Supply-Side Equilibria in Recommender Systems

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Abstract

Digital recommender systems such as Spotify and Netflix affect not only consumer behavior but also *producer incentives*: producers seek to supply content that will be recommended by the system. But what content will be produced? We investigate the supply-side equilibria in content recommender systems. We model users and content as D-dimensional vectors, and the system as recommending the content that has the highest dot product with each user. The main features of our model are that the producer decision space is *high-dimensional* and the user base is *heterogeneous*. At equilibrium, producers can thus either specialize their content to subsets of users or create mainstream content at equilibrium.

The potential for specialization by producers gives rise to new qualitative phenomena compared to classical one-dimensional models. First, we show that producers can create different types of content at equilibrium, which leads to *the formation of genres*. Using a duality argument, we derive necessary and sufficient conditions for genre formation to occur. Then, we characterize the distribution of content across genres and within each genre at equilibrium, focusing on concrete settings with two populations of users. Lastly, we show that producers can achieve *positive profit at equilibrium*, which is typically impossible under perfect competition. We derive sufficient conditions for this to occur, and show it is closely connected to specialization of content. In all of our results, the interplay between the geometry of the users and the structure of producer costs influences the structure of the supply-side equilibria. At a conceptual level, our work serves as a starting point to investigate how recommender systems shape supply-side competition, and for understanding what new phenomena arise in multi-dimensional competitive settings.

Contents

1 Introduction

Content recommender systems have disrupted the production of digital goods such as movies, music, and news. In the music industry, artists have changed the length and structure of songs in response to Spotify's algorithm and payment structure [Hod21]. In the movie industry, personalization has led to low-budget films catering to specific audiences [McD19], in some cases constructing data-driven "taste communities" [Ada18]. Across industries, recommender systems shape how producers decide what content to create, influencing the supply side of the digital goods market. This raises the questions: *What factors drive and influence the supply-side marketplace? What content will be produced at equilibrium?*

Intuitively, supply-side effects are induced by the multi-sided interaction between producers, the recommender system algorithm, and users. Users tend to follow recommendations when deciding what content to consume [Urs18]—thus, recommendations influence how many users consume each digital good and impact the profit (or utility) generated by each content producer. As a result, content producers shape their content to maximize appearance in recommendations—this creates competition between the producers, which can be modeled as a game. However, understanding such producer-side effects has been difficult, both empirically and theoretically. This is a pressing problem, as these gaps in understanding have hindered the regulation of digital marketplaces [Sti19].

This paper provides a theoretical model and toolkit to formalize supply-side effects. At a high level, there are two primary challenges complicating theoretical analyses. (1) Digital goods such as movies have many attributes and thus must be embedded in a *multi-dimensional* continuous space, leading to a large producer action space. Such multi-dimensionality is a departure from traditional economic models of price and spatial competition. (2) A core aspect of such marketplaces is that producers specialize—producing content that some users especially enjoy but that others do not. However, incentives to specialize depend on the geometry of the *heterogeneous user base* and the cost structure for producing goods (i.e., whether it is more expensive to produce items that are good in multiple dimensions). They further require a model which allows producers to, at equilibrium, produce different items. Together, these features create the potential for supply-side equilibria to exhibit rich economic phenomena, but pose a challenge to both modeling and analysis.

Contributions. We introduce a simple game-theoretic model for supply-side competition in recommender systems, and we analyze the equilibria of this model. Our model captures the multi-dimensional space of producer decisions, rich structures of production costs, and general configurations of users. The resulting equilibria elucidate what content producers are incentivized to create as user geometry and producer costs vary. Using this model, we analyze economic phenomena associated with recommender systems: we focus on when producers create specialized content, and the resulting effects on user and producer utility. Our analysis of supply-side competition thus takes a step towards elucidating how personalization shapes the marketplace of digital goods. We further hope that the simplicity of our model allows it to serve as a theoretical foundation for future work.

Our model. We summarize our model, deferring a formalization to Section 2. We represent users and digital goods as vectors in $\mathbb{R}^D_{\geq 0}$, so that the value that a user with vector u derives from a digital good p is equal to the inner product $\langle u, p \rangle$. The platform has N users, where user i is associated with vector $u_i \in \mathbb{R}^D_{\geq 0}$. The platform has $P \geq 2$ producers, where producer j *chooses* a single digital good p_j to create. The recommender system shows each user the good with maximum value to them: user i is recommended the digital good created by producer $j^*(i)$, where $j^*(i) = \arg\max_{1 \leq i \leq P} \langle u_i, p_j \rangle$.

The goal of a producer to maximize their profit, which is equal to the number of users who are recommended their content minus the cost of producing the content. We assume that the cost is a fixed (one-time) cost, since we focus on marketplaces for digital goods such as movies that are costly to produce but cheap to distribute. We consider producer cost functions of the form $c(p) := \|p\|^\beta$, where $\|\cdot\|$ is an arbitrary norm and the exponent β is at least 1.

We study the Nash equilibria of the resulting game between producers. A salient feature of our model is that the producer utilities are inherently *discontinuous*, since the function $\arg\max_{1\le i\le P}\langle u_i, p_j\rangle$ changes discontinuously with the producer vectors p_j. As a result of these discontinuities, pure strategy equilibria do not exist (Proposition 6). We thus must focus on mixed strategy equilibria. Using the technology of discontinuous games [Ren99], we establish the existence of **symmetric mixed Nash equilibria** (Proposition 7), and take these as our object of study. These symmetric equilibria can be represented as a distribution μ over $\mathbb{R}^D_{\ge 0}$ and are thus more tractable than general asymmetric equilibria. We can nonetheless capture asymmetric concepts (such as producer specialization) by examining the support of the equilibrium distribution, as we describe in Section 1.1.

Focus of our work. We initiate the study of how supply-side competition between producers impacts the quality and type of content that they are incentivized to create. Underlying the producer decision is that they can win a user either by exhibiting higher *quality* (vector norm) or by exhibiting better *personalization* (cosine similarity). These two levers impact the producer profit in fundamentally different ways. Improving quality simultaneously improves the value for all users (and thus improves the chance of winning each of them), but increases cost of production. On the other hand, since users are heterogeneous, improving personalization for one user can worsen personalization for other users, but does not impact cost of production. Given the inherent tensions between personalization for different users, a producer's choice of how to personalize content depends on the content created by other producers, user heterogeneity, and the structure of producer costs.

We investigate whether producers decide to specialize their content to individual users or subgroups of users, or cater their content to an average user at equilibrium. Which behavior occurs at equilibrium depends on the geometry of the users as well as the cost function parameters. This raises the questions:

> *Under what conditions do producers create specialized content versus mainstream content at equilibrium? What are the economic consequences of specialization?*

To mathematically study these questions, we investigate the structure of the symmetric mixed Nash equilibria μ. We focus on how the market specifics—e.g. the geometry of the users $u_1, \ldots, u_N$ and the producer cost function parameters $\|\cdot\|$ and β—impact the content created at equilibrium.

We first characterize when specialization occurs in a marketplace (Section 1.1). We formalize this in terms of the support of the equilibrium distribution—whether vectors in the support all point in the same direction (so there is a single "genre" of content) or whether vectors span multiple directions (so multiple "genres" form). We provide a tight characterization of when such genre formation occurs (Theorem 1). Next, we analyze fine-grained properties of the equilibrium distribution in concrete settings (see Section 1.2), providing tools to recover the exact support and density of the equilibrium μ. Finally, we illustrate economic consequences of specialization (Section 1.3), showing that it enables producers to achieve *positive profit at equilibrium* (Proposition 4). This means that despite competition between producers, the marketplace still exhibits monopoly-like behavior.

En route, we develop technical tools to analyze the complex, high-dimensional behavior of producers, which may be of broader interest. First and foremost, to prove Theorem 1, we draw a connection to minimax theory in optimization. In particular, we show that the existence of a single-genre equilibrium is equivalent to strong duality holding for a certain optimization program that we define. This allows us to leverage techniques from optimization theory to provide a necessary and sufficient condition for genre formation. Second, to analyze properties of equilibria in concrete instances, we provide a decoupling lemma in terms of the equilibrium's support and its one-dimensional marginals. This produces one-dimensional functional equations that make solving for the underlying equilibrium more tractable.

Model discussion. Our model and research questions relate to classical models of competition in economic theory. However, particular aspects of recommender systems—high-dimensionality of the producer action space, rich structure of producer costs, and role of user geometry—require a new model. While classical models of price competition[1] focus on how producers set prices (a one-dimensional quantity), the decisions make by the producers in our model are multi-dimensional. Moreover, classical models of spatial competition[2] focus on how producers choose a direction (typically in $\mathbb{R}^1$ or $\mathbb{S}^1$) and face costs based on Euclidean distance; in contrast, producers in our model jointly select the direction and magnitude of their content and face production costs that exhibit rich tradeoffs between excelling in different dimensions.

Our model is one of the simplest possible that captures high-dimensional content and heterogeneous users. Bilinear (dot product) utilities are a simple assumption on user values motivated by standard algorithms used by recommendation systems: for example, matrix completion assumes that the user values are inner products between preference vectors and content attributes vectors [KBV09]. Our producer cost function is stylized but flexible, as it accommodates arbitrary norms, and suffices to study which cost structures induce specialized vs. homogeneous goods (Theorem 1).

We hope that the simplicity of our model, and its ability to capture specialization, makes it a useful starting point for further studying the impact of recommender systems on production. We consider perfect recommendations that match each user to their favorite content, which could be later relaxed to accomodate imperfect information. We also assume that producers earn fixed per-user revenue, which could be later relaxed to let producers set prices. However, our current model suffices to study the emergence of specialization and its economic consequences, which already provides a rich set of questions and is the focus of our work.

1.1 Specialization and the formation of genres

Specialization is when different producers create goods tailored to different users. To formalize this, we need to disentangle two forms of differentiation: (1) differentiation along direction (genre), and (2) differentiation along magnitude (quality level). We define specialization as differentiation along genres, and not as differentiation along quality level. To focus on the former, we define **genres** as the set of *directions* that arise at a symmetric mixed Nash equilibrium μ:

$$\text{Genre}(\mu) := \left\{ \frac{p}{\|p\|} \mid p \in \text{supp}(\mu) \right\} \tag{1}$$

When an equilibrium has a single genre, all producers cater to an average user, and only a single type of content appears on the platform. On the other hand, when an equilibrium has multiple genres, many types of digital content are likely to appear on the platform. Note that we formalize the concept of "genres" in terms of the *support* of the symmetric mixed equilibrium distribution. This captures the set of content that may arise on the platform in some realization of randomness of the producers' strategies. This formalization of genre formation captures that producers may create different goods without requiring us to reason about asymmetric equilibria.

[1] See Baye and Kovenock [BK08] for a textbook treatment.

[2] See Anderson, de Palma, and Thisse [AdPT92] for a textbook treatment.

In order to investigate whether specialization occurs in a given marketplace, we investigate when the set of genres $\text{Genre}(\mu)$ of an equilibrium μ contains more than one direction. We distinguish between two regimes of marketplaces based on whether or not a single-genre equilibrium exists:

1. A marketplace is in the *single-genre regime* if there exists an equilibrium μ such that $|\text{Genre}(\mu)| = 1$. All producers thus create content of the same genre.
2. A marketplace is in the *multi-genre regime* if all equilibria μ satisfy $|\text{Genre}(\mu)| > 1$. Producers thus necessarily differentiate in the genre of content that they produce.

To understand these regimes, we ask: *what conditions on the user vectors $u_1, \ldots, u_N$ and the cost function parameters $\|\cdot\|$ and β determine which regime the marketplace is in?*

Characterization of single-genre and multi-genre regimes. We first provide a tight geometric characterization of when a marketplace is in the single-genre regime versus in the multi-genre regime. Specifically, genres emerge when a certain set $\mathcal{S}^\beta$ is sufficiently different from its convex hull $\bar{\mathcal{S}}^\beta$:

Theorem 1. *Let $\mathbf{U} := [u_1; \cdots; u_N]$ be the $N \times D$ matrix of user vectors, let $\mathcal{S}$ be $\left\{\mathbf{U}p \mid \|p\| \leq 1, p \in \mathbb{R}^D_{\geq 0}\right\}$, and let $\mathcal{S}^\beta$ be the image of $\mathcal{S}$ under taking coordinate-wise powers (i.e. if $(z_1, \ldots, z_N) \in \mathcal{S}$ then $(z_1^\beta, \ldots, z_N^\beta) \in \mathcal{S}^\beta$). Then, there is a symmetric equilibrium μ with $|\text{Genre}(\mu)| = 1$ if and only if*

$$\max_{y \in \mathcal{S}^\beta} \prod_{i=1}^N y_i = \max_{y \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^N y_i. \tag{2}$$

Otherwise, all symmetric equilibria have multiple genres. Moreover, if (2) holds for some β, it also holds for every $\beta' \leq \beta$.

Theorem 1 relates the existence of a single-genre equilibrium to the convexity of the set $\mathcal{S}^\beta$. The condition in (2) always holds if $\mathcal{S}^\beta$ is convex, but is strictly speaking weaker than convexity. Interestingly, the condition depends on the geometry of the user embeddings and the cost function but *not* on the number of producers. We give concrete examples of $\mathcal{S}^\beta$ in Figure 5 (Section 3).

Theorem 1 further shows that the boundary between the single-genre and multi-genre regimes can be represented by a *threshold* β^*, where single-genre equilibria exist exactly when $\beta \leq \beta^*$. Conceptually, larger β make producer costs more superlinear, which eventually discentivizes producers from attempting to perform well on all dimensions at once. To further understand the boundary β^*, we establish several corollaries:

- First, for cost functions $c(p) = \|p\|_q^\beta$, we show that $\beta^* \geq q$ for any set of user vectors (Corollary 5). Thus, the threshold β^* relates closely to the convexity of the cost function and whether the cost function is superlinear.
- Second, we consider 2 users with cosine similarity $\cos(\theta^*)$ and cost function $c(p) = \|p\|_2^\beta$, and show that $\beta^* = \frac{2}{1-\cos(\theta^*)}$ (Corollary 6). The threshold β^* thus increases as the users become closer together, because it is easier to simultaneously cater to all users.

Finally, we consider general configurations of users and cost functions, and upper-bound on β^*:

Corollary 1. *Let $\|\cdot\|_*$ denote the dual norm of $\|\cdot\|$, defined to be $\|p\|_* = \max_{\|p\|=1, p \in \mathbb{R}^D_{\geq 0}} \langle q, p \rangle$. Let $Z := \|\sum_{n=1}^N \frac{u_n}{\|u_n\|_*}\|_*$. Then,*

$$\beta^* \leq \frac{\log(N)}{\log(N) - \log(Z)}. \tag{3}$$

In equation (3), the threshold β^* increases as Z increases. As an example, consider the cost function $c(p) = \|p\|_2^\beta$. We see that if the user vectors point in the same direction, then $Z = N$ and the right-hand side of (3) is ∞. On the other hand, if $u_1, \ldots, u_n$ are orthogonal, then $Z = \sqrt{N}$ and the right-side of (3) is 2, which exactly matches the bound in Corollary 5. In fact, for *random* vectors $u_1, \ldots, u_N$ drawn from a truncated gaussian distribution, we see that $Z = \tilde{O}(\sqrt{N})$ in expectation, in which case the right-hand side of (3) is close to 2 as long as N is large. Thus, for many choices of user vectors, even small values of β are enough to induce multiple genres.

Computational aspects of Theorem 1. Interestingly, both terms in equation (2) are convex optimization programs. For the right-hand side, taking the logarithm reveals that it is the maximum of a concave function on a convex set. For the left-hand side, it is the βth power of the maximizer on S (which is convex), so it can also be cast as the maximum of a concave function on a convex set. This suggests that it might be possible to efficiently compute equation (2), and we defer the design of optimization algorithms to future work.

Location of single-genre equilibrium. We next study where is the single-genre equilibrium is located, in cases where it exists. As a consequence of the proof of Theorem 1, we can show that the location of the single-genre equilibrium maximizes the *Nash social welfare* [Nas50] of the users.

Corollary 2. *If there exists μ with $|\mathrm{Genre}(\mu)| = 1$, then the corresponding producer direction maximizes Nash social welfare of the users:*

$$\mathrm{Genre}(\mu) = \underset{\|p\|=1 | p \in \mathbb{R}^D_{\geq 0}}{\arg\max} \sum_{i=1}^{N} \log(\langle p, u_i \rangle). \tag{4}$$

Corollary 2 reveals that producers competing with each other for users implicitly maximizes this measure of welfare of users at equilibrium. Moreover, the properties of the Nash social welfare thus imply that the single-genre equilibria strike a compromise between fairness (balancing values of different users) and efficiency (the sum of the values achieved across all users). Finally, note the objective in equation (4) is concave in p and so can be computed.

Proof techniques for Theorem 1. Since the single-genre equilibrium does not admit a straightforward closed-form solution, we must implicitly reason about its existence when proving Theorem 1. To do so, we draw a connection to minimax theory in optimization. We show that the existence of a single-genre equilibrium is equivalent to strong duality holding for the following minmax problem:

$$\inf_{y \in \mathcal{S}^\beta} \sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^{N} \frac{y'_i}{y_i}. \tag{5}$$

While the objective is convex in y and linear (concave) in y', the constraints on y and y' through the set $\mathcal{S}^\beta$ can be non-convex. Nonetheless, if the set $\mathcal{S}^\beta$ is sufficiently convex, then strong duality will hold.

Our proof formalizes this argument. First, we characterize the cumulative distribution function of the quality level of a single-genre equilibria as $F(q) \propto q^\beta$ (Lemma 2). Then we show that y corresponds to an equilibrium direction if $\sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^{N} \frac{y'_i}{y_i} \leq N$, which allows us to establish the connection between single-genre equilibria and strong duality of equation (5) (Lemma 3). Finally, we analyze when strong duality holds for equation (5) by convexifying the optimization program.

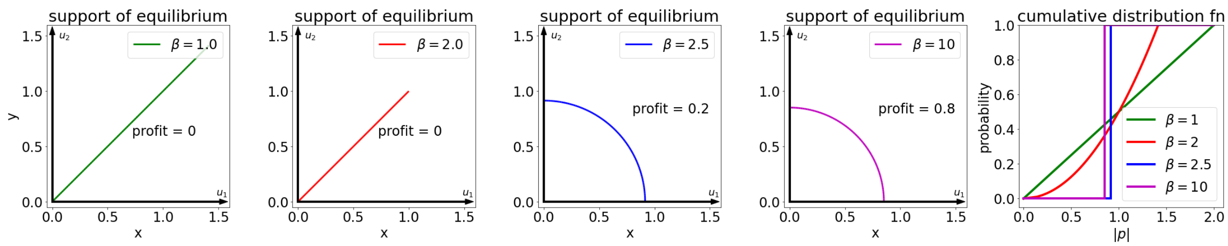

Figure 1: A symmetric equilibrium for different settings of β, for 2 users located at the standard basis vectors e_1 and e_2, $P = 2$ producers, and producer cost function $c(p) = \|p\|_2^\beta$. The first 4 plots show the support of the equilibrium μ. As β increases, there is a phase transition from a single-genre equilibrium to an equilibrium with infinitely many genres (Theorem 2); the profit also transitions from zero to positive (Propositions 4-5). The last plot shows the cumulative distribution function of $\|p\|$ where $p \sim \mu$, which is a step function for the multi-genre equilibria.

1.2 Equilibrium structure for two equally sized populations of users

We next works towards concretely characterizing the equilibria μ, in terms of the set of genres $\text{Genre}(\mu)$ and the density of μ. To do so, we develop machinery to analyze the structure of the equilibrium distribution μ and apply it to several concrete settings. In this section, we focus on the case of two equally sized populations and producer cost functions of the form $c(p) = \|p\|_2^\beta$.

We establish *structural properties* of the equilibria and also provide *closed-form expressions* for the equilibria in special cases, with several economic consequences. Our results establish that in the multi-genre regime, for any finite P all equilibria have infinitely many genres. In the limit as $P \to \infty$, however, the infinite-genre equilibria converge to just two genres, not necessarily located at the user vectors (Theorem 3). We depict equilibria for both finite and infinite P in Figures 1-3.

Structural properties of the equilibria. We first establish properties about the *support* of the equilibrium distributions μ. First, we show that the support of the equilibrium distribution μ is 1-dimensional in that it cannot contain an ϵ-ball for any ϵ:

Proposition 1. *Suppose that there are N users split equally between two linearly independently vectors $u_1, u_2 \in \mathbb{R}_{\geq 0}^2$, and let $\theta^* := \cos^{-1}\left(\frac{\langle u_1, u_2\rangle}{\|u_1\|_2 \|u_2\|}\right)$. Let the cost function be $c(p) = \|p\|_2^\beta$, and let $P \geq 2$. Let μ be a symmetric Nash equilibrium such that the distributions $\langle u_1, p\rangle$ and $\langle u_2, p\rangle$ over $\mathbb{R}_{\geq 0}$ are absolutely continuous. As long as $\beta \neq 2$ or $\theta^* \neq \pi/2$, the support of μ does not contain an ℓ_2-ball of radius ϵ for any $\epsilon > 0$.*[3]

Proposition 1 demonstrates that the support of μ must be a union of 1-dimensional curves. In the single-genre regime, the support is always a line segment through the origin. In the multi-genre regime, however, the support can be curves with different shapes (see Figure 2 for specific examples). We will later characterize where these curves are increasing or decreasing in terms of the location of the curve, the angle $\theta^* = \cos^{-1}\left(\frac{\langle u_1, u_2\rangle}{\|u_1\| \|u_2\|}\right)$, and the cost function parameter β (Lemma 11).

We next show that all equilibria must have either one or infinitely many genres, dictated by whether β is above or below the critical value β^* (see Figure 1):

Theorem 2. *[Informal version of Theorem 4] Suppose that there are N users split equally between two linearly independently vectors $u_1, u_2 \in \mathbb{R}_{\geq 0}^D$, and let $\theta^* := \cos^{-1}\left(\frac{\langle u_1, u_2\rangle}{\|u_1\|_2 \|u_2\|}\right)$. Let the cost function*

[3] The case of $\beta = 2$ and $\theta^* = \pi/2$ is degenerate and permits a range of possible equilibria.

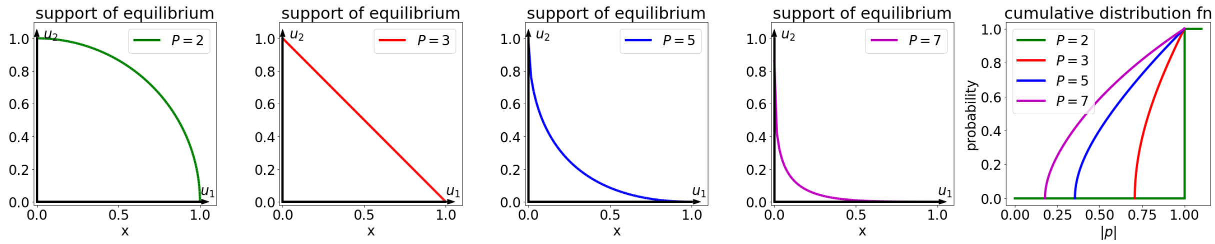

Figure 2: A symmetric equilibrium for different settings of P, for 2 users located at the standard basis vectors e_1 and e_2, for producer cost function $c(p) = \|p\|_2^\beta$ with $\beta = 2$ (see Proposition 3). The first 4 plots show the support of an equilibrium μ. As P increases, the support goes from concave, to a line segment, to convex. The last plot shows the cumulative distribution function of $\|p\|$ for $p \sim \mu$. The distribution for lower P stochastically dominates the distribution for higher values of P.

be $c(p) = \|p\|_2^\beta$. Let μ be a symmetric Nash equilibrium. Under mild conditions on μ, there are two regimes based on β and θ^*:

1. *If $\beta < \beta^* = \frac{2}{1-\cos(\theta^*)}$, then μ satisfies $|\mathrm{Genre}(\mu)| = 1$.*
2. *If $\beta > \beta^* = \frac{2}{1-\cos(\theta^*)}$, then μ satisfies $|\mathrm{Genre}(\mu)| = \infty$.*

Theorem 2 provides a tight characterization of when specialization occurs in a marketplace: specialization occurs *if and only if* β is above β^*. More specifically, the first part of Theorem 2 strengthens Theorem 1 to show that *all* equilibria are single-genre when $\beta < \beta^*$, which means that producers are *never* incentivized to specialize in this regime.

In the multi-genre regime, Theorem 2 shows that producers interestingly do not fully personalize content to either of the two users u_1 and u_2, or even choose between finitely many types of content. Rather, producers choose infinitely many types of content that balance the interests of the two populations in different ways. The lack of coordination between producers—as captured by a symmetric mixed Nash equilibrium—is what drives this result. Producers do not know exactly what content other producers will create in a given realization of the randomness, which results in a diversity of content on the platform.

Closed-form equilibria in special cases. We next concretely compute the equilibria μ in several special instances that permit closed-form solutions. For ease of notation, we assume these populations each consist of a *single* user (these results can be easily adapted to the case of $N/2$ users in each population). First, we consider the standard basis vectors for $P = 2$ producers (see Figure 1).

Proposition 2. *Suppose that there are 2 users located at the standard basis vectors $e_1, e_2 \in \mathbb{R}^2$, and the cost function is $c(p) = \|p\|_2^\beta$. For $P = 2$ and $\beta \geq \beta^* = 2$, there is an equilibrium μ supported on the quarter-circle of radius $(2\beta^{-1})^{\frac{1}{\beta}}$, where the angle $\theta \in [0, \pi/2]$ has density $f(\theta) = 2\cos(\theta)\sin(\theta)$.*

Since all (x, y) in the support have the same radius, producers always expend the same cost regardless of the realization of randomness in their strategy. Since $c(p) = \|p\|_2^\beta$, producers pay a cost of $2\beta^{-1}$. The cost of production therefore goes to 0 as $\beta \to \infty$. This enables producers achieving *positive profit* at equilibrium (see Corollary 8) as we describe in more detail in Section 1.3.

We next vary the number of producers P while fixing $\beta = 2$ (see Figure 2).

Proposition 3. *Suppose that there are 2 users located at the standard basis vectors $e_1, e_2 \in \mathbb{R}^2$, with cost function $c(p) = \|p\|_2^\beta$ and $\beta = 2$. Then, there is an equilibrium μ with support equal to*

$$\left\{\left(x, (1 - x^{\frac{2}{P-1}})^{\frac{P-1}{2}}\right) \mid x \in [0, 1]\right\}, \tag{6}$$

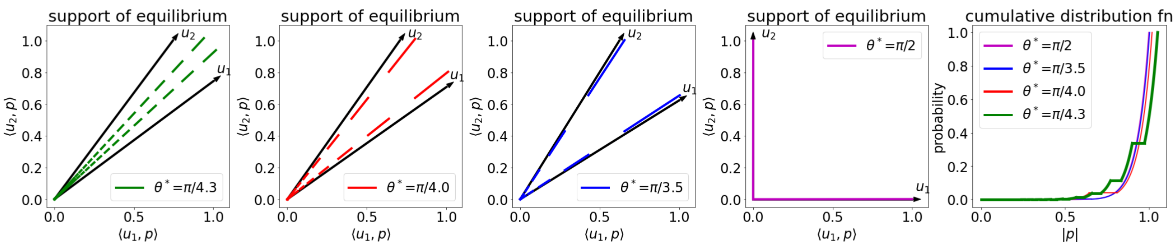

Figure 3: A symmetric equilibrium for different settings of θ^*, for 2 users located at u_1 and u_2 such that $\theta^* = \cos^{-1}\left(\frac{\langle u_1, u_2\rangle}{\|u_1\|\|u_2\|}\right)$, for producer cost function $c(p) = \|p\|_2^\beta$ with $\beta = 7$, and for $P = \infty$ producers (see Theorem 3). The first 4 plots show the support of the equilibrium in a reparameterized space: note that the the x-axis is $\langle u_1, p\rangle$ and the y-axis is $\langle u_2, p\rangle$. The last plot shows the cumulative distribution function of the conditional quality distribution (i.e. the distribution of the maximum quality level along a genre). The support of this distribution has gaps and consists of countably infinite disjoint intervals.

and where the distribution of x has cdf equal to $\min(1, x^{2/(P-1)})$.

