Supply-Side Equilibria in Recommender Systems

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Abstract

Digital recommender systems such as Spotify and Netflix affect not only consumer behavior but also *producer incentives*: producers seek to supply content that will be recommended by the system. But what content will be produced? In this paper, we investigate the supply-side equilibria in content recommender systems. We model users and content as D-dimensional vectors, and recommend the content that has the highest dot product with each user. The main features of our model are that the producer decision space is *high-dimensional* and the user base is *heterogeneous*. This gives rise to new qualitative phenomena at equilibrium: First, *the formation of genres*, where producers specialize to compete for subsets of users. Using a duality argument, we derive necessary and sufficient conditions for this specialization to occur. Second, we show that producers can achieve *positive profit at equilibrium*, which is typically impossible under perfect competition. We derive sufficient conditions for this to occur, and show it is closely connected to specialization of content. In both results, the interplay between the geometry of the users and the structure of producer costs influences the structure of the supply-side equilibria. At a conceptual level, our work serves as a starting point to investigate how recommender systems shape supply-side competition between producers.

1 Introduction

Content recommender systems have disrupted the production of digital goods such as movies, music, and news. In the music industry, artists have changed the length and structure of songs in response to Spotify's algorithm and payment structure (Hodgson, 2021). In the movie industry, personalization has led to low-budget films catering to specific audiences (McDonald, 2019), in some cases constructing data-driven "taste communities" (Adalian, 2018). Across industries, recommender systems are shaping how producers decide what content to create, influencing the supply side of the digital goods market. This raises the questions: *What factors drive and influence these supply-side effects? What content will be produced at equilibrium?*

Intuitively, supply-side effects are induced by the multi-sided interaction between producers, recommender system algorithm, and users. Users tend to follow recommendations when deciding what content to consume (Ursu, 2018)—thus, recommendations influence how many users consume each digital good and impact the profit (or utility) generated by each content producer. As a result, content producers shape their content to maximize appearance in recommendations. This creates competition between the producers, which can be modeled as a game.

The supply-side competition for digital goods has two salient features that make it richer than classical price competition (Baye and Kovenock, 2008). First, digital goods such as movies have many attributes and thus must be embedded in a high-dimensional space. This *high dimensionality* increases the space of decisions for producers. In particular, they choose both the magnitude of the vector (which captures the quality level) and its direction (which captures the genre). Second, the platform has a *heterogeneous user base* and recommendations are personalized to each user. Producers thus must decide balance the preferences of these users in order to maximize the profit that they derive. These two features lead supply-side equilibria to exhibit rich economic phenomena, and the goal of our work is to characterize these phenomena.

Our contribution. In this paper, we present a theoretical framework for supply-side equilibria, focusing on the role of the infinite, high-dimensional space of decisions available to producers. In particular, producers choose both the magnitude and the direction of their content. The possibility of producers differentiating from each other on these two axes leads to two fundamentally new phenomena: *the formation of genres* and *positive profit at equilibrium*. We investigate what properties of the user base and producer costs determine when these phenomena occur.

In our model, each user and good is associated with a vector in $\mathbb{R}^D$. The producer's decision of what digital good to produce thus corresponds to choosing a D-dimensional vector. Each user is recommended the good with maximum dot product—a producer can therefore win a user by exhibiting either higher *quality* (vector norm), or better *personalization* (cosine similarity). Producers receive profit equal to the number of users who are recommended their content minus the cost of producing the content. We study the Nash equilibria of the resulting game between producers.

The first phenomenon that we characterize is the formation of genres. Content can differ in either quality level or type; to focus on the latter, we define *genres* as the set of directions that arise at an equilibrium. When an equilibrium has a single genre, all producers cater to an average user, and only a single type of content appears on the platform. On the other hand, when an equilibrium has multiple genres, many types of digital content are likely to appear on the platform. In Theorem 1, we provide necessary and sufficient conditions for all equilibria to have multiple genres.

Theorem 1 reveals that the emergence of multiple genres depends on the geometry of the users and the structure of producer costs. When users are close together, there is a single genre equilibrium where all producers cater to an average user. However, when users are sufficiently different, producers create specialized content for different users. How different the users need to be depends on how much the producer cost function implicitly incentivizes specialization, which we characterize in Section 3.2.

The second phenomenon that we pinpoint is the presence of strictly positive equilibrium utilities. At first glance, one might imagine that competition drives the equilibrium utility level down to 0, as with perfect competition in pricing markets (Baye and Kovenock, 2008). Indeed, at a single-genre equilibrium, where producers all compete along the same direction, the equilibrium utility is necessarily zero. However, we show that the utility can be nonzero at multi-genre equilibria, and provide sufficient conditions under which all equilibria have *strictly positive utility* (Theorem 2). Intuitively, specialization of content limits intra-genre competition, which leads to some monopoly-like behavior emerging within each genre.

Technical machinery. We develop new technical tools to analyze the complex, high-dimensional behavior of producers, which may be of broader interest. First and foremost, to prove Theorem 1, we draw a connection to minimax theory in optimization. In particular, we show that the existence of a single-genre equilibrium is equivalent to strong duality holding for a certain optimization program that we define. This allows us to leverage techniques from optimization theory to provide a necessary and sufficient condition for genre formation. Second, due to the inherent discontinuities in the producer utility function, we rely on the technology of discontinuous games (Reny, 1999) to establish the existence of symmetric mixed equilibria. Third, we investigate the formation of genres—which intuitively captures heterogeneity across producers—without having to reason about asymmetric equilibria. To do this, we formalize the concept of "genres" in terms of the support of the symmetric mixed equilibrium distribution.

1.1 Related Work

Our work is related to research on *societal effects in recommender systems*, *models of competition in economics*, and *strategic effects induced by algorithmic decisions*.

Societal effects of recommender systems. Ben-Porat and Tennenholtz (2018) also studied supply-side effects from a theoretical perspective, with a focus on mitigating strategic effects by content producers. Ben-Porat et al. (2020), building on Basat et al. (2017), also studied supply-side equilibria with a focus on convergence of learning dynamics for producers. The main difference from our work is that producers in these models choose a topic from a *finite* set of options; in contrast, our model captures the infinite, high-dimensional producer decision space that drives the emergence of genres. Moreover, we focus on the structure of equilibria rather than the convergence of learning.

In concurrent and independent work, Hron et al. (2022) study a related model for supply-side competition in recommender systems where producers choose content embeddings in $\mathbb{R}^D$. One main difference is that, rather than having a cost on producer content, they constrain producer vectors to the ℓ_2 unit ball (this corresponds to our model when $\beta \to \infty$ and the norm is the ℓ_2-norm, although the limit behaves differently than finite β); additionally, Hron et al. (2022) incorporate a softmax decision rule to capture exploration and user non-determinism, whereas we focus entirely on hardmax recommendations. Thus, our model focuses on the role of producer costs while Hron et al. (2022)'s focuses on the role of the recommender environment. At a technical level, Hron et al. (2022) study the existence of different types of equilibria and the use of behaviour models for auditing, whereas we analyze the economic phenomena exhibited by symmetric mixed strategy Nash equilibria (with a focus on genre formation and equilibrium producer profit).

Other phenomena studied are the emergence of filter bubbles (Flaxman et al., 2016), the ability of users to reach different content (Dean et al., 2020), the shaping of user preferences (Adomavicius et al., 2013), and stereotyping (Guo et al., 2021).

Models of competition in microeconomics. There has been a vast literature on different models for competition in microeconomic theory. These models include *price competition* (e.g. Bertrand competition (Baye and Kovenock, 2008)), where producers set a *price*, but not what good to produce; *spatial competition* (e.g. Hotelling's location model (Hotelling, 1981) and Salop's circle model (Berry, 1979)), where firms choose the location of their content[1]; and *supply function equilibria* (e.g. (Grossman, 1981)), where the producer chooses a function from quantity to prices. In contrast, producers in our model decide on the content that they produce—in particular, they jointly choose both the genre and quality level of the content.

In the pure characteristics model ((Berry, 1994)), attributes of users and producers are also embedded in $\mathbb{R}^D$ like in our model. However, this line of work focuses on demand estimation for a fixed set of content, rather than analyzing the content that arises at equilibria in the marketplace.

Strategic effects induced by algorithmic decisions. Brückner et al. (2012) and Hardt et al. (2016) initiated the study of *strategic classification*, where algorithmic decisions induce participants to strategically change their features to improve their outcomes. The models for participant behavior in this line of work (e.g. Kleinberg and Raghavan (2019); Jagadeesan et al. (2021); Ghalme et al. (2021)) generally do not capture competition between participants. One exception is Liu et al. (2022), where participants compete to appear higher in a single ranked list. In contrast, the participants in our model simultaneously compete for users with *heterogeneous* preferences. In a review of the fair recommendation literature, Patro et al. (2022) call for the need to study strategic effects.

[1] In the context of our model, this is similar to producers choosing the genre but not the quality level of content. Moreover, these models classically focus on 1-dimension (either $\mathbb{R}^1$ or $\mathbb{S}^1$) rather than higher dimensions.

2 Model

Consider a platform with $N \geq 1$ users and $P \geq 2$ producers. Each user i is associated with a D-dimensional embedding $u_i \in \mathbb{R}^D_{\geq 0}$ that captures their preferences. We assume the coordinates of each embedding are nonnegative and that each embedding is nonzero.

While user vectors are fixed, producers *choose* what content to create. Each producer j creates a single digital good, which is associated with a content vector $p_j \in \mathbb{R}^D_{\geq 0}$. The value of good p to user u is $\langle u, p \rangle$.

Personalized recommendations. After the producers create content, the platform offers personalized recommendations to each user. We consider a stylized model where the platform has complete knowledge of the user and content vectors. The platform recommends to each user the content of maximal value to them, assigning them to the producer who created this content. Mathematically, the platform assigns a user u to the producer j^*, where $j^*(u;\ p_{1:P}) = \arg\max_{1 \leq j \leq P} \langle u, p_j \rangle$. If there are ties, the platform sets $j^*(u;\ p_{1:P})$ to be a randomly chosen producer in the argmax.

