Simple Analysis of Sparse, Sign-Consistent JL

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Linear dimensionality reduction: ℓ_2-to-ℓ_2

Informal goal: Project vectors in $\mathbb{R}^n$ to $\mathbb{R}^m$ (for $m << n$) with a linear map while "preserving geometry" (i.e. $\|f(x) - f(y)\|_2 \approx \|x - y\|_2$).

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Many applications:

- Feature hashing (Weinberger et al. '09, Dahlgaard et al. '17, Freksen et al. '18, etc.)
- Numerical linear algebra (Clarkson and Woodruff '12, Nelson and Nguyen '14, etc.)
- Approximate nearest neighbors (Ailon and Chazelle '09, etc.)
- k-means/k-medians (Makarychev, Makarychev, Razenshteyn '18)

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- <u>Compression in the brain</u> (Allen-Zhu, Gelashvili, Micali, Shavit '15)

Mathematical framework

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Use a probability distribution $\mathcal{M}$ over linear maps $\mathbb{R}^n \to \mathbb{R}^m$ ($m << n$).

Geometry-preserving property: for *each* $x \in \mathbb{R}^n$

$$\mathbb{P}_{M \in \mathcal{M}}[(1-\epsilon)\|x\|_2 \leq \|Mx\|_2 \leq (1+\epsilon)\|x\|_2] > 1-\delta.$$

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Fundamental result of linear dimensionality reduction:

Lemma (Distributional Johnson-Lindenstrauss Lemma)

Can obtain $m = \Theta(\epsilon^{-2}\log(1/\delta))$ using $\mathcal{M}$ with i.i.d gaussian entries.

This dimension is actually optimal for any distribution over linear maps (Kane et al. '11, Jayram and Woodruff '11).

Optimality for N-point version (Larsen and Nelson '17)

Application to information compression in the brain

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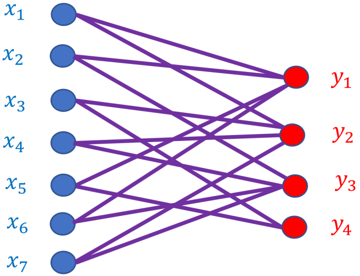
Convergent pathways compress information w/o losing the ability to perform computations.

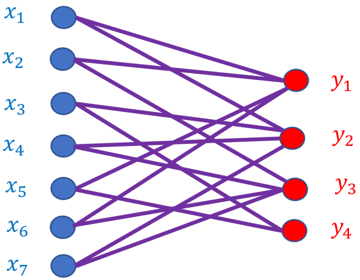


A model (Ganguli, Sompolinsky '12)

- Source information: $x \in \mathbb{R}^n$
- Target information: $y \in \mathbb{R}^m$
- Synaptic connections: a random matrix $M \in \mathbb{R}^{m \times n}$

Biological Constraints on Supp($\mathcal{M}$)

(As per Ganguli and Sompolinsky's model)



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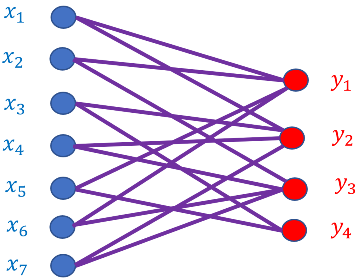

Sparsity: every column has $\leq s$ nonzero entries.

- Neurons connected to few post-synaptic neurons

Sign-consistency: in each column, nonzero entries are *all positive* or *all negative*

- Neurons are excitatory or inhibitory

JL for sparse matrices

Sparsity is also more generally useful for reducing projection time.