For different values of P, the support of the equilibrium μ follows different curves connecting $[1, 0]$ and $[0, 1]$. The curve is concave for $P = 2$, a line segment for $P = 3$, and convex for all $P \geq 4$. Indeed, as P increases, the support converges towards the union of the two coordinate axes.

Infinite-producer limit. Motivated by this final observation, we investigate equilibria in a "limiting marketplace" where $P \to \infty$. In this case, a *two-genre* equilibrium exists, regardless of the geometry of the 2 users, and we characterize the equilibrium distribution μ (see Figure 3). Studying the limit as $P \to \infty$ is subtle: the distribution of any given producer approaches a point mass at 0 as $P \to \infty$, but the *winners' distributions* $\arg\max_{1 \leq j \leq P} \langle u_i, p_j\rangle$ approach a non-degenerate limiting measure as $P \to \infty$. See Definition 1 in Section 4 for a formal treatment of the infinite producer marketplace.

We characterize the infinite-producer equilibrium in terms of two properties—the genres and the conditional quality distributions (i.e., the distribution of $\|p\|$) given the genres:

Theorem 3. *[Informal version of Theorem 5] Suppose that there are 2 users located at two linearly independently vectors $u_1, u_2 \in \mathbb{R}^D_{\geq 0}$, let $\theta^* := \cos^{-1}\left(\frac{\langle u_1, u_2\rangle}{\|u_1\|_2\|u_2\|}\right)$ be the angle between them. Suppose we have cost function $c(p) = \|p\|_2^\beta$, $\beta > \beta^* = \frac{2}{1-\cos(\theta^*)}$, and $P = \infty$ producers. Then, there exists an equilibrium with two genres:*

$$\left\{[\cos(\theta^G + \theta_{min}), \sin(\theta^G + \theta_{min})], [\cos(\theta^* - \theta^G + \theta_{min}), \sin(\theta^* - \theta^G + \theta_{min})]\right\}$$

where $\theta^G := \arg\max_{\theta \leq \theta^*/2}\left(\cos^\beta(\theta) + \cos^\beta(\theta^* - \theta)\right)$ and $\theta_{min} := \min\left(\cos^{-1}\left(\frac{\langle u_1, e_1\rangle}{\|u_1\|}\right), \cos^{-1}\left(\frac{\langle u_2, e_1\rangle}{\|u_2\|}\right)\right)$.

For each genre, the norm $\|p\|$ of the winning producer p has cdf given by a countably-infinite piecewise function, where each piece is either constant or grows proportionally to $\|p\|^{2\beta}$.

Specific examples of equilibria for $P = \infty$ are given in Figure 3. Theorem 3 reveals that while finite genre equilibria do not exist for any finite P, they do exist in the infinite-producer limit. The resulting equilibrium distribution has complex structure–for instance, the support consists of countably infinite disjoint line segments.

Proof techniques for analyzing the equilibrium distribution. To prove our results in this section, our first step is establish a useful characterization of equilibria that enables us to separately account for the geometry of the users and the number of producers. This takes the form of necessary and sufficient conditions that decouple in terms of two quantities: a set of *marginal distributions* H_i, and the *support* $S \subseteq \mathbb{R}^N_{\geq 0}$.

Lemma 1. *Let $\mathbf{U} = [u_1; u_2; \ldots; u_N]$ be the $N \times D$ matrix of users vectors. Given a set $S \subseteq \mathbb{R}^N_{\geq 0}$ and distributions $H_1, \ldots, H_N$ over $\mathbb{R}_{\geq 0}$, suppose that the following conditions hold:*

(C1) Every $z^* \in S$ is a maximizer of the equation:

$$\max_{z \in \mathbb{R}^D_{\geq 0}} \sum_{i=1}^{N} H_i(z_i) - c_{\mathbf{U}}(z), \tag{7}$$

where $c_{\mathbf{U}}(z) := \min \left\{ c(p) \mid p \in \mathbb{R}^D_{\geq 0}, \mathbf{U}p = z \right\}$.

(C2) There exists a random variable Z with support S, such that the marginal distribution Z_i has cdf equal to $H_i(z)^{1/(P-1)}$.

(C3) Z is distributed as $\mathbf{U}Y$ with $Y \sim \mu$, for some distribution μ over $\mathbb{R}^D_{\geq 0}$.

Then, the distribution μ from (C3) is a symmetric mixed Nash equilibrium. Moreover, every symmetric mixed Nash equilibrium μ is associated with some $(H_1, \ldots, H_N, S)$ that satisfy (C1)-(C3).

In Lemma 1, the set S captures the support of the realized user utilities $[\langle u_1, p\rangle, \ldots, \langle u_N, p\rangle]$ for $p \sim \mu$. The distribution H_i captures the distribution of the maximum user utility $\max_{1 \leq j \leq P-1} \langle u_i, p_j\rangle$ for user u_i.

The conditions in Lemma 1 help us identify and analyze the equilibria in concrete instantations. Let us describe how we leverage these conditions in the case of two users (the same techniques generalize to 2 equally sized populations of users).

- Condition (C1) places conditions on H_1, H_2, and S in terms of the induced cost function $c_{\mathbf{U}}$. We use the first-order and second-order conditions of equation (7) at $z = [z_1, z_2]$ to determine the necessary densities $h_1(z_1)$ and $h_2(z_2)$ of H_1 and H_2 for z to be in the support S.
- Condition (C2) restricts the relationship between H_1, H_2, and S for a given value of P, which we instantiate in two different ways, depending on whether the support is a single curve or whether the distribution μ has finitely many genres.
- Condition (C3) holds essentially without loss of generality when u_1 and u_2 are linearly independent.

The proofs of our results in this section boil down to leveraging these conditions.

1.3 Economic implications: positive producer profit at equilibrium

Having studied the phenomenon of specialization, we next study its economic consequences on the resulting marketplace of digital goods. We show a surprising result: that producers can achieve *positive profit at equilibrium*, even though producers are competing with each other. In particular, we show sufficient conditions under which all equilibria have strictly positive profit.

Proposition 4. *Suppose that*

$$\max_{\|p\| \leq 1} \min_{i=1}^{N} \left\langle p, \frac{u_i}{||u_i||} \right\rangle < N^{-P/\beta}. \tag{8}$$

Then for any symmetric equilibrium μ, the profit $\mathcal{P}^{eq}(\mu)$ is strictly positive.

Proposition 4 provides insight into how the geometry of the users and structure of producer costs impact whether producers can achieve positive profit. To interpret Proposition 4, let us examine the quantity $Q := \max_{\|p\| \le 1} \min_{i=1}^{N} \langle p, \frac{u_i}{\|u_i\|} \rangle$ that appears on the left-hand side of (8). Intuitively, Q captures how easy it is to produce content that appeals simultaneously to all users. It is larger when the users are close together and smaller when they are spread out. For any set of vectors we see that $Q \le 1$, with strict inequality if the set of vectors is non-degenerate. The right-hand side of (8), on the other hand, goes to 1 as $\beta \to \infty$. Thus, for any non-degenerate set of users, if β is sufficiently large, the condition in Proposition 4 is met and producer profit is strictly positive. The value of β at which positive profit is guaranteed by Proposition 4 decreases as the user vectors become more spread out.

Connection to specialization. Although Theorem 4 does not explicitly consider specialization, we show that specialization is nonetheless central to achieving positive profit at equilibrium. To show this, we draw a distinction between profit in the single-genre regime (where there is no specialization) and the multi-genre regime (where this is specialization). This parallels the classical distinction in economics between markets with homogeneous goods and markets with differentiated goods.

Let's first examine the single-genre regime: here, producers do not earn profit at equilibrium.

Proposition 5. *If μ is a single-genre equilibrium, then the profit $\mathcal{P}^{eq}(\mu)$ is equal to* 0.

The single-genre regime bears resemblance to marketplaces with homogeneous goods where firms compete based on price (see [BK08] for a textbook treatment). In these marketplaces, if a firm sets their price above above the zero profit level, they can be undercut by other firms and lose their users. The possibility of undercutting intuitively drives the profit of the firm down to zero at equilibrium, as long as there are at least 2 firms. Similarly, in the marketplace that we study, when there is no specialization, producers all compete along the same direction—this drives producer profit to zero.

In contrast, multi-genre equilibria can exhibit positive equilibrium profit (Proposition 4), which bears resemblance to a marketplace with differentiated goods (see [AdPT92] for a textbook treatment). Product differentiation in these markets intuitively reduces competition between firms, since firms compete for different users. This leads to local monopolies where firms can set prices above the zero profit level. Similarly, in the marketplace that we study, specialization by producers leads to product differentiation and thus induces monopoly-like behavior where the profit is positive.

Thus, our results formalize how recommendation systems interpolate between marketplaces between homogeneous goods and marketplaces with differentiated goods, depending on the content production cost structure, preferences of users, and number of producers. An empirical analysis could quantify where on this spectrum a given recommender system is located, and regulatory policy could seek to shift a recommender system towards one of the regimes.

When is the marketplace saturated? At a conceptual level, the equilibrium profit of a marketplace provides insight about market competitiveness. Zero profit suggests that competition has driven producers to expend their full cost budget on improving product quality. Positive profit, on the other hand, suggests that the market is not yet fully saturated and new producers have incentive to enter the marketplace. Viewed in this context, Theorem 4 shows *the marketplace of digital goods may need far more than 2 producers in order to be saturated.*

While a marketplace with 2 producers may not be fully competitive, the equilibrium profit does nonetheless approach 0 in the limit as the number of producers in the marketplace goes to ∞. To see this, note that the equilibrium profit is always bounded by the ratio N/P of the number of users to the number of producers: this is because the cumulative profit of all producers is at most N and all producers achieve the same profit at a symmetric equilibrium. Perfect competition is therefore recovered in the limit as $P \to \infty$.

1.4 Discussion and Open Questions

Above, we presented a model for supply-side competition in recommender systems. The structure of production costs and the heterogeneity of users enable us to capture marketplaces that exhibit a wide range of levels of specialization. Despite this richness, our model is also mathematically tractable enough to fully characterize when specialization occurs in a given marketplace and to uncover other structural properties about the nature of specialization. More broadly, we hope that our work serves as a starting point to mathematically investigate the structure of supply-side equilibria. We propose several directions for future work that may benefit from leveraging tools from the theoretical computer science and probability theory literatures.

One direction for future work is to further examine the economic consequences of specialization. Several of our results take a step towards this goal: Corollary 2 illustrates that single-genre equilibria occur at the direction that maximizes the Nash user welfare, and Propositions 4 shows that specialization can lead to positive producer profit. These results leave open the question of how the welfare of users and producers relate to one another.

Open Question 1. *Quantify the aggregate utility that users derive at equilibrium. How does it vary between the single-genre and multi-genre regimes? What is the relationship between user utility and producer profit?*

Intuitively, specialization has two opposing forces: it may reduce user welfare, since positive profit implies that that the quality of content is lower than the zero profit level, but it also enables better personalization for users. Characterizing user utility at equilibrium would elucidate whether specialization ends up helping producers at the expense of users, or whether specialization helps all market participants.

Another direction for future work is to further characterize the equilibrium structure. Our analysis in Section 1.2 (e.g. Theorem 2) provides insight into the equilibrium structure in the case of two homogeneous users: we showed that finite genre equilibria do not exist outside of the single-genre regime (Theorem 2), and we provided closed-form expression for the equilibria in special cases (Propositions 2-3, and Theorem 3). It would be interesting to examine whether these insights extend to general configurations of users.

Open Question 2. *Characterize the equilibrium structure for general geometries of users, beyond the two homogeneous populations case. Is there always an equilibrium with finitely many genres when $P = \infty$? Are there finite-genre equilibria (beyond single-genre equilibria) when $P < \infty$?*

Characterizing multi-genre equilibria would provide insight into the nature of specialization in supply-side markets.

Another natural direction is investigating when the equilibria can be computed, either exactly or approximately. The characterization of single-genre equilibria in Corollary 2 reveals that they can be computed efficiently. However, it is not clear when multi-genre equilibria can be efficiently computed—for instance, even the special case of two homogeneous populations does not admit an obvious closed form, and it is not clear in general how to compute the set of genres even when they are finite.

Open Question 3. *Under what conditions on the user geometry and cost functions can be the equilibria be computed efficiently? In cases where there are finitely many genres, is there a polynomial-time algorithm for computing them?*

Apart from computing equilibria, it would be interesting to provide insight into when and how quickly a marketplace converges to an equilibrium. Although we assumed that the platforms and producers know the marketplace specifics, in reality, they might be learning about the user vectors or each others' actions through repeated interactions. It would be interesting to theoretically analyze these learning dynamics, for instance in cases where learners employ no-regret algorithms.

Open Question 4. *Suppose that the platforms and producers do not have full knowledge of the marketplace and instead perform no-regret learning. When do such algorithms converge, and what is the resulting steady-state?*

Finally, our model focuses entirely on *noiseless* consumption patterns, where the platform recommended each user their favorite content and users exactly consumed the content recommended by the platform. In practice, the platform has incomplete data about users and content, and users may not follow platform recommendations.

Open Question 5. *Investigate how noise in recommendations and user consumption affect the supply-side equilibria.*

Addressing these questions would further elucidate the market effects induced by supply-side competition, and inform our understanding of the societal effects of recommender systems.

1.5 Related Work

Our work is related to research on *societal effects in recommender systems*, *models of competition in economics*, and *strategic effects induced by algorithmic decisions*.

Supply-side effects of recommender systems. Ben-Porat and Tennenholtz [BT18] also studied supply-side effects from a theoretical perspective, with a focus on mitigating strategic effects by content producers. Ben-Porat, Rosenberg, and Tennenholtz [BRT20], building on Basat, Tennenholtz, and Kurland [BTK17], also studied supply-side equilibria with a focus on convergence of learning dynamics for producers. The main difference from our work is that producers in these models choose a topic from a *finite* set of options; in contrast, our model captures the infinite, multi-dimensional producer decision space that drives the emergence of genres. Moreover, we focus on the structure of equilibria rather than the convergence of learning.

In concurrent and independent work, Hron et al. [HKJ+22] study a related model for supply-side competition in recommender systems where producers choose content embeddings in $\mathbb{R}^D$. One main difference is that, rather than having a cost on producer content, they constrain producer vectors to the ℓ_2 unit ball (this corresponds to our model when $\beta \to \infty$ and the norm is the ℓ_2-norm, although the limit behaves differently than finite β); additionally, Hron et al. incorporate a softmax decision rule to capture exploration and user non-determinism, whereas we focus entirely on hardmax recommendations. Thus, our model focuses on the role of producer costs while Hron et al.'s focuses on the role of the recommender environment. At a technical level, Hron et al. study the existence of different types of equilibria and the use of behaviour models for auditing, whereas we analyze the economic phenomena exhibited by symmetric mixed strategy Nash equilibria, with a focus on genre formation and equilibrium producer profit.

Other work has studied the emergence of filter bubbles [FGR16], the ability of users to reach different content [DRR20], the shaping of user preferences [ABC+13], and stereotyping [GKJ+21].

Models of competition in microeconomics. There has been a vast literature on different models for competition in microeconomic theory. For example, this includes *price competition*, where producers set a *price*, but do not decide what good to produce (e.g. Bertrand competition, see [BK08] for a textbook treatment). In contrast with these models, the producer's decision in our model is multi-dimensional and cannot be captured by a 1-dimensional quantity such as price.

Another line of work on *product selection* has investigated how producers choose content at equilibrium (see [AdPT92] for a textbook treatment); however, these models do not seem to capture the rich structure of producer action space, production costs, and user geometry that we focus on in this work. For example, spatial location models [Hot81; Ber79] classically focus on 1 dimension (either $\mathbb{R}^1$ or $\mathbb{S}^1$). This is similar to producers choosing the genre in our model but not the quality level of content. Since these models additionally allow producers to set a price, it may be tempting to draw an analogy between the quality level $\|p\|$ in our model and price in these models. However, this analogy breaks down because the production cost can be highly nonlinear in the content p. The nonlinear structure of our cost function was necessary to capture tradeoffs between excelling on different dimensions.

Other related work has investigated *supply function equilibria* (e.g. [Gro81]), where the producer chooses a function from quantity to prices, rather than what content to produce, and the *pure characteristics model* (e.g. [Ber94]), where attributes of users and producers are also embedded in $\mathbb{R}^D$ like in our model, but which focuses on demand estimation for a fixed set of content, rather than analyzing the content that arises at equilibrium in the marketplace.

Strategic effects induced by algorithmic decisions. Brückner, Kanzow, and Scheffer [BKS12] and Hardt et al. [HMP+16] initiated the study of *strategic classification*, where algorithmic decisions induce participants to strategically change their features to improve their outcomes. The models for participant behavior in this line of work (e.g. Kleinberg and Raghavan [KR19], Jagadeesan, Mendler-Dünner, and Hardt [JMH21], and Ghalme et al. [GNE+21]) generally do not capture competition between participants. One exception is Liu, Garg, and Borgs [LGB22], where participants compete to appear higher in a single ranked list. In contrast, the participants in our model simultaneously compete for users with *heterogeneous* preferences.

2 Details of Model and Preliminaries

Consider a platform with $N \geq 1$ users and $P \geq 2$ producers. Each user i is associated with a D-dimensional embedding $u_i \in \mathbb{R}^D_{\geq 0} \setminus \left\{\vec{0}\right\}$ that captures their preferences. We assume the coordinates of each embedding are nonnegative and that each embedding is nonzero.

While user vectors are fixed, producers *choose* what content to create. Each producer j creates a single digital good, which is associated with a content vector $p_j \in \mathbb{R}^D_{\geq 0}$. The value of good p to user u is $\langle u, p \rangle$.

Personalized recommendations. After the producers create content, the platform offers personalized recommendations to each user. We consider a stylized model where the platform has complete knowledge of the user and content vectors. The platform recommends to each user the content of maximal value to them, assigning them to the producer who created this content. Mathematically, the platform assigns a user u to the producer j^*, where $j^*(u;\ p_{1:P}) = \arg\max_{1 \leq j \leq P} \langle u, p_j \rangle$. If there are ties, the platform sets $j^*(u;\ p_{1:P})$ to be a randomly chosen producer in the argmax.

Producer cost function. Each producer faces a *fixed cost* for producing content p, which depends on the magnitude of p. Since the good is digital and thus cheap to replicate, the production cost does not scale with the number of users.

The cost function $c(p)$ takes the form $\|p\|^\beta$, where $\|\cdot\|$ is any norm and the exponent β is at least 1. The magnitude $\|p\|$ captures the *quality* of the content: in particular, if a producer chooses content λp, they win at least as many users as if they choose $\lambda' p$ for $\lambda' < \lambda$. (This relies on the fact that all vectors are in the positive orthant.) The norm and β together encode the cost of producing a content vector v, and reflect cost tradeoffs for excelling in different dimensions (for example, producing a movie that is both a drama and a comedy). Large β, for instance, means that this cost grows superlinearly. In Section 3, we will see that these tradeoffs capture the extent to which producers are incentivized to specialize.

Producer profit. A producer receives profit equal to the number of users who are recommended their content minus the cost of producing the content. For producer j, let $p_{-j} = [p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_P]$ denote the content produced by all of the other producers. The profit of producer j is equal to:

$$\mathcal{P}(p_j; p_{-j}) = \mathbb{E}\Big[\Big(\sum_{i=1}^{n} 1[j^*(u_i;\ p_{1:P}) = j]\Big) - \|p_j\|^\beta\Big], \tag{9}$$

where the expectation comes from the randomness over platform recommendations in the case of ties.

2.1 Equilibrium concept and existence of equilibrium

We study the equilibria of the game between producers. Each producer j chooses a (random) strategy over content, given by a probability measure μ_j over the content embedding space $\mathbb{R}^D_{\geq 0}$. The strategies $(\mu_1, \ldots, \mu_P)$ form a *Nash equilibrium* if no producer—given the strategies of other producers—can chose a different strategy where they achieve higher expected profit. We call $\mu_{1:P}$ a *pure strategy equilibrium* if each μ_j contains only one vector in its support; otherwise, we call it a *mixed strategy equilibrium.*

A salient feature of this game is that there are discontinuities in the producer utility function in equation (9). In particular, whether a producer wins a given user is not continuous in the content embedding p_j, since the winner is chosen by a hard $\arg\max$ rule. As a result, no pure strategy equilibrium exists:

Proposition 6. *For any set of users and any $\beta \geq 1$, a pure strategy equilibrium does not exist.*

The proof of Proposition 6 in Appendix A.1 leverages that if two producers are tied, then a producer can increase their utility by infinitesimally increasing the magnitude of their content.

Although pure strategy equilibria are not guaranteed to exist, a *symmetric mixed equilibrium* (where $\mu_1 = \ldots = \mu_P$) necessarily exists. Since all producers play the same strategy at a symmetric equilibrium, we represent it with a single distribution μ over the content embedding space $\mathbb{R}^D_{\geq 0}$.

Proposition 7. *A symmetric mixed equilibrium μ exists.*

To prove Proposition 7, we leverage the technology of mixed equilibria in discontinuous games [Ren99]. The existence of a *symmetric* equilibrium follows from the symmetries of producer utility function. We defer the full proof to Appendix A.2.

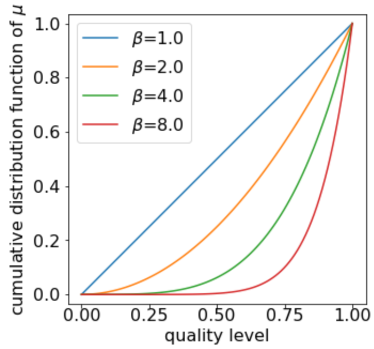

Figure 4: Cumulative distribution function (cdf) of the symmetric equilibrium μ for 1-dimensional setup (Example 1) with $P = 2$ producers. The equilibrium μ interpolates from a uniform distribution to a point mass as the exponent β increases.

Every symmetric equilibrium μ necessarily exhibits randomization across different content embeddings, since pure strategy equilibria do not exist. We in fact show that μ cannot contain point masses:

Proposition 8. *If μ is a symmetric mixed equilibrium, then μ is atomless.*

Proposition 8 implies that μ has *infinite support*. The randomness in μ can come from randomness over *magnitudes* $\|p\|$ as well as randomness over *directions* $\frac{p}{\|p\|}$.

2.2 Warmup: Homogeneous Users

To gain intuition for the structure of μ, let's focus on a simple one-dimensional setting with one user. We show that the equilibria take the following form (see Figure 2.2):

Example 1 (1-dimensional setup)**.** *Let $D = 1$, and suppose that there is a single user $u_1 = 1$. Suppose the cost function is $c(p) = |p|^\beta$. The unique symmetric mixed equilibrium μ is supported on the full interval $[0, 1]$ and has cumulative distribution function $F(p) = \left(\frac{p}{N}\right)^{\beta/(P-1)}$. We defer the derivation to Appendix A.4.*

Since $D = 1$ in Example 1, content is specified by a single value $p \in \mathbb{R}^{\geq 0}$. Since the user will be assigned to the content with the highest value of p, we can interpret p as the *quality level* of the content. For a producer, setting p to be larger increases the likelihood of being assigned to users, at the expense of a greater cost of production.

The equilibrium changes substantially with the parameters β and P. First, for any fixed P, the equilibrium distribution for higher values of β stochastically dominates the equilibrium distribution for lower values of β (see Figure 2.2). The intuition is that higher levels of β lower production costs for realized quality levels, which means that producers must produce higher quality content at equilibrium. Similarly, for any fixed value of β, Example 1 demonstrates that the equilibrium distribution for lower values of P stochastically dominates the equilibrium distribution for higher values of P. This is because when more producers enter the marketplace, any given producer is less likely to win users (i.e. a producer only wins a user with probability $1/P$ if all producers choose the same vector), so they cannot expend as high of a production cost.

We next translate these insights about the equilibria for one-dimensional marketplaces to higher-dimensional marketplaces with a population of *homogeneous* users. If all users are embedded at the same vector $u \in \mathbb{R}^D_{\geq 0}$, then the producer's decision about what direction of content to choose is trivial: they would choose a direction in $\arg\max_{\|p\|=1} \langle p, u \rangle$. As a result, the producer's decision again boils down to a one-dimensional decision: choosing the quality level $\|p\|$ of the content.

Corollary 3. *Suppose that there is a single population of N users, all of whose embeddings are at the same vector u. Then, there is a symmetric mixed Nash equilibrium μ supported on the one-dimensional set $\left\{ qp^* \mid q \in [0, N^{\frac{1}{\beta}}] \right\}$ where $p^* \in \arg\max_{\|p\|=1} \langle p, u \rangle$. The cumulative distribution function of $q = \|p\| \sim \mu$ is $F(q) = \left(\frac{q}{N}\right)^{\beta/(P-1)}$.*

Corollary 3 relies on the fact that when users are homogeneous, there is no tension between catering to one user and catering to other users.

2.3 Formation of genres

In contrast, when users are heterogeneous, there are inherent tensions between catering to one user and catering to other users. As a result, the producer make nontrivial choices not only about the quality level of the content (Section 2.2), but also the *genre* of content as reflected by its direction in $\mathbb{R}^d$. A producer might exhibit *specialization* by creating a digital good that caters to a single user or a subgroup of users. Alternatively, all producers might still produce the same genre of content at equilibrium and thus only exhibit differentiation on the axis of quality.

To formalize specialization, we examine the set of content that appears in the support of the equilibrium distribution. Recall that for a symmetric mixed equilibrium μ, we define *the set of genres of* μ as:

$$\text{Genre}(\mu) := \left\{ \frac{p}{\|p\|} \mid p \in \text{supp}(\mu) \right\}, \tag{10}$$

which corresponds to the set of directions that appear in the support of μ. We normalize by $\|p\|$ to separate out the quality level (norm) of the digital good from the "type" (direction) of the good.

3 When do all equilibria have multiple genres?

We examine the question: *when does specialization occur in a marketplace?* In Section 3.1, we give necessary and sufficient conditions for all equilibria to have multiple genres (Theorem 1) and we show several corollaries of this result. We prove Theorem 1 in Section 3.2, and we prove the corollaries of Theorem 1 in Section 3.3. At a conceptual level, our results provide insight into whether the producers create specialized content at equilibrium, or create generic content that caters to the average user in the marketplace.