Producer cost function. Each producer faces a *fixed cost* for producing content p, which depends on the magnitude of p. Since the good is digital, the production cost does not scale with the number of users.

The cost function $c(p)$ takes the form $\|p\|^\beta$, where $\|\cdot\|$ is any norm and the exponent β is at least 1. The magnitude $\|p\|$ captures the *quality* of the content: in particular, if a producer chooses content λp, they win at least as many users as if they choose $\lambda' p$ for $\lambda' < \lambda$. (This relies on the fact that all vectors are in the positive orthant.) The norm and β together encode the cost of producing a content vector v, and reflect cost tradeoffs for excelling in different dimensions (for example, producing a movie that is both a drama and a comedy). Large β, for instance, means that this cost grows superlinearly. In Section 3, we will see that these tradeoffs capture the extent to which producers are incentivized to specialize.

Producer profit. A producer receives profit equal to the number of users who are recommended their content minus the cost of producing the content. For producer j, let $p_{-j} = [p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_P]$ denote the content produced by all of the other producers. The profit of producer j is equal to:

$$\mathcal{P}(p_j; p_{-j}) = \mathbb{E}\Big[\Big(\sum_{i=1}^{n} 1[j^*(u_i;\ p_{1:P}) = j]\Big) - \|p_j\|^\beta\Big], \tag{1}$$

where the expectation comes from the randomness over platform recommendations in the case of ties.

2.1 Equilibrium concept and existence of equilibrium

We study the equilibria of the game between producers. Each producer j chooses a (random) strategy over content, given by a probability measure μ_j over the content embedding space $\mathbb{R}^D_{\geq 0}$. The strategies $(\mu_1, \ldots, \mu_P)$ form a *Nash equilibrium* if no producer—given the strategies of other producers—can chose a different strategy where they achieve higher expected profit. We call $\mu_{1:P}$ a *pure strategy equilibrium* if each μ_j contains only one vector in its support; otherwise, we call it a *mixed strategy equilibrium.*

A salient feature of this game is that there are discontinuities in the producer utility function in equation (1). In particular, whether a producer wins a given user is not continuous in the content embedding p_j, since the winner is chosen by a hard $\arg\max$ rule. As a result, no pure strategy equilibrium exists:

Proposition 1. *For any set of users and any $\beta \geq 1$, a pure strategy equilibrium does not exist.*

The proof of Proposition 1 in Appendix A.1 leverages that if two producers are tied, then a producer can increase their utility by infinitesimally increasing the magnitude of their content.

Although pure strategy equilibria are not guaranteed to exist, a *symmetric mixed equilibrium* (where $\mu_1 = \ldots = \mu_P$) necessarily exists. Since all producers play the same strategy at a symmetric equilibrium, we represent it with a single distribution μ over the content embedding space $\mathbb{R}^D_{\geq 0}$.

Proposition 2. *A symmetric mixed equilibrium μ exists.*

To prove Proposition 2, we leverage the technology of mixed equilibria in discontinuous games (Reny, 1999). The existence of a *symmetric* equilibrium follows from the symmetries of producer utility function. We defer the full proof to Appendix A.2.

Every symmetric equilibrium μ necessarily exhibits randomization across different content embeddings, since pure strategy equilibria do not exist. We in fact show that μ is *atomless* (Proposition 5), so in particular has infinite support.

2.2 One-dimensional setup

To provide intuition for the supply-side equilibria defined in Section 2.1, we consider a simple one-dimensional setting. We summarize results here and defer the derivation to Appendix A.4.

Example 1 (1-dimensional setup)**.** *Let $D = 1$, and suppose that there is a single user $u_1 = 1$. Suppose the cost function is $c(p) = |p|^\beta$. The unique symmetric mixed equilibrium μ is supported on the full interval $[0, 1]$ and has cumulative distribution function $F(p) = p^{\beta/(P-1)}$.*

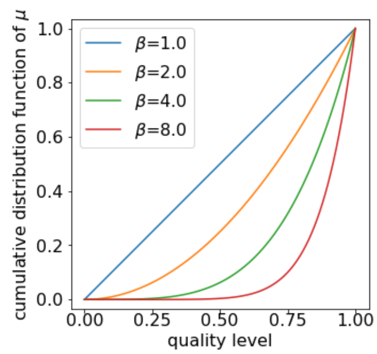

Figure 1: Cumulative distribution function (cdf) of the symmetric equilibrium μ for 1-dimensional setup (Example 1) with $P = 2$ producers. The equilibrium μ interpolates from a uniform distribution to a point mass as the exponent β increases.

Since $D = 1$ in Example 1, content is specified by a single value $p \in \mathbb{R}^{\geq 0}$. Since the user will be assigned to the content with the highest value of p, we can interpret p as the *quality level* of the content. For a producer, setting p to be larger increases the likelihood of being assigned to users, at the expense of a greater cost of production.

3 Formation of genres

When $D > 1$, producers choose not only the quality level of the content (Section 2.2) but also the *genre* of content as reflected by its direction in $\mathbb{R}^d$. A producer might exhibit *specialization* by creating a digital good that caters to a single user or group of users. Alternatively, all producers might still produce the same genre of content at equilibrium and thus only exhibit differentiation on the axis of quality.

To formalize specialization, we examine the set of content that appears in the support of the equilibrium distribution. More formally, for a symmetric mixed equilibrium μ, we define *the set of genres of μ* as:

$$\text{Genre}(\mu) := \left\{ \frac{p}{\|p\|} \mid p \in \text{supp}(\mu) \right\}, \tag{2}$$

which corresponds to the set of directions that appear in the support of μ. We normalize by $\|p\|$ to separate out the quality level (norm) of the digital good from the "type" (direction) of the good.

To gain intuition for the set of genres at equilibrium, we consider an example with 2 users and vary the cost exponent β. We defer the derivation to Appendix B.5.

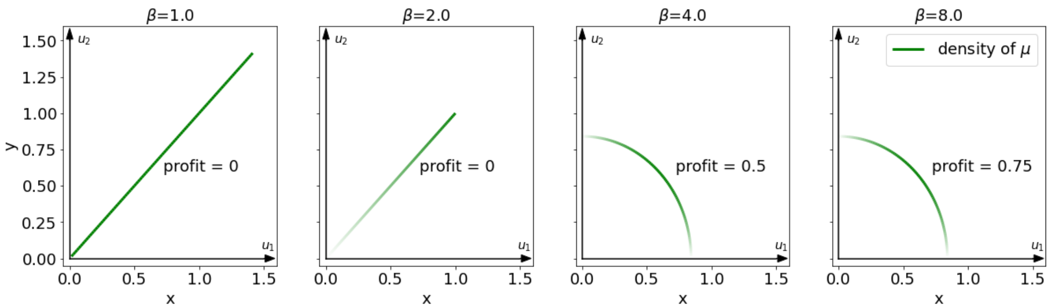

Figure 2: A symmetric equilibrium for different settings of β, for the setup in Example 2 with $P = 2$ producers. The opacity of the line illustrates the density of μ along the support. As β increases, there is a transition from a single-genre equilibrium to an equilibrium with infinitely many genres (studied in Section 3). The profit (studied in Section 4) also transitions from zero to positive.

Example 2. *Suppose that $D = 2$, $P = 2$, and there are $N = 2$ users given by the standard basis vectors e_1 and e_2. Let the cost $c(p) = \|p\|_2^\beta$. If $1 \le \beta \le 2$, then there exists a single-genre equilibrium, where* $\text{Genre}(\mu)$ *contains the single element $\frac{e_1+e_2}{\sqrt{2}}$. If $\beta > 2$, then* $\text{Genre}(\mu)$ *contains more than one direction for every equilibrium μ—in fact,* $\text{Genre}(\mu)$ *contains infinitely many genres in this example. Interestingly, the infinite-genre equilibria (unlike the single-genre equilibria) do not exhibit differentiation along magnitude.*

The equilibrium in this example is depicted in Figure 2 for different values of β. When β is sufficiently high, so that excelling on multiple dimensions is challenging, the cost function sufficiently favors specialization such that all equilibria have multiple genres.

The question that we resolve in this section is: *when do all equilibria have multiple genres?* We first present a general result (Theorem 1) that gives necessary and sufficient conditions for this to occur. We then study several corollaries of Theorem 1, which indicate when the norm and exponent incentivize specialization (Section 3.2). Finally, we provide a proof sketch of Theorem 1 that leverages duality of an optimization program that we construct (Section 3.3). At a conceptual level, our results provide insight into whether the producers create specialized content at equilibrium, or create generic content that caters to the average user in the marketplace.

3.1 Necessary and sufficient conditions for all equilibria to have multiple genres

To better understand the emergence of genres, we derive a necessary and sufficient condition for all equilibria to contain multiple genres. To formalize the condition, let $\mathbf{U} = [u_1; \cdots ; u_N]$ be the $N \times D$ matrix of user vectors, and let $\mathcal{S}$ denote the image of the unit ball under $\mathbf{U}$:

$$\mathcal{S} := \left\{ \mathbf{U}p \mid \|p\| \le 1, p \in \mathbb{R}^D_{\ge 0} \right\} \tag{3}$$

Each element of $\mathcal{S}$ is an N-dimensional vector, which represents the user utilities for some unit-norm producer p. Additionally, let $\mathcal{S}^\beta$ be the image of $\mathcal{S}$ under taking coordinate-wise powers, i.e. if $(z_1, \ldots, z_N) \in \mathcal{S}$ then $(z_1^\beta, \ldots, z_N^\beta) \in \mathcal{S}^\beta$.

