Understanding Sparse JL for Feature Hashing

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NeurIPS 2019

Dimensionality reduction (ℓ_2-to-ℓ_2)

Informal goal: Project vectors in $\mathbb{R}^n$ to $\mathbb{R}^m$ (for $m << n$) with a linear map while "preserving geometry" (i.e. Euclidean norm distances).

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Many applications:

- Document classification tasks (Weinberger et al. '09, etc)
- Support Vector Machines (Paul et al. '14)
- k-means/k-medians (Makarychev, Makarychev, Razenshteyn '18)
- Nearest neighbors (Ailon, Chazelle '09, Har-Peled et al. '14, Wei '19)
- Numerical linear algebra (Clarkson and Woodruff '12, Nelson and Nguyen '14, etc.)

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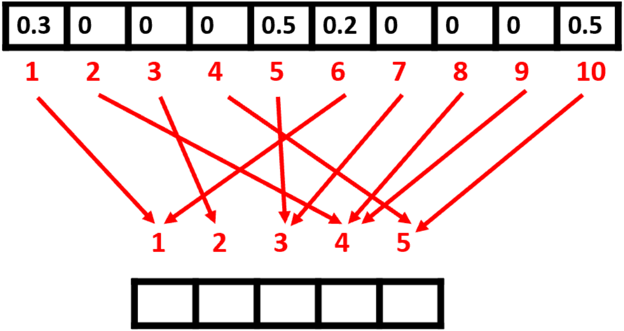

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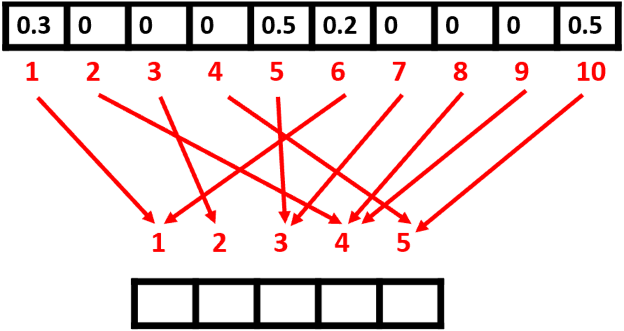

But how should collisions be handled?

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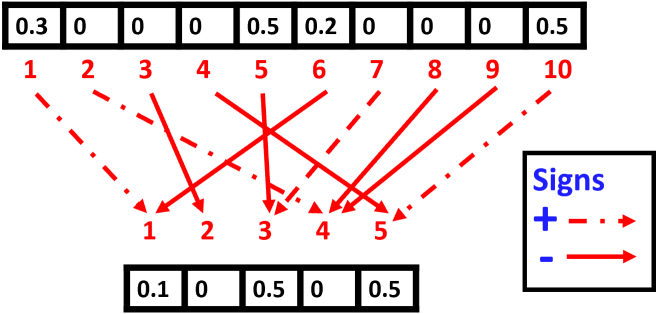

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Analysis of sparse JL on feature vectors in terms of s.
$\implies$ *Sparse JL with ≥ 4 hash functions can have much better norm-preserving properties on feature vectors than feature hashing.*

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- Can apply to distances between vectors since f is linear.

Choosing s and m

Notation: ϵ is target error, δ is target failure probability.

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\|x\|_2 \leq \|f(x)\|_2 \leq (1+\epsilon)\|x\|_2] > 1-\delta$.

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$m = O(\min(2\epsilon^{-2}/\delta, \epsilon^{-2} \log(1/\delta) e^{\Theta(\epsilon^{-1} \log(1/\delta)/s)})$ (Cohen '16).

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$\implies$ *Characterization of sparse JL performance in terms of ϵ, δ, and ℓ_∞-to-ℓ_2 norm ratio for a general # of hash functions s*

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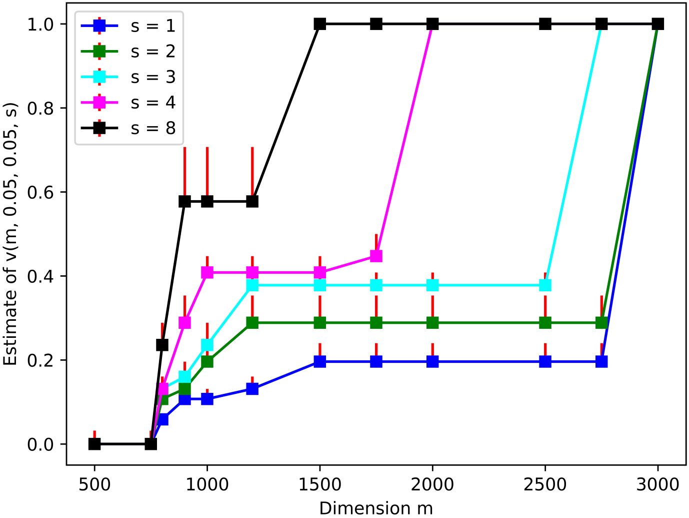