3.1 Boundary between single-genre and multi-genre regimes

Let $\mathbf{U} = [u_1; \cdots ; u_N]$ be the $N \times D$ matrix of user vectors, and let $\mathcal{S}$ denote the image of the unit ball under $\mathbf{U}$:

$$\mathcal{S} := \left\{ \mathbf{U}p \mid \|p\| \leq 1, p \in \mathbb{R}^D_{\geq 0} \right\} \tag{11}$$

Each element of $\mathcal{S}$ is an N-dimensional vector, which represents the user utilities for some unit-norm producer p. Additionally, let $\mathcal{S}^\beta$ be the image of $\mathcal{S}$ under taking coordinate-wise powers, i.e. if $(z_1, \ldots, z_N) \in \mathcal{S}$ then $(z_1^\beta, \ldots, z_N^\beta) \in \mathcal{S}^\beta$. Theorem 1, restated below, shows that the condition for multiple genres to emerge is that $\mathcal{S}^\beta$ is sufficiently different from its convex hull $\bar{\mathcal{S}}^\beta$.

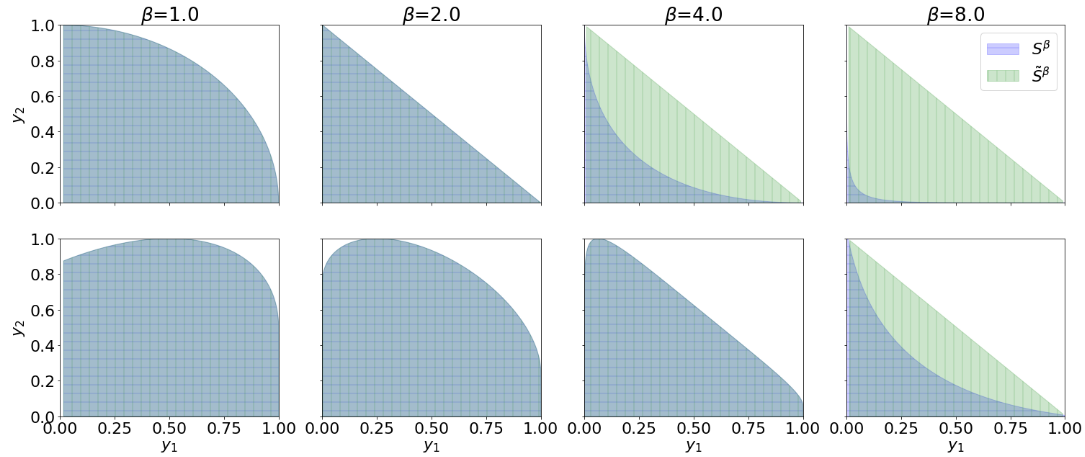

Figure 5: The sets $\mathcal{S}^\beta$ and $\bar{\mathcal{S}}^\beta$ for two configurations of user vectors (rows) and settings of β (columns). The user vectors are $[1,0],[0,1]$ (top, same as Figure 1) and $[1,0],[0.5,0.87]$ (bottom). $\mathcal{S}^\beta$ transitions from convex to non-convex as β increases, though the transition point depends on the user vectors.

Theorem 1. *Let* $\mathbf{U} := [u_1; \cdots; u_N]$ *be the* $N \times D$ *matrix of user vectors, let* $\mathcal{S}$ *be* $\left\{\mathbf{U}p \mid \|p\| \leq 1, p \in \mathbb{R}^D_{\geq 0}\right\}$, *and let* $\mathcal{S}^\beta$ *be the image of* $\mathcal{S}$ *under taking coordinate-wise powers (i.e. if* $(z_1, \ldots, z_N) \in \mathcal{S}$ *then* $(z_1^\beta, \ldots, z_N^\beta) \in \mathcal{S}^\beta$*). Then, there is a symmetric equilibrium* μ *with* $|\mathrm{Genre}(\mu)| = 1$ *if and only if*

$$\max_{y \in \mathcal{S}^\beta} \prod_{i=1}^N y_i = \max_{y \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^N y_i. \tag{2}$$

Otherwise, all symmetric equilibria have multiple genres. Moreover, if (2) *holds for some* β*, it also holds for every* $\beta' \leq \beta$.

As a special case, the condition in Theorem 1 always holds if $\mathcal{S}^\beta$ is convex. In Figure 5, we display the sets $\mathcal{S}^\beta$ and $\bar{\mathcal{S}}^\beta$ for different configurations of user vectors and different settings of β. Observe that $\mathcal{S}^\beta$ depends on both the geometry of the user embeddings and the structure of the cost function (though interestingly *not* on the number of producers). Conceptually, once β is sufficiently large, the cost function incentivizes a sufficient degree of specialization that it is no longer in the best interest of producers to all compete along the same direction.

Interestingly, both terms in equation (2) correspond to convex optimization problems. To see this, first consider the right-hand side, which can be reformulated as:

$$\max_{y \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^N y_i = \exp\left(\max_{y \in \bar{\mathcal{S}}^\beta} \sum_{i=1}^N \log(y_i)\right).$$

Since $\max_{y \in \bar{\mathcal{S}}^\beta} \sum_{i=1}^N \log(y_i)$ is the maximum of a concave function on a convex set. Now, consider the left-hand side:

$$\max_{y \in \mathcal{S}^\beta} \prod_{i=1}^N y_i = \exp\left(\beta \cdot \max_{y \in \mathcal{S}} \sum_{i=1}^N \log(y_i)\right).$$

Since $\max_{y \in \mathcal{S}} \sum_{i=1}^N \log(y_i)$ is the maximum of a concave function over a convex set.

To further understand the condition in equation (2), we consider a series of special cases that provide intuition for when single-genre equilibria exist. First, let us consider $\beta = 1$, in which case the cost function is a norm. Then $\mathcal{S}^1 = \mathcal{S}$ is convex, so a single-genre equilibrium always exists.

Corollary 4. *The threshold β^* is always at least* 1*. That is, if $\beta = 1$, then there always exists a single-genre equilibrium.*

The economic intuition behind Corollary 4 is that norms incentivize averaging rather than specialization.

We next take a closer look at how the choice of norm affects the emergence of genres. Within the family of ℓ_q norms, the threshold β^* is equal to q.

Corollary 5. *Let the cost function be $c(p) = \|p\|_q^\beta$. For any set of user vectors, it holds that $\beta^* \geq q$. If the user vectors are equal to the standard basis vectors $\{e_1, \ldots, e_D\}$, then β^* is equal to q.*

Corollary 5 implies the cost function must be sufficiently nonconvex for all equilibria to be multi-genre. For example, for the ℓ_∞-norm, where producers only pay for the highest magnitude coordinate, it is never possible to incentivize specialization: there exists a single-genre equilibrium regardless of β. On the other hand, for norms where costs aggregate nontrivially across dimensions, specialization is possible.

In addition to the choice of norm, the geometry of the user vectors also influences whether multiple genres emerge. Corollary 6 (deferred to Section 4) shows that $\beta^* = \frac{2}{1-\cos(\theta^*)}$ in the case of 2 users with cosine similarity $\cos(\theta)$ and cost $c(p) = \|p\|_2^\beta$, the threshold $\beta^* = \frac{2}{1-\cos(\theta^*)}$. The threshold β^* increases as the angle θ^* between the users decreases, interpolating from 2 when the users are orthogonal to ∞ when the users point in the same direction.

We also consider general user geometries: Corollary 1, restated below, shows a natural sufficient condition under which all equilibria have multiple genres.

Corollary 1. *Let $\|\cdot\|_*$ denote the dual norm of $\|\cdot\|$, defined to be $\|p\|_* = \max_{\|p\|=1, p \in \mathbb{R}^D_{\geq 0}} \langle q, p \rangle$. Let $Z := \|\sum_{n=1}^N \frac{u_n}{\|u_n\|_*}\|_*$. Then,*

$$\beta^* \leq \frac{\log(N)}{\log(N) - \log(Z)}. \tag{3}$$

In equation (3), the threshold for β increases as Z increases. As an example, consider the cost function $c(p) = \|p\|_2^\beta$. We see that if the user vectors point in the same direction, then $Z = N$ and the right-hand side of (3) is ∞. On the other hand, if $u_1, \ldots, u_n$ are orthogonal, then $Z = \sqrt{N}$ and the right-side of (3) is 2, which exactly matches the bound in Corollary 5. In fact, for *random* vectors $u_1, \ldots, u_N$, we see that $Z = \tilde{O}(\sqrt{N})$ in expectation, in which case the right-hand side of (3) is close to 2 as long as N is large. Thus, for many choices of user vectors, even small values of β are enough to induce multiple genres.

Finally, we investigate *where* the single-genre equilibrium occurs, in the case where it does exist. Corollary 2, restated below, follows as a consequence of the proof of Theorem 1.

Corollary 2. *If there exists μ with $|\text{Genre}(\mu)| = 1$, then the corresponding producer direction maximizes Nash social welfare of the users:*

$$\text{Genre}(\mu) = \underset{\|p\|=1 | p \in \mathbb{R}^D_{\geq 0}}{\arg\max} \sum_{i=1}^N \log(\langle p, u_i \rangle). \tag{4}$$

Corollary 2 demonstrates that the single-genre equilibrium directions maximizes the Nash social welfare [Nas50] for users. Interestingly, this measure of welfare for *users* is implicitly maximized by *producers* competing with each other in the marketplace. Properties of the Nash social welfare are thus inherited by single-genre equilibria. In particular, since the Nash social welfare corresponds to the logarithm of the geometric mean of the user values, the Nash social welfare strikes a compromise between fairness (balancing values of different users) and efficiency (the sum of the values achieved across all users)—this means that the single-genre equilibria exhibit the same tradeoff between fairness and efficiency.

We note that this welfare result relies on the assumption that all producers choose the same direction of content. In particular, at multi-genre equilibria, the Nash social welfare could be even higher due to specialization leading to personalization. On the other hand, the reduced amount of competition at multi-genre equilibria may end up lowering the quality of goods. We defer an in-depth analysis of the welfare implications of supply-side competition to future work (see Open Question 1).

3.2 Proof of Theorem 1

To prove Theorem 1, we relate the existence of a single equilibrium to strong duality of a certain optimization program. We then use convexification to establish necessary and sufficient conditions under which strong duality holds.

Suppose that there exists an equilibrium μ such that $\text{Genre}(\mu) = \{p^*\}$ contains a single direction. Then μ is fully determined by the distribution over quality level $\|p\|$ where $p \sim \mu$; therefore, let F denote the cdf of $\|p\|$ for $p \sim \mu$. We can derive a closed-form expression for F; in fact, we show that it is identical to the cdf of the 1-dimensional setup in Example 1.

Lemma 2. *Suppose that μ is a symmetric equilibrium such that* $\text{Genre}(\mu)$ *contains a single vector. Let F be the cdf of the distribution over $\|p\|$ where $p \sim \mu$. Then, it holds that:*

$$F(r) = \min\left(1, \left(\frac{r^\beta}{N}\right)^{1/(P-1)}\right). \tag{12}$$

The intuition for Lemma 2 is that a single-genre equilibrium essentially reduces the producer's decision to a 1-dimensional space, and so inherits the structure of the 1-dimensional equilibrium.

For μ to be an equilibrium, no alternative q should do better than $p \sim \mu$, which yields the following necessary and sufficient condition after plugging into the profit function (9):

$$\sup_q \left(\sum_{i=1}^N \frac{1}{N}\left(\frac{\langle q, u_i\rangle}{\langle p^*, u_i\rangle}\right)^\beta - \|q\|^\beta\right) = \mathbb{E}_{p'\sim\mu}\left[\sum_{i=1}^N \frac{1}{N}\left(\frac{\langle p', u_i\rangle}{\langle p^*, u_i\rangle}\right)^\beta - \|p'\|^\beta\right] \tag{13}$$

The term $\frac{1}{N}(\cdot)^\beta$ is the probability $(F(\cdot))^{P-1}$ that q outperforms the max of $P-1$ samples from μ.

We next change variables according to $y_i = \langle p^*, u_i\rangle^\beta$ and $y'_i = \langle \frac{q}{\|q\|}, u_i\rangle^\beta$ and simplify to see that μ is an equilibrium if and only if $\sup_{y'\in\mathcal{S}^\beta} \sum_{i=1}^n \frac{y'_i}{y_i} = N$. Thus, there exists a single-genre equilibrium if and only if

$$\inf_{y\in\mathcal{S}^\beta} \sup_{y'\in\mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{y_i} = N. \tag{14}$$

While the left-hand side of equation (14) is challenging to reason about directly, we show that the dual $\sup_{y'\in\mathcal{S}^\beta} \inf_{y\in\mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{y_i}$ is in fact equal to N. Therefore, the existence of a single-genre equilibrium boils down to a minimax theorem.

Lemma 3 (Informal)**.** *There exists a symmetric equilibrium μ with* $|\text{Genre}(\mu)| = 1$ *if and only if:*

$$\inf_{y\in\mathcal{S}^\beta}\left(\sup_{y'\in\mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{y_i}\right) = \sup_{y\in\mathcal{S}^\beta}\left(\inf_{y\in\mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{y_i}\right). \tag{15}$$

By Sion's min-max theorem, we can flip sup and inf in a convexified version of the left-hand side of (15). The remainder of the proof will boil down to relating the resulting expression to the right-hand side of equation (15).

To formalize the lemmas in the proof sketch, we will define a set $\mathcal{S}_{>0}$ which deletes all points with a zero coordinate from $\mathcal{S}$. More formally:

$$\mathcal{S}_{>0} := \left\{\mathbf{U}p \mid \|p\| \leq 1, p \in \mathbb{R}^D_{\geq 0}\right\} \cap \mathbb{R}^N_{>0}.$$

For notational convenience, we also define:

$$\mathcal{B} := \left\{p \in \mathbb{R}^D_{\geq 0} \mid \|p\| \leq 1\right\},$$

$$\mathcal{B}_{>0} := \left\{p \in \mathbb{R}^D_{\geq 0} \mid \|p\| \leq 1, \langle p, u_i \rangle > 0 \forall i\right\},$$

which are both convex sets. We further define:

$$\mathcal{D} := \left\{p \in \mathbb{R}^D_{\geq 0} \mid \|p\| = 1\right\}$$

and

$$\mathcal{D}_{>0} := \left\{p \in \mathbb{R}^D_{\geq 0} \mid \|p\| = 1, \langle p, u_i \rangle > 0 \forall i\right\}.$$

Note that it follows from definition that:

$$\mathcal{S} = \{\mathbf{U}p \mid p \in \mathcal{B}\}$$

$$\mathcal{S}_{>0} = \{\mathbf{U}p \mid p \in \mathcal{B}_{>0}\}$$

The proof will proceed by proving Lemma 2 and Lemma 3, and then proving Theorem 1 from these lemmas. In Section 3.2.1, we prove a useful auxiliary lemma about single-genre equilibria; in Section 3.2.2, we prove Lemma 2; in Appendix 3.2.3, we formalize and prove Lemma 3; and in Section 3.2.5, we prove Theorem 1.

3.2.1 Auxiliary lemma

We show that at a single-genre equilibrium, it must hold that the direction vector has nonzero inner product with every user.

Lemma 4. *Suppose that μ is a symmetric equilibrium such that* $\text{Genre}(\mu)$ *contains a single vector p^*. Then $p^* \in span(u_1, \ldots, u_N)$ (which also means that $\langle p^*, u_i \rangle > 0$ for all i.)*

Proof. Assume for sake of contradiction that $\langle p^*, u_i \rangle = 0$ for some i. Suppose that $p' \in \text{supp}(\mu)$, and consider the vector $p' + \epsilon \frac{u_i}{||u_i||}$. We see that $p' + \epsilon \frac{u_i}{||u_i||}$ wins user u_i with probability 1 whereas p' wins user u_i with probability $1/P$. The probability that $p + \epsilon u_i$ wins any other user is also at least the probability that p' wins u_i. By leveraging this discontinuity, we see there exists ϵ such that $\mathcal{P}(p' + \epsilon \frac{u_i}{||u_i||}; [\mu, \ldots, \mu]) > \mathcal{P}(p'; [\mu, \ldots, \mu]) + (1 - \frac{1}{P})$ which is a contradiction. □

3.2.2 Proof of Lemma 2

We restate and prove Lemma 2.

Lemma 2. *Suppose that μ is a symmetric equilibrium such that* $\text{Genre}(\mu)$ *contains a single vector. Let F be the cdf of the distribution over $\|p\|$ where $p \sim \mu$. Then, it holds that:*

$$F(r) = \min\left(1, \left(\frac{r^\beta}{N}\right)^{1/(P-1)}\right). \quad (12)$$

Proof. Next, we show that $F(r) = 0$ only if $r = 0$. Since the distribution μ is atomless (by Proposition 8), we can view the support as a closed set. Let $r_{\min}$ be the minimum magnitude element in the support of μ. Since μ is atomless, this means that with probability 1, every producer will have magnitude greater than $r_{\min}$. This, coupled with Lemma 4, means that the producer the expected number of users achieved at $r_{\min}p$ is 0, and $\mathcal{P}(r_{\min}p; [\mu, \ldots \mu]) = -r_{\min}^{\beta}$. However, since $r_{\min}p \in \text{supp}(\mu)$, it must hold that:

$$-r_{\min}^{\beta} = \mathcal{P}(r_{\min}p; [\mu, \ldots, \mu]) \geq \mathcal{P}(\vec{0}; [\mu, \ldots, \mu]) \geq 0.$$

This means that $r_{\min} = 0$.

Next, we show that the equilibrium profit at $(\mu, \ldots, \mu)$ is equal to 0. To see this, suppose that if the producer chooses $\vec{0}$. Since μ is atomless and since $\langle p^*, u_i \rangle > 0$ for all i (by Lemma 4), we see that if a producer chooses $\vec{0} \in \text{supp}(\mu)$, they receive 0 users in expectation. This means that $\mathcal{P}(\vec{0}; [\mu, \ldots, \mu]) = 0$ as desired.

Next, we show that $F(r) = \left(\frac{r^\beta}{N}\right)^{1/(P-1)}$ for any $rp^* \in \text{supp}(\mu)$. To show this, notice that the producer must earn the same profit—here, zero profit—for any $p \in \text{supp}(\mu)$. This means that for any $rp^* \in \text{supp}(\mu)$, it must hold that $NF(r)^{P-1} - r^\beta = 0$. Solving, we see that $F(r) = \left(\frac{r^\beta}{N}\right)^{1/(P-1)}$.

Finally, we show that the support of F is exactly $[0, N^{1/\beta}]$. First, we already showed that $r_{\min} = 0$ which means that 0 is the minimum magnitude element in the support. Moreover, $r = N^{1/\beta}$ must be the maximum magnitude element in the support since it is the unique value for which $F(r) = 1$. Now, assume for sake of contradiction that the support does not contain some interval $(x, x + \epsilon)$ within $[0, N^{1/\beta}]$. Given the structure of $F(r)$ above, this means that the cdf jumps from $F(x)$ to $F(x + \epsilon)$ anyway so there would be atoms (but there are no atoms by Proposition 8). This proves that the support is $[0, N^{1/\beta}]$.

In conclusion, we have shown that $F(r) = \left(\frac{r^\beta}{N}\right)^{1/(P-1)}$ for any $r \in [0, N^{1/\beta}]$. The min with 1 comes from the fact that $F(r) = 1$ for $r \geq N^{1/\beta}$. □

3.2.3 Formal Statement and Proof of Lemma 3

We formalize and prove Lemma 3, which illustrates the connection between the existence of a single-genre equilibrium and minimax duality.

Lemma 5 (Formalization of Lemma 3)**.** *There exists a symmetric equilibrium μ with $|\text{Genre}(\mu)| = 1$ if and only if:*

$$\inf_{p^* \in \mathcal{B}_{>0}} \sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} = \sup_{y' \in \mathcal{S}^\beta} \inf_{p^* \in \mathcal{B}_{>0}} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta}. \tag{16}$$

It turns out to be more convenient to use a (slightly less intuitive) variant of Lemma 5 to prove Theorem 1. We state and prove Lemma 6 below.

Lemma 6. *There exists a symmetric equilibrium μ with $|\text{Genre}(\mu)| = 1$ if and only if:*

$$\inf_{p^* \in \mathcal{B}_{>0}} \sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} \leq N. \tag{17}$$

The main ingredient in the proof of Lemma 6 is the following characterization of a single-genre equilibrium in a given direction.

Lemma 7. *There is a symmetric equilibrium μ with* $\text{Genre}(\mu) = \{p^*\}$ *if and only if:*

$$\sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} \leq N. \tag{18}$$

Proof. First, by Lemma 4, we see that the denominator is nonzero for every term in the sum, so equation (18) is well-defined.

If μ is a single-genre equilibrium, then the cdf of the magnitudes follows the form in Lemma 2. Thus, it suffices to identify necessary and sufficient conditions for that solution (that we call μ_{p^*}) to be a symmetric equilibrium.

The solution μ_{p^*} is an equilibrium if and only if no alternative q should do better than $p \sim \mu$. The profit level at μ_{p^*} is 0 by the structure of the cdf. Putting this all together, we see a necessary and sufficient for μ_{p^*} to be an equilibrium is:

$$\sup_{q \in \mathbb{R}^D_{\geq 0}} \left(\sum_{i=1}^{N} F\left(\frac{\langle q, u_i \rangle}{\langle p^*, u_i \rangle} \right)^{P-1} - \|q\|^\beta \right) \leq 0,$$

where the term $\frac{1}{N}(\cdot)^\beta$ is the probability $(F(\cdot))^{P-1}$ that q outperforms the max of $P-1$ samples from μ. Using the structure of the cdf, we can write this as:

$$\sup_{q \in \mathbb{R}^D_{\geq 0}} \left(\sum_{i=1}^{N} \min\left(1, \frac{1}{N} \left(\frac{\langle q, u_i \rangle}{\langle p^*, u_i \rangle} \right)^\beta \right) - \|q\|^\beta \right) \leq 0.$$

We can equivalently write this as:

$$\sup_{q \in \mathbb{R}^D_{\geq 0}} \left(\frac{1}{||q||^\beta} \sum_{i=1}^{N} \min\left(1, \frac{1}{N} \left(\frac{\langle q, u_i \rangle}{\langle p^*, u_i \rangle} \right)^\beta \right) - 1 \right) \leq 0,$$

which we can equivalently write as

$$\sup_{q \in \mathcal{D}} \sup_{r > 0} \left(\frac{1}{r^\beta} \sum_{i=1}^{N} \min\left(1, \frac{r^\beta}{N} \left(\frac{\langle q, u_i \rangle}{\langle p^*, u_i \rangle} \right)^\beta \right) - 1 \right) \leq 0.$$

For any direction q, if we disregard the first min with 1, the expression would be constant in r. With the minimum, the objective $\left(\frac{1}{r^\beta} \sum_{i=1}^{N} \min\left(1, \frac{1}{N} \left(\frac{\langle q, u_i \rangle}{\langle p^*, u_i \rangle} \right)^\beta \right) - 1 \right)$ is weakly decreasing in r. Thus, $\sup_{r>0} \left(\frac{1}{r^\beta} \sum_{i=1}^{N} \min\left(1, \frac{1}{N} \left(\frac{\langle q, u_i \rangle}{\langle p^*, u_i \rangle} \right)^\beta \right) - 1 \right)$ is attained as $r \to 0$. In fact, the maximum is attained at a value r if $r\langle q, u_i \rangle < N^{1/\beta} \langle p^*, u_i \rangle$ for all i. This holds for *some* $r > 0$ since $\langle p^*, u_i \rangle > 0$ for all i by Lemma 4. Thus we can equivalently formulate the condition as:

$$\sup_{q \in \mathcal{D}} \left(\left(\sum_{i=1}^{N} \frac{1}{N} \left(\frac{\langle q, u_i \rangle}{\langle p^*, u_i \rangle} \right)^\beta \right) - 1 \right) \leq 0,$$

which we can write as:

$$\sup_{q \in \mathcal{D}} \sum_{i=1}^{N} \left(\frac{\langle q, u_i \rangle}{(\langle p^*, u_i \rangle)} \right)^\beta \leq N.$$

This is equivalent to:

$$\sup_{q \in \mathcal{B}} \sum_{i=1}^{N} \left(\frac{\langle q, u_i \rangle}{(\langle p^*, u_i \rangle)} \right)^{\beta} \leq N.$$

A change of variables gives us the desired formulation. $\square$

Now, we can deduce Lemma 6.

Proof of Lemma 6. First, suppose that equation (17) does not hold. Then it must be true that:

$$\sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} > N$$

for every direction $p^* \in \mathcal{D}_{>0}$. This means that no direction in $\mathcal{D}_{>0}$ can be a single-genre equilibrium. We can further rule out directions in $\mathcal{D} \setminus \mathcal{D}_{>0}$ by applying Lemma 4.

Now, suppose that equation (17) does hold. It is not difficult to see that the optimum $\inf_{p^* \in \mathcal{B}_{>0}} \sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta}$ is attained at some direction $p^* \in \mathcal{D}_{>0}$. Applying Lemma 7, we see that there exists a single-genre equilibrium in the direction p^*. $\square$

3.2.4 Proof of Lemma 5

To prove Lemma 5 from Lemma 6, we require the following additional lemma that helps us analyze the right-hand side of equation (16).

Lemma 8. *For any set $\mathcal{R} \subseteq \mathbb{R}^N_{>0}$, it holds that:*

$$\sup_{y' \in \mathcal{R}} \inf_{y \in \mathcal{R}} \sum_{i=1}^{N} \frac{y'_i}{y_i} = N.$$

Proof. By taking $y' = y$, we see that:

$$\sup_{y' \in \mathcal{R}} \inf_{y \in \mathcal{R}} \sum_{i=1}^{N} \frac{y'_i}{y_i} \leq N.$$

To show equality, notice by AM-GM that:

$$\sum_{i=1}^{N} \frac{y'_i}{y_i} \geq N \left(\prod_{i=1}^{n} \frac{y'_i}{y_i} \right)^{1/N} = N \left(\frac{\prod_{i=1}^{n} y'_i}{\prod_{i=1}^{N} y_i} \right)^{1/N}.$$

We can take $y' = \arg\max_{y'' \in \mathcal{R}} \prod_{i=1}^{n} y''_i$, and obtain a lower bound of N as desired. (If the arg max does not exist, then note that if we take y' where $\prod_{i=1}^{n} y'_i$ is sufficiently close to the optimum $\sup_{y'' \in \mathcal{R}} \prod_{i=1}^{n} y''_i$, we have that $\inf_{y \in \mathcal{R}} \left(\frac{\prod_{i=1}^{n} y'_i}{\prod_{i=1}^{N} y_i} \right)^{1/N}$ is sufficiently close to 1 as desired.) $\square$

Now we are ready to prove Lemma 5.

Proof of Lemma 5. First, we see that:

$$\begin{aligned}
N &= \sup_{y' \in \mathcal{S}^\beta_{>0}} \inf_{y \in \mathcal{S}^\beta_{>0}} \sum_{i=1}^N \frac{y'_i}{y_i} \\
&= \sup_{y' \in \mathcal{S}^\beta} \inf_{y \in \mathcal{S}^\beta_{>0}} \sum_{i=1}^N \frac{y'_i}{y_i} \\
&= \sup_{y' \in \mathcal{S}^\beta} \inf_{p^* \in \mathcal{B}_{>0}} \sum_{i=1}^N \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta},
\end{aligned}$$

where the first equality follows from Lemma 8.