The condition for multiple genres to emerge is that $\mathcal{S}^\beta$ is sufficiently different from its convex hull $\bar{\mathcal{S}}^\beta$:

Theorem 1. *There is a symmetric equilibrium μ with* $|\text{Genre}(\mu)| = 1$ *if and only if*

$$\max_{y \in \mathcal{S}^\beta} \prod_{i=1}^N y_i = \max_{y \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^N y_i. \tag{4}$$

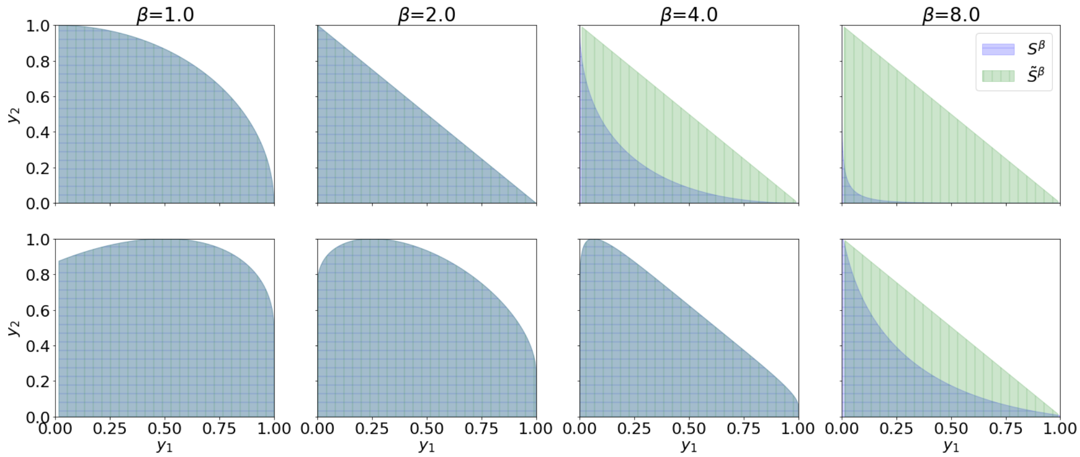

Figure 3: The sets $\mathcal{S}^\beta$ and $\bar{\mathcal{S}}^\beta$ for two configurations of user vectors (rows) and settings of β (columns). The user vectors are $[1,0],[0,1]$ (top, same as Figure 2) and $[1,0],[0.5,0.87]$ (bottom). $\mathcal{S}^\beta$ transitions from convex to non-convex as β increases, though the transition point depends on the user vectors.

Otherwise, all symmetric equilibria have multiple genres.

As a special case, the condition in Theorem 1 always holds if $\mathcal{S}^\beta$ is convex. In Figure 3, we display the sets $\mathcal{S}^\beta$ and $\bar{\mathcal{S}}^\beta$ for different configurations of user vectors and different settings of β. Observe that $\mathcal{S}^\beta$ depends on both the geometry of the user embeddings and the structure of the cost function (though interestingly *not* on the number of producers). Conceptually, once β is sufficiently large, the cost function incentivizes a sufficient degree of specialization that it is no longer in the best interest of producers to all compete along the same direction.

3.2 Consequences of Theorem 1

To further understand the condition in equation (4), we consider a series of special cases that provide intuition for when single-genre equilibria exist. We defer proofs of these results to Appendix C.

First, let us consider $\beta = 1$, in which case the cost function is a norm. Then $\mathcal{S}^1 = \mathcal{S}$ is convex, so a single-genre equilibrium always exists.

Corollary 1. *If $\beta = 1$, then there always exists a symmetric equilibrium μ with $|\text{Genre}(\mu)| = 1$.*

The economic intuition behind Corollary 1 is that convex cost functions incentivize averaging rather than specialization.

We next take a closer look at how the choice of norm affects the emergence of genres. Within the family of ℓ_q norms, a single-genre equilibrium exists for the standard basis vectors when $\beta \leq q$:

Corollary 2. *Let the cost function be $c(p) = \|p\|_q^\beta$. If $\beta \leq q$. For the user vectors $\{e_1, \ldots, e_D\}$, there exists a symmetric equilibrium μ with $|\text{Genre}(\mu)| = 1$.*

Corollary 2 again holds because $\mathcal{S}^\beta$ is convex for $\beta \leq q$. For the ℓ_∞-norm, where producers only pay for the highest magnitude coordinate, it is never possible to incentivize specialization: there exists a single-genre equilibrium regardless of β. On the other hand, for norms where costs aggregate nontrivially across dimensions, specialization is possible.

In addition to the choice of norm, the geometry of the user vectors also influences whether multiple genres emerge. To illustrate this, we show a natural sufficient condition under which all equilibria have multiple genres.

Corollary 3. *Let* $\|\cdot\|_*$ *denote the dual norm of* $\|\cdot\|$*, defined to be* $\|p\|_* = \max_{\|p\|=1, p \in \mathbb{R}^D_{\geq 0}} \langle q, p \rangle$*. Let* $Z := \| \sum_{n=1}^N \frac{u_n}{\|u_n\|_*} \|_*$ *and suppose the following condition on* β *holds:*

$$Z < N^{1-\frac{1}{\beta}}, \text{ or equivalently } \beta > \frac{\log(N)}{\log(N) - \log(Z)}. \tag{5}$$

Then, every symmetric equilibrium μ *satisfies* $|\mathrm{Genre}(\mu)| > 1$*.*

In equation (5), the threshold for β increases as Z increases. As an example, consider the cost function $c(p) = \|p\|_2^\beta$. We see that if the user vectors point in the same direction, then $Z = N$ and the right-hand side of (5) is ∞. On the other hand, if $u_1, \ldots, u_n$ are orthogonal, then $Z = \sqrt{N}$ and the right-side of (5) is 2, which exactly matches the lower bound in Corollary 2. In fact, for *random* vectors $u_1, \ldots, u_N$, we see that $Z = \tilde{O}(\sqrt{N})$ in expectation, in which case the right-hand side of (5) is close to 2 as long as N is large. Thus, for many choices of user vectors, even small values of β are enough to induce multiple genres.

Finally, we investigate *where* the single-genre equilibrium occurs, in the case where it does exist. As a consequence of the proof of Theorem 1, we can show the following:

Corollary 4. *If there exists* μ *with* $|\mathrm{Genre}(\mu)| = 1$*, then*

$$\mathrm{Genre}(\mu) \in \underset{\|p\|=1 | p \in \mathbb{R}^D_{\geq 0}}{\arg\max} \sum_{n=1}^N \log(\langle p, u_n \rangle). \tag{6}$$

Corollary 4 demonstrates that the producers cater their content to an "average user" of sorts. However, this average does not generally equal to the arithmetic mean of the users.

3.3 Proof sketch of Theorem 1

To prove Theorem 1, we relate the existence of a single equilibrium to strong duality of a certain optimization program. We then use convexification to establish necessary and sufficient conditions under which strong duality holds.

Suppose that there exists an equilibrium μ such that $\mathrm{Genre}(\mu) = \{p^*\}$ contains a single direction. Then μ is fully determined by the distribution over quality level $\|p\|$ where $p \sim \mu$; therefore, let F denote the cdf of $\|p\|$ for $p \sim \mu$. We can derive a closed-form expression for F; in fact, we show that it is identical to the cdf of the 1-dimensional setup in Example 1.

Lemma 1. *Suppose that* μ *is a symmetric equilibrium such that* $\mathrm{Genre}(\mu)$ *contains a single vector. Let* F *be the cdf of the distribution over* $\|p\|$ *where* $p \sim \mu$*. Then, it holds that:*

$$F(r) = \min\left(1, \left(\frac{r^\beta}{N}\right)^{1/(P-1)}\right). \tag{7}$$

The intuition for Lemma 1 is that a single-genre equilibrium essentially reduces the producer's decision to a 1-dimensional space, and so inherits the structure of the 1-dimensional equilibrium.

For μ to be an equilibrium, no alternative q should do better than $p \sim \mu$, which yields the following necessary and sufficient condition after plugging into the profit function (1):

$$\sup_q \left(\sum_{i=1}^N \frac{1}{N} \left(\frac{\langle q, u_i \rangle}{\langle p^*, u_i \rangle} \right)^\beta - \|q\|^\beta \right) = \mathbb{E}_{p' \sim \mu} \left[\sum_{i=1}^N \frac{1}{N} \left(\frac{\langle p', u_i \rangle}{\langle p^*, u_i \rangle} \right)^\beta - \|p'\|^\beta \right] \tag{8}$$

The term $\frac{1}{N}(\cdot)^\beta$ is the probability $(F(\cdot))^{P-1}$ that q outperforms the max of $P-1$ samples from μ.

We next change variables according to $y_i = \langle p^*, u_i \rangle^\beta$ and $y'_i = \langle \frac{q}{||q||}, u_i \rangle^\beta$ and simplify to see that μ is an equilibrium if and only if $\sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^n \frac{y'_i}{y_i} = N$. Thus, there exists a single-genre equilibrium if and only if

$$\inf_{y \in \mathcal{S}^\beta} \sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{y_i} = N. \tag{9}$$

While the left-hand side of equation (9) is challenging to reason about directly, we show that the dual $\sup_{y' \in \mathcal{S}^\beta} \inf_{y \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{y_i}$ is in fact equal to N. Therefore, the existence of a single-genre equilibrium boils down to a minimax theorem.

Lemma 2 (Informal)**.** *There exists a symmetric equilibrium μ with $|\text{Genre}(\mu)| = 1$ if and only if:*

$$\inf_{y \in \mathcal{S}^\beta} \left(\sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{y_i} \right) = \sup_{y \in \mathcal{S}^\beta} \left(\inf_{y \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{y_i} \right). \tag{10}$$

By Sion's min-max theorem, we can flip sup and inf in a convexified version of the left-hand side of (10). The remainder of the proof boils down to relating the resulting expression to the right-hand side of equation (10). We defer the full proof of Theorem 1 to Appendix B.

4 Strictly positive equilibrium profit

In this section, we examine the question: *when can a producer make a profit?* At a symmetric equilibria μ, all producers receive the same expected profit given by:

$$\mathcal{P}^{\text{eq}}(\mu) := \mathbb{E}_{p_1, \ldots, p_P \sim \mu}[\mathcal{P}(p_1; p_{-1})] = \mathbb{E}\Big[\Big(\sum_{i=1}^N 1[j^*(u_i;\ p_{1:P}) = 1]\Big) - \|p_1\|^\beta\Big], \tag{11}$$

where expectation in the last term is taken over $p_1, \ldots, p_P \sim \mu$ as well as randomness in recommendations.

