln(1/\delta), s)$, where $f'(m, \epsilon, p, s)$ is defined to be:

$$
\begin{cases}
1 & \text{if } m \geq \min\left(2\epsilon^{-2}e^{p}, \epsilon^{-2}pe^{\Theta\left(\max\left(1, \frac{p\epsilon^{-1}}{s}\right)\right)}\right) \\
\Theta\left(\sqrt{\epsilon s}\frac{\sqrt{\ln(\frac{m\epsilon^2}{p})}}{\sqrt{p}}\right) & \text{else, if } m \geq \max\left(\Theta(\epsilon^{-2}p), s \cdot e^{\Theta\left(\max\left(1, \frac{p\epsilon^{-1}}{s}\right)\right)}\right) \\
 & \text{and } m \leq \epsilon^{-2}e^{\Theta(p)} \\
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Uniform: Mildly correlate hash functions so $h_j(i) \neq h_k(i)$.

Example (Uniform Sparse JL)

Uniformly choose s nonzero entries in each column; i.i.d signs for nonzero entries.

Block: Take $h_i : \{1, \ldots, n\} \to \{(m/s)(i-1)+1, \ldots, (m/s)(i)\}$

Example (Block Sparse JL)

Choose one nonzero coordinate per m/s-length block per column; i.i.d signs for nonzero entries.

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Sparse JL distributions are state-of-the-art sparse random projections.

High-level approach of our analysis

(r, i)th coordinate is $\eta_{r,i}\sigma_{r,i}/\sqrt{s}$, where $\eta_{r,i} \in \{0, 1\}$, $\sigma_{r,i} \in \{-1, 1\}$

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$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{1 \le i \ne j \le n} \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}x_i x_j$$

This random variable has been repeatedly analyzed in the literature.

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But... existing bounds are limited to $s = 1$ (Freksen et al., etc.)

High-level approach of our analysis

(r, i)th coordinate is $\eta_{r,i}\sigma_{r,i}/\sqrt{s}$, where $\eta_{r,i} \in \{0, 1\}$, $\sigma_{r,i} \in \{-1, 1\}$

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Need tight bounds on $\mathbb{E}[R(x_1, \ldots, x_n)^p]$ on S_v at every threshold v.

Bounding moments of $R(x_1, \ldots, x_n)$ at every threshold v

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We use a non-combinatorial approach with Rademacher-specific bounds.

Overview of our approach

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{m} Z_r(x_1, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{m} \left(\sum_{1 \le i \neq j \le n} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} x_i x_j \right).$$

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2. View $Z_r(v, \ldots, v, 0, \ldots, 0)$ as a quadratic form of ± 1 rvs
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1. Create <u>tractable</u> versions of estimates in (Latała '97, '99)
   Structure of $Z_r(x_1, \ldots, x_n)$ is helpful
2. Combine over $r \in [m]$ using (Latała '97)