Now, let's combine this with Lemma 6 to see that a necessary and sufficient condition for the existence of a single-genre equilibrium is:

$$\inf_{p^* \in \mathcal{B}_{>0}} \sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} \leq \sup_{y' \in \mathcal{S}^\beta} \inf_{p^* \in \mathcal{B}_{>0}} \sum_{i=1}^N \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} \tag{19}$$

Weak duality tells us that $\inf_{p^* \in \mathcal{B}_{>0}} \sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} \geq \sup_{y' \in \mathcal{S}^\beta} \inf_{p^* \in \mathcal{B}_{>0}} \sum_{i=1}^N \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta}$, so equation (19) is equivalent to:

$$\inf_{p^* \in \mathcal{B}_{>0}} \sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} = \sup_{y' \in \mathcal{S}^\beta} \inf_{p^* \in \mathcal{B}_{>0}} \sum_{i=1}^N \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta}.$$

□

3.2.5 Finishing the proof of Theorem 1

Proof of Theorem 1. Recall that by Lemma 6, a single genre equilibrium exists if and only if equation (17) is satisfied.

We can rewrite the left-hand side of equation (17) as follows:

$$\inf_{p^* \in \mathcal{B}_{>0}} \left(\sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{\langle p^*, u_i \rangle^\beta} \right) = \inf_{p^* \in \mathcal{B}_{>0}} \left(\sup_{y' \in \bar{\mathcal{S}}^\beta} \sum_{i=1}^N \frac{y'_i}{\langle p^*, u_i \rangle^\beta} \right),$$

since the objective is linear in y'. Now, observing that the objective is convex in p and concave in y', we can apply Sion's min-max theorem[4] to see that:

$$\inf_{p^* \in \mathcal{B}_{>0}} \left(\sup_{y' \in \bar{\mathcal{S}}^\beta} \sum_{i=1}^N \frac{y'_i}{\langle p^*, u_i \rangle^\beta} \right) = \sup_{y' \in \bar{\mathcal{S}}^\beta} \left(\inf_{p^* \in \mathcal{B}_{>0}} \sum_{i=1}^N \frac{y'_i}{\langle p^*, u_i \rangle^\beta} \right) = \sup_{y' \in \bar{\mathcal{S}}^\beta} \left(\inf_{y \in \mathcal{S}^\beta_{>0}} \sum_{i=1}^N \frac{y'_i}{y_i} \right).$$

Thus, we have the following necessary and sufficient condition for a single-genre equilibrium to exist:

$$\sup_{y' \in \bar{\mathcal{S}}^\beta} \left(\inf_{y \in \mathcal{S}^\beta_{>0}} \sum_{i=1}^N \frac{y'_i}{y_i} \right) \leq N. \tag{20}$$

[4] Note that $\bar{\mathcal{S}}^\beta$ is compact and convex and $\mathcal{B}_{>0}$ is convex (but not compact). We apply the non-compact formulation of Sion's min-max theorem in [Ha81].

First, we show that if (2) does not hold, then there does not exist a single-genre equilibrium. Let $y' = \arg\max_{y'' \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^n y''_i$. (The maximum exists because $\prod_{i=1}^n y''_i$ is a continuous function and $\bar{\mathcal{S}}^\beta$ is a compact set.) We see that:

$$\sum_{i=1}^N \frac{y'_i}{y_i} \geq N \left(\frac{\prod_{i=1}^n y'_i}{\prod_{i=1}^n y_i} \right)^{1/N} \geq N \left(\frac{\max_{y'' \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^n y''_i}{\max_{y'' \in \mathcal{S}^\beta_{>0}} \prod_{i=1}^n y''_i} \right)^{1/N} = N \left(\frac{\max_{y'' \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^n y''_i}{\max_{y'' \in \mathcal{S}^\beta} \prod_{i=1}^n y''_i} \right)^{1/N} > N,$$

which proves that:

$$\inf_{p^* \in \mathcal{B}_{>0}} \left(\sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{\langle p^*, u_i \rangle^\beta} \right) = \sup_{y' \in \bar{\mathcal{S}}^\beta} \left(\inf_{y \in \mathcal{S}^\beta_{>0}} \sum_{i=1}^N \frac{y'_i}{y_i} \right) > N.$$

Thus equation (20) is not satisfied and a single-genre equilibrium does not exist as desired.

Next, we show that if (2) holds, then there exists a single-genre equilibrium. Let $y^* = \arg\max_{y'' \in \mathcal{S}^\beta} \prod_{i=1}^n y''_i = \arg\max_{y'' \in \mathcal{S}^\beta} \sum_{i=1}^n \log(y''_i)$. (The maximum exists because $\prod_{i=1}^n y''_i$ is a continuous function and $\mathcal{S}^\beta$ is a compact set.) By assumption, we see that y^* is also the maximizer over $\bar{S}^\beta$. We further see that $y^* \in \mathcal{S}^\beta_{>0}$. Using convexity of $\bar{S}^\beta$, this means that for any $y' \in \bar{S}^\beta$, it must hold that $\langle y' - y^*, \nabla \left(\sum_{i=1}^n \log(y^*_i) \right) \rangle \leq 0$. We can write this as:

$$\langle y' - y^*, \nabla \sum_{i=1}^n \frac{1}{y^*_i} \rangle \leq 0.$$

This can be written as:

$$\sum_{i=1}^n \frac{y'_i - y^*_i}{y^*_i} \leq 0,$$

which implies that:

$$\sum_{i=1}^n \frac{y'_i}{y^*_i} \leq N.$$

Thus, we have that

$$\sup_{y' \in \bar{\mathcal{S}}^\beta} \left(\inf_{y \in \mathcal{S}^\beta_{>0}} \sum_{i=1}^N \frac{y'_i}{y^*_i} \right) \leq N,$$

and thus equation (20) is satisfied so a single-genre equilibrium does not exist as desired.

Next, we show that if all equilibria have multiple genres for some β, then all equilibria have multiple genres for all $\beta' \geq \beta$. $\beta' \leq \beta$. Notice that equation 2 can equivalently be restated as:

$$\max_{y \in \mathcal{S}} \prod_{i=1}^N y_i = \max_{y \in \bar{\mathcal{S}}^\beta} \left(\prod_{i=1}^N y_i \right)^{1/\beta}. \tag{21}$$

It thus suffices to show that:

$$\max_{y \in \bar{\mathcal{S}}^\beta} \left(\prod_{i=1}^N y_i \right)^{1/\beta} \leq \max_{y \in \bar{\mathcal{S}}^{\beta'}} \left(\prod_{i=1}^N y_i \right)^{1/\beta'}$$

for all $\beta' \geq \beta$. To see this, let y denote the maximizer of $\max_{y \in \bar{\mathcal{S}}^\beta} \left(\prod_{i=1}^N y_i \right)^{1/\beta}$ (this is achieved since we are taking a maximum of a continuous function over a compact set). By definition, we see that y can be written as a convex combination $\sum_{j=1}^{P} \lambda_j (x_i^j)^\beta$ where $x^1, \ldots, x^P$ denote vectors in $\mathcal{S}$ and where $\sum_{j=1}^{P} \lambda_j = 1$. In this notation, we see that:

$$\max_{y \in \bar{\mathcal{S}}^\beta} \left(\prod_{i=1}^{N} y_i \right)^{1/\beta} = \left(\prod_{i=1}^{N} \left(\sum_{j=1}^{P} \lambda_j (x_i^j)^\beta \right)^{1/\beta} \right)$$

By taking y to be $\sum_{j=1}^{P} \lambda_j (x_i^j)^{\beta'}$, we see that:

$$\max_{y \in \bar{\mathcal{S}}^{\beta'}} \left(\prod_{i=1}^{N} y_i \right)^{1/\beta'} \geq \left(\prod_{i=1}^{N} \left(\sum_{j=1}^{P} \lambda_j (x_i^j)^{\beta'} \right)^{1/\beta'} \right).$$

Notice that for any $1 \leq i \leq N$, it holds that:

$$\left(\sum_{j=1}^{P} \lambda_j (x_i^j)^{\beta'} \right) = \left(\sum_{j=1}^{P} \lambda_j ((x_i^j)^\beta)^{\beta'/\beta} \right) \geq \left(\sum_{j=1}^{P} \lambda_j ((x_i^j)^\beta) \right)^{\beta'/\beta},$$

where the last inequality follows from convexity of $f(c) = c^{\beta'/\beta}$ for $\beta' \geq \beta$. Putting this all together, we see that:

$$\max_{y \in \bar{\mathcal{S}}^{\beta'}} \left(\prod_{i=1}^{N} y_i \right)^{1/\beta'} \geq \left(\prod_{i=1}^{N} \left(\sum_{j=1}^{P} \lambda_j (x_i^j)^{\beta'} \right)^{1/\beta'} \right) \geq \left(\prod_{i=1}^{N} \left(\sum_{j=1}^{P} \lambda_j (x_i^j)^{\beta} \right)^{1/\beta} \right) = \max_{y \in \bar{\mathcal{S}}^{\beta}} \left(\prod_{i=1}^{N} y_i \right)^{1/\beta}$$

as desired. □

3.3 Proofs of corollaries of Theorem 1

First, we prove Corollary 4, restated below.

Corollary 4. *The threshold β^* is always at least 1. That is, if $\beta = 1$, then there always exists a single-genre equilibrium.*

Proof. When $\beta = 1$, we see that $\mathcal{S}^\beta = \mathcal{S}^1$ is a linear transformation of a convex set (the unit ball restricted to $\mathbb{R}^D_{\geq 0}$), so it is convex. This means that $\bar{\mathcal{S}}^\beta = \mathcal{S}^\beta$, and so (2) is trivially satisfied. By Theorem 1, there exists a single-genre equilibrium. □

Next, we prove Corollary 5, restated below.

Corollary 5. *Let the cost function be $c(p) = \|p\|_q^\beta$. For any set of user vectors, it holds that $\beta^* \geq q$. If the user vectors are equal to the standard basis vectors $\{e_1, \ldots, e_D\}$, then β^* is equal to q.*

Proof. We split the proof into two steps: (1) showing that $\beta^* \geq q$ for any set of user vectors and (2) showing that $\beta^* \leq q$ for the standard basis vectors.

Showing that $\beta^* \geq q$ for any set of users. To show that $\beta^* \geq q$, by Theorem 1, it suffices to show that equation (2) is satisfied at $\beta = q$. Suppose that the right-hand side of (2):

$$\max_{y \in \bar{S}^\beta} \prod_{i=1}^{N} y_i$$

is maximized at some $y^* \in \bar{S}^\beta$. It suffices to construct $\tilde{y} \in \mathcal{S}^\beta$ such that

$$\prod_{i=1}^{N} \tilde{y}_i \geq \prod_{i=1}^{N} y_i^* \tag{22}$$

To construct $\tilde{y}$, we introduce some notation. By the definition of a convex hull, we can write y^* as

$$y^* = \sum_{k=1}^{m} \lambda_k y^k,$$

where $y^1, \ldots, y^m \in \mathcal{S}^\beta$ and where $\lambda_1, \ldots, \lambda_m \in [0,1]$ are such that $\sum_{k=1}^m \lambda_k = 1$. Let $p^1, \ldots, p^m \in \mathbb{R}^D_{\geq 0}$ be such that $\|p^k\|_q \leq 1$ for all $1 \leq k \leq m$ and y^k is given by the β-coordinate-wise powers of $\mathbf{U}p_k$. Now, we let $y = \mathbf{U}\tilde{p}$ where the dth coordinate of $\tilde{p}$ is given by:

$$\tilde{p}_d := \left(\sum_{k=1}^{m} \lambda_k ((p^k)_d)^q\right)^{1/q}.$$

It follows from definition that:

$$\|\tilde{p}\|_q = \left(\sum_{d=1}^{D}\sum_{k=1}^{m} \lambda_k ((p^k)_d)^q\right)^{1/q} = \left(\sum_{k=1}^{m} \lambda_k \sum_{d=1}^{D} ((p^k)_d)^q\right)^{1/q} \leq \left(\sum_{k=1}^{m} \lambda_k \|p^k\|_q^q\right)^{1/q} \leq 1,$$

which means that $\tilde{y} \in \mathcal{S}^\beta$.

The remainder of the proof boils down to showing (22). It suffices to show that for every $1 \leq i \leq N$, it holds that $\tilde{y}_i \geq y_i^*$. Notice that:

$$y_i^* = \sum_{k=1}^{m} \lambda_k (y^k)_i = \sum_{k=1}^{m} \lambda_k \langle u_i, p^k \rangle^q = \sum_{k=1}^{m} \lambda_k \left(\sum_{d=1}^{D} (u_i)_d (p^k)_d\right)^q,$$

and

$$\tilde{y}_i = \langle u_i, \tilde{p} \rangle^q = \left(\sum_{d=1}^{D} (u_i)_d \tilde{p}_d\right)^q = \left(\sum_{d=1}^{D} (u_i)_d \left(\sum_{k=1}^{m} \lambda_k ((p^k)_d)^q\right)^{1/q}\right)^q.$$

Thus, it suffices to show the following inequality:

$$\sum_{d=1}^{D} (u_i)_d \left(\sum_{k=1}^{m} \lambda_k ((p^k)_d)^q\right)^{1/q} \geq \left(\sum_{k=1}^{m} \lambda_k \left(\sum_{d=1}^{D} (u_i)_d (p^k)_d\right)^q\right)^{1/q}. \tag{23}$$

The high-level idea is that the proof boils down to the triangle inequality for an appropriately chosen norm over $\mathbb{R}^m$. For $z \in \mathbb{R}^m$, we let:

$$\|z\|_\lambda := \left(\sum_{k=1}^{m} \lambda_k z^q\right)^{1/q}.$$

To see that this is a norm, note that $(\sum_{k=1}^m \lambda_k z^q)^{1/q} = \left(\sum_{k=1}^m (\lambda_k^{1/q} z)^q\right)^{1/q}$. The norm properties of this function are implied by the norm properties of $\|\cdot\|_q$. By the triangle inequality, we see that:

$$\begin{aligned}
\sum_{d=1}^D (u_i)_d \left(\sum_{k=1}^m \lambda_k ((p^k)_d)^q\right)^{1/q} &= \sum_{d=1}^D (u_i)_d \|[p_d^1, \ldots, p_d^m]\|_\lambda \\
&\geq \| \sum_{d=1}^D (u_i)_d [p_d^1, \ldots, p_d^m]\|_\lambda \\
&= \left(\sum_{k=1}^m \lambda_k \left(\sum_{d=1}^D (u_i)_d (p^k)_d\right)^q\right)^{1/q}
\end{aligned}$$

which implies equation (23).

Showing that $\beta^* \leq q$ for the standard basis vectors. By Theorem 1, it suffices to show, for any $\beta > q$, that equation (2) is not satisfied. First, we compute the left-hand side of equation (2):

$$\max_{y \in \mathcal{S}^\beta} \prod_{i=1}^N y_i = \left(\max_{x \in \mathbb{R}^D_{\geq 0}, \|x\|_q = 1} \prod_{i=1}^D x_i\right)^\beta = \left(\frac{1}{D}\right)^{\beta/q} < \left(\frac{1}{D}\right).$$

where the last line follows from AM-GM. Now, we compute the right-hand side:

$$\max_{y \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^N y_i.$$

Consider $y^* = \left[\frac{1}{D}, \ldots, \frac{1}{D}\right]$. Notice that y is a convex combination of the standard basis vectors—all of which are in $\mathcal{S}$ and actually in $\mathcal{S}^\beta$ too—so $y \in \bar{\mathcal{S}}^\beta$. This means that

$$\max_{y \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^N y_i \geq \prod_{i=1}^N y_i^* = \left(\frac{1}{D}\right).$$

This proves that:

$$\max_{y \in \mathcal{S}^\beta} \prod_{i=1}^N y_i < \max_{y \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^N y_i,$$

so equation (2) is not satisfied as desired.

□

We prove Corollary 1, restated below.

Corollary 1. *Let $\|\cdot\|_*$ denote the dual norm of $\|\cdot\|$, defined to be $\|p\|_* = \max_{\|p\|=1, p \in \mathbb{R}^D_{\geq 0}} \langle q, p \rangle$. Let $Z := \|\sum_{n=1}^N \frac{u_n}{\|u_n\|_*}\|_*$. Then,*

$$\beta^* \leq \frac{\log(N)}{\log(N) - \log(Z)}. \tag{3}$$

Proof. WLOG assume that the users to have unit dual norm. By Theorem 1, it suffices to show that:

$$\max_{y \in \mathcal{S}^\beta} \prod_{i=1}^N y_i < \max_{y \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^N y_i.$$

First, let's lower bound the right-hand side. Consider the point $y = \frac{1}{N} \sum_{i'=1}^N z^{i'}$ where $z^{i'}$ is defined to be the β-coordinate-wise power of $\mathbf{U}\left(\arg\max_{||p||=1} \langle p, u_i \rangle\right)$. This means that

$$y_i \geq \frac{1}{N} z_i^i = \frac{1}{N} \left(\max_{||p||=1} \langle p, u_i \rangle \right)^\beta = \frac{||u_i||_*^\beta}{N} = \frac{1}{N}.$$

This means that:

$$\max_{y \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^N y_i \geq \frac{1}{N^N}.$$

Next, let's upper bound the left-hand side. By AM-GM, we see that:

$$\max_{y \in \mathcal{S}^\beta} \prod_{i=1}^N y_i = \max_{||p||=1, p \in \mathbb{R}^D_{\geq 0}} \left(\prod_{i=1}^N \langle p, u_i \rangle \right)^\beta \leq \left(\frac{\sum_{i=1}^N \langle p, u_i \rangle}{N} \right)^{N\beta} \leq \left(\frac{\langle p, \sum_{i=1}^N u_i \rangle}{N} \right)^{N\beta} \leq \frac{\left(\| \sum_{i=1}^N u_i \|_* \right)^{N\beta}}{N^{N\beta}}.$$

Putting this all together, we see that it suffices for:

$$\frac{1}{N^N} > \frac{\left(\| \sum_{i=1}^N u_i \|_* \right)^{N\beta}}{N^{N\beta}},$$

which we can rewrite as:

$$N^{\beta - 1} > \left(\| \sum_{i=1}^N u_i \|_* \right)^\beta$$

which we can rewrite as:

$$N^{1 - 1/\beta} > \| \sum_{i=1}^N u_i \|_*.$$

□

We prove Corollary 2, restated below.

Corollary 2. *If there exists μ with $|\text{Genre}(\mu)| = 1$, then the corresponding producer direction maximizes Nash social welfare of the users:*

$$\text{Genre}(\mu) = \arg\max_{||p||=1 | p \in \mathbb{R}^D_{\geq 0}} \sum_{i=1}^N \log(\langle p, u_i \rangle). \tag{4}$$

Proof. Corollary 2 follows as a consequence of the proof of Theorem 1. We apply Lemma 7 to see that if μ is a single-genre equilibrium with $\text{Genre}(\mu) = \{p^*\}$, then:

$$\sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} \leq N.$$

We see that:

$$N \geq \sup_{y' \in \mathcal{S}^\beta} \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} \geq N \sup_{y' \in \mathcal{S}^\beta} \left(\frac{\prod_{i=1}^N y'_i}{\prod_{i=1}^N (\langle p^*, u_i \rangle)^\beta} \right)^{1/N} \geq N \left(\frac{\sup_{y' \in \mathcal{S}^\beta} \prod_{i=1}^N y'_i}{\prod_{i=1}^N (\langle p^*, u_i \rangle)^\beta} \right)^{1/N}.$$

This implies that:

$$\prod_{i=1}^N y_i = \prod_{i=1}^N (\langle p^*, u_i \rangle)^\beta \geq \sup_{y' \in \mathcal{S}^\beta} \prod_{i=1}^N y'_i,$$

where $y \in \mathcal{S}^\beta$ is defined so that $y_i = \langle p^*, u_i \rangle^\beta$. This implies that:

$$p^* \in \operatorname*{arg\,max}_{||p|| \leq 1, p \in \mathbb{R}^D_{\geq 0}} \sum_{i=1}^N \log(\langle p, u_i \rangle) = \operatorname*{arg\,max}_{||p|| = 1, p \in \mathbb{R}^D_{\geq 0}} \sum_{i=1}^N \log(\langle p, u_i \rangle)$$

as desired. □

4 Equilibrium structure for two populations of users

Beyond separating into single-genre and multi-genre regimes, the equilibria in our model possess a number of other rich structural properties. To uncover these properties, we work towards concretely characterizing the equilibria μ in terms of both the set of genres $\text{Genre}(\mu)$ and the density of μ itself. We focus on the concrete instance of two equally sized populations of users with cost functions of the form $c(p) = \|p\|_2^\beta$.

In Section 4.1, we develop machinery to characterize equilibria and we provide an overview of how we apply it to the setting of two populations of users. In Section 4.2, we show structural properties of equilibria, and in Section 4.3, we compute closed-form equilibria in special cases. We prove our results in Sections 4.4-4.6.

4.1 Overview of proof techniques

The first ingredient in our proofs is the necessary and sufficient conditions for μ to be an equilibria given by Lemma 1, restated below.

Lemma 1. *Let $\mathbf{U} = [u_1; u_2; \ldots; u_N]$ be the $N \times D$ matrix of users vectors. Given a set $S \subseteq \mathbb{R}^N_{\geq 0}$ and distributions $H_1, \ldots, H_N$ over $\mathbb{R}_{\geq 0}$, suppose that the following conditions hold:*

(C1) Every $z^* \in S$ is a maximizer of the equation:

$$\max_{z \in \mathbb{R}^D_{\geq 0}} \sum_{i=1}^N H_i(z_i) - c_{\mathbf{U}}(z), \tag{7}$$

where $c_{\mathbf{U}}(z) := \min \{ c(p) \mid p \in \mathbb{R}^D_{\geq 0}, \mathbf{U}p = z \}$.

(C2) There exists a random variable Z with support S, such that the marginal distribution Z_i has cdf equal to $H_i(z)^{1/(P-1)}$.

(C3) Z is distributed as $\mathbf{U}Y$ with $Y \sim \mu$, for some distribution μ over $\mathbb{R}^D_{\geq 0}$.

Then, the distribution μ from (C3) is a symmetric mixed Nash equilibrium. Moreover, every symmetric mixed Nash equilibrium μ is associated with some $(H_1, \ldots, H_N, S)$ that satisfy (C1)-(C3).

Proof. To prove Lemma 1, we reparameterize from content vectors in $\mathbb{R}^D_{\geq 0}$ to realized user values in $\mathbb{R}^N_{\geq 0}$. That is, we transform the content vector $p \in \mathbb{R}^D_{\geq 0}$ into the vector of realized user values given by $z = \mathbf{U}p$. This reparameterization allows us to cleanly reason about the number of users that a producer wins: a producer wins a user u_i if and only if they have the highest value in the ith coordinate of z. In this parametrization, the cost of production can be computed through an induced function $c_{\mathbf{U}}$ given by $c_{\mathbf{U}}(z) := \min\left\{c(p) \mid p \in \mathbb{R}^D_{\geq 0}, z = \mathbf{U}p\right\}$ if $z \in \left\{\mathbf{U}p \mid p \in \mathbb{R}^D_{\geq 0}\right\}$.

In this reparameterization, the producer profit takes a clean form. If producer 1 chooses $z \in \mathbb{R}^N$, and other producers follow a distribution μ_Z over $\mathbb{R}^N$, then the expected profit of producer 1 is:

$$\sum_{i=1}^{N} H_i(z_i) - c_{\mathbf{U}}(z),$$

where $H_i(\cdot)$ is the cumulative distribution function of the maximum realized user value over the other $P-1$ producers, i.e. of the random variable $\max_{2 \leq j \leq P}(z_j)_i$ where $z_2, \ldots, z_P \sim \mu_Z$.

Recall that a distribution μ corresponds to a symmetric mixed Nash equilibrium if and only if every z in the support $S := \operatorname{supp}(\mu_Z)$ is a maximizer of equation (7) (where μ_Z is the distribution over $\mathbf{U}p$ for $p \sim \mu$).

□

The conditions in Lemma 1 disentangle the role of the number of producers P from the the role of the user geometry $u_1, \ldots, u_n$. In particular, the conditions (C1) and (C3) are independent of the number of producers P but depend on the user geometry $\{u_1, \ldots, u_N\}$ through the cost function $c_{\mathbf{U}}$. This allows us to cleanly investigate the conditions that $\{u_1, \ldots, u_N\}$ places on the support S and the distributions $H_1, \ldots, H_N$. The condition (C2), on the other hand, depends on the number of producers P but is independent of the user geometry. This condition turns out to be mathematically trickier to analyze in general, but we show that it can be reduced to tractable formulations when S has sufficient structure.

In the rest of the section, we focus on instantiating Lemma 1 for the case of two equally sized homogeneous populations of users. More formally, let there be N users split equally between two linearly independently vectors $u_1, u_2 \in \mathbb{R}^D_{\geq 0}$, and let $\theta^* := \cos^{-1}\left(\frac{\langle u_1, u_2 \rangle}{\|u_1\|_2 \|u_2\|}\right)$ be the angle between the user vectors. Let the cost function be $c(p) = \|p\|_2^\beta$. It is more mathematically convenient to instantiate Lemma 1 for *two normalized users* with a rescaled cost function, and it is easy to show that these settings are exactly equivalent:

Claim 1. *A distribution μ is an equilibria for a marketplace with 2 populations of users of size $N/2$ located at vectors u_1 and u_2 and with producer cost function $c(p) = \|p\|_2^\beta$ if and only if μ is an equilibria for a marketplace with 2 users located at vectors $\frac{u_1}{\|u_1\|}$ and $\frac{u_2}{\|u_2\|}$ and with producer cost function $c(p) = \frac{2}{N}\|p\|_2^\beta$.*

Thus, we focus on marketplaces with 2 users located at vectors u_1 and u_2 such that $\|u_1\| = \|u_2\| = 1$ and with producer cost function $c(p) = \alpha\|p\|_2^\beta$ for $\alpha > 0$. This means that the set S in Lemma 1 is a subset of $\mathbb{R}^2_{\geq 0}$ and there are two relevant marginal distributions H_1 and H_2. In the following subsubsection, we provide an overview of how we leverage conditions (C1)-(C3) in the concrete instance of two users.

4.1.1 Leveraging (C1)

To leverage (C1), we use the first-order and second-order conditions for z to be a maximizer of equation (7). In order to obtain useful closed-form expressions, we explicitly compute the induced cost function in terms of the angle θ^* between the user vectors.

Lemma 9. *Let there be 2 users located at $u_1, u_2 \in \mathbb{R}^D_{\geq 0}$ such that $\|u_1\| = \|u_2\| = 1$, and let $\theta^* := \cos^{-1}(\langle u_1, u_2 \rangle) > 0$ be the angle between the user vectors. Let the cost function be $c(p) = \alpha \|p\|_2^\beta$ for $\alpha > 0$. For any $z \in \{\mathbf{U}p \mid p \in \mathbb{R}^D_{\geq 0}\}$, the induced cost function is given by:*

$$c_{\mathbf{U}}(z) = \alpha \sin^{-\beta}(\theta^*) \left(z_1^2 + z_2^2 - 2z_1 z_2 \cos(\theta^*)\right)^{\frac{\beta}{2}}.$$

First-order condition. The first order condition implies that we can compute the densities h_1 and h_2 of H_1 and H_2 in terms of the $c_{\mathbf{U}}$. The densities $h_1(z_1)$ and $h_2(z_2)$ depend on the gradient $\nabla_z c_{\mathbf{U}}$ and *both* coordinates z_1 and z_2.