Intuitively, the equilibrium profit of a marketplace provides insight about market competitiveness. Zero profit suggests that competition has driven producers to expend their full cost budget on improving product quality. Positive profit, on the other hand, suggests that the market is not yet fully saturated and new producers have incentive to enter the marketplace.

We illustrate how the behavior of $\mathcal{P}^{\text{eq}}(\mu)$ fundamentally differs in one dimension and in higher dimensions. We first verify equilibrium profit is always zero in one dimension (Proposition 3), which is in line with classical intuition. Then, we show that the equilibrium profit need not be zero in in $D > 1$ dimensions (Theorem 2): in particular, we provide a sufficient condition for positive profit in terms of the user geometry and cost function. We also show that single-genre equilibria necessarily result in zero profit (Proposition 4). This suggests that the driving force for positive profit is that producers form multiple genres, which reduces the level of competition within each genre.

4.1 Equilibrium profit in one dimension

We first verify that the classical results from price competition (Baye and Kovenock, 2008) apply to our model in the one-dimensional case. Specifically, we show that the equilibrium profit is guaranteed to be 0.

Proposition 3. *Consider a 1-dimensional setup with N users $u_1 = u_2 = \ldots = u_N = 1$. Suppose that the cost function is $c(p) = |p|^\beta$. At the unique symmetric equilibrium μ, the profit $\mathcal{P}^{eq}(\mu)$ is equal to* 0.

Proposition 3 shows that even when there are only 2 producers, the supply-side market is competitive enough for each producer to maximally increase the quality of their product. Intuitively, all producers compete along the same axis (quality level), which drives the quality level up until producers cannot afford to increase it anymore without receiving negative profit. We defer the proof to Appendix D.1.

4.2 Equilibrium profit in high dimensions

At multi-genre equilibria, however, the equilibrium profit is no longer guaranteed to be 0. To see this, let us revisit Example 2 for the regime where all equilibria have multiple genres.

Example 2 (continued). *Recall that in Example 2 (Figure 2), multiple genres emerge once $\beta > 2$ and the corresponding set of genres is the unit circle. In this case, the profit is $\mathcal{P}^{eq}(\mu) = 1 - \frac{2}{\beta}$.*

The intuition is that (after sampling the randomness in μ), different producers are likely to produce content in different directions. This reduces the amount of competition along any single direction of content. As a result, producers are no longer forced to maximize quality, thus enabling them to generate a strictly positive profit.

This finding can be generalized to sets of many users and producers and to arbitrary norms. We provide the following sufficient condition under which the profit at equilibrium is strictly positive:

Theorem 2. *Suppose that*

$$\max_{\|p\| \le 1} \min_{i=1}^{N} \left\langle p, \frac{u_i}{||u_i||} \right\rangle < N^{-P/\beta}. \tag{12}$$

Then for any symmetric equilibrium μ, the profit $\mathcal{P}^{eq}(\mu)$ is strictly positive.

Let us examine the quantity $Q := \max_{\|p\| \le 1} \min_{i=1}^{N} \langle p, \frac{u_i}{||u_i||} \rangle$ that appears on the left-hand side of (12). Intuitively, Q captures how easy it is to produce content that appeals simultaneously to all users. It is larger when the users are close together and smaller when they are spread out. For any set of vectors we see that $Q \le 1$, and as long as the vectors is nondegenerate, Q is strictly less than 1.

The right-hand side of (12), on the other hand, goes to 1 as $\beta \to \infty$. Thus, for any non-degenerate set of users, if β is sufficiently large, the condition in Theorem 2 will be met. In other words, if the cost function sufficiently incentivizes specialization, then the profit $U(\mu)$ at any equilibrium μ is strictly positive. Moreover, the value of β at which the condition is met increases as Q increases (in particular, for sets of user vectors that are closer together).

Proof sketch of Theorem 2. Without loss of generality, we assume user vectors have unit norm $\|u_i\|$. Given an equilibrium μ, we will construct an explicit vector p that generates positive profit. The vector p is of the form $Q \left(\max_{p' \in \text{supp}(\mu)} ||p'||\right) \cdot u_{i^*}$ for some $i^* \in [1, N]$.

Cluster the set of unit vectors p into N groups $G_1, \ldots, G_N$, based on the user for whom they generate the lowest value. That is, each vector p belongs to the group G_i where $i = \text{argmin}_{1 \le i' \le N} \langle p, u_{i'} \rangle$. This means that if all producers choose directions in G_i, then the maximum value received by user u_i is

$$\begin{aligned}
\max_{1 \le j \le P} \langle p_j, u_i \rangle &\le \left(\max_{p' \in \text{supp}(\mu)} ||p'|| \right) \cdot \max_{1 \le j \le P} \langle \frac{p_j}{\|p_j\|}, u_i \rangle \\
&\le \left(\max_{p' \in \text{supp}(\mu)} ||p'|| \right) \max_{\|p\| \le 1} \min_{i=1}^{N} \langle p, u_i \rangle \\
&= Q \left(\max_{p' \in \text{supp}(\mu)} ||p'|| \right).
\end{aligned}$$

Let G_{i^*} be the group with highest probability of appearing in μ. Then with decent probability (at least $(1/N)^{P-1}$), all of the other producers will choose directions in G_{i^*} and p wins user u_{i^*}, so the profit is at least $(1/N)^{P-1} - Q^\beta \left(\max_{p' \in \text{supp}(\mu)} ||p'||\right)^\beta$. We defer the full proof to Appendix D.2.

Relationship with genre formation. Although the proof of Theorem 2 does not explicitly consider genre formation, we show that the presence of multiple genres at equilibrium is nonetheless central to Theorem 2. To illustrate this, we show that at a single-genre equilibrium, the profit is zero whenever there are at least $P \geq 2$ producers. The intuition from the 1-dimensional setup in Section 4.1 carries over, since the producers again compete only along the axis of quality.

Proposition 4. *If μ is a single-genre symmetric equilibrium, then the profit $\mathcal{P}^{eq}(\mu)$ is equal to* 0.

When Proposition 4 is viewed in the context of Theorem 2, we see that (12) cannot hold when single-genre equilibria exist. This aligns with the fact that (12) intuitively corresponds to users being sufficiently far apart and β being sufficiently large to incentivize specialization.

Summary. Theorem 2 indicates that the supply-side marketplace in high dimensions may not always be as competitive as in the single-dimension case. Essentially, if producers exhibit sufficient specialization, then they are able to generate a strictly positive profit, which is indicative of monopoly-like behavior. This suggests that the market is not saturated and new producers might be incentivized to enter.

5 Discussion and Future Work

We present a theoretical framework for investigating supply-side equilibria in recommender systems. Relative to classical economic models, our model captures the heterogeneity of users and the high-dimensionality of producer decisions. Our results reveal that whether or not genres form or producers generate positive profits depends on the geometry of the users and the the cost function.

Our work serves as a starting point to investigate the structure of supply-side equilibria in greater generality. We propose several directions for future work:

Open Question 1. *Characterize what sets of genres* $\text{Genre}(\mu)$ *arise at equilibria μ. When (if ever) is this set finite but not of size one?*

Characterizing multi-genre equilibria would provide insight into specialization in supply-side markets, and in particular whether or not genres can be conceptualized as discrete categories.

Open Question 2. *Quantify the aggregate utility that users derive at equilibrium. How does it vary with the number of producers? What is the relationship between user utility and producer profit?*

Quantifying user utility would elucidate whether specialization ends up helping producers at the expense of users, or whether specialization helps all market participants.

Open Question 3. *Investigate how noise in recommendations affect the supply-side equilibria.*

In practice, recommender systems have incomplete data about market participants—analyzing the role of noise would help elucidate how this affects market equilibria.

Addressing these questions would further elucidate the market effects induced by supply-side competition and more broadly, inform the understanding of the societal effects of recommender systems.

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A Proofs for Section 2

A.1 Proof of Proposition 1

We restate and prove Proposition 1.

Proposition 1. *For any set of users and any $\beta \geq 1$, a pure strategy equilibrium does not exist.*

Proof of Proposition 1. Assume for sake of contradiction that the solution $p_1, \ldots, p_P$ is a pure strategy equilibrium. We divide into two cases: (1) there exist $1 \leq j' \neq j \leq P$ and i such that $\langle p_j, u_i \rangle = \langle p_{j'}, u_i \rangle$, (2) there does not exist j, j' and i such that $\langle p_j, u_i \rangle = \langle p_{j'}, u_i \rangle$.

Case 1: there exist $1 \leq j' \neq j \leq P$ and i such that $\langle p_j, u_i \rangle = \langle p_{j'}, u_i \rangle$. Let producer j and producer j' be such that $\langle p_j, u_i \rangle = \langle p_{j'}, u_i \rangle$. The idea is that the producer j can leverage the discontinuity in their profit function (1) at p_j. In particular, consider the vector $p_j + \epsilon u_i$. The number of users that they receive as $\epsilon \to_+ 0$ is *strictly greater* than at p_j. The cost, on the other hand, is continuous in ϵ. This demonstrates that there exists $\epsilon > 0$ such that:

$$\mathcal{P}(p_j + \epsilon u_i; p_{-j}) > \mathcal{P}(p_j; p_{-j})$$

as desired. This is a contradiction.

Case 2: there does not exist j, j' and i such that $\langle p_j, u_i \rangle = \langle p_{j'}, u_i \rangle$. Consider a producer j who wins a nonzero number of users in expectation. Let $\mathcal{N}$ be the set of $1 \leq i \leq N$ such that $\langle p_j, u_i \rangle > \langle p_{j'}, u_i \rangle$ for all $j' \neq j$. This is nonempty since the producer wins a nonzero number of users in expectation. We leverage that the profit function of producer j is continuous at p_j. There exists $\epsilon > 0$ such that $\langle p_j(1-\epsilon), u_i \rangle > \langle p_{j'}, u_i \rangle$ for all $j' \neq j$ and all i, so that:

$$\mathcal{P}(p_j(1-\epsilon); p_{-j}) > \mathcal{P}(p_j; p_{-j})$$

as desired. This is a contradiction.