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$$f'(m, \epsilon, p, s) = \begin{cases} 1 & \textit{High } m\textit{, full norm preservation} \\ \Theta\left(\sqrt{\epsilon s}\frac{\sqrt{\ln(\frac{m\epsilon^2}{p})}}{\sqrt{p}}\right) & \textit{Medium } m\textit{, middle regime} \\ \Theta\left(\sqrt{\epsilon s}\min\left(\frac{\ln(\frac{m\epsilon}{p})}{p}, \frac{\sqrt{\ln(\frac{m\epsilon^2}{p})}}{\sqrt{p}}\right)\right) & \textit{Medium } m\textit{, middle regime} \\ 0 & \textit{Small } m\textit{, no norm preservation} \end{cases}$$

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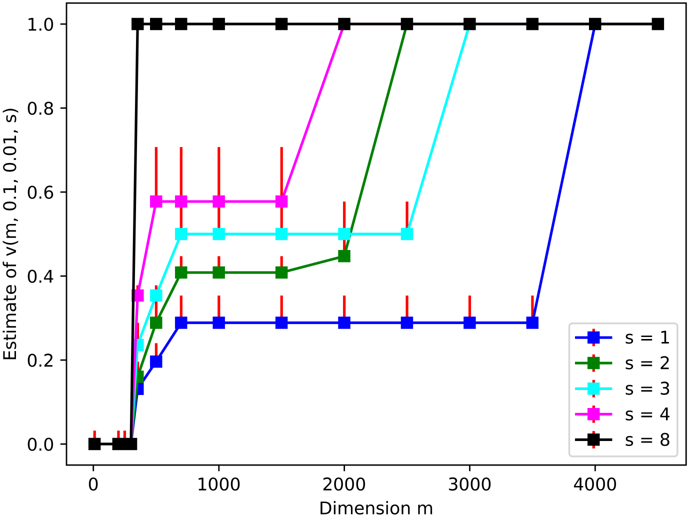

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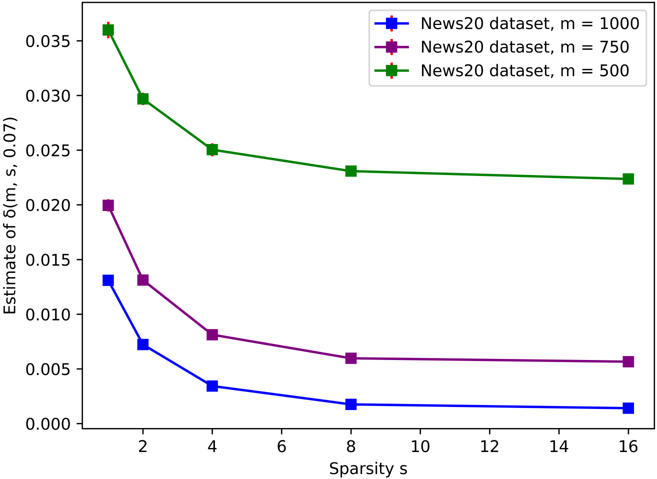

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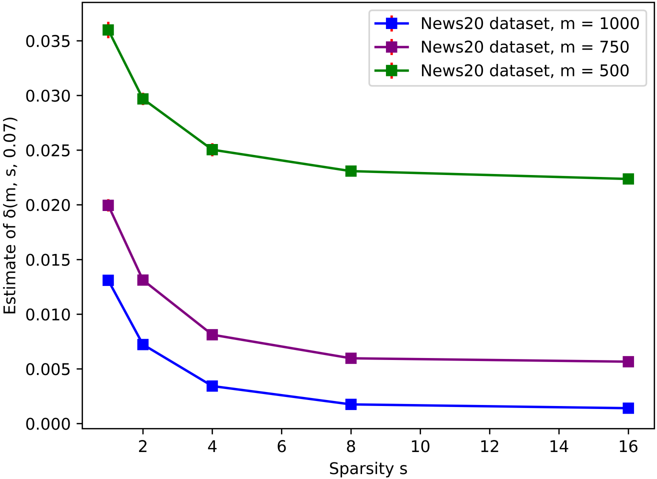

Use sparse JL with more than one hash function!!

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(r,i)th coordinate is $\eta_{r,i}\sigma_{r,i}/\sqrt{s}$, where $\eta_{r,i} \in \{0,1\}$, $\sigma_{r,i} \in \{-1,1\}$

High-level approach of our analysis

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Need tight bounds on $\mathbb{E}[R(x_1, \ldots, x_n)^p]$ on S_v at every threshold v.

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We use a non-combinatorial approach with Rademacher-specific bounds.

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Hope to see you at the poster session!!!