Lemma 10. *Let there be 2 users located at $u_1, u_2 \in \mathbb{R}^D_{\geq 0}$ such that $\|u_1\| = \|u_2\| = 1$, and let $\theta^* := \cos^{-1}(\langle u_1, u_2 \rangle) > 0$ be the angle between the user vectors. Let the cost function be $c(p) = \alpha \|p\|_2^\beta$ for $\alpha > 0$. For any $z \in \{\mathbf{U}p \mid p \in \mathbb{R}^D_{\geq 0}\}$, the first-order condition of equation (7) can be written as:*

$$\begin{bmatrix} h_1(z_1) \\ h_2(z_2) \end{bmatrix} = \nabla_z(c_{\mathbf{U}}(z)).$$

More specifically, it holds that:

$$\begin{bmatrix} h_1(z_1) \\ h_2(z_2) \end{bmatrix} = \beta\alpha \sin^{-\beta}(\theta^*) \left(z_1^2 + z_2^2 - 2z_1 z_2 \cos(\theta^*)\right)^{\frac{\beta}{2}-1} \begin{bmatrix} z_1 - z_2\cos(\theta^*) \\ z_2 - z_1 \cos(\theta^*) \end{bmatrix},$$

and if we represent $z = \mathbf{U}[r\cos(\theta), r\sin(\theta)]$, then it also holds that:

$$\begin{bmatrix} h_1(z_1) \\ h_2(z_2) \end{bmatrix} = \beta\alpha r^{\beta-1} \begin{bmatrix} \frac{\sin(\theta^*-\theta)}{\sin(\theta^*)} \\ \frac{\sin(\theta)}{\sin(\theta^*)} \end{bmatrix}.$$

Second-order condition. When we also take advantage of the second-order condition, we can identify the "direction" that the support must point at $z \in S$ terms of the location of z, the cost function parameter β, and the angle θ^* between the two populations of users.

Lemma 11. *Let there be 2 users located at $u_1, u_2 \in \mathbb{R}^D_{\geq 0}$ such that $\|u_1\| = \|u_2\| = 1$, and let $\theta^* := \cos^{-1}(\langle u_1, u_2 \rangle) > 0$ be the angle between the user vectors. Let the cost function be $c(p) = \alpha \|p\|_2^\beta$ for $\alpha > 0$. Suppose that condition (C1) is satisfied for (H_1, H_2, S). If S contains a curve of the form $\{(z_1, g(z_1)) \mid x \in I\}$ for any open interval I and any differentiable function g, then for any $z_1 \in I$, it holds that:*

$$g'(z_1) \cdot \left(\frac{\beta-2}{\beta}\cos(\theta^* - 2\theta) - \cos(\theta^*)\right) \leq 0.$$

Lemma 11 demonstrates that if $\left(\frac{\beta-2}{\beta}\cos(\theta^* - 2\theta) - \cos(\theta^*)\right) > 0$, then the curve g must be decreasing, and if $\left(\frac{\beta-2}{\beta}\cos(\theta^* - 2\theta) - \cos(\theta^*)\right) < 0$, then the curve g must be increasing. This characterizes the "direction" of the curve in terms of the location z_1.

4.1.2 Leveraging (C3)

The condition (C3) always holds, as long as condition (C1) holds. Since the two vectors u_1 and u_2 are linearly independent, the matrix U is invertible, so we can define μ to be the distribution $\mathbf{U}$. The only condition comes p being restricted to $\mathbb{R}^D_{\geq 0}$ rather than $\mathbb{R}^D$. This means that S must be contained in the convex cone generated by $[1, \cos(\theta^*)]$ and $[\cos(\theta^*), 1]$. This restriction on S is already implicitly implied by (C1): it is not difficult to see that all maximizers of (7) will be contained in this convex cone.

4.1.3 Leveraging (C2)

To leverage (C2), we obtain a functional equation that restricts the relationship between H_1, H_2, and S for a given value of P, and we instantiate this in two ways. First, when the support is a curve $(z_1, g(z_1))$, the marginal distributions Z_1 and Z_2 are related by a change of variables formula given by $Z_2 \sim g(Z_1)$. This translates into a condition on H_1 and H_2 that depends on the derivative g' and the number of producers P. Second, if the equilibrium were to contain finitely many genres, there would be a pair of functional equations relating the cdfs H_1 and H_2, the distribution over quality levels within each genre, and the number of producers P. We describe each of these settings in more detail below.

Case 1: support is a single curve. The first case where we instantiate (C2) is when S is equal to $\{(z_1, g(z_1)) \mid x \in M\}$ where M is a (well-behaved) subset of $\mathbb{R}_{\geq 0}$. Let h_1^* and h_2^* be the densities of the marginal distributions Z_1 and Z_2 respectively. Since $Z_2 \sim g(Z_1)$, the change of variables formula implies that the densities h_1^* and h_2^* are related as follows:

$$h_1^*(z_1) = h_2^*(g(z_1))|g'(z_1)|, \tag{24}$$

In order to use equation (24), we need to translate it into a condition on the distributions H_1 and H_2. Let h_1 and h_2 be the densities of H_1 and H_2 respectively. Then equation (24) can reformulated as:

$$\frac{h_1(x)}{(H_1(x))^{\frac{P-2}{P-1}}} = \frac{h_2(g(x))}{(H_2(g(x)))^{\frac{P-2}{P-1}}}|g'(x)|. \tag{25}$$

Equation (25) reveals that the constraint induced by the number of producers P can be messy in general, since it involves both the densities h_1 and h_2 and the cdfs H_1 and H_2. Intuitively, these complexities arise because H_i^* and H_i are related by a $(P-1)$th degree polynomial (put differently, the maximum of $P-1$ i.i.d. draws of a random variable does not generally have a clean structure). Nonetheless, equation (25) does simplify into a tractable form in special cases. For example, if $P = 2$, then the dependence on H_1 and H_2 vanishes. As another example, if g is *increasing*, then $H_1(x) = H_2(g(x))$ for any $P \geq 2$, so the dependence on H_1 and H_2 again vanishes.

Case 2: two-genre equilibria. The second case where we instantiate (C2) is when S is a subset of the union of two lines: that is,

$$S \subseteq \{(z_1, c_1 \cdot z_1) \mid z_1 \in \mathbb{R}_{\geq 0}\} \cup \{(z_1, c_2 \cdot z_1) \mid z_1 \in \mathbb{R}_{\geq 0}\},$$

where $\cos(\theta^*) \leq c_1, c_2 \leq \frac{1}{\cos(\theta^*)}$. Since linear transformations preserve lines through the origin, this means that the support of the distribution μ of $U^{-1}Z$ is also contained in the union of two lines through the origin: thus $|\text{Genre}(\mu)| \leq 2$.

A distribution Z can be entirely specified by the probabilities $\alpha_1 + \alpha_2$ that it places on each of the two lines and the conditional distribution of Z_1 along each of the lines (this in particular determines the conditional distribution of Z_2 along the lines). More specifically, the probabilities $\alpha_1 + \alpha_2$ will correspond to

$$\alpha_1 := \mathbb{P}_Z[Z \in \{(z_1, c_1 \cdot z_1) \mid z_1 \in \mathbb{R}_{\geq 0}\}]$$
$$\alpha_2 := \mathbb{P}_Z[Z \in \{(z_1, c_2 \cdot z_1) \mid z_1 \in \mathbb{R}_{\geq 0}\}],$$

and F_1 and F_2 will correspond to the cdfs of the conditional distributions

$$F_1 \sim Z_1 \mid Z \in \{(z_1, c_1 \cdot z_1\}$$
$$F_2 \sim Z_1 \mid Z \in \{(z_1, c_2 \cdot z_1\}$$

respectively. The (unique) distribution Z associated with $\alpha_1, \alpha_2, F_1, F_2$ satisfies (C2) if and only if the following pairs of functional equations are satisfied:

$$(\alpha_1 F_1(z_1) + \alpha_2 F_2(z_1)) = (H_1(z_1))^{\frac{1}{P-1}} \text{ and } \left(\alpha_1 F_1(c_1^{-1} z_2) + \alpha_2 F_2(c_2^{-1} z_2)\right) = (H_2(z_2))^{\frac{1}{P-1}}. \quad (26)$$

The functional equations can be solved to determine if there is a valid solution.

4.2 Structural properties of equilibria

Our first structural result is a characterization of the boundary between the single-genre and multi-genre regimes.

Corollary 6. *Suppose that there are N users split equally between two linearly independently vectors $u_1, u_2 \in \mathbb{R}^D_{\geq 0}$, and let $\theta^* := \cos^{-1}\left(\frac{\langle u_1, u_2 \rangle}{\|u_1\|_2 \|u_2\|}\right)$. Let the cost function be $c(p) = \|p\|_2^\beta$. Then it holds that:*

$$\beta^* = \frac{2}{1 - \cos(\theta^*)}.$$

Corollary 6 demonstrates that the threshold β^* increases as the angle θ^* between the users decreases, interpolating from 2 when the users are orthogonal to ∞ when the users point in the same direction.

We next go beyond single-genre equilibria and investigate general properties of the support of the equilibrium. Proposition 1 (restated below) shows that the support of the equilibrium distribution cannot contain any 2-dimensional ball.

Proposition 1. *Suppose that there are N users split equally between two linearly independently vectors $u_1, u_2 \in \mathbb{R}^2_{\geq 0}$, and let $\theta^* := \cos^{-1}\left(\frac{\langle u_1, u_2 \rangle}{\|u_1\|_2 \|u_2\|}\right)$. Let the cost function be $c(p) = \|p\|_2^\beta$, and let $P \geq 2$. Let μ be a symmetric Nash equilibrium such that the distributions $\langle u_1, p \rangle$ and $\langle u_2, p \rangle$ over $\mathbb{R}_{\geq 0}$ are absolutely continuous. As long as $\beta \neq 2$ or $\theta^* \neq \pi/2$, the support of μ does not contain an ℓ_2-ball of radius ϵ for any $\epsilon > 0$.*[5]

Proposition 1 demonstrates that the support of the equilibria is a union of 1-dimensional curves. This significantly restricts the set of possible distributions that we need to search when trying to characterize equilibria for two homogeneous populations of users.

Our next structural result illustrates that there is a *phase transition* at β^* at which the equilibrium transitions from single-genre to infinitely many genres (see Figure 1).

[5]The case of $\beta = 2$ and $\theta^* = \pi/2$ is degenerate and permits a range of possible equilibria.

Theorem 4. *[Formal version of Theorem 2] Suppose that there are N users split equally between two linearly independently vectors $u_1, u_2 \in \mathbb{R}^D_{\geq 0}$, and let $\theta^* := \cos^{-1}\left(\frac{\langle u_1, u_2 \rangle}{\|u_1\|_2 \|u_2\|}\right)$. Let the cost function be $c(p) = \|p\|_2^\beta$. Let μ be a a distribution on $\mathbb{R}^d$ such that the distributions $\langle u_1, p\rangle$ and $\langle u_2, p\rangle$ over $\mathbb{R}_{\geq 0}$ over $\mathbb{R}_{\geq 0}$ for $p \sim \mu$ are absolutely continuous and twice continuously differentiable within their supports. There are two regimes based on β and θ^*:*

1. *If $\beta < \beta^* = \frac{2}{1-\cos(\theta^*)}$ and if μ is a symmetric mixed equilibrium, then μ satisfies $|\text{Genre}(\mu)| = 1$.*
2. *If $\beta > \beta^* = \frac{2}{1-\cos(\theta^*)}$, if $|\text{Genre}(\mu)| < \infty$, and if the conditional distribution of $\|p\|$ along each genre is continuously differentiable, then μ is not an equilibrium.*

The proof of Theorem 4 also boils down to leveraging the machinery given by Lemma 1. For the first part, Condition (C1) helps us show that the support S can be specified by $(w, g(w))$ for an increasing function w: in particular, Lemma 10 enables us to show that S must be one-to-one, and Lemma 11 enables us to pin down the sign of g'. Using condition (C2), which simplifies since g is increasing, we show a functional equation in terms of g that has a unique solution at the single-genre equilibrium. For the second part, we use Lemma 10 to rule out all finite-genre equilibria except for two-genre equilibria. We can show that $H_1(w)$ and $H_2(w)$ grow proportionally to w^β. Then, we can implement this knowledge of H_1 and H_2 into the finite genre formulation of condition (C2) in equation (26) and show that no solutions to the functional equation exist for finite P.

Theorem 4 provides a tight characterization of how many genres emerge at equilibrium. The first part of Theorem 4 shows that specialization occurs in a marketplace if and only if β is above β^* (subject to some mild continuity conditions). This strengthens the results from Section 3: in particular, all equilibria when $\beta < \beta^*$ are single-genre, which rules out the possibility of *any* equilibrium exhibiting genre formation. The second part of Theorem 4 shows that finite genre equilibria, apart from the single-genre equilibria, do not exist (subject to some mild continuity conditions). In the multi-genre regime, producers thus do not fully personalize their content to users. Moreover, producer choices at equilibrium reflect infinitely many types of content that balance the interests of the two populations in different ways—the nature of specialization is therefore complex. However, we note that focusing solely on the *number* of genres is somewhat limiting: in the limit as $P \to \infty$, the support of the equilibria distribution collapses onto two genres, as we show in the next section (Theorem 3).

4.3 Closed-form equilibria in special cases

Going beyond the number of genres, we characterize the equilibrium in special cases, uncovering striking features of the equilibrium distribution.

First, we consider a concrete market instance where user are located at the standard basis vectors and there are 2 producers. In this market, the threshold β^* between the single-genre and multi-genre regimes is equal to 2 (see Corollary 5). Proposition 2, restated below, characterizes equilibria in the multi-genre regime.

Proposition 2. *Suppose that there are 2 users located at the standard basis vectors $e_1, e_2 \in \mathbb{R}^2$, and the cost function is $c(p) = \|p\|_2^\beta$. For $P = 2$ and $\beta \geq \beta^* = 2$, there is an equilibrium μ supported on the quarter-circle of radius $(2\beta^{-1})^{\frac{1}{\beta}}$, where the angle $\theta \in [0, \pi/2]$ has density $f(\theta) = 2\cos(\theta)\sin(\theta)$.*

The machinery given by Lemma 1 enables us to systematically identify the equilibrium in the concrete market instance of Proposition 2. Condition (C1) is simple along the quarter circle: by Lemma 10, the densities $h_1(u)$ and $h_2(v)$ are *proportional* to u and v. Since the support of a single curve and $P = 2$, condition (C2) can be simplified to a clean condition on the densities h_1 and h_2

given by (24). (To prove Proposition 2, we only need to *verify* that the equilibrium μ in Proposition 2 which is easier; we defer a full proof of Proposition 2 to Section 4.6.)

As shown in Figure 1, the support of the equilibrium distribution is a quarter circle with radius $\left(\frac{2}{\beta}\right)^{1/\beta}$. The cumulative distribution function of $\|p\|$ for $p \sim \mu$ is thus a point mass (see the last panel of Figure 1). Moreover, the radius $\left(2\beta^{-1}\right)^{\frac{1}{\beta}}$ is equal to 1 at $\beta = 2$, decreases to around 0.8, and then increases back to 1 in the limit as $\beta \to \infty$. Nonetheless, the fast rate of decay of $c(p)$ means the production cost still approaches 0 in the limit. Lastly, the distribution Θ of the distribution of the angle θ has much of its mass concentrated in the middle: the density at the endpoints $\theta = 0$ and $\theta = \pi/2$ is equal to 0.

We next consider how the number of producers affects the equilibria. Proposition 3, restated below, characterizes equilibria for every $P \geq 2$ for the standard basis vectors with $\beta = 2$.

Proposition 3. *Suppose that there are 2 users located at the standard basis vectors $e_1, e_2 \in \mathbb{R}^2$, with cost function $c(p) = \|p\|_2^\beta$ and $\beta = 2$. Then, there is an equilibrium μ with support equal to*

$$\left\{\left(x, (1 - x^{\frac{2}{P-1}})^{\frac{P-1}{2}}\right) \mid x \in [0,1]\right\}, \tag{6}$$

and where the distribution of x has cdf equal to $\min(1, x^{2/(P-1)})$.

Again, the machinery given by Lemma 1 enables us to systematically identify the equilibrium in the concrete market instance of Proposition 3. Since we need to consider $P \neq 2$, the condition (C2) does not take as clean of a form: as shown by (25), it depends on both the densities h_1 and h_2 along with the cdfs H_1 and H_2. Nonetheless, in the special case of $\beta = 2$, we can compute the cdf in closed-form: Lemma 10 implies that the density $h_1(z_1)$ is entirely specified by z_1 and does not depend on z_2, so we can integrate over the density to explicitly compute the cdf. We can obtain the equilibria in Proposition 3 as a solution to a differential equation. (To prove Proposition 2, we again only need to *verify* that the equilibrium μ in Proposition 3 which is easier; we defer a full proof of Proposition 3 to Section 4.6.)

Proposition 3 implies that the set of genres is always all of the directions in $\mathbb{R}_{\geq 0}$ (see Figure 2). Interestingly, the support $\text{supp}(\mu)$ of the equilibrium distribution is also always one-to-one: for every value of x, there is at most one value of y such that $(x, y) \in \text{supp}(\mu)$, and vice versa for y. However, the distribution of quality levels differs significantly with P (see the last panel of Figure 2). The equilibrium distribution of $\|p\|$ for $p \sim \mu$ for higher values of P stochastically dominates the equilibrium distribution of $\|p\|$ for $p \sim \mu$ for lower values of P, which aligns with intuition from the 1-dimensional case in Example 1. However, in contrast with the one-dimensional case, the shape of the support changes dramatically with P: for $P = 2$, it forms a quarter circle; for $P = 3$, it forms a line segment, and for $P \geq 4$, it becomes convex (see the first three panels of Figure 2). The support of the equilibrium distribution appears to approach the two standard basis vectors as P becomes larger.

Motivated by this collapse onto the standard basis vectors, we consider the equilibria that arise in the limit as $P \to \infty$. In fact, we show equilibria in this limiting marketplace that permit clean closed-form solutions not only for the standard basis vectors, but also for general configurations of two user vectors.

Informally speaking, $P = \infty$ corresponds to the limiting marketplace as the number of producers goes to ∞. Taking this limit is subtle, because the distribution of any single producer approaches a point mass at 0 in the limit. This is because the expected number of users that any producer wins is N/P, which approaches 0 in the limit, so the production cost that a producer can afford to expend approaches 0 in the limit. Nonetheless, the distribution of the *winning* producer turns out to be non-degenerate. To get intuition from this, let's revisit the one-dimensional setup of Example 1. The cumulative distribution function $F(p) = \left(\frac{p}{N}\right)^{\beta/(P-1)}$ of a single producer as $P \to \infty$ approaches $F(p) = 1$ for any $p > 0$ (this corresponds to a point mass at 0). On the other hand, the cumulative distribution function of the *winning* producer $F^{\max}(p) = \left(\frac{p}{N}\right)^{\beta P/(P-1)}$ approaches $\left(\frac{p}{N}\right)^{\beta}$, which is a well-defined function.

We thus focus on the winning producers when defining what an equilibrium means for $P = \infty$. Since our characterization result (Theorem 3) focuses on finite-genre equilibria, we restrict our formal definition to case of finite genres.

Definition 1 (Finite-genre equilibria for $P = \infty$)**.** *Let $u_1, \ldots, u_N \in \mathbb{R}^D_{\geq 0}$ be a set of users and let $c(p) = \|p\|_2^\beta$ be the cost function. A set of genres $d_1, \ldots, d_G \in \mathbb{R}^D_{\geq 0}$ such that $\|d_i\|_2 = 1$ for all $1 \leq g \leq G$, a set of conditional quality distributions $F_1, F_2, \ldots, F_G$ over $\mathbb{R}_{\geq 0}$, and a set of weights $\alpha_1, \ldots, \alpha_G \geq 0$ such that $\sum_{g=1}^G \alpha_g = 1$ forms a finite-genre equilibrium if the following condition holds for*

$$p^* \in \arg\max_{p \in \mathbb{R}^D_{\geq 0}} \left(\sum_{i=1}^N \left(\prod_{g=1}^G \left(F_g \left(\frac{\langle u_i, p \rangle}{\langle u_i, d_g \rangle} \right) \right)^{\alpha_i} \right) - c(p) \right) \tag{27}$$

for any $p^* = q_i d_i$ such that $1 \leq i \leq G$ and $q_i \in supp(F_i)$.

The motivation for Definition 1 is that it is a natural limit of conditions (C1)-(C3) in Lemma 1 as $P \to \infty$. More specifically, if P were finite, we would be able to establish a one-to-one correspondence between sets $(d_1, \ldots, d_G, F_1, \ldots, F_G, \alpha_1, \ldots, \alpha_G)$ and distributions μ with finitely many genres. In particular, let $(d_1, \ldots, d_G, F_1, \ldots, F_G, \alpha_1, \ldots, \alpha_G)$ correspond to the distribution μ of a single producer given by the following: with probability α_i, choose the vector $q_g d_g$ where q_g is drawn from a distribution with cdf $(F_g(\cdot))^{1/(P-1)}$. If we let:

$$H_i(z_i) = \left(\sum_{g=1}^G \alpha_g F_g \left(\frac{\langle u_i, p \rangle}{\langle u_i, d_g \rangle} \right)^{1/(P-1)} \right)^{P-1},$$

then conditions (C3) and (C2) are satisfied by the distribution μ and the distribution $\mathbf{U}p$ for $p \sim \mu$. Condition (C1) boils down to requiring that:

$$\max_{p \in \mathbb{R}^D_{\geq 0}} \left(\sum_{i=1}^N H_i(\langle u_i, p \rangle) - c(p) \right)$$

is maximized for any $p \in \text{supp}(\mu)$, which can be rewritten as requiring that any $p^* \in \text{supp}(\mu)$ satisfies:

$$p^* \in \arg\max_{p \in \mathbb{R}^D_{\geq 0}} \left(\sum_{i=1}^N \left(\sum_{g=1}^G \alpha_g F_g \left(\frac{\langle u_i, p \rangle}{\langle u_i, d_g \rangle} \right)^{1/(P-1)} \right)^{P-1} - c(p) \right). \tag{28}$$

Taking a limit as $P \to \infty$, we see that

$$\left(\sum_{g=1}^G \alpha_g F_g \left(\frac{\langle u_i, p \rangle}{\langle u_i, d_g \rangle} \right)^{1/(P-1)} \right)^{P-1} \to \prod_{g=1}^G \left(F_g \left(\frac{\langle u_i, p \rangle}{\langle u_i, d_g \rangle} \right) \right)^{\alpha_i}.$$

Thus, in the limit, equation (28) approaches equation (27).

Using the formalization in Definition 1 of equilibria for $P = \infty$, we investigate the case of two homogeneous populations of users, and we characterize two-genre equilibria.

Theorem 5. *[Formal version of Theorem 3] Suppose that there are 2 users located at two linearly independently vectors $u_1, u_2 \in \mathbb{R}^D_{\geq 0}$, let $\theta^* := \cos^{-1}\left(\frac{\langle u_1, u_2 \rangle}{\|u_1\|_2\|u_2\|}\right) < 0$ be the angle between them. Suppose we have cost function $c(p) = \|p\|_2^\beta$, $\beta > \beta^* = \frac{2}{1-\cos(\theta^*)}$, and $P = \infty$ producers. Then, the genres d_1, d_2, conditional quality distributions $F_1 = F$ and $F_2 = F$, and weights $\alpha_1 = \alpha_2 = 2$ form an equilibrium[6], where*

$$\{d_1, d_2\} := \left\{[\cos(\theta^G + \theta_{min}), \sin(\theta^G + \theta_{min})], [\cos(\theta^* - \theta^G + \theta_{min}), \sin(\theta^* - \theta^G + \theta_{min})]\right\}$$

such that $\theta^G := \arg\max_{\theta \leq \theta^*/2}\left(\cos^\beta(\theta) + \cos^\beta(\theta^* - \theta)\right)$ *and* $\theta_{min} := \min\left(\cos^{-1}\left(\frac{\langle u_1, e_1 \rangle}{\|u_1\|}\right), \cos^{-1}\left(\frac{\langle u_2, e_1 \rangle}{\|u_2\|}\right)\right)$, *and where*

$$F(q) := \begin{cases} C_2^{(2n+2)\beta} & \text{if } q \in C_1^{1/\beta} C_2^{2n+1}\,[C_2, 1] \text{ for } n \geq 0 \\ C_1^{-2} C_2^{-2n\beta} q^{2\beta} & \text{if } q \in C_1^{1/\beta} C_2^{2n}\,[C_2, 1] \text{ for } n \geq 0 \\ 1 & \text{if } q \geq C_1^{1/\beta}, \end{cases},$$

such that the constants are defined by $C_1 := \frac{\sin(\theta^*)\cos(\theta^G)}{\sin(\theta^* - \theta_G)}$ *and* $C_2 := \frac{\cos(\theta^* - \theta^G)}{\cos(\theta^G)}$.

Machinery that is conceptually similar to Lemma 1 enables us to systematically identify the particular equilibrium within the family of two-genre equilibrium. The first-order condition (Lemma 10) given by condition (C1) helps identify the location of the genre directions, and this further enables us to compute the cdfs H_1 and H_2. At this stage, the proof boils down to solving for the conditional quality distributions F_1 and F_2. We obtain an infinite-producer limit of the functional equations in (26) which can be solved directly. (To prove Theorem 5, we again only need to *verify* that the equilibrium μ in Theorem 5 which is easier; we defer a full proof of Theorem 5 to Section 4.6.)

Theorem 5 reveals that finite-genre equilibria re-emerge in the limit as $P \to \infty$, although they do not exist for any finite P. For users located at the standard basis vectors, Theorem 5 formalizes the intuition from Proposition 3 (and Figure 2) that the equilibria converges to a distribution supported on the standard basis vectors. This means that at $P = \infty$, producers either entirely personalize their content to the first user or entirely personalize their content to the second user, but do not try to appeal to both users at the same time.

Interestingly, the set of genres is *not* equal to the set of two users unless users are orthogonal. As shown in Figure 3, the two-genres are located within the interior of the convex cone formed by the two users. This means that producers always attempt to cater their content to both users at the same time, although they either place a greater weight on one user or place a greater weight on the other user, depending on which genre they choose. Moreover, the balance between personalization and catering to an average user changes for different values of β. When β approaches the single-genre boundary, θ^G and $\theta^* - \theta^G$ both collapse onto the single-genre direction $\theta^*/2$. On the other hand, when β approaches ∞, θ^G approaches 0 and $\theta^* - \theta^G$ approaches θ^*, so the genres converge onto the two users.

Beyond the structure of the genres, the distribution of content across genres exhibits striking behavior. Observe that the conditional quality distributions of each genre (see the last panel of Figure 3) has gaps in its support. In fact, the cdf given by a countably-infinite piecewise function, where each piece is either constant or grows proportionally to $q^{2\beta}$. The level of "bumpiness" of the cdf decreases as θ^* increases: for the limiting case of $\theta = \pi/2$, it converges to the smooth function $F(q) = q^{2\beta}$. Interestingly, for $\theta^* < \pi/2$, the regions of zero density of each of the two genres are

[6] See Definition 1 for a formalization of equilibrium for $P = \infty$.

actually staggered. More formally, let's consider the "support"

$$S' := \{q_1[\cos(\theta_G), \sin(\theta_G)] \mid q_1 \in \text{supp}(F_1)\} \cup \{q_2[\cos(\theta^* - \theta_G), \sin(\theta^* - \theta_G)] \mid q_2 \in \text{supp}(F_2)\}$$

of the equilibrium. The reparameterized version of the support (see the first three panels of Figure 3) given by:

$$S := \left\{\mathbf{U}p \mid p \in S'\right\}$$

is *one-to-one*: that is, for any value of z_1, there is at most one value of z_2 such that $(z_1, z_2) \in S$ (and a similar property holds when the roles of z_1 and z_2 are swapped). This parallels the one-to-one property that we showed in Proposition 3 for the standard basis vectors with finite values of P. An interesting direction for future work would be to uncover when this one-to-one property holds in general.