$\square$

A.2 Proof of Proposition 2

We restate and prove Proposition 2.

Proposition 2. *A symmetric mixed equilibrium μ exists.*

Proof of Proposition 2. We apply a standard existence result of symmetric, mixed strategy equilibria in discontinuous games (see Corollary 5.3 of (Reny, 1999)). We adopt the terminology of that paper and refer the reader to (Reny, 1999) for a formal definition of the conditions. It suffices to show that: (1) the producer action space is convex and compact, (2) the payoff function is continuous in μ, and (3) the game is diagonally payoff secure.

Producer action space is convex and compact. In the current game, the producer action space is not compact. However, we show that we can define a slightly modified game, where the producer action space is convex and compact, without changing the equilibrium of the game. For the remainder of the proof, we analyze this modified game.

In particular, each producer must receive at least 0 profit at equilibrium since $\mathcal{P}(\vec{0}; p_{-1}) \geq 0$ regardless of the actions p_{-1} taken by other producers. If a producer chooses p such that $\|p\| > N^{1/\beta}$, then their utility will be strictly negative. Thus, we can restrict to $\left\{p \in \mathbb{R}^D_{\geq 0} \mid \|p\| \leq 2U^{1/\beta}\right\}$ which a convex compact set. We add a factor of 2 slack to guarantee that any best-response by a producer will be in the *interior* of the action space and not on the boundary.

Establishing continuity. We show the payoff function $\mathcal{P}(\mu; [\mu, \ldots, \mu])$ (where μ is a distribution over the producer action space) is continuous in μ. Here we slightly abuse notation since $\mathcal{P}$ is technically defined over pure strategies in (1). We implicitly extend the definition to mixed strategies by considering expected profit. Using the fact that each producer receives a $1/P$ fraction of users in expectation at a symmetric solution, we see that:

$$\mathcal{P}(\mu; [\mu, \ldots, \mu]) = \frac{N}{P} - \int \|p\|^\beta d\mu,$$

which implies continuity.

Establishing diagonal payoff security. We construct, for each relevant payoff in the closure of the graph of the game's diagonal payoff function, an action that diagonal payoff secures that payoff. More formally, let (μ^*, α^*) be in the closure of the graph of the game's diagonal payoff function, and suppose that $(\mu^*, \ldots, \mu^*)$ is not an equilibrium. It suffices to construct μ' that diagonal payoff secures α^*. First, we see that $\alpha^* = \mathcal{P}(\mu, \ldots, \mu)$ by the continuity of the payoff function described above. Since $(\mu^*, \ldots, \mu^*)$ is not an equilibrium, there exists $p \in \left\{p \in \mathbb{R}^D_{\geq 0} \mid \|p\| \leq N^{1/\beta}\right\}$ such that

$$\mathcal{P}(v; [\mu, \ldots, \mu]) > \mathcal{P}(\mu; [\mu, \ldots, \mu]).$$

Since the inequality is strict, there exists $\epsilon > 0$ such that:

$$\mathcal{P}(p(1+\epsilon); [\mu, \ldots, \mu]) > \mathcal{P}(\mu; [\mu, \ldots, \mu]).$$

We claim that μ' taken to be the point mass at $p(1+\epsilon)$ will diagonally payoff secure (μ^*, u^*). It suffices to show that exists an open neighborhood around μ' such that $\mathcal{P}(\mu'', [\mu, \ldots, \mu]) > \mathcal{P}(\mu, [\mu, \ldots, \mu])$ for all μ'' in the open neighborhood. To see this, we observe that for sufficiently small $\epsilon' > 0$, for any μ'' that changes an ϵ' measure of μ' arbitrarily, it holds that $\mathcal{P}(\mu''; [\mu, \ldots, \mu]) > \mathcal{P}(\mu; [\mu, \ldots, \mu])$; moreover, for sufficiently small $\epsilon' > 0$, it holds that $\mathcal{P}(\mu''; [\mu, \ldots, \mu]) > \mathcal{P}(\mu; [\mu, \ldots, \mu])$ for any μ'' that is a point mass at $p(1+\epsilon) + \epsilon' v$ where v is a unit vector. □

A.3 Statement and proof of Proposition 5

We state and prove Proposition 5.

Proposition 5. *If μ is a symmetric mixed equilibrium, then μ is atomless.*

In this proof, we consider the payoff function $\mathcal{P}(\mu_1; [\mu_2, \ldots, \mu_P])$ (where μ is a distribution over the producer action space) defined to be the expected profit attained if a producer plays μ_1 when other producers play $\mu_2, \ldots, \mu_P$. Strictly speaking, this is an abuse of notation since $\mathcal{P}$ is technically defined over pure strategies in (1). We implicitly extend the definition to mixed strategies by considering *expected* profit.

Proof of Proposition 5. Let μ be a symmetric equilibrium, and assume for sake of contradiction that there is an atom at $p \in \mathbb{R}^d$ with probability mass $\alpha > 0$. It suffices to construct a vector p' that achieves profit

$$\mathcal{P}(p'; [\mu, \ldots, \mu]) > \mathcal{P}(\vec{0}; [\mu, \ldots, \mu]) = \mathcal{P}(\mu; [\mu, \ldots, \mu]).$$

Consider the vector $p' = p + \epsilon u_1$ for some $\epsilon > 0$. For any given realization of actions by other producers, and for any given user, the vector p' never wins the user with lower probability than the vector p. We construct an event and a user where the vector p' wins the user with strictly higher probability than the vector p. Let E be the event that all of the other producers choose the p vector. This event happens with probability α^{P-1}. Conditioned on E, the vector p' wins user u_1; on the other hand, the vector p wins user u_1 with probability $1/P$. Since the cost function is continuous in ϵ, there exists ϵ such that $\mathcal{P}(p; [\mu, \ldots, \mu]) > \mathcal{P}(\vec{0}; [\mu, \ldots, \mu]) = \mathcal{P}(\mu; [\mu, \ldots, \mu])$. This is a contradiction. □

A.4 Derivation of Example 1

To see that the cumulative distribution function is $F(p) = \min(1, p^{\beta/P-1})$, we use the fact that every equilibrium is by definition a single-genre equilibrium in 1 dimension and apply Lemma 1.

B Proof of Theorem 1

Before diving into the proof of Theorem 1, let's recall the statement of Theorem 1. Let $\mathbf{U} = [u_1; \cdots; u_N]$ be the $N \times D$ matrix of user vectors, and let $\mathcal{S}$ denote the image of the unit ball under $\mathbf{U}$:

$$\mathcal{S} = \left\{ \mathbf{U}p \mid \|p\| \leq 1, p \in \mathbb{R}^D_{\geq 0} \right\}.$$

Each element of $\mathcal{S}$ is an N-dimensional vector, which represents the user utilities for some unit-norm producer p. Additionally, let $\mathcal{S}^\beta$ be the image of $\mathcal{S}$ under taking coordinate-wise powers, i.e. if $(z_1, \ldots, z_N) \in \mathcal{S}$ then $(z_1^\beta, \ldots, z_N^\beta) \in \mathcal{S}^\beta$. Let $\bar{\mathcal{S}}^\beta$ be the convex hull of $\mathcal{S}^\beta$.

Theorem 1. *There is a symmetric equilibrium μ with $|\mathrm{Genre}(\mu)| = 1$ if and only if*

$$\max_{y \in \mathcal{S}^\beta} \prod_{i=1}^N y_i = \max_{y \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^N y_i. \tag{4}$$

Otherwise, all symmetric equilibria have multiple genres.

For technical reasons, we also define a set $\mathcal{S}_{>0}$ which deletes all points with a zero coordinate from $\mathcal{S}$. More formally:

$$\mathcal{S}_{>0} := \left\{ \mathbf{U}p \mid \|p\| \leq 1, p \in \mathbb{R}^D_{\geq 0} \right\} \cap \mathbb{R}^N_{>0}.$$

For notational convenience, we also define:

$$\mathcal{B} := \left\{ p \in \mathbb{R}^D_{\geq 0} \mid \|p\| \leq 1, \langle p, u_i \rangle > 0 \forall i \right\},$$

$$\mathcal{B}_{>0} := \left\{ p \in \mathbb{R}^D_{\geq 0} \mid \|p\| \leq 1, \langle p, u_i \rangle > 0 \forall i \right\},$$

which are both convex sets. We further define:

$$\mathcal{D} := \left\{ p \in \mathbb{R}^D_{\geq 0} \mid \|p\| = 1 \right\}$$

and

$$\mathcal{D}_{>0} := \left\{ p \in \mathbb{R}^D_{\geq 0} \mid \|p\| = 1, \langle p, u_i \rangle > 0 \forall i \right\}.$$

Note that it follows from definition that:

$$\mathcal{S}_{>0} = \{ \mathbf{U}p \mid p \in \mathcal{B} \}$$

$$\mathcal{S}_{>0} = \{ \mathbf{U}p \mid p \in \mathcal{B}_{>0} \}$$

The proof will proceed by proving Lemma 1 and Lemma 2, and then proving Theorem 1 from these lemmas. In Appendix B.1, we prove a useful auxiliary lemma about single-genre equilibria; in Appendix B.2, we prove Lemma 1; in Appendix B.3, we formalize and prove Lemma 2; and in Appendix B.4, we prove Theorem 1.

B.1 Auxiliary lemma

We show that at a single-genre equilibrium, it must hold that the direction vector has nonzero inner product with every user.