Informal Construction (Sparse JL)

Uniformly choose s nonzero entries per column; i.i.d signs for nonzero entries

Can set $m = \Theta(\epsilon^{-2} \log(1/\delta))$ and $s = \Theta(\epsilon^{-1} \log(1/\delta)$ (Kane and Nelson, J. ACM '12)

Can set $m = \min(2\epsilon^{-2}/\delta, \Theta(\epsilon^{-2} \log(1/\delta) B))$ and $s = \Theta(\epsilon^{-1} \log(1/\delta)/\log B)$ (Cohen, SODA '16)

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$\mathcal{M}$ is defined so the (r, i)th entry is $\sigma_i \eta_{r,i}/\sqrt{s}$ where:

- *σ_i are i.i.d. Rademachers (random signs)*
- *$\eta_{r,i}$ are $\{0, 1\}$ rvs s.t. $\sum_{r=1}^{m} \eta_{r,i} = s$ and w/ mild assumptions*

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Can set $m = \Theta(\epsilon^{-2} \log^2(1/\delta))$, $s = \Theta(\epsilon^{-1} \log(1/\delta))$ (Allen-Zhu, Gelashvili, Micali, and Shavit, PNAS '15)

This work

Simplify and generalize the analysis of sparse, sign-consistent JL.

Theorem (Informal)

For any $e \leq B \leq 1/\delta$*, can set* $m = \Theta(\epsilon^{-2}\log^2(1/\delta)B/\log^2(B))$ *and* $s = \Theta(\epsilon^{-1}\log(1/\delta)/\log B)$ *for sparse, sign-consistent JL.*

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Remainder of the talk will focus on the proof method.

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For sparse, sign-consistent JL:

$$Z := \|Mx\|_2^2 - 1 = \frac{1}{s} \sum_{i \neq j} \sum_{r=1}^{m} \sigma_i \sigma_j \eta_{r,i} \eta_{r,j} x_i x_j.$$

Analyzing the moments of Z

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My key ingredient: more precise quadratic form bounds

Expressing Z as a Rademacher quadratic form

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But, we show using $\mathbb{E}_g \left[\left(g^T A_\eta g \right)^p \right]$ is too weak for this setting!

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Thus, we need a Rademacher-specific bound for $\mathbb{E}_\sigma\left[\left(\sigma^T A_\eta \sigma\right)^p\right]$.

A digression on Rademachers vs. gaussians for linear forms

$\|Y\|_p := (\mathbb{E}[Y^p])^{1/p}$; σ_i i.i.d. Rademachers; g_i i.i.d gaussians;
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Turns out $\|\sum_{i=1}^n a_i \sigma_i\|_p \sim \left|\sum_{i=1}^p a_i\right| + \sqrt{p}\sqrt{\sum_{i>p} a_i^2}$ (Hitzchenko '93).

Generalizing to quadratic forms of Rademachers

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Lemma

If $(A_{i,j})$ is a symmetric $n \times n$ matrix with zero diagonal and p even, then

$$\left\| \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} \sigma_i \sigma_j \right\|_p \lesssim \left(\sum_{i=1}^{\min(p,n)} \sum_{j=1}^{\min(p,n)} |A_{i,j}| \right) + \sqrt{p} \sqrt{\sum_{i=1}^{n} \left\| \sum_{j>p} A_{i,j} \sigma_j \right\|_p^2}.$$

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(Tight bound on $\left\| \sigma^T A \sigma \right\|_p$ (Latała '99) messy when A a *random matrix*.)

Follow-up work that uses intuition from these methods

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"Understanding Sparse JL for Feature Hashing" (NeurIPS 2019)

I study sparse JL on feature vectors.

- Model: limit to vectors x with "small" ℓ_∞-to-ℓ_2 norm ratio
- $s = 1$ understood (Weinberger et al '09, Dahlgaard et al. '17, Freksen et al. '18, etc.)

My main result: Generalization to $s > 1$.

- Tight tradeoff between ℓ_∞-to-ℓ_2 ratio, s, m, ϵ, and δ for sparse JL
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Similarly unclear how to adapt combinatorics; gaussian bounds too weak.

Tractable Rademacher-specific bounds are the key technical tool.

Conclusion

- Simplified and generalized the analysis of sparse, sign-consistent JL (Allen-Zhu, Gelashvili, Micali, Shavit '15).
- Specifically obtained dimensionality-sparsity tradeoffs $m = \Theta(\epsilon^{-2} \log^2(1/\delta) B / \log^2(B))$ and $s = \Theta(\epsilon^{-1} \log(1/\delta) / \log B)$.
- Introduced a simple moment bound for Rademacher quadratic forms which enables a simpler analysis of sparse, sign-consistent JL, and could be of broader use.