4.4 Proofs of auxiliary lemmas from Section 4.1

We prove Lemma 9.

Proof of Lemma 9. It suffices to show that if $z_1 = \langle u_1, p\rangle$ and $z_2 = \langle u_2, p\rangle$, then:

$$\|p\|^2 = \frac{z_1^2 + z_2^2 - 2z_1 z_2 \cos(\theta^*)}{\sin^2(\theta^*)} \tag{29}$$

WLOG, let $u_1 = e_1$ and let $u_2 = [\cos(\theta^*), \sin(\theta^*)]$. We see that:

$$\begin{aligned}
\frac{z_1^2 + z_2^2 - 2z_1 z_2 \cos(\theta^*)}{\sin^2(\theta^*)} &= \frac{p_1^2 + (p_1\cos(\theta^*) + p_2\sin(\theta^*))^2 - 2p_1(p_1\cos(\theta^*) + p_2\sin(\theta^*))\cos(\theta^*)}{\sin^2(\theta^*)} \\
&= \frac{p_1^2\sin^2(\theta^*) + p_2^2\sin^2(\theta^*)}{\sin^2(\theta^*)} \\
&= p_1^2 + p_2^2 \\
&= \|p\|_2^2,
\end{aligned}$$

which proves equation (29). □

We prove Lemma 10.

Proof of Lemma 10. Since μ is a symmetric mixed equilibrium, z must be a maximizer of equation (7). The equation

$$\begin{bmatrix} h_1(z_1) \\ h_2(z_2) \end{bmatrix} = \nabla_z(c_{\mathbf{U}}(z))$$

is the first-order condition and thus holds for every z is in the support of μ.

Next, we show that:

$$\nabla_z(c_{\mathbf{U}}(z)) = \beta\alpha^\beta \sin^{-\beta}(\theta^*)\left(\left(z_1^2 + z_2^2 - 2z_1 z_2\cos(\theta^*)\right)^{\frac{\beta}{2}-1}\right)\begin{bmatrix} z_1 - z_2\cos(\theta^*) \\ z_2 - z_1\cos(\theta^*) \end{bmatrix}.$$

By applying Lemma 9, we see that:

$$\begin{aligned}
\nabla_z(c_{\mathbf{U}}(z)) &= \nabla_z\left(\alpha^\beta \sin^{-2\beta}(\theta^*)\left(z_1^2 + z_2^2 - 2z_1 z_2\cos(\theta^*)\right)^{\frac{\beta}{2}}\right) \\
&= \alpha^\beta \sin^{-\beta}(\theta^*) \cdot \nabla_z\left(\left(z_1^2 + z_2^2 - 2z_1 z_2\cos(\theta^*)\right)^{\frac{\beta}{2}}\right) \\
&= \beta\alpha^\beta \sin^{-\beta}(\theta^*)\left(\left(z_1^2 + z_2^2 - 2z_1 z_2\cos(\theta^*)\right)^{\frac{\beta}{2}-1}\right)\begin{bmatrix} z_1 - z_2\cos(\theta^*) \\ z_2 - z_1\cos(\theta^*) \end{bmatrix},
\end{aligned}$$

as desired.

Finally, we show that

$$\nabla_z(c_{\mathbf{U}}(z)) = \beta\alpha^\beta \sin^{-\beta}(\theta^*) \left(z_1^2 + z_2^2 - 2z_1 z_2 \cos(\theta^*)\right)^{\frac{\beta}{2}-1} \begin{bmatrix} z_1 - z_2\cos(\theta^*) \\ z_2 - z_1 \cos(\theta^*) \end{bmatrix}.$$

We see that:

$$\begin{aligned}
\nabla_z(c_{\mathbf{U}}(z)) &= \beta\alpha^\beta \sin^{-\beta}(\theta^*) \left(\left(z_1^2 + z_2^2 - 2z_1 z_2 \cos(\theta^*)\right)^{\frac{\beta}{2}-1} \right) \begin{bmatrix} z_1 - z_2\cos(\theta^*) \\ z_2 - z_1 \cos(\theta^*) \end{bmatrix} \\
&= \beta\alpha^\beta r^{\beta-2} \begin{bmatrix} \frac{z_1 - z_2\cos(\theta^*)}{\sin^2(\theta^*)} \\ \frac{z_2 - z_1\cos(\theta^*)}{\sin^2(\theta^*)} \end{bmatrix} \\
&= \beta\alpha^\beta r^{\beta-1} \begin{bmatrix} \frac{\cos(\theta) - \cos(\theta^* - \theta)\cos(\theta^*)}{\sin^2(\theta^*)} \\ \frac{\cos(\theta^* - \theta) - \cos(\theta)\cos(\theta^*)}{\sin^2(\theta^*)} \end{bmatrix} \\
&= \beta\alpha^\beta r^{\beta-1} \begin{bmatrix} \frac{\cos(\theta * - (\theta^* - \theta)) - \cos(\theta^* - \theta)\cos(\theta^*)}{\sin^2(\theta^*)} \\ \frac{\sin(\theta^*)\sin(\theta)}{\sin^2(\theta^*)} \end{bmatrix} \\
&= \beta\alpha^\beta r^{\beta-1} \begin{bmatrix} \frac{\sin(\theta^*)\sin(\theta^* - \theta)}{\sin^2(\theta^*)} \\ \frac{\sin(\theta)}{\sin(\theta^*)} \end{bmatrix} \\
&= \beta\alpha^\beta r^{\beta-1} \begin{bmatrix} \frac{\sin(\theta^* - \theta)}{\sin(\theta^*)} \\ \frac{\sin(\theta)}{\sin(\theta^*)} \end{bmatrix},
\end{aligned}$$

as desired.

□

We now prove Lemma 11. To prove Lemma 11, we use the following sublemma.

Lemma 12. *Let there be 2 users located at $u_1, u_2 \in \mathbb{R}^D_{\geq 0}$ such that $\|u_1\| = \|u_2\| = 1$, and let $\theta^* := \cos^{-1}(\langle u_1, u_2 \rangle) > 0$ be the angle between the user vectors. Let the cost function be $c(p) = \alpha\|p\|_2^\beta$ for $\alpha > 0$. If z is of the form $[r\cos(\theta), r\cos(\theta^* - \theta)]$ for $\theta \in [0, \theta^*]$, then the sign of $\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2}$ is equal to the sign of:*

$$\frac{\beta - 2}{\beta}\cos(\theta^* - 2\theta) - \cos(\theta^*).$$

Proof of Lemma 12. By construction, we see that $z \in \left\{\mathbf{U}p \mid p \in \mathbb{R}^D_{\geq 0}\right\}$. We can apply Lemma 10 to see that

$$\begin{aligned}
\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2} &= \frac{\partial^2}{\partial z_1 \partial z_2} \left(\sin^{-2\beta}(\theta^*) \left(z_1^2 + z_2^2 - 2z_1 z_2 \cos(\theta^*)\right)^{\frac{\beta}{2}} \right) \\
&= \frac{\partial}{\partial z_2} \left(\beta\alpha^\beta \sin^{-\beta}(\theta^*) \left(z_1^2 + z_2^2 - 2z_1 z_2 \cos(\theta^*)\right)^{\frac{\beta}{2}-1} (z_1 - z_2\cos(\theta^*)) \right) \\
&= \beta\alpha^\beta \sin^{-\beta}(\theta^*) \frac{\partial}{\partial z_2} \left(\left(z_1^2 + z_2^2 - 2z_1 z_2 \cos(\theta^*)\right)^{\frac{\beta}{2}-1} (z_1 - z_2\cos(\theta^*)) \right).
\end{aligned}$$

This is the same sign as:

$$\frac{\partial}{\partial z_2}\left(\left(z_1^2+z_2^2-2z_1z_2\cos(\theta^*)\right)^{\frac{\beta}{2}-1}(z_1-z_2\cos(\theta^*))\right)$$
$$=(\beta-2)\left(z_1^2+z_2^2-2z_1z_2\cos(\theta^*)\right)^{\frac{\beta}{2}-2}(z_1-z_2\cos(\theta^*))(z_2-z_1\cos(\theta^*))-\left(z_1^2+z_2^2-2z_1z_2\cos(\theta^*)\right)^{\frac{\beta}{2}-1}\cos(\theta^*)$$
$$=\left(z_1^2+z_2^2-2z_1z_2\cos(\theta^*)\right)^{\frac{\beta}{2}-2}\left((\beta-2)(z_1-z_2\cos(\theta^*))(z_2-z_1\cos(\theta^*))-\cos(\theta^*)\left(z_1^2+z_2^2-2z_1z_2\cos(\theta^*)\right)\right)$$

This is the same sign as:

$$(\beta-2)(z_1-z_2\cos(\theta^*))(z_2-z_1\cos(\theta^*))-\cos(\theta^*)\left(z_1^2+z_2^2-2z_1z_2\cos(\theta^*)\right).$$

Let's represent z as $[r\cos(\theta), r\cos(\theta^*-\theta)]$. The above expression is the same sign as:

$$\begin{aligned}&(\beta-2)(\cos(\theta)-\cos(\theta^*-\theta)\cos(\theta^*))(\cos(\theta^*-\theta)-\cos(\theta)\cos(\theta^*))-\cos(\theta^*)\sin^2(\theta^*)\\ &=(\beta-2)(\sin(\theta^*)\sin(\theta^*-\theta))(\sin(\theta)\sin(\theta^*))-\cos(\theta^*)\sin^2(\theta^*)\\ &=\sin^2(\theta^*)\left((\beta-2)\sin(\theta^*-\theta)\sin(\theta)-\cos(\theta^*)\right).\end{aligned}$$

This is the same sign as:

$$(\beta-2)\sin(\theta^*-\theta)\sin(\theta)-\cos(\theta^*)=(\frac{\beta}{2}-1)(\cos(\theta^*-2\theta)-\cos(\theta^*))-\cos(\theta^*)=\left(\frac{\beta}{2}-1\right)(\cos(\theta^*-2\theta)-\frac{\beta}{2}\cos(\theta^*).$$

This is the same sign as:

$$\frac{\beta-2}{\beta}\cos(\theta^*-2\theta)-\cos(\theta^*).$$

□

We prove Lemma 11.

Proof of Lemma 11. By Lemma 12, we see that $\left(\frac{\beta-2}{\beta}\cos(\theta^*-2\theta)-\cos(\theta^*)\right)$ has the same sign as $\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1\partial z_2}$. Thus it suffices to show that $g'(z_1)\cdot\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1\partial z_2}\le 0$. When $\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1\partial z_2}=0$, the condition in the proposition statement is trivially satisfied. We thus assume for the remainder of the proof that $\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1\partial z_2}\neq 0$.

The second-order condition for z to be a maximizer of equation (7) is the following:

$$\begin{bmatrix}h_1'(z_1) & 0\\ 0 & b \quad h_2'(z_2)\end{bmatrix}-\nabla^2 c_{\mathbf{U}}(z)\preceq 0. \tag{30}$$

Let's apply Lemma 10, to see that:

$$h_1(x)=\frac{\partial c_{\mathbf{U}}([x,g(x)])}{\partial z_1}.$$

Since this holds in a neighborhood of z_1, we see that:

$$h_1'(z_1)=\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1^2}+g'(z_1)\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1\partial z_2}.$$

An analogous argument, coupled with the inverse function theorem, shows that:

$$h_2'(z_2)=\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_2^2}+\frac{1}{g'(z_1)}\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1\partial z_2}.$$

Plugging this into equation (30), we obtain:

$$\begin{aligned}
0 &\succeq \begin{bmatrix} h_1'(z_1) & 0 \\ 0 & b \quad h_2'(z_2) \end{bmatrix} - \nabla^2 c_{\mathbf{U}}(z) \\
&= \begin{bmatrix} \frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1^2} + g'(z_1)\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2} & 0 \\ 0 & \frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_2^2} + \frac{1}{g'(z_1)}\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2} \end{bmatrix} - \nabla^2 c_{\mathbf{U}}(z) \\
&= \begin{bmatrix} g'(z_1)\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2} & -\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2} \\ -\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2} & \frac{1}{g'(z_1)}\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2} \end{bmatrix} \\
&= \frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2} \begin{bmatrix} g'(z_1) & -1 \\ -1 & \frac{1}{g'(z_1)}. \end{bmatrix}
\end{aligned}$$

When $\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2} = 0$, the condition in the proposition statement is trivially satisfied. Since we've assumed that $\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2} \neq 0$, the eigenvectors are $[1, g'(u)]$ which has eigenvalue 0 and $[-g'(u), 1]$ which has eigenvalue

$$\frac{(g'(z_1))^2 + 1}{g'(z_1)} \cdot \frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2}.$$

The sign of that eigenvalue is equal to the sign of $g'(z_1) \cdot \frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2}$. Since the matrix must be negative semidefinite, we see that $g'(z_1) \cdot \frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2} \leq 0$.

□

4.5 Proofs of structural properties of equilibria

We prove Corollary 6, restated below:

Corollary 6. *Suppose that there are N users split equally between two linearly independently vectors $u_1, u_2 \in \mathbb{R}_{\geq 0}^D$, and let $\theta^* := \cos^{-1}\left(\frac{\langle u_1, u_2 \rangle}{\|u_1\|_2 \|u_2\|}\right)$. Let the cost function be $c(p) = \|p\|_2^\beta$. Then it holds that:*

$$\beta^* = \frac{2}{1 - \cos(\theta^*)}.$$

Proof. By Claim 1, we can assume that there are 2 normalized users $\|u_1\| = \|u_2\|$. We further assume WLOG that $u_1 = e_1$.

We claim that if there is a single-genre equilibrium, it must be in the direction of $[\cos(\theta^*/2), \sin(\theta^*/2)]$. By Corollary 2, if there is a single-genre equilibrium in a direction p, then it must maximize $\log(\langle p, u_1 \rangle) + \log(\langle p, u_1 \rangle)$. Let's let $p = [\cos(\theta), \sin(\theta)]$. Then, we see that:

$$\log(\langle p, u_1 \rangle) + \log(\langle p, u_2 \rangle) = \log(\cos(\theta)) + \log(\cos(\theta^* - \theta)) = \log\left(\frac{\cos(\theta^*) + \cos(\theta^* - 2\theta)}{2}\right),$$

which is uniquely maximized at $\theta = \theta^*/2$ as desired.

We first show that $\beta^* \leq \frac{2}{1-\cos(\theta^*)}$. Assume for sake of contradiction that there is a single-genre equilibrium. The above argument shows that it must be in the direction of $[\cos(\theta^*/2), \sin(\theta^*/2)]$. By Lemma 2, we know that the support of the equilibrium distribution is a line segment. If $\beta > \frac{2}{1-\cos(\theta^*)}$, we see that

$$\frac{\beta - 2}{\beta} \cos(\theta^* - 2\theta) - \cos(\theta^*) = 1 - \frac{2}{\beta} - \cos(\theta^*) < 0.$$

By Lemma 1 and Lemma 11, we see that the single-genre line $(z, g(z))$ must have $g'(z_1) \leq 0$ in its support, which is a contradiction.

We next show that $\beta^* \leq \frac{2}{1-\cos(\theta^*)}$. It suffices to show that the single-genre distribution in the direction of $[\cos(\theta^*/2), \sin(\theta^*/2)]$ with cdf given by $F(q) = \left(\frac{q^\beta}{2}\right)^{1/(P-1)}$. We apply Claim 1; it suffices to verify condition (C1). Notice that

$$H_1(w) = H_2(w) = \left(\frac{w^\beta}{2\cos^\beta(\theta^*/2)}\right).$$

Thus, equation (7) can be written as:

$$\max_z \left(\min(1, \frac{z_1^\beta}{2\cos^\beta(\theta^*/2)}) + \min(1, \frac{z_2^\beta}{2\cos^\beta(\theta^*/2)}) - c_{\mathbf{U}}(z)\right).$$

It suffices to show that that for all z, it holds that:

$$z_1^\beta + z_2^\beta - 2\cos^\beta(\theta^*/2)\left(\frac{z_1^2 + z_2^2 - 2z_1 z_2 \cos(\theta^*)}{\sin^2(\theta^*)}\right)^\beta \leq 0.$$

Let $z = [r\cos(\theta), r\cos(\theta^* - \theta)]$. Then this reduces to:

$$\cos^\beta(\theta) + \cos^\beta(\theta^* - \theta) \leq 2\cos^\beta(\theta^*/2) \leq 0.$$

We observe that $\cos^\beta(\theta) + \cos^\beta(\theta^* - \theta)$ is maximized at $\theta = \theta^*/2$, which proves the desired statement.

□

We prove Proposition 1, restated below:

Proposition 1. *Suppose that there are N users split equally between two linearly independently vectors $u_1, u_2 \in \mathbb{R}^2_{\geq 0}$, and let $\theta^* := \cos^{-1}\left(\frac{\langle u_1, u_2\rangle}{\|u_1\|_2\|u_2\|}\right)$. Let the cost function be $c(p) = \|p\|_2^\beta$, and let $P \geq 2$. Let μ be a symmetric Nash equilibrium such that the distributions $\langle u_1, p\rangle$ and $\langle u_2, p\rangle$ over $\mathbb{R}_{\geq 0}$ are absolutely continuous. As long as $\beta \neq 2$ or $\theta^* \neq \pi/2$, the support of μ does not contain an ℓ_2-ball of radius ϵ for any $\epsilon > 0$.*[7]

Proof of Proposition 1. Assume for sake of contradiction that the support of μ contains an ℓ_2-ball of radius $\epsilon_1 > 0$. We apply Lemma 1 and show that condition (C1) is violated. Since μ contains a ball of ϵ_1-radius ball, we know that the distribution Z over Up over $p \sim \mu$ contains an ℓ_2 ball of radius $\epsilon_2 > 0$. Let this ball be B. Notice that Z_1 and Z_2 are absolutely continuous by assumption, Z_1 and Z_2 have bounded support, and the function $m \mapsto m^{P-1}$ is Lipschitz on any bounded interval: this means that H_1 and H_2 are also absolutely continuous. This means that densities exist a.e. For $(z_1, z_2) \in B$, we can apply the first-order condition in Lemma 10 to obtain that:

$$h_1(z_1) = \frac{\partial c_{\mathbf{U}}(z)}{\partial z_1}$$

We see that this needs to be satisfied for $z = [z_1, m]$ where $m \in (z_2 - \epsilon', z_2 + \epsilon')$. This means that the mapping $m \mapsto \frac{\partial c_{\mathbf{U}}([z_1, m])}{\partial z_1}$ needs to be a constant on $m \in (z_2 - \epsilon', z_2 + \epsilon')$. This means that the derivative of this mapping with respect to z_2 needs to be 0, so:

$$\frac{\partial^2 c_{\mathbf{U}}([z_1, z_2])}{\partial z_1 \partial z_2} = 0 \tag{31}$$

[7] The case of $\beta = 2$ and $\theta^* = \pi/2$ is degenerate and permits a range of possible equilibria.

for all $z \in B$.

We apply Lemma 12 to show that equation (31) cannot be zero on all of B. For all z that satisfy equation (31), Lemma 12 implies if we represent z as $\mathbf{U}[r\cos(\theta), r\sin(\theta)]$, then

$$\frac{\beta-2}{\beta}\cos(\theta^*-2\theta)=\cos(\theta^*).$$

If equation (31) holds for all $z \in B$, then it must hold at all θ within some nonempty interval. This is a contradiction as long as $\beta \neq 2$ or $\theta^* \neq \pi/2$.

For the special case where $\beta = 2$ and $\theta^* = \pi/2$, □

We next prove Theorem 4, restated below:

Theorem 4. *[Formal version of Theorem 2] Suppose that there are N users split equally between two linearly independently vectors $u_1, u_2 \in \mathbb{R}^D_{\geq 0}$, and let $\theta^* := \cos^{-1}\left(\frac{\langle u_1, u_2\rangle}{\|u_1\|_2\|u_2\|}\right)$. Let the cost function be $c(p) = \|p\|_2^\beta$. Let μ be a a distribution on $\mathbb{R}^d$ such that the distributions $\langle u_1, p\rangle$ and $\langle u_2, p\rangle$ over $\mathbb{R}_{\geq 0}$ over $\mathbb{R}_{\geq 0}$ for $p \sim \mu$ are absolutely continuous and twice continuously differentiable within their supports. There are two regimes based on β and θ^*:*

1. *If $\beta < \beta^* = \frac{2}{1-\cos(\theta^*)}$ and if μ is a symmetric mixed equilibrium, then μ satisfies $|\mathrm{Genre}(\mu)| = 1$.*
2. *If $\beta > \beta^* = \frac{2}{1-\cos(\theta^*)}$, if $|\mathrm{Genre}(\mu)| < \infty$, and if the conditional distribution of $\|p\|$ along each genre is continuously differentiable, then μ is not an equilibrium.*

We split into two propositions: together, these propositions directly imply Theorem 4.

Proposition 9. *Consider the setup in Theorem 4. If $\beta < \beta^* = \frac{2}{1-\cos(\theta^*)}$ and μ is a symmetric mixed equilibrium, then μ satisfies $|\mathrm{Genre}(\mu)| = 1$.*

Proposition 10. *Consider the setup in Theorem 4. If $\beta > \beta^* = \frac{2}{1-\cos(\theta^*)}$, if $|\mathrm{Genre}(\mu)| < \infty$, and if the conditional distribution of $\|p\|$ along each genre is continuous differentiable, then μ is not an equilibrium.*

First, we prove Proposition 9.

Proof of Proposition 9. By Claim 1, it suffices to focus on the case of 2 normalized users. By Lemma 1, it suffices to study (H_1, H_2, S) that satisfy (C1), (C2), and (C3).

Let $\mathrm{supp}(H_1) = I_1$ and let $\mathrm{supp}(H_2) = I_2$. Note that since the distributions are twice continuously differentiable, we know that the densities h_1 and h_2 exist and are continuously differentiable a.e on I_1 and I_2 respectively. We break the proof into several steps.

Step 1: there exists a one-to-one function g such that $S = \{(w, g(w)) \mid w \in I_1\}$ and where g is continuously differentiable and strictly increasing. We first show that $\frac{\partial^2 c_{\mathbf{U}}(z)}{\partial z_1 \partial z_2} < 0$ everywhere. By Lemma 12, it suffices to show that $\frac{\beta-2}{\beta}\cos(\theta^*-2\theta) - \cos(\theta^*) < 0$. To see this, notice that

$$\frac{\beta-2}{\beta}\cos(\theta^*-2\theta)-\cos(\theta^*) < 0 \leq \frac{\beta-2}{\beta}-\cos(\theta^*) = 1-\cos(\theta^*)-\frac{2}{\beta} < 0$$

because $\beta < \frac{2}{1-\cos(\theta^*)}$.

We now show that the support S is equal to $\{(w, g(w)) \mid w \in I_1\}$ for some one-to-one function $g : I_1 \to I_2$. To show this, it suffices to show that the support does contain both (z_1, z_2) and (z_1, z_2')

for $z_2 \neq z_2'$ (and, analogously, the support does not contain both (z_1', z_2) and (z_1, z_2) for $z_1 \neq z_1'$). Notice that for any fixed value of z_1, the function $z_2 \mapsto \frac{\partial c_{\mathbf{U}}([z_1,z_2])}{\partial z_1}$ is strictly decreasing. If (z_1, z_2) and (z_1, z_2') are both in the support, then by Lemma 10, it must be true that:

$$h_1(z_1) = \frac{\partial c_{\mathbf{U}}([z_1, z_2])}{\partial z_1} = \frac{\partial c_{\mathbf{U}}([z_1, z_2'])}{\partial z_1}.$$

However, since $z_2 \mapsto \frac{\partial c_{\mathbf{U}}([z_1,z_2])}{\partial z_1}$ is strictly decreasing, this means that $z_2 = z_2'$ as desired.

We can thus implicitly define the function g by the (unique) value such that:

$$Q(w, g(w)) - h_1(w) = 0$$

where

$$Q(z_1, z_2) := \frac{\partial c_{\mathbf{U}}([z_1, z_2])}{\partial z_1}.$$

Uniqueness follows from the fact that Q is a strictly decreasing function in its second argument, since $\frac{\partial Q(w,g(w))}{\partial z_2} = \frac{\partial^2 c_{\mathbf{U}}([w,g(w)])}{\partial z_1 \partial z_2} < 0$ as we showed previously. Since $h_1(w)$ is continuously differentiable and since:

$$\frac{\partial Q(w, g(w))}{\partial z_2} \neq 0$$

for $w \in I_1$, we can apply the implicit function theorem to see that $g(w)$ is continuously differentiable for $w \in I_1$.

We next show that g is increasing on I_1. Within the interior of I_1, by Lemma 11 along with the fact that $\frac{\beta-2}{\beta}\cos(\theta^* - 2\theta) - \cos(\theta^*) < 0$ everywhere, we see that g is a strictly increasing function on each contiguous portion of I_1. It thus suffices to show I_1 is an interval and that there are no gaps. If there is a gap, there must be a gap for both z_1 *and* z_2 at the same point z since the support is on-to-one and closed. However, if z is right above the gap, the producer would obtain higher utility by choosing $(1-\epsilon)z$ for sufficiently small ϵ to ensure that $(1-\epsilon)z$ is within the gap on both coordinates. This means that I_1 is an interval, which proves g is an increasing function.

Step 2: differential equation. We show that

$$g'(w)g(w) - g'(w)w\cos(\theta^*) = w - g(w)\cos(\theta^*), \tag{32}$$

for all $w \in \text{supp}(H_1)$.

First, we derive the the condition that we described in equation (25) and further simplify it using that g is increasing. Let $H_1^*(w) = H_1(w)^{\frac{1}{P-1}}$ and $H_2^*(w) = H_2(w)^{\frac{1}{P-1}}$. The densities h_1^* and h_2^* take the following form:

$$h_1^*(w) = (H_1^*)'(w) = \frac{1}{P-1} h_1(w) H_1(w)^{-\frac{P-2}{P-1}}$$
$$h_2^*(w) = (H_2^*)'(w) = \frac{1}{P-1} h_2(w) H_2(w)^{-\frac{P-2}{P-1}}.$$

In order for there to exist a distribution μ that satisfies condition (C2), it must hold that $H_1^*(w) = H_2^*(g(w))$ because g is increasing. (This also means that $H_1(w) = H_2(g(w))$.) This means that $h_1^*(w) = h_2^*(g(w))g'(w)$ and $H_1(w) = H_2(g(w))$. Plugging this into the above expressions for h_1^* and h_2^*, this means that:

$$h_1(w) = (P-1)h_1^*(w)H_1(w)^{\frac{P-2}{P-1}} = (P-1)g'(w)h_2^*(g(w))H_2(g(w))^{\frac{P-2}{P-1}} = h_2(w)g'(w).$$

This means that

$$g'(w) = \frac{h_1(w)}{h_2(w)} = \frac{w - g(w)\cos(\theta^*)}{g(w) - w\cos(\theta^*)},$$

where the last line follows from Lemma 10. This gives us the desired differential equation.