Lemma 3. *Suppose that μ is a symmetric equilibrium such that* $\text{Genre}(\mu)$ *contains a single vector p^*. Then $p^* \in span(u_1, \ldots, u_N)$ (which also means that $\langle p^*, u_i \rangle > 0$ for all i.)*

Proof. Assume for sake of contradiction that $\langle p^*, u_i \rangle = 0$ for some i. Suppose that $p' \in \text{supp}(\mu)$, and consider the vector $p' + \epsilon \frac{u_i}{||u_i||}$. We see that $p' + \epsilon \frac{u_i}{||u_i||}$ wins user u_i with probability 1 whereas p' wins user u_i with probability $1/P$. The probability that $p + \epsilon u_i$ wins any other user is also at least the probability that p' wins u_i. By leveraging this discontinuity, we see there exists ϵ such that $\mathcal{P}(p' + \epsilon \frac{u_i}{||u_i||}; [\mu, \ldots, \mu]) > \mathcal{P}(p'; [\mu, \ldots, \mu]) + (1 - \frac{1}{P})$ which is a contradiction. □

B.2 Proof of Lemma 1

We restate and prove Lemma 1.

Lemma 1. *Suppose that μ is a symmetric equilibrium such that* $\text{Genre}(\mu)$ *contains a single vector. Let F be the cdf of the distribution over $\|p\|$ where $p \sim \mu$. Then, it holds that:*

$$F(r) = \min\left(1, \left(\frac{r^\beta}{N}\right)^{1/(P-1)}\right). \tag{7}$$

Proof. Next, we show that $F(r) = 0$ only if $r = 0$. Since the distribution μ is atomless (by Proposition 5), we can view the support as a closed set. Let $r_{\min}$ be the minimum magnitude element in the support of μ. Since μ is atomless, this means that with probability 1, every producer will have magnitude greater than $r_{\min}$. This, coupled with Lemma 3, means that the producer the expected number of users achieved at $r_{\min} p$ is 0, and $\mathcal{P}(r_{\min} p; [\mu, \ldots \mu]) = -r_{\min}^\beta$. However, since $r_{\min} p \ \in \text{supp}(\mu)$, it must hold that:

$$-r_{\min}^\beta = \mathcal{P}(r_{\min} p; [\mu, \ldots, \mu]) \geq \mathcal{P}(\vec{0}; [\mu, \ldots, \mu]) \geq 0.$$

This means that $q_{\min} = 0$.

Next, we show that the equilibrium profit at $(\mu, \ldots, \mu)$ is equal to 0. To see this, suppose that if the producer chooses $\vec{0}$. Since μ is atomless and since $\langle p^*, u_i \rangle > 0$ for all i (by Lemma 3), we see that if a producer chooses $\vec{0} \in \text{supp}(\mu)$, they receive 0 users in expectation. This means that $\mathcal{P}(\vec{0}; [\mu, \ldots, \mu]) = 0$ as desired.

Now, let's use the fact that the producer must earn the same profit—here, zero profit—for any $p \in \text{supp}(\mu)$. This means that for any $rp^* \in \text{supp}(\mu)$, it must hold that $NF(r)^{P-1} - r^\beta = 0$. Solving, we see that $F(r) = \left(\frac{r^\beta}{N}\right)^{1/(P-1)}$. The support cannot contain gaps, because that would mean that the cdf would jump and there would be atoms (but there are no atoms by Proposition 5). Thus, the support is a closed interval. Since $F(r) = 1$ at the top of the interval by definition, we have a min with 1 in the lemma. □

B.3 Formal Statement and Proof of Lemma 2

We formalize and prove Lemma 2, which illustrates the connection between the existence of a single-genre equilibrium and minimax duality.

Lemma 4 (Formalization of Lemma 2). *There exists a symmetric equilibrium μ with $|\text{Genre}(\mu)| = 1$ if and only if:*

$$\inf_{p^* \in \mathcal{B}_{>0}} \sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} = \sup_{y' \in \mathcal{S}^\beta} \inf_{p^* \in \mathcal{B}_{>0}} \sum_{i=1}^N \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta}. \tag{13}$$

It turns out to be more convenient to use a (slightly less intuitive) variant of Lemma 4 to prove Theorem 1. We state and prove Lemma 5 below.

Lemma 5. *There exists a symmetric equilibrium μ with $|\text{Genre}(\mu)| = 1$ if and only if:*

$$\inf_{p^* \in \mathcal{B}_{>0}} \sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} \leq N. \tag{14}$$

The main ingredient in the proof of Lemma 5 is the following characterization of a single-genre equilibrium in a given direction.

Lemma 6. *There is a symmetric equilibrium μ with $\text{Genre}(\mu) = \{p^*\}$ if and only if:*

$$\sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{(\langle p^*, u_i \rangle)^\beta} \leq N. \tag{15}$$

Proof. First, by Lemma 3, we see that the denominator is nonzero for every term in the sum, so equation (15) is well-defined.

If μ is a single-genre equilibrium, then the cdf of the magnitudes follows the form in Lemma 1. Thus, it suffices to identify necessary and sufficient conditions for that solution (that we call μ_{p^*}) to be a symmetric equilibrium.

The solution μ_{p^*} is an equilibrium if and only if no alternative q should do better than $p \sim \mu$. The profit level at μ_{p^*} is 0 by the structure of the cdf. Putting this all together, we see a necessary and sufficient for μ_{p^*} to be an equilibrium is:

$$\sup_{q \in \mathbb{R}^D_{\geq 0}} \left(\sum_{i=1}^N F\left(\frac{\langle q, u_i \rangle}{\langle p^*, u_i \rangle} \right)^{P-1} - \|q\|^\beta \right) \leq 0,$$

where the term $\frac{1}{N}(\cdot)^\beta$ is the probability $(F(\cdot))^{P-1}$ that q outperforms the max of $P-1$ samples from μ. Using the structure of the cdf, we can write this as:

$$\sup_{q \in \mathbb{R}^D_{\geq 0}} \left(\sum_{i=1}^N \min\left(1, \frac{1}{N} \left(\frac{\langle q, u_i \rangle}{\langle p^*, u_i \rangle} \right)^\beta \right) - \|q\|^\beta \right) \leq 0.$$

We can equivalently write this as:

$$\sup_{q \in \mathbb{R}^D_{\geq 0}} \left(\frac{1}{||q||^\beta} \sum_{i=1}^N \min\left(1, \frac{1}{N} \left(\frac{\langle q, u_i \rangle}{\langle p^*, u_i \rangle} \right)^\beta \right) - 1 \right) \leq 0,$$

which we can equivalently write as

$$\sup_{q \in \mathcal{D}} \sup_{r > 0} \left(\frac{1}{r^\beta} \sum_{i=1}^N \min\left(1, \frac{r^\beta}{N} \left(\frac{\langle q, u_i \rangle}{\langle p^*, u_i \rangle} \right)^\beta \right) - 1 \right) \leq 0.$$

For any direction q, if we disregard the first min with 1, the expression would be constant in r. With the minimum, the objective $\left(\frac{1}{r^\beta}\sum_{i=1}^N \min\left(1, \frac{1}{N}\left(\frac{\langle q,u_i\rangle}{\langle p^*,u_i\rangle}\right)^\beta\right) - 1\right)$ is weakly decreasing in r. Thus, $\sup_{r>0}\left(\frac{1}{r^\beta}\sum_{i=1}^N \min\left(1, \frac{1}{N}\left(\frac{\langle q,u_i\rangle}{\langle p^*,u_i\rangle}\right)^\beta\right) - 1\right)$ is attained as $r \to 0$. In fact, the maximum is attained at a value r if $r\langle q, u_i\rangle < N^{1/\beta}\langle p^*, u_i\rangle$ for all i. This holds for *some* $r > 0$ since $\langle p^*, u_i\rangle > 0$ for all i by Lemma 3. Thus we can equivalently formulate the condition as:

$$\sup_{q\in\mathcal{D}}\left(\left(\sum_{i=1}^N \frac{1}{N}\left(\frac{\langle q,u_i\rangle}{\langle p^*,u_i\rangle}\right)^\beta\right) - 1\right) \leq 0,$$

which we can write as:

$$\sup_{q\in\mathcal{D}}\sum_{i=1}^N \left(\frac{\langle q,u_i\rangle}{(\langle p^*,u_i\rangle)^\beta}\right)^\beta \leq N.$$

This is equivalent to:

$$\sup_{q\in\mathcal{B}}\sum_{i=1}^N \left(\frac{\langle q,u_i\rangle}{(\langle p^*,u_i\rangle)^\beta}\right)^\beta \leq N.$$

A change of variables gives us the desired formulation. □

Now, we can deduce Lemma 5.

Proof of Lemma 5. First, suppose that equation (14) does not hold. Then it must be true that:

$$\sup_{y'\in\mathcal{S}^\beta}\sum_{i=1}^N \frac{y'_i}{(\langle p^*,u_i\rangle)^\beta} > N$$

for every direction $p^* \in \mathcal{D}_{>0}$. This means that no direction in $\mathcal{D}_{>0}$ can be a single-genre equilibrium. We can further rule out directions in $\mathcal{D}\setminus\mathcal{D}_{>0}$ by applying Lemma 3.

Now, suppose that equation (14) does hold. It is not difficult to see that the optimum $\inf_{p^*\in\mathcal{B}_{>0}}\sup_{y'\in\mathcal{S}^\beta}\sum_{i=1}^N \frac{y'_i}{(\langle p^*,u_i\rangle)^\beta}$ is attained at some direction $p^* \in \mathcal{D}_{>0}$. Applying Lemma 6, we see that there exists a single-genre equilibrium in the direction p^*. □

B.3.1 Proof of Lemma 4

To prove Lemma 4 from Lemma 5, we require the following additional lemma that helps us analyze the right-hand side of equation (13).