Step 3: solving the differential equation. We claim that the only valid solution to the differential equation (32) is $g(w) = w$. To see this, let $f(w) = \frac{g(w)}{w}$. This means that $wf(w) = g(w)$ and thus $f(w) + wf'(w) = g'(w)$. Plugging this into equation (32) and simplifying we obtain a separable differential equation. The solutions to this differential equation are $f(w) = 1$ and the following:

$$f_K^*(w) = K - \log(w) = \frac{1}{2}\left((1 + \cos(\theta^*))\log(1 + f(w)) - (1 + \cos(\theta^*))\log(1 - f(w))\right)$$

for some constant K. Notice that for f_K^* to even be well-defined, we know that $f_K^*(w) < 1$ everywhere.

Assume for sake of contradiction that there exists an equilibrium with support given by $\{(w, g(w)) \mid w \in I\}$ for $g(w) \neq w$. Then we know that $g(w) = f_K^*(w) \cdot x$ for some K. In order for this solution to even be well-defined, it would imply that $f_K^*(w) < 1$ everywhere. This implies that $g(w) < w$, for all $w \in I_1$. However, we know that the function g^{-1} must satisfy the differential equation too (and $g^{-1}(w) \neq w$), so by an analogous argument, we know that $g^{-1}(w) < w$ for all $w \in I_2$, which means that $w < g(w)$. This is a contradiction.

We can thus conclude that since $g(x) = x$, we have that $|\text{Genre}(\mu)| = 1$ as desired.

□

Now, we prove Proposition 10.

Proof of Proposition 10. By Claim 1, it suffices to focus on the case of 2 normalized users. We further assume WLOG that $u_1 = e_1$ and $u_2 = [\cos(\theta^*), \sin(\theta^*)]$. Since $\beta > \frac{2}{1-\cos(\theta^*)}$, we know by Corollary 6 that there is no single-genre equilibrium. Assume for sake of contradiction that there exists a *finite*-genre equilibrium μ with $|\text{Genre}(\mu)| \geq 2$. By Lemma 1, we know that there exists H_1, H_2 and S associated with μ that satisfy (C1)-(C3). Our proof boils down to two steps: (1) showing that $|\text{Genre}(\mu)| = 2$ and that $H_1(z) = H_2(z) = cz^\beta$ for some constant $c > 0$, and (2) showing that there is no two-genre distribution μ that realizes these settings of H_1 and H_2.

Step 1: $|\text{Genre}(\mu)| = 2$ and $H_1(z) = H_2(z) = cz^\beta$ for some constant $c > 0$. First, we observe that the following quantity:

$$\text{Genre}_Z(S) := \left\{\frac{1}{c_{\mathbf{U}}(z)}[z_1, z_2] \mid z \in S\right\}$$

is exactly analogous to Genre(μ). In particular, vectors in $\text{Genre}_Z(S)$ are of the form $[\cos(\theta), \cos(\theta^* - \theta)]$ by the normalization by $c_{\mathbf{U}}(z)$. A vector $[\cos(\theta), \cos(\theta^* - \theta)]$ is in $\text{Genre}_Z(S)$ if and only if $[\cos(\theta), \sin(\theta)$ is in Genre(μ). Finite genres in terms of $\text{Genre}_Z(S)$ thus exactly. corresponds to finite genres in terms of Genre(μ).

We show that every $z \in \text{Genre}_Z(S)$ is on the form $[\cos(\theta), \cos(\theta^* - \theta)]$ for $\theta \in (0, \theta^*)$. It is easy to see that $\theta \in [0, \theta^*]$. It suffices to show that $\theta \neq 0$ and $\theta \neq \theta*$. We show that $\theta \neq 0$ (the case of $\theta \neq \theta*$ follows from an analogous argument). In this case, we see that there must be some set of the form $\{[r, r\cos(\theta^*)] \mid r \in [r_{\min}, r_{\max}]\}$ for $r_{\min} < r_{\max}$ that is subset of S. If $\theta^* = \pi/2$, then this would mean the distribution given by H_2 would have a point mass at 0, which is clearly not possible at equilibrium. Otherwise, if $\theta^* < \pi/2$, we apply (C1) and Lemma 10, and we see that

$h_2(r\cos(\theta^*)) = 0$. However, this is a contradiction, since there is positive probability mass on some line segment on this genre by assumption.

Now, we observe that the support of the cdfs H_1 and H_2 must be *intervals* of the form $[0, z_1^{\max}]$ and $[0, z_2^{\max}]$. First, assume for sake of contradiction that there is a gap in the support, then consider z_1^* such that $[z_1, z_2] \in S$ where z_1 is above the gap. Notice that $[z_1, z_2] \in S$ must be located on a genre $\theta \in (0, \theta^*)$. We can thus reduce z_1 and hold z_2 fixed, while keeping $H_1(z_1) + H_2(z_2)$ fixed, and reducing the cost $c_{\mathbf{U}}(z)$. This is a contradiction. Thus, the support of H_1 must be an interval. Next, let $z_1^{min} = \min(\text{supp}(H_1))$, and assume for sake of contradiction that $z_1^{min} > 0$, notice that some $[z_1^{min}, z_2] \in S$ must be located on a genre $\theta \in (0, \theta^*)$. We can thus reduce z_1^{min} and hold z_2 fixed, while keeping $H_1(z_1) + H_2(z_2)$ fixed, and reducing the cost $c_{\mathbf{U}}(z)$. This is a contradiction, so $z_1^{min} = 0$.

Next, we show that for any two genres θ_1 and θ_2 in the set of genres, it must hold that

$$\frac{\sin(\theta^* - \theta_1)}{\cos^{\beta-1}(\theta_1)} = \frac{\sin(\theta^* - \theta_2)}{\cos^{\beta-1}(\theta_2)} \text{ and } \frac{\sin(\theta_1)}{\cos^{\beta-1}(\theta^* - \theta_1)} = \frac{\sin(\theta_2)}{\cos^{\beta-1}(\theta^* - \theta_2)} \tag{33}$$

To prove this, suppose that $|\text{Genre}_Z(S)| = G$ and label the genres by the indices $1, \ldots, G$ arbitrarily. For $z_1 \in \text{supp}(H_1)$ let $T(z_1) \subseteq \{1, \ldots, G\}$ be the set of genres j where there exists z_2 such that $(z_1, z_2) \in S$ and $[z_1, z_2]$ is on genre j. Since the density h_1 is continuous, we know that by Lemma 10 that $h_1(z) = \frac{\partial c_{\mathbf{U}}(z)}{\partial z_1}$. This means that $\frac{\partial c_{\mathbf{U}}([z_1, z_2])}{\partial z_1}$ must be equal for all z_2 associated with the genres in $T(z_1)$. It is not difficult to show that if two genres j and j' have equal values of $\frac{\partial c_{\mathbf{U}}([z_1, z_2])}{\partial z_1}$ for some z_1, then it must hold if θ and θ' denote the associated genres, then:

$$\frac{\sin(\theta^* - \theta)}{\cos^{\beta-1}(\theta)} = \frac{\sin(\theta^* - \theta')}{\cos^{\beta-1}(\theta')}.$$

We can chain together these equalities to see that

$$\frac{\sin(\theta^* - \theta_1)}{\cos^{\beta-1}(\theta_1)} = \frac{\sin(\theta^* - \theta_2)}{\cos^{\beta-1}(\theta_2)}.$$

for any two genres. We can also repeat this same argument for z_2 and conclude that

$$\frac{\sin(\theta_1)}{\cos^{\beta-1}(\theta^* - \theta_1)} = \frac{\sin(\theta_2)}{\cos^{\beta-1}(\theta^* - \theta_2)}$$

for any two genres as well.

We show that there exist exactly 2 genres given by $\theta_1 < \theta_2$. Using Lemma 12, we see that for any θ, there are at most two values of $\theta' \neq \theta_1$ such that equation (33) can hold. Moreover, by Lemma 11, one of these values lies within the region where g' would have to be negative (which is not possible). Thus, there are at most two genres, and Lemma 12 further tells us that they lie on opposite sides of $\theta^*/2$.

Finally, we show that $H_1(z_1) = c_1 z_1^\beta$ and $H_2(z_2) = c_2 z_2^\beta$. We can apply Lemma 10 to get a formulation for the densities h_1 and h_2 and integrate to obtain that H_1 and H_2 are proportional to w^β. WLOG assume that $c_1 \geq c_2$.

Step 2: there is no two-genre distribution μ that realizes H_1 and H_2. Condition (C2) gives us functional equations that the distribution μ must satisfy for $P < \infty$. We show that these functional equations have no solution. As in the previous step, let the genres be $\{\theta_1, \theta_2\}$ where

$\theta_1 < \theta^*/2 < \theta_2$. Let F_1 be the cdf of the magnitude of the genre given by θ_1, and let F_2 be the cdf of the magnitude of the genre given by θ_2. We observe that:

$$\left(\alpha_1 F_1\left(\frac{z_1}{\cos(\theta_1)}\right) + \alpha_2 F_2\left(\frac{z_1}{\cos(\theta_2)}\right)\right)^{P-1} = c_1 z_1^{\beta}$$
$$\left(\alpha_1 F_1\left(\frac{z_2}{\cos(\theta^* - \theta_1)}\right) + \alpha_2 F_2\left(\frac{z_2}{\cos(\theta^* - \theta_2)}\right)\right)^{P-1} = c_2 z_2^{\beta}.$$

It is more convenient to write the equations as follows:

$$\alpha_1 F_1\left(\frac{z_1}{\cos(\theta_1)}\right) + \alpha_2 F_2\left(\frac{z_1}{\cos(\theta_2)}\right) = c_1 z_1^{\frac{\beta}{P-1}}$$
$$\alpha_1 F_1\left(\frac{z_2}{\cos(\theta^* - \theta_1)}\right) + \alpha_2 F_2\left(\frac{z_2}{\cos(\theta^* - \theta_2)}\right) = c_2 z_2^{\frac{\beta}{P-1}}.$$

For any z_1 within the support of H_1 and z_2 within the support of H_2

$$\frac{\alpha_1}{\cos(\theta_1)} f_1\left(\frac{z_1}{\cos(\theta_1)}\right) + \frac{\alpha_2}{\cos(\theta_2)} f_2\left(\frac{z_1}{\cos(\theta_2)}\right) = c_1 \frac{\beta}{P-1} z_1^{\frac{\beta}{P-1}-1}. \tag{34}$$

$$\frac{\alpha_1}{\cos(\theta^* - \theta_1)} f_1\left(\frac{z_2}{\cos(\theta^* - \theta_1)}\right) + \frac{\alpha_2}{\cos(\theta^* - \theta_2)} f_2\left(\frac{z_2}{\cos(\theta^* - \theta_2)}\right) = c_2 \frac{\beta}{P-1} z_2^{\frac{\beta}{P-1}-1}. \tag{35}$$

The first case is where the r_1^{max} and r_2^{max} are such that $\arg\max_i r_i^{max} \cos(\theta_i)$ and $\arg\max_i r_i^{max} \cos(\theta^* - \theta_i)$ are not both maximized by $i = 2$. We see that $\arg\max_i r_i^{max} \cos(\theta_i)$ must be maximized by $i = 1$. Let $z_1^{max} = r_1^{max} \cos(\theta_1)$, which we see is in the support of H_1. We see that $\frac{z_1^{max}}{\cos(\theta_2)}$ is not in the support of H_2 by assumption. This means that the density f_2 of F_2 at $\frac{z_1^{max}}{\cos(\theta_2)}$ is equal to 0 and, moreover, there exists $z_1^* < z_1^{max}$ sufficiently close to z_1^{max} such that z_1^* is in the support of H_1 and $\frac{z_1^*}{\cos(\theta_2)}$ is not in the support of F_2. At z_1^*, by equation (34), we see that:

$$\frac{\alpha_1}{\cos(\theta_1)} f_1\left(\frac{z_1^*}{\cos(\theta_1)}\right) = \frac{\alpha_1}{\cos(\theta_1)} f_1\left(\frac{z_1^*}{\cos(\theta_1)}\right) + \frac{\alpha_2}{\cos(\theta_2)} f_2\left(\frac{z_1^*}{\cos(\theta_2)}\right) = c_1 \frac{\beta}{P-1} (z_1^*)^{\frac{\beta}{P-1}-1}.$$

Now, let's let z_2^* be such that:

$$z_2^* := z_1^* \frac{\cos(\theta^* - \theta_1)}{\cos(\theta_1)}.$$

At z_2^*, we see that the left-hand side of equation (35) satisfies

$$
\begin{aligned}
&\frac{\alpha_1}{\cos(\theta^* - \theta_1)} f_1\left(\frac{z_2^*}{\cos(\theta^* - \theta_1)}\right) + \frac{\alpha_2}{\cos(\theta^* - \theta_2)} f_2\left(\frac{z_2^*}{\cos(\theta^* - \theta_2)}\right) \\
&\geq \frac{\alpha_1}{\cos(\theta^* - \theta_1)} f_1\left(\frac{z_2^*}{\cos(\theta^* - \theta_1)}\right) \\
&= \frac{\cos(\theta_1)}{\cos(\theta^* - \theta_1)} \left(\frac{\alpha_1}{\cos(\theta_1)} f_1\left(\frac{z_1^*}{\cos(\theta_1)}\right)\right) \\
&= \frac{\cos(\theta_1)}{\cos(\theta^* - \theta_1)} \left(c_1 \frac{\beta}{P-1} (z_1^*)^{\frac{\beta}{P-1}-1}\right) \\
&= \frac{\cos(\theta_1)}{\cos(\theta^* - \theta_1)} \left(c_1 \frac{\beta}{P-1} \left(z_2^* \frac{\cos(\theta_1)}{\cos(\theta^* - \theta_1)}\right)^{\frac{\beta}{P-1}-1}\right) \\
&= c_1 (z_2^*)^{\frac{\beta}{P-1}-1} \frac{\beta}{P-1} \left(\frac{\cos(\theta_1)}{\cos(\theta^* - \theta_1)}\right)^{\frac{\beta}{P-1}} \\
&> c_2 \frac{\beta}{P-1} (z_2^*)^{\frac{\beta}{P-1}-1}.
\end{aligned}
$$

However, this is a contradiction since (35) must hold.

The other case is that $i = 2$ maximizes both $\arg\max_i r_i^{max} \cos(\theta_i)$ and $\arg\max_i r_i^{max} \cos(\theta^* - \theta_i)$. Let (z_1^m, z_2^m) be the maximum coordinates achieved by the genre $i = 1$. For $z_1 > z_1^m$ and $z > z_2^m$, we can apply the equation given by (25) to see that the genre must be $\theta_2 = \theta^*/2$, which is a contradiction.

□

4.6 Proofs of closed-form equilibria in special cases

A sublemma that is useful in the proofs of Proposition 3 and Proposition 2 is the following.

Lemma 13. *For any $\beta \geq 2$, the expression*

$$\max_{z_1, z_2 \geq 0} \left(\left(\min\left(\left(\frac{2}{\beta}\right)^{-2/\beta} z_1^2, 1\right) + \min\left(\left(\frac{2}{\beta}\right)^{-2/\beta} z_2^2, 1\right)\right) - (z_1^2 + z_2^2)^{\beta/2}\right)$$

is maximized for any (z_1, z_2) such that $z_1^2 + z_2^2 = \left(\frac{2}{\beta}\right)^{2/\beta}$.

Proof. First, for z_1, z_2 such that $z_1^2 + z_2^2 = \left(\frac{2}{\beta}\right)^{2/\beta}$, we have that

$$\left(\frac{2}{\beta}\right)^{-2/\beta} (z_1^2 + z_2^2) - (z_1^2 + z_2^2)^{\beta/2} = 1 - \frac{2}{\beta}.$$

It thus suffices to prove that:

$$\left(\min\left(\left(\frac{2}{\beta}\right)^{-2/\beta} z_1^2, 1\right) + \min\left(\left(\frac{2}{\beta}\right)^{-2/\beta} z_2^2, 1\right)\right) - (z_1^2 + z_2^2)^{\beta/2} \leq 1 - \frac{2}{\beta}$$

for any $z_1, z_2 \geq 0$. It suffices to prove the stronger statement that:

$$\left(\frac{2}{\beta}\right)^{-2/\beta} (z_1^2 + z_2^2) - (z_1^2 + z_2^2)^{\beta/2} \leq 1 - \frac{2}{\beta}$$

Let $c = z_1^2 + z_2^2$; then we can rewrite the desired condition as:

$$\max_{c \geq 0} \left(\left(\frac{2}{\beta} \right)^{-2/\beta} c^2 - c^\beta \right) \leq 1 - \frac{2}{\beta}.$$

A first-order condition tells us for $\beta \geq 2$, that $\left(\frac{2}{\beta}\right)^{-2/\beta} c^2 - c^\beta$ is maximized at $c = \left(\frac{2}{\beta}\right)^{1/\beta}$, which proves the desired statement. $\square$

We prove Proposition 2, restated below.

Proposition 2. *Suppose that there are 2 users located at the standard basis vectors $e_1, e_2 \in \mathbb{R}^2$, and the cost function is $c(p) = \|p\|_2^\beta$. For $P = 2$ and $\beta \geq \beta^* = 2$, there is an equilibrium μ supported on the quarter-circle of radius $(2\beta^{-1})^{\frac{1}{\beta}}$, where the angle $\theta \in [0, \pi/2]$ has density $f(\theta) = 2\cos(\theta)\sin(\theta)$.*

Proof. By Lemma 1, it suffices to prove that (C1)-(C3) hold for H_1, H_2, and S associated with the distribution μ in the statement of the proposition. Conditions (C2) and (C3) follow by construction of μ, so it suffices to prove (C1).

First, we claim that

$$H_1(z_1) = \left(\frac{2}{\beta}\right)^{-2/\beta} z_1^2, \text{ and } H_2(z_2) = \left(\frac{2}{\beta}\right)^{-2/\beta} z_2^2.$$

We show that $H_2(z_2) = \left(\frac{2}{\beta}\right)^{-2/\beta} z_2^2$ (an analogous argument applies to H_1). We see that H_2 is supported on $\left[0, \left(\frac{2}{\beta}\right)^{1/\beta}\right]$ by construction, so it suffices to show that

$$h_2(z_2) = 2\left(\frac{2}{\beta}\right)^{-2/\beta} z_2$$

on this interval. Since $z_2 = \left(\frac{2}{\beta}\right)^{1/\beta} \sin(\theta)$, by the change of variables formula for $P = 2$, we see that

$$h_2(z_2)\left(\frac{2}{\beta}\right)^{1/\beta} \cos(\theta) = f(\theta) = 2\sin(\theta)\cos(\theta).$$

We can solve and obtain:

$$h_2(z_2) = 2\left(\frac{2}{\beta}\right)^{-1/\beta} \sin(\theta) = 2\left(\frac{2}{\beta}\right)^{-2/\beta} z_2,$$

as desired.

Now, we prove (C1). Applying Lemma 9, we see that:

$$H_1(z_1) + H_2(z_2) - c_{\mathbf{U}}(z) = \left(\min\left(\left(\frac{2}{\beta}\right)^{-2/\beta} z_1^2, 1\right) + \min\left(\left(\frac{2}{\beta}\right)^{-2/\beta} z_2^2, 1\right)\right) - (z_1^2 + z_2^2)^{\beta/2}.$$

Thus, equation (7) can be written as:

$$\max_{z_1, z_2 \geq 0} \left(\left(\min\left(\left(\frac{2}{\beta}\right)^{-2/\beta} z_1^2, 1\right) + \min\left(\left(\frac{2}{\beta}\right)^{-2/\beta} z_2^2, 1\right)\right) - (z_1^2 + z_2^2)^{\beta/2}\right) \tag{36}$$

We wish to show equation (36) is maximized whenever $z \in S$. Since $z_1^2 + z_2^2 = \left(\frac{2}{\beta}\right)^{2/\beta}$ for any $z \in S$, this follows from Lemma 13. $\square$

We prove Proposition 3, restated below.

Proposition 3. *Suppose that there are 2 users located at the standard basis vectors $e_1, e_2 \in \mathbb{R}^2$, with cost function $c(p) = \|p\|_2^\beta$ and $\beta = 2$. Then, there is an equilibrium μ with support equal to*

$$\left\{\left(x, (1 - x^{\frac{2}{P-1}})^{\frac{P-1}{2}}\right) \mid x \in [0, 1]\right\}, \tag{6}$$

and where the distribution of x has cdf equal to $\min(1, x^{2/(P-1)})$.

Proof of Proposition 3. By Lemma 1, it suffices to prove that (C1)-(C3) hold for H_1, H_2, and S for the distribution μ given in the statement of the proposition. Conditions (C2) and (C3) follow by construction of μ, so it suffices to prove (C1).

First, we claim that $H_1(z_1) = z_1^2$ and $H_1(z_1) = z_2^2$. We see that since the cdf of p_1 for $p \sim \mu$ is z_1, we know that $H_1(z_1) = z_1^2$ by construction. For z_2, first we note that the cdf of p_2 for $p_2 \sim \mu$ is given by:

$$\mathbb{P}_{p_2 \sim \mu}[p_2 \leq p_2'] = \mathbb{P}_{p_1 \sim \mu}\left[p_1 \geq (1 - (p_2')^{\frac{2}{P-1}})^{\frac{P-1}{2}}\right] = 1 - (1 - (p_2')^{\frac{2}{P-1}}) = (p_2')^{\frac{2}{P-1}}.$$

By definition, this means that $H_2(z_2) = z_2^2$ as desired.

Now, we prove (C1). Applying Lemma 9, we see that:

$$H_1(z_1) + H_2(z_2) - c_{\mathbf{U}}(z) = \left(\min\left(z_1^2, 1\right) + \min\left(z_2^2, 1\right)\right) - (z_1^2 + z_2^2).$$

Thus, equation (7) can be written as:

$$\max_{z_1, z_2 \geq 0} \left(\min\left(z_1^2, 1\right) + \min\left(\left(z_2^2, 1\right)\right) - (z_1^2 + z_2^2)^{\beta/2}\right) \tag{37}$$

We wish to show equation (37) is maximized whenever $z \in S$. Since $z_1^2 + z_2^2 = 1$ for any $z \in S$, this follows from Lemma 13 applied to $\beta = 2$. $\square$

Theorem 5. *[Formal version of Theorem 3] Suppose that there are 2 users located at two linearly independently vectors $u_1, u_2 \in \mathbb{R}^D_{\geq 0}$, let $\theta^* := \cos^{-1}\left(\frac{\langle u_1, u_2 \rangle}{\|u_1\|_2 \|u_2\|}\right) < 0$ be the angle between them. Suppose we have cost function $c(p) = \|p\|_2^\beta$, $\beta > \beta^* = \frac{2}{1 - \cos(\theta^*)}$, and $P = \infty$ producers. Then, the genres d_1, d_2, conditional quality distributions $F_1 = F$ and $F_2 = F$, and weights $\alpha_1 = \alpha_2 = 2$ form an equilibrium*[8]*, where*

$$\{d_1, d_2\} := \left\{[\cos(\theta^G + \theta_{min}), \sin(\theta^G + \theta_{min})], [\cos(\theta^* - \theta^G + \theta_{min}), \sin(\theta^* - \theta^G + \theta_{min})]\right\}$$

such that $\theta^G := \arg\max_{\theta \leq \theta^*/2} \left(\cos^\beta(\theta) + \cos^\beta(\theta^* - \theta)\right)$ *and* $\theta_{min} := \min\left(\cos^{-1}\left(\frac{\langle u_1, e_1 \rangle}{\|u_1\|}\right), \cos^{-1}\left(\frac{\langle u_2, e_1 \rangle}{\|u_2\|}\right)\right)$, *and where*

$$F(q) := \begin{cases} C_2^{(2n+2)\beta} & \text{if } q \in C_1^{1/\beta} C_2^{2n+1} [C_2, 1] \text{ for } n \geq 0 \\ C_1^{-2} C_2^{-2n\beta} q^{2\beta} & \text{if } q \in C_1^{1/\beta} C_2^{2n} [C_2, 1] \text{ for } n \geq 0 \\ 1 & \text{if } q \geq C_1^{1/\beta}, \end{cases},$$

such that the constants are defined by $C_1 := \frac{\sin(\theta^*) \cos(\theta^G)}{\sin(\theta^* - \theta_G)}$ *and* $C_2 := \frac{\cos(\theta^* - \theta^G)}{\cos(\theta^G)}$.

[8]See Definition 1 for a formalization of equilibrium for $P = \infty$.

Proof of Theorem 5. WLOG, we assume that $\|u_1\| = \|u_2\| = 1$. It suffices to verify that the genres, conditional quality distributions, and weights satisfy (27). Motivated by Lemma 1, we define:

$$H_1(z_1) = \sqrt{F_1\left(\frac{z_1}{\langle u_1, d_1\rangle}\right) F_2\left(\frac{z_1}{\langle u_1, d_2\rangle}\right)}$$
$$H_2(z_2) = \sqrt{F_1\left(\frac{z_2}{\langle u_1, d_1\rangle}\right) F_2\left(\frac{z_2}{\langle u_1, d_2\rangle}\right)}.$$

We define the support S to be

$$S := \{[\langle u_1, qd_1\rangle, \langle u_2, qd_1\rangle] \mid q_1 \in \text{supp}(F_1)\} \cup \{[\langle u_1, qd_1\rangle, \langle u_2, qd_1\rangle] \mid q_2 \in \text{supp}(F_2)\}.$$

Using this notation, we can rewrite (27) as requiring that:

$$\max_z \left(H_1(z_1) + H_2(z_2) - c_{\mathbf{U}}(z)\right) \tag{38}$$

is maximized for every $z \in S$.

First, we show that

$$\sin(\theta^G)\cos^{\beta-1}(\theta^G) = \sin(\theta^* - \theta^G)\cos^{\beta-1}(\theta^* - \theta^G) \tag{39}$$

This immediately follows from using that $\theta^G \in \arg\max_\theta \left(\cos^\beta(\theta) + \cos^\beta(\theta^* - \theta)\right)$ and applying the first-order condition.