Lemma 7. *For any set $\mathcal{R} \subseteq \mathbb{R}^N_{>0}$, it holds that:*

$$\sup_{y'\in\mathcal{R}}\inf_{y\in\mathcal{R}}\sum_{i=1}^N \frac{y'_i}{y_i} = N.$$

Proof. By taking $y' = y$, we see that:

$$\sup_{y'\in\mathcal{R}}\inf_{y\in\mathcal{R}}\sum_{i=1}^N \frac{y'_i}{y_i} \leq N.$$

To show equality, notice by AM-GM that:

$$\sum_{i=1}^{N} \frac{y'_i}{y_i} \geq N \left(\prod_{i=1}^{n} \frac{y'_i}{y_i} \right)^{1/N} = N \left(\frac{\prod_{i=1}^{n} y'_i}{\prod_{i=1}^{N} y_i} \right)^{1/N}.$$

We can take $y' = \arg\max_{y'' \in \mathcal{R}} \prod_{i=1}^{n} y''_i$, and obtain a lower bound of N as desired. (If the arg max does not exist, then note that if we take y' where $\prod_{i=1}^{n} y'_i$ is sufficiently close to the optimum $\sup_{y'' \in \mathcal{R}} \prod_{i=1}^{n} y''_i$, we have that $\inf_{y \in \mathcal{R}} \left(\frac{\prod_{i=1}^{n} y'_i}{\prod_{i=1}^{N} y_i} \right)^{1/N}$ is sufficiently close to 1 as desired.) □

Now we are ready to prove Lemma 4.

Proof of Lemma 4. First, we see that:

$$\begin{aligned} N &= \sup_{y' \in \mathcal{S}^{\beta}_{>0}} \inf_{y \in \mathcal{S}^{\beta}_{>0}} \sum_{i=1}^{N} \frac{y'_i}{y_i} \\ &= \sup_{y' \in \mathcal{S}^{\beta}} \inf_{y \in \mathcal{S}^{\beta}_{>0}} \sum_{i=1}^{N} \frac{y'_i}{y_i} \\ &= \sup_{y' \in \mathcal{S}^{\beta}} \inf_{p^* \in \mathcal{B}_{>0}} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^{\beta}}, \end{aligned}$$

where the first equality follows from Lemma 7.

Now, let's combine this with Lemma 5 to see that a necessary and sufficient condition for the existence of a single-genre equilibrium is:

$$\inf_{p^* \in \mathcal{B}_{>0}} \sup_{y' \in \mathcal{S}^{\beta}} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^{\beta}} \leq \sup_{y' \in \mathcal{S}^{\beta}} \inf_{p^* \in \mathcal{B}_{>0}} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^{\beta}} \tag{16}$$

Weak duality tells us that $\inf_{p^* \in \mathcal{B}_{>0}} \sup_{y' \in \mathcal{S}^{\beta}} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^{\beta}} \geq \sup_{y' \in \mathcal{S}^{\beta}} \inf_{p^* \in \mathcal{B}_{>0}} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^{\beta}}$, so equation (16) is equivalent to:

$$\inf_{p^* \in \mathcal{B}_{>0}} \sup_{y' \in \mathcal{S}^{\beta}} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^{\beta}} = \sup_{y' \in \mathcal{S}^{\beta}} \inf_{p^* \in \mathcal{B}_{>0}} \sum_{i=1}^{N} \frac{y'_i}{(\langle p^*, u_i \rangle)^{\beta}}.$$

□

B.4 Proof of Theorem 1

Proof of Theorem 1. Recall that by Lemma 5, a single genre equilibrium exists if and only if equation (14) is satisfied.

We can rewrite the left-hand side of equation (14) as follows:

$$\inf_{p^* \in \mathcal{B}_{>0}} \left(\sup_{y' \in \mathcal{S}^{\beta}} \sum_{i=1}^{N} \frac{y'_i}{\langle p^*, u_i \rangle^{\beta}} \right) = \inf_{p^* \in \mathcal{B}_{>0}} \left(\sup_{y' \in \bar{\mathcal{S}}^{\beta}} \sum_{i=1}^{N} \frac{y'_i}{\langle p^*, u_i \rangle^{\beta}} \right),$$

since the objective is linear in y'. Now, observing that the objective is convex in p and concave in y', we can apply Sion's min-max theorem[2] to see that:

$$\inf_{p^* \in \mathcal{B}_{>0}} \left(\sup_{y' \in \bar{\mathcal{S}}^\beta} \sum_{i=1}^N \frac{y'_i}{\langle p^*, u_i \rangle^\beta} \right) = \sup_{y' \in \bar{\mathcal{S}}^\beta} \left(\inf_{p^* \in \mathcal{B}_{>0}} \sum_{i=1}^N \frac{y'_i}{\langle p^*, u_i \rangle^\beta} \right) = \sup_{y' \in \bar{\mathcal{S}}^\beta} \left(\inf_{y \in \mathcal{S}^\beta_{>0}} \sum_{i=1}^N \frac{y'_i}{y_i} \right).$$

Thus, we have the following necessary and sufficient condition for a single-genre equilibrium to exist:

$$\sup_{y' \in \bar{\mathcal{S}}^\beta} \left(\inf_{y \in \mathcal{S}^\beta_{>0}} \sum_{i=1}^N \frac{y'_i}{y_i} \right) \leq N. \tag{17}$$

First, we show that if (4) does not hold, then there does not exist a single-genre equilibrium. Let $y' = \arg\max_{y'' \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^n y''_i$. (The maximum exists because $\prod_{i=1}^n y''_i$ is a continuous function and $\bar{\mathcal{S}}^\beta$ is a compact set.) We see that:

$$\sum_{i=1}^N \frac{y'_i}{y_i} \geq N \left(\frac{\prod_{i=1}^n y'_i}{\prod_{i=1}^n y_i} \right)^{1/N} \geq N \left(\frac{\max_{y'' \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^n y''_i}{\max_{y'' \in \mathcal{S}^\beta_{>0}} \prod_{i=1}^n y''_i} \right)^{1/N} = N \left(\frac{\max_{y'' \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^n y''_i}{\max_{y'' \in \mathcal{S}^\beta} \prod_{i=1}^n y''_i} \right)^{1/N} > N,$$

which proves that:

$$\inf_{p^* \in \mathcal{B}_{>0}} \left(\sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y'_i}{\langle p^*, u_i \rangle^\beta} \right) = \sup_{y' \in \bar{\mathcal{S}}^\beta} \left(\inf_{y \in \mathcal{S}^\beta_{>0}} \sum_{i=1}^N \frac{y'_i}{y_i} \right) > N.$$

Thus equation (17) is not satisfied and a single-genre equilibrium does not exist as desired.

Next, we show that if (4) holds, then there exists a single-genre equilibrium. Let $y^* = \arg\max_{y'' \in \mathcal{S}^\beta} \prod_{i=1}^n y''_i = \arg\max_{y'' \in \mathcal{S}^\beta} \sum_{i=1}^n \log(y''_i)$. (The maximum exists because $\prod_{i=1}^n y''_i$ is a continuous function and $\mathcal{S}^\beta$ is a compact set.) By assumption, we see that y^* is also the maximizer over $\bar{S}^\beta$. We further see that $y^* \in \mathcal{S}^\beta_{>0}$. Using convexity of $\bar{\mathcal{S}}^\beta$, this means that for any $y' \in \bar{\mathcal{S}}^\beta$, it must hold that $\langle y' - y^*, \nabla \left(\sum_{i=1}^n \log(y^*_i) \right) \rangle \leq 0$. We can write this as:

$$\langle y' - y^*, \nabla \sum_{i=1}^n \frac{1}{y^*_i} \rangle \leq 0.$$

This can be written as:

$$\sum_{i=1}^n \frac{y'_i - y^*_i}{y^*_i} \leq 0,$$

which implies that:

$$\sum_{i=1}^n \frac{y'_i}{y^*_i} \leq N.$$

Thus, we have that

$$\sup_{y' \in \bar{\mathcal{S}}^\beta} \left(\inf_{y \in \mathcal{S}^\beta_{>0}} \sum_{i=1}^N \frac{y'_i}{y^*_i} \right) \leq N,$$

and thus equation (17) is satisfied so a single-genre equilibrium does not exist as desired. □

[2]Note that $\bar{\mathcal{S}}^\beta$ is compact and convex and $\mathcal{B}_{>0}$ is convex (but not compact). We apply the non-compact formulation of Sion's min-max theorem in (Ha, 1981).

B.5 Derivation of Example 2

For $\beta \leq 2$, we can apply Corollary 2 to see that a single-genre equilibrium exists and apply Corollary 4 to pin down where the single-genre equilibrium is.

For $\beta > 2$, we can again apply Corollary 3 to see that a single-genre equilibrium does not exist. It is not difficult to show that there is an equilibrium μ with $\text{supp}(\mu) = \{[r\cos(\theta), r\sin(\theta)] \mid \theta \in [0, \pi/2]\}$ where the radius $r = \left(\frac{2}{\beta}\right)^{1/\beta}$. The cumulative distribution function of the angle θ is $F(\theta) = \sin^2(\theta)$. To see that the producer profit is $1 - \frac{2}{\beta}$, notice that each producer in expectation wins one user and pays cost $\frac{2}{\beta}$.

C Proof of Corollaries in Section 3.2

First, we restate and prove Corollary 1.

Corollary 1. *If $\beta = 1$, then there always exists a symmetric equilibrium μ with $|\text{Genre}(\mu)| = 1$.*

Proof. When $\beta = 1$, we see that $\mathcal{S}^\beta = \mathcal{S}^1$ is a linear transformation of a convex set (the unit ball restricted to $\mathbb{R}^D_{\geq 0}$), so it is convex. This means that $\bar{\mathcal{S}}^\beta = \mathcal{S}^\beta$, and so (4) is trivially satisfied. By Theorem 1, there exists a single-genre equilibrium. □

Next, we restate and prove Corollary 2.

Corollary 2. *Let the cost function be $c(p) = \|p\|_q^\beta$. If $\beta \leq q$. For the user vectors $\{e_1, \ldots, e_D\}$, there exists a symmetric equilibrium μ with $|\text{Genre}(\mu)| = 1$.*

Proof. We show that the set $\mathcal{S}^\beta$ is convex. Let's consider $y, y' \in \mathcal{S}^\beta$. There exists $p \in \mathbb{R}^D_{\geq 0}$ such that $||p|| \leq 1$ and y is given by the β-coordinate-wise powers of $\mathbf{U}p$; similarly, there exists $p' \in \mathbb{R}^D_{\geq 0}$ such that $||p'|| \leq 1$ and y' is given by the β-coordinate-wise powers of $\mathbf{U}p'$. For any $\lambda \in [0, 1]$, we wish to show that $\lambda y + (1-\lambda)y' \in \mathcal{S}^\beta$. Equivalently, we wish to show that there exists $p'' \in \mathbb{R}^D_{\geq 0}$ with $||p''|| \leq 1$ where y'' given by the β-coordinate-wise powers of $\mathbf{U}p''$ is equal to $\lambda y + (1-\lambda)y' \in \mathcal{S}^\beta$. Notice that:

$$(\lambda y + (1-\lambda)y')_d = (\lambda p_d^\beta + (1-\lambda)(p')_d^\beta).$$

We define p'' to be:

$$p''_d = (\lambda p_d^\beta + (1-\lambda)(p')_d^\beta)^{1/\beta}.$$

Since $\beta \leq q$, we see that

$$\sum_{d=1}^D (p''_d)^q = \sum_{d=1}^D (\lambda p_d^\beta + (1-\lambda)(p')_d^\beta)^{q/\beta} \leq \lambda \sum_{d=1}^D p_d^q + (1-\lambda)\sum_{d=1}^D (p')_d^q) \leq 1,$$

using the convexity given by the fact that $\beta \leq q$. □

Next, we restate and prove Corollary 3.

Corollary 3. *Let $\|\cdot\|_*$ denote the dual norm of $\|\cdot\|$, defined to be $\|p\|_* = \max_{\|p\|=1, p\in\mathbb{R}^D_{\geq 0}} \langle q, p\rangle$. Let $Z := \|\sum_{n=1}^N \frac{u_n}{\|u_n\|_*}\|_*$ and suppose the following condition on β holds:*

$$Z < N^{1-\frac{1}{\beta}}, \text{ or equivalently } \beta > \frac{\log(N)}{\log(N) - \log(Z)}. \tag{5}$$

Then, every symmetric equilibrium μ satisfies $|\text{Genre}(\mu)| > 1$.

Proof. WLOG assume that the users to have unit dual norm. By Theorem 1, it suffices to show that:

$$\max_{y \in \mathcal{S}^\beta} \prod_{i=1}^N y_i < \max_{y \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^N y_i.$$

First, let's lower bound the right-hand side. Consider the point $y = \frac{1}{N} \sum_{i'=1}^N z^{i'}$ where $z^{i'}$ is defined to be the β-coordinate-wise power of $\mathbf{U}\left(\arg\max_{||p||=1} \langle p, u_i \rangle\right)$. This means that

$$y_i \geq \frac{1}{N} z_i^i = \frac{1}{N} \left(\max_{||p||=1} \langle p, u_i \rangle \right)^\beta = \frac{||u_i||_*^\beta}{N} = \frac{1}{N}.$$

This means that:

$$\max_{y \in \bar{\mathcal{S}}^\beta} \prod_{i=1}^N y_i \geq \frac{1}{N^N}.$$

Next, let's upper bound the left-hand side. By AM-GM, we see that:

$$\max_{y \in \mathcal{S}^\beta} \prod_{i=1}^N y_i = \max_{||p||=1, p \in \mathbb{R}^D_{\geq 0}} \left(\prod_{i=1}^N \langle p, u_i \rangle \right)^\beta \leq \left(\frac{\sum_{i=1}^N \langle p, u_i \rangle}{N} \right)^{N\beta} \leq \left(\frac{\langle p, \sum_{i=1}^N u_i \rangle}{N} \right)^{N\beta} \leq \frac{\left(\| \sum_{i=1}^N u_i \|_* \right)^{N\beta}}{N^{N\beta}}.$$

Putting this all together, we see that it suffices for:

$$\frac{1}{N^N} > \frac{\left(\| \sum_{i=1}^N u_i \|_* \right)^{N\beta}}{N^{N\beta}},$$

which we can rewrite as:

$$N^{\beta-1} > \left(\| \sum_{i=1}^N u_i \|_* \right)^\beta$$

which we can rewrite as:

$$N^{1-1/\beta} > \| \sum_{i=1}^N u_i \|_*.$$

□

Finally, we restate and prove Corollary 4.

Corollary 4. *If there exists μ with $|\text{Genre}(\mu)| = 1$, then*

$$\text{Genre}(\mu) \in \arg\max_{\|p\|=1 | p \in \mathbb{R}^D_{\geq 0}} \sum_{n=1}^N \log(\langle p, u_n \rangle). \tag{6}$$

Proof. We apply Lemma 6 to see that if μ is a single-genre equilibrium with $\text{Genre}(\mu) = \{p^*\}$, then:

$$\sup_{y' \in \mathcal{S}^\beta} \sum_{i=1}^N \frac{y_i'}{(\langle p^*, u_i \rangle)^\beta} \leq N.$$

We see that:

$$N \leq \sup_{y' \in \mathcal{S}^\beta} \frac{y_i'}{(\langle p^*, u_i \rangle)^\beta} \geq N \sup_{y' \in \mathcal{S}^\beta} \left(\frac{\prod_{i=1}^N y_i'}{\prod_{i=1}^N (\langle p^*, u_i \rangle)^\beta} \right)^{1/N} \geq N \left(\frac{\sup_{y' \in \mathcal{S}^\beta} \prod_{i=1}^N y_i'}{\prod_{i=1}^N (\langle p^*, u_i \rangle)^\beta} \right)^{1/N}.$$

This implies that:

$$\prod_{i=1}^{N} y_i = \prod_{i=1}^{N} (\langle p^*, u_i \rangle)^\beta \geq \sup_{y' \in \mathcal{S}^\beta} \prod_{i=1}^{N} y'_i,$$

where $y \in \mathcal{S}^\beta$ is defined so that $y_i = \langle p^*, u_i \rangle$. This implies that:

$$p^* \in \underset{||p|| \leq 1, p \in \mathbb{R}^D_{\geq 0}}{\arg\max} \sum_{i=1}^{N} \log(\langle p, u_i \rangle) = \underset{||p|| = 1, p \in \mathbb{R}^D_{\geq 0}}{\arg\max} \sum_{i=1}^{N} \log(\langle p, u_i \rangle)$$

as desired. □

D Proofs for Section 4

D.1 Proof of Proposition 3

Proof of Proposition 3. Since μ is an equilibrium, all choices p in the support of μ achieve profit equal to the equilibrium profit. We apply Lemma 1 to see that the cdf of μ is $F(p) = \min\left(1, \left(\frac{p^\beta}{N}\right)^{1/(P-1)}\right)$, which shows that $p = 0$ is in $\text{supp}(\mu)$. For this choice of p, the cost is 0, but we also never win any users, so the profit is also zero, as claimed. □

D.2 Proof of Theorem 2

Proof of Theorem 2. Without loss of generality, we assume user vectors have unit norm $\|u_i\|$. Given an equilibrium μ, we will construct an explicit vector p that generates positive profit. This proves that the equilibrium profit is positive because no vector can achieve higher than the equilibrium profit. The vector p is of the form $(Q\left(\max_{p' \in \text{supp}(\mu)} ||p'||\right) + \epsilon) \cdot u_{i^*}$ for some $i^* \in [1, N]$.

Cluster the set of unit vectors p into N groups $G_1, \ldots, G_N$, based on the user for whom they generate the lowest value. That is, each vector p belongs to the group G_i where $u_i = \text{argmin}_{1 \leq i' \leq N} \langle p, u_{i'} \rangle$. This means that if all producers choose (unit vector) directions in G_i, then the maximum value received by user u_i is

$$\max_{1 \leq j \leq P} \langle p_j, u_i \rangle \leq \max_{||p|| \leq 1} \min_{i=1}^{N} \langle p, u_i \rangle = Q. \tag{18}$$

Let G_{i^*} be the group with highest probability of appearing in μ, that is $i^* \in \arg\max_i \mathbb{P}_{v \sim \mu}\left[\frac{v}{||v||} \in G_i\right]$.

Let E be the event that all of the other $P-1$ producers choose directions in G_{i^*}. The event E happens with probability at least $\mathbb{P}_{v \sim \mu}\left[\frac{v}{||v||} \in G_{i^*}\right] \geq (1/N)^{P-1}$. Since the value received by the user is linear in the magnitude of the producer action, we see that the maximum possible value that could be received by user u_i from the other producers is $Q\left(\max_{p' \in \text{supp}(\mu)} ||p'||\right)$. On the other hand, the action p results in value $(Q\left(\max_{p' \in \text{supp}(\mu)} ||p'||\right) + \epsilon)$ for u_{i^*}, so it wins u_{i^*} with probability 1 on the event E. This means that the expected profit obtained by p is at most

$$\left(\frac{1}{N}\right)^{P-1} - \left(Q\left(\max_{p' \in \text{supp}(\mu)} ||p'||\right) + \epsilon\right)^\beta.$$

Taking a limit as $\epsilon \rightarrow_+ 0$, we obtain the profit can be set arbitrarily close to:

$$\left(\frac{1}{N}\right)^{P-1} - \left(Q\left(\max_{p' \in \text{supp}(\mu)} ||p'||\right)\right)^\beta. \tag{19}$$

It suffices to bound $\max_{p' \in \text{supp}(\mu)} ||p'||$. The action $p'' \in \arg\max_{p' \in \text{supp}(\mu)} ||p'||$ produces a profit of at most $N - \left(\max_{p \in \text{supp}(\mu)} ||p||\right)^{\beta}$. Thus, $\left(\max_{p \in \text{supp}(\mu)} ||p||\right)^{\beta} \leq N$, so $\left(\max_{p \in \text{supp}(\mu)} ||p||\right) \leq N^{1/\beta}$.

Plugging this into (19), we see that there exist actions that produces profit arbitrarily close to

$$\left(\frac{1}{N}\right)^{P-1} - NQ^{\beta}.$$

Thus, a strictly positive profit will be obtained if:

$$Q < \left(\frac{1}{N}\right)^{P/\beta},$$

as desired. $\square$