For the remainder of the proof, we define:

$$c := \frac{\sin(\theta^* - \theta^G)}{\sin(\theta^*)\cos^{\beta-1}(\theta^G)} = \frac{\sin(\theta^G)}{\sin(\theta^*)\cos^{\beta-1}(\theta^* - \theta^G)},$$

Computing H_1 and H_2. We show that:

$$H_1(z_1) = \min\left(cz_1^\beta, 1\right) \text{ and } H_2(z_2) = \min\left(1, cz_2^\beta\right).$$

We show that

$$H_1(z_1) = \min\left(cz_1^\beta, 1\right), \tag{40}$$

and observe that the expression for H_2 follows from an analogous argument. By definition, we see that:

$$\begin{aligned} H_1(z_1) &= \sqrt{F_1\left(\frac{z_1}{\langle u_1, d_1\rangle}\right) F_2\left(\frac{z_1}{\langle u_1, d_2\rangle}\right)} \\ &= \sqrt{F\left(\frac{z_1}{\langle u_1, d_1\rangle}\right) F\left(\frac{z_1}{\langle u_1, d_2\rangle}\right)}. \end{aligned}$$

We know that either (1) $\langle u_1, d_1\rangle = \langle u_2, d_2\rangle = \cos(\theta^G)$ and $\langle u_1, d_2\rangle = \langle u_2, d_1\rangle = \cos(\theta^* - \theta^G)$, or (2) $\langle u_1, d_2\rangle = \langle u_2, d_1\rangle = \cos(\theta^G)$ and $\langle u_1, d_1\rangle = \langle u_2, d_2\rangle = \cos(\theta^* - \theta^G)$. WLOG, we assume that (1) holds. This means that:

$$\begin{aligned} H_1(z_1) &= \sqrt{F\left(\frac{z_1}{\langle u_1, d_1\rangle}\right) F\left(\frac{z_1}{\langle u_1, d_2\rangle}\right)} \\ &= \sqrt{F\left(\frac{z_1}{\cos(\theta^G)}\right) F\left(\frac{z_1}{\cos(\theta^* - \theta^G)}\right)} \end{aligned}$$

Let's reparameterize and let:

$$q_1 = \frac{z_1}{\cos(\theta^* - \theta^G)}.$$

This means that:

$$H_1(q_1 \cos(\theta^* - \theta^G)) = \sqrt{F(q_1) F\left(q_1 \frac{\cos(\theta^* - \theta^G)}{\cos(\theta^G)}\right)}.$$

Equation (40) reduces to

$$\sqrt{F(q_1) F\left(q_1 \frac{\cos(\theta^* - \theta^G)}{\cos(\theta^G)}\right)} = \min\left(1, c\cos(\theta^* - \theta^G)^\beta q_1^\beta\right).$$

which simplifies to

$$\sqrt{F(q_1) F\left(q_1 \frac{\cos(\theta^* - \theta^G)}{\cos(\theta^G)}\right)} = \min\left(1, \frac{\sin(\theta^G)\cos(\theta^* - \theta^G)}{\sin(\theta^*)} q_1^\beta\right)$$

which simplifies to

$$\sqrt{F(q_1) F\left(q_1 C_2\right)} = \min\left(1, C_3^{-1} q_1^\beta\right) \tag{41}$$

We verify equation (41) by doing casework on q_1. Note that $C_1^{1/\beta} = C_3^{1/\beta} C_2$. If $q_1 \geq C_3^{1/\beta} C_2^{-1/\beta}$, then we see that $F(q_1) = F(q_1 C_2) = 1$ and the equation holds. In fact, if $q_1 \geq C_3^{1/\beta} C_2^{1-1/\beta}$, then we see that $F(q_1) = 1$ and

$$F(q_1 C_2) = C_3^{-2} C_2^{2-2\beta} (q_1 C_2)^{2\beta} = C_3^{-2} C_2^2 q_1^{2\beta}$$

, so equation (41) is satisfied. Otherwise, if $q_1 = C_3^{1/\beta} C_2 C_2^{2n} \gamma$ for $n \geq 0$ and $\gamma \in [C_2, 1]$, then

$$F(q_1) = C_3^{-2} C_2^{-2\beta - 2n\beta} q^{2\beta}$$

and

$$F(q_1 C_2) = C_2^{(2n+2)\beta},$$

so:

$$\sqrt{F(q_1) F\left(q_1 C_2\right)} = \sqrt{C_3^{-2} C_2^{-2\beta - 2n\beta} C_2^{(2n+2)\beta}} = \sqrt{C_3^{-2} C_2^2 q^{2\beta}} = C_3^{-1} C_2 q_1^\beta$$

as desired. Finally, if $q_1 = C_1^{1/\beta} C_2^{1-1/\beta} C_2^{2n+1} \gamma$ for $n \geq 0$ and $\gamma \in [C_2, 1]$, then

$$F(q_1) = C_2^{(2n+2)\beta}$$

and

$$F(q_1 C_2) = C_3^{-2} C_2^{-(2n+4)\beta} q^{2\beta},$$

so:

$$\sqrt{F(q_1) F\left(q_1 C_2\right)} = \sqrt{C_2^{(2n+2)\beta} C_3^{-2} C_2^{(2n+4)\beta} q^{2\beta} C_2^{2\beta}} = C_3^{-1} q^\beta.$$

This proves the desired formulas for H_1 and an analogous argument applies to H_2.

Showing equation (38) is maximized at every $z \in S$. We need to show that for every $z \in S$, it holds that:

$$H_1(z_1) + H_2(z_2) - c_{\mathbf{U}}(z) = \max_{z'}(H_1(z_1') + H_2(z_2') - c_{\mathbf{U}}(z')).$$

Plugging in our expressions above, our goal is to show:

$$\min(1, cz_1^{\beta}) + \min(1, cz_2^{\beta}) - c_{\mathbf{U}}(z) = \max_{z'}(H_1(z_1') + H_2(z_2') - c_{\mathbf{U}}(z'))$$

for every $z \in S$.

We split into two steps: first, we show that

$$\min(1, cz_1^{\beta}) + \min(1, cz_2^{\beta}) - c_{\mathbf{U}}(z) = 0 \tag{42}$$

for every $z \in S$, and next we show that:

$$\max_{z'}(H_1(z_1') + H_2(z_2') - c_{\mathbf{U}}(z')) \leq 0. \tag{43}$$

To show (42), let's first consider $[z_1, z_2] = [r\cos(\theta^G), r\cos(\theta^G - \theta^*)] \in S$. Then we see that:

$$\begin{aligned}
\min(1, cz_1^{\beta}) + \min(1, cz_2^{\beta}) - c_{\mathbf{U}}(z) &= cz_1^{\beta} + cz_2^{\beta} - c_{\mathbf{U}}(z) \\
&= r^{\beta}\left(c\cos^{\beta}(\theta^G) + c\cos^{\beta}(\theta^* - \theta^G) - 1\right)
\end{aligned}$$

Thus, it suffices to show that:

$$\cos^{\beta}(\theta^G) + \cos^{\beta}(\theta^* - \theta^G) = \frac{1}{c}. \tag{44}$$

We now show equation (44):

$$\begin{aligned}
\cos^{\beta}(\theta^G) + \cos^{\beta}(\theta^* - \theta^G) &=_{(A)} \frac{\cos(\theta^G)\cos^{\beta-1}(\theta^* - \theta^G)\sin(\theta^* - \theta^G)}{\sin(\theta^G)} + \cos^{\beta}(\theta^* - \theta^G) \\
&= \frac{\cos^{\beta-1}(\theta^* - \theta^G)}{\sin(\theta^G)}\left(\cos(\theta^G)\sin(\theta^* - \theta^G) + \cos(\theta^* - \theta^G)\sin(\theta^G)\right) \\
&= \frac{\cos^{\beta-1}(\theta^* - \theta^G)}{\sin(\theta^G)}\sin(\theta^*) \\
&= \frac{1}{c}.
\end{aligned}$$

where (A) follows from applying equation (39). Let's now consider let's first consider $[z_1, z_2] = [r\cos(\theta^G - \theta^*), r\cos(\theta^*)] \in S$. Then, we see that

$$\min(1, cz_1^{\beta}) + \min(1, cz_2^{\beta}) - c_{\mathbf{U}}(z) = cz_1^{\beta} + cz_2^{\beta} - c_{\mathbf{U}}(z) = r^{\beta}\left(c\cos^{\beta}(\theta^G) + c\cos^{\beta}(\theta^* - \theta^G) - 1\right) = 0,$$

where the last equality follows from equation (44). This establishes equation (43).

Now, we show equation (43). Let's represent z' as $\mathbf{U}[r'\cos(\theta), r'\sin(\theta)]$. Then this becomes:

$$c(r')^{\beta}\cos^{\beta}(\theta) + c(r')^{\beta}\cos^{\beta}(\theta^* - \theta) \leq (r')^{\beta}.$$

Dividing by r'^{β}, we obtain:

$$\cos^{\beta}(\theta) + \cos^{\beta}(\theta^* - \theta) \leq \frac{1}{c}.$$

To show this, observe that:

$$\cos^{\beta}(\theta) + \cos^{\beta}(\theta^* - \theta) \leq \cos^{\beta}(\theta^G) + \cos^{\beta}(\theta^* - \theta^G) = \frac{1}{c}.$$

where the first inequality follows from the fact that θ^G is a maximizer of $\cos^{\beta}(\theta) + \cos^{\beta}(\theta^* - \theta)$ by definition, and the second equality follows from equation (44). This establishes equation (43).

This proves that equation (44) is maximized at every $z \in S$, and thus the conditions of Definition 1 are satisfied. $\square$

5 Strictly positive equilibrium profit

In this section, we examine the question: *when can a producer make a profit?* At a symmetric equilibria μ, all producers receive the same expected profit given by:

$$\mathcal{P}^{\text{eq}}(\mu) := \mathbb{E}_{p_1,\ldots,p_P \sim \mu}[\mathcal{P}(p_1; p_{-1})] = \mathbb{E}\Big[\Big(\sum_{i=1}^{N} 1[j^*(u_i;\ p_{1:P}) = 1]\Big) - \|p_1\|^{\beta}\Big], \tag{45}$$

where expectation in the last term is taken over $p_1, \ldots, p_P \sim \mu$ as well as randomness in recommendations.

Intuitively, the equilibrium profit of a marketplace provides insight about market competitiveness. Zero profit suggests that competition has driven producers to expend their full cost budget on improving product quality. Positive profit, on the other hand, suggests that the market is not yet fully saturated and new producers have incentive to enter the marketplace.

We illustrate how the behavior of $\mathcal{P}^{\text{eq}}(\mu)$ fundamentally differs for homogeneous users and heterogeneous users. We first verify equilibrium profit is always zero for homogeneous users (Corollary 7), which is in line with classical intuition. Then, we show that the equilibrium profit need not be zero for heterogeneous users (Theorem 4): in particular, we provide a sufficient condition for positive profit in terms of the user geometry and cost function. We also show that single-genre equilibria necessarily result in zero profit (Proposition 5). This suggests that the driving force for positive profit is that producers form multiple genres, which reduces the level of competition within each genre.

5.1 Equilibrium profit for homogeneous users

We first verify that the classical results from price competition [BK08] apply to our model in the one-dimensional case. Specifically, we show that the equilibrium profit is guaranteed to be 0.

Proposition 11. *Consider a 1-dimensional setup with N users $u_1 = u_2 = \ldots = u_N = 1$. Suppose that the cost function is $c(p) = |p|^{\beta}$. At the unique symmetric equilibrium μ, the profit $\mathcal{P}^{eq}(\mu)$ is equal to* 0.

Proof. Since μ is an equilibrium, all choices p in the support of μ achieve profit equal to the equilibrium profit. We apply Lemma 2 to see that the cdf of μ is $F(p) = \min\left(1, \left(\frac{p^{\beta}}{N}\right)^{1/(P-1)}\right)$, which shows that $p = 0$ is in $\text{supp}(\mu)$. For this choice of p, the cost is 0, but the producer also never wins any users, so the profit is also zero, as claimed. $\square$

Proposition 11 shows that even when there are only 2 producers, the supply-side market is competitive enough for each producer to maximally increase the quality of their product. Intuitively, all producers compete along the same axis (quality level), which drives the quality level up until producers cannot afford to increase it anymore without receiving negative profit.

This zero profit result also carries over a homogeneous population of users, regardless of the embedding dimension.

Corollary 7. *Suppose that there is a single population of N users, all of whose embeddings are at the same vector $u \in \mathbb{R}^D_{\geq 0}$. At any symmetric equilibrium μ, the profit $\mathcal{P}^{eq}(\mu)$ is equal to* 0.

The intuition is that the equilibria for one homogeneous population of users are essentially the same as the equilibria in one dimension, as we established in Corollary 7.

Proof. A producer's decision about what direction of content to produce is trivial: they will always choose a direction in $\arg\max_{\|p\|=1}\langle p, u\rangle$. Even if there are multiple directions that maximize $\langle p, u\rangle$, they result in the same user utility. Thus, we can still apply :emma 2 to see that the cdf of μ is $F(p) = \min\left(1, \left(\frac{p^\beta}{N}\right)^{1/(P-1)}\right)$, which shows that $p = 0$ is in $\text{supp}(\mu)$, and the profit is 0. □

5.2 Equilibrium profit in multiple dimensions

When users are heterogeneous, the equilibrium profit is no longer guaranteed to be 0. To see this, let us revisit two users located at the standard basis vectors and cost function $c(p) = \|p\|_2^\beta$. Using the equilibrium characterization in Proposition 2, we can obtain the following characterization of profit.

Corollary 8. *Suppose that there are 2 users located at the standard basis vectors $e_1, e_2 \in \mathbb{R}^2$, and the cost function is $c(p) = \|p\|_2^\beta$. For $P = 2$ and $\beta \geq \beta^* = 2$, there is an equilibrium μ where*

$$\mathcal{P}^{eq}(\mu) = 1 - \frac{2}{\beta}.$$

Corollary 8 shows that there exist equilibria that exhibit strictly positive profit for any $\beta \geq 2$. The intuition is that (after sampling the randomness in μ), different producers are likely to produce content in different directions. This reduces the amount of competition along any single direction of content. As a result, producers are no longer forced to maximize quality, thus enabling them to generate a strictly positive profit.

This finding can be generalized to sets of many users and producers and to arbitrary norms. Theorem 4, restated below, provides the following sufficient condition under which the profit at equilibrium is strictly positive.

Proposition 4. *Suppose that*

$$\max_{\|p\|\leq 1} \min_{i=1}^{N} \left\langle p, \frac{u_i}{||u_i||}\right\rangle < N^{-P/\beta}. \tag{8}$$

Then for any symmetric equilibrium μ, the profit $\mathcal{P}^{eq}(\mu)$ is strictly positive.

Proof. Without loss of generality, we assume user vectors have unit norm $\|u_i\|$. Given an equilibrium μ, we will construct an explicit vector p that generates positive profit. This proves that the equilibrium profit is positive because no vector can achieve higher than the equilibrium profit. The vector p is of the form $\left(Q\left(\max_{p' \in \text{supp}(\mu)} ||p'||\right) + \epsilon\right) \cdot u_{i^*}$ for some $i^* \in [1, N]$.

Cluster the set of unit vectors p into N groups $G_1, \ldots, G_N$, based on the user for whom they generate the lowest value. That is, each vector p belongs to the group G_i where $u_i =$

$\text{argmin}_{1 \le i' \le N} \langle p, u_{i'} \rangle$. This means that if all producers choose (unit vector) directions in G_i, then the maximum value received by user u_i is

$$\max_{1 \le j \le P} \langle p_j, u_i \rangle \le \max_{\|p\| \le 1} \min_{i=1}^{N} \langle p, u_i \rangle = Q. \tag{46}$$

Let G_{i^*} be the group with highest probability of appearing in μ, that is $i^* \in \arg\max_i \mathbb{P}_{v \sim \mu}\left[\frac{v}{||v||} \in G_i\right]$.

Let E be the event that all of the other $P-1$ producers choose directions in G_{i^*}. The event E happens with probability at least $\mathbb{P}_{v \sim \mu}\left[\frac{v}{||v||} \in G_{i^*}\right] \ge (1/N)^{P-1}$. Since the value received by the user is linear in the magnitude of the producer action, we see that the maximum possible value that could be received by user u_i from the other producers is $Q\left(\max_{p' \in \text{supp}(\mu)} ||p'||\right)$. On the other hand, the action p results in value $(Q\left(\max_{p' \in \text{supp}(\mu)} ||p'||\right) + \epsilon)$ for u_{i^*}, so it wins u_{i^*} with probability 1 on the event E. This means that the expected profit obtained by p is at most

$$\left(\frac{1}{N}\right)^{P-1} - \left(Q\left(\max_{p' \in \text{supp}(\mu)} ||p'||\right) + \epsilon\right)^{\beta}.$$

Taking a limit as $\epsilon \to_+ 0$, we obtain the profit can be set arbitrarily close to:

$$\left(\frac{1}{N}\right)^{P-1} - \left(Q\left(\max_{p' \in \text{supp}(\mu)} ||p'||\right)\right)^{\beta}. \tag{47}$$

It suffices to bound $\max_{p' \in \text{supp}(\mu)} ||p'||$. The action $p'' \in \arg\max_{p' \in \text{supp}(\mu)} ||p'||$ produces a profit of at most $N - \left(\max_{p \in \text{supp}(\mu)} ||p||\right)^{\beta}$. Thus, $\left(\max_{p \in \text{supp}(\mu)} ||p||\right)^{\beta} \le N$, so $\left(\max_{p \in \text{supp}(\mu)} ||p||\right) \le N^{1/\beta}$.

Plugging this into (47), we see that there exist actions that produces profit arbitrarily close to

$$\left(\frac{1}{N}\right)^{P-1} - NQ^{\beta}.$$

Thus, a strictly positive profit will be obtained if:

$$Q < \left(\frac{1}{N}\right)^{P/\beta},$$

as desired. □

Let us examine the quantity $Q := \max_{\|p\| \le 1} \min_{i=1}^{N} \langle p, \frac{u_i}{||u_i||} \rangle$ that appears on the left-hand side of (8). Intuitively, Q captures how easy it is to produce content that appeals simultaneously to all users. It is larger when the users are close together and smaller when they are spread out. For any set of vectors we see that $Q \le 1$, and as long as the vectors is nondegenerate, Q is strictly less than 1.

The right-hand side of (8), on the other hand, goes to 1 as $\beta \to \infty$. Thus, for any non-degenerate set of users, if β is sufficiently large, the condition in Theorem 4 will be met. In other words, if the cost function sufficiently incentivizes specialization, then the profit $U(\mu)$ at any equilibrium μ is strictly positive. Moreover, the value of β at which the condition is met increases as Q increases (in particular, for sets of user vectors that are closer together).

Relationship with genre formation. Although the proof of Theorem 4 does not explicitly consider genre formation, we show that the presence of multiple genres at equilibrium is nonetheless central to Theorem 4. To illustrate this, we show in Proposition 5, restated below, that at a single-genre equilibrium, the profit is zero whenever there are at least $P \geq 2$ producers.

Proposition 5. *If μ is a single-genre equilibrium, then the profit $\mathcal{P}^{eq}(\mu)$ is equal to* 0.

The proof from the 1-dimensional setup in Section 5.1 carries over, since the producers again compete only along the axis of quality. When Proposition 5 is viewed in the context of Theorem 4, we see that (8) cannot hold when single-genre equilibria exist. This aligns with the fact that equation (8) intuitively corresponds to users being sufficiently far apart and β being sufficiently large to incentivize specialization.

The distinction between profit in the single-genre regime (where there is no specialization) and the multi-genre regime (where this is specialization) parallels the classical distinction in economics between markets with homogeneous goods and markets with differentiated goods.

- The single-genre regime bears resemblance to marketplaces with homogeneous goods where firms compete based on price (see [BK08] for a textbook treatment). In these marketplaces, the possibility of undercutting intuitively drives the profit of the firm down to zero at equilibrium, as long as there are at least 2 firms. Similarly, in the marketplace that we study, producers all compete along the same direction when there is no specialization. The possibility of being outcompeted along quality drives producer profit to zero. The analogy is not exact: in our model producers still play a distribution of quality and thus might be out-competed in a given realization.
- The multi-genre equilibria regime bears resemblance to a marketplace with differentiated goods (see [AdPT92] for a textbook treatment). Product differentiation leads to local monopolies where firms can set prices above the zero profit level. Similarly, in the marketplace that we study, specialization by producers leads to product differentiation and thus induces monopoly-like behavior. More specifically, specialization limits competition within each genre and can enable producers to set the quality of their goods below the zero profit level. This suggests that the market is not saturated and new producers might be incentivized to enter.

Our results thus formalize how recommendation systems interpolate between marketplaces between homogeneous goods and marketplaces with differentiated goods, depending on the content production cost structure, preferences of users, and number of producers.

A Proofs for Section 2

A.1 Proof of Proposition 6

We restate and prove Proposition 6.

Proposition 6. *For any set of users and any $\beta \geq 1$, a pure strategy equilibrium does not exist.*

Proof of Proposition 6. Assume for sake of contradiction that the solution $p_1, \ldots, p_P$ is a pure strategy equilibrium. We divide into two cases: (1) there exist $1 \leq j' \neq j \leq P$ and i such that $\langle p_j, u_i \rangle = \langle p_{j'}, u_i \rangle$, (2) there does not exist j, j' and i such that $\langle p_j, u_i \rangle = \langle p_{j'}, u_i \rangle$.

Case 1: there exist $1 \leq j' \neq j \leq P$ and i such that $\langle p_j, u_i \rangle = \langle p_{j'}, u_i \rangle$. Let producer j and producer j' be such that $\langle p_j, u_i \rangle = \langle p_{j'}, u_i \rangle$. The idea is that the producer j can leverage the discontinuity in their profit function (9) at p_j. In particular, consider the vector $p_j + \epsilon u_i$. The number of users that they receive as $\epsilon \to_+ 0$ is *strictly greater* than at p_j. The cost, on the other hand, is continuous in ϵ. This demonstrates that there exists $\epsilon > 0$ such that:

$$\mathcal{P}(p_j + \epsilon u_i; p_{-j}) > \mathcal{P}(p_j; p_{-j})$$

as desired. This is a contradiction.

Case 2: there does not exist j, j' and i such that $\langle p_j, u_i \rangle = \langle p_{j'}, u_i \rangle$. Consider a producer j who wins a nonzero number of users in expectation. Let $\mathcal{N}$ be the set of $1 \leq i \leq N$ such that $\langle p_j, u_i \rangle > \langle p_{j'}, u_i \rangle$ for all $j' \neq j$. This is nonempty since the producer wins a nonzero number of users in expectation. We leverage that the profit function of producer j is continuous at p_j. There exists $\epsilon > 0$ such that $\langle p_j(1-\epsilon), u_i \rangle > \langle p_{j'}, u_i \rangle$ for all $j' \neq j$ and all i, so that:

$$\mathcal{P}(p_j(1-\epsilon); p_{-j}) > \mathcal{P}(p_j; p_{-j})$$

as desired. This is a contradiction.

□

A.2 Proof of Proposition 7

We restate and prove Proposition 7.

Proposition 7. *A symmetric mixed equilibrium μ exists.*

Proof of Proposition 7. We apply a standard existence result of symmetric, mixed strategy equilibria in discontinuous games (see Corollary 5.3 of [Ren99]). We adopt the terminology of that paper and refer the reader to [Ren99] for a formal definition of the conditions. It suffices to show that: (1) the producer action space is convex and compact, (2) the payoff function is continuous in μ, and (3) the game is diagonally payoff secure.

Producer action space is convex and compact. In the current game, the producer action space is not compact. However, we show that we can define a slightly modified game, where the producer action space is convex and compact, without changing the equilibrium of the game. For the remainder of the proof, we analyze this modified game.

In particular, each producer must receive at least 0 profit at equilibrium since $\mathcal{P}(\vec{0}; p_{-1}) \geq 0$ regardless of the actions p_{-1} taken by other producers. If a producer chooses p such that $\|p\| > N^{1/\beta}$, then their utility will be strictly negative. Thus, we can restrict to $\left\{p \in \mathbb{R}_{\geq 0}^D \mid \|p\| \leq 2U^{1/\beta}\right\}$ which a convex compact set. We add a factor of 2 slack to guarantee that any best-response by a producer will be in the *interior* of the action space and not on the boundary.

Establishing continuity. We show the payoff function $\mathcal{P}(\mu; [\mu, \ldots, \mu])$ (where μ is a distribution over the producer action space) is continuous in μ. Here we slightly abuse notation since $\mathcal{P}$ is technically defined over pure strategies in (9). We implicitly extend the definition to mixed strategies by considering expected profit. Using the fact that each producer receives a $1/P$ fraction of users in expectation at a symmetric solution, we see that:

$$\mathcal{P}(\mu; [\mu, \ldots, \mu]) = \frac{N}{P} - \int \|p\|^{\beta} d\mu,$$

which implies continuity.

Establishing diagonal payoff security. We construct, for each relevant payoff in the closure of the graph of the game's diagonal payoff function, an action that diagonal payoff secures that payoff. More formally, let (μ^*, α^*) be in the closure of the graph of the game's diagonal payoff function, and suppose that $(\mu^*, \ldots, \mu^*)$ is not an equilibrium. It suffices to construct μ' that diagonal payoff secures α^*. First, we see that $\alpha^* = \mathcal{P}(\mu, \ldots, \mu)$ by the continuity of the payoff function described above. Since $(\mu^*, \ldots, \mu^*)$ is not an equilibrium, there exists $p \in \left\{p \in \mathbb{R}^D_{\geq 0} \mid \|p\| \leq N^{1/\beta}\right\}$ such that

$$\mathcal{P}(v; [\mu, \ldots, \mu]) > \mathcal{P}(\mu; [\mu, \ldots, \mu]).$$

Since the inequality is strict, there exists $\epsilon > 0$ such that:

$$\mathcal{P}(p(1+\epsilon); [\mu, \ldots, \mu]) > \mathcal{P}(\mu; [\mu, \ldots, \mu]).$$

We claim that μ' taken to be the point mass at $p(1+\epsilon)$ will diagonally payoff secure (μ^*, u^*). It suffices to show that exists an open neighborhood around μ' such that $\mathcal{P}(\mu'', [\mu, \ldots, \mu]) > \mathcal{P}(\mu, [\mu, \ldots, \mu])$ for all μ'' in the open neighborhood. To see this, we observe that for sufficiently small $\epsilon' > 0$, for any μ'' that changes an ϵ' measure of μ' arbitrarily, it holds that $\mathcal{P}(\mu''; [\mu, \ldots, \mu]) > \mathcal{P}(\mu; [\mu, \ldots, \mu])$; moreover, for sufficiently small $\epsilon' > 0$, it holds that $\mathcal{P}(\mu''; [\mu, \ldots, \mu]) > \mathcal{P}(\mu; [\mu, \ldots, \mu])$ for any μ'' that is a point mass at $p(1+\epsilon) + \epsilon' v$ where v is a unit vector. □

A.3 Proof of Proposition 8

In this proof, we consider the payoff function $\mathcal{P}(\mu_1; [\mu_2, \ldots, \mu_P])$ (where μ is a distribution over the producer action space) defined to be the expected profit attained if a producer plays μ_1 when other producers play $\mu_2, \ldots, \mu_P$. Strictly speaking, this is an abuse of notation since $\mathcal{P}$ is technically defined over pure strategies in (9). We implicitly extend the definition to mixed strategies by considering *expected* profit.

Proof of Proposition 8. Let μ be a symmetric equilibrium, and assume for sake of contradiction that there is an atom at $p \in \mathbb{R}^d$ with probability mass $\alpha > 0$. It suffices to construct a vector p' that achieves profit

$$\mathcal{P}(p'; [\mu, \ldots, \mu]) > \mathcal{P}(\vec{0}; [\mu, \ldots, \mu]) = \mathcal{P}(\mu; [\mu, \ldots, \mu]).$$

Consider the vector $p' = p + \epsilon u_1$ for some $\epsilon > 0$. For any given realization of actions by other producers, and for any given user, the vector p' never wins the user with lower probability than the vector p. We construct an event and a user where the vector p' wins the user with strictly higher probability than the vector p. Let E be the event that all of the other producers choose the p vector. This event happens with probability α^{P-1}. Conditioned on E, the vector p' wins user u_1; on the other hand, the vector p wins user u_1 with probability $1/P$. Since the cost function is continuous in ϵ, there exists ϵ such that $\mathcal{P}(p; [\mu, \ldots, \mu]) > \mathcal{P}(\vec{0}; [\mu, \ldots, \mu]) = \mathcal{P}(\mu; [\mu, \ldots, \mu])$. This is a contradiction. □

A.4 Derivation of Example 1

To see that the cumulative distribution function is $F(p) = \min(1, p^{\beta/P-1})$, we use the fact that every equilibrium is by definition a single-genre equilibrium in 1 dimension and apply Lemma 2.

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