

# Improved Bayes Risk Can Yield Reduced Social Welfare Under Competition

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## Abstract

As the scale of machine learning models increases, trends such as scaling laws anticipate consistent downstream improvements in predictive accuracy. However, these trends take the perspective of a single model-provider in isolation, while in reality providers often compete with each other for users. In this work, we demonstrate that competition can fundamentally alter the behavior of these scaling trends, even causing overall predictive accuracy across users to be non-monotonic or decreasing with scale. We define a model of competition for classification tasks, and use data representations as a lens for studying the impact of increases in scale. We find many settings where improving data representation quality (as measured by Bayes risk) decreases the overall predictive accuracy across users (i.e., social welfare) for a marketplace of competing model-providers. Our examples range from closed-form formulas in simple settings to simulations with pretrained representations on CIFAR-10. At a conceptual level, our work suggests that favorable scaling trends for individual model-providers need not translate to downstream improvements in social welfare in marketplaces with multiple model providers.

## 1 Introduction

Scaling trends in machine learning suggest that increasing the scale of a system consistently improves predictive accuracy. For example, scaling laws illustrate that increasing the number of model parameters [KMH<sup>+</sup>20; SK20; BDK<sup>+</sup>21] and amount of data [HBM<sup>+</sup>22] can reliably improve model performance, leading to better representations and thus better predictions for downstream tasks [HKH<sup>+</sup>21].

However, these scaling laws typically take the perspective of a single model-provider in isolation, when in reality, model-providers often compete with each other for users. For example, in digital marketplaces, multiple online platforms may provide similar services (e.g., Google search vs. Bing, Spotify vs. Pandora, Apple Maps vs. Google, and chatGPT vs. Claude) and thus compete for users on the basis of prediction quality. A distinguishing feature of competing platforms is that users can switch between platforms and select a platform that offers them the highest predictive accuracy for their specific requests. This breaks the direct connection between predictive accuracy of a single platform in isolation and social welfare across competing platforms, and raises the question: *what happens to scaling laws when model-providers compete with each other?*

We show that the typical intuition about scaling laws can fundamentally break down under competition. Surprisingly, even monotonicity can be violated: increasing scale can *decrease* the overall predictive accuracy (social welfare) for users. More specifically, we study increases to scale through the lens of data representations, motivated by how increasing scale generally improves

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\*Equal contribution

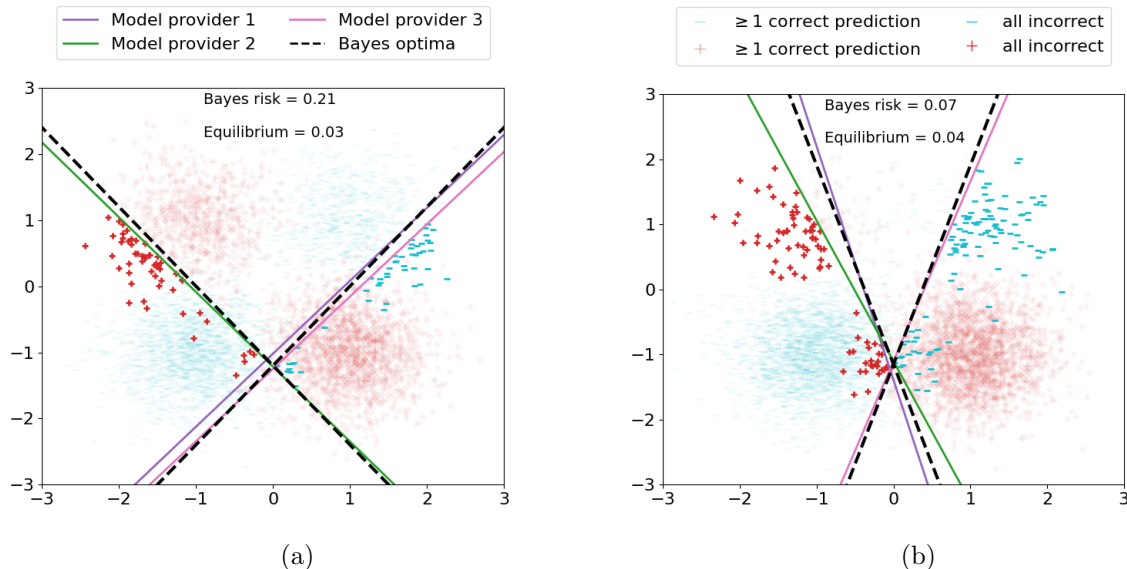


Figure 1: Comparison of equilibrium loss on two data distributions, one with high Bayes risk (left) and one with lower Bayes risk (right). Each plot shows the linear predictors chosen at equilibrium under competition between three model-providers (solid lines), along with two approximately Bayes-optimal predictors (dashed lines). The equilibrium social loss is lower in the left plot than the right plot, even though the Bayes risk is much higher. The intuition is that approximate Bayes optima disagree on more data points in the left plot than in the right plot; thus, users have a greater likelihood of at least one predictor offering them a correct prediction, which increases the overall predictive accuracy for users (i.e., the social welfare).

representation quality [BCV13]. We exhibit several binary classification tasks where better data representations (as measured by Bayes risk) *decrease* the overall predictive accuracy (social welfare) for users, when varying data representations along several different axes.

The basic intuition for this non-monotonicity is illustrated in Figure 1. When data representations are low quality, any predictor will be incorrect on a large fraction of users, and near-optimal predictors may disagree on large subpopulations of users. Model providers are thus incentivized to choose complementary predictors that cater to different subpopulations (market segments), thus improving the overall predictive accuracy for users. In contrast, when representations are high quality, each optimal predictor is incorrect on only a small fraction of users, and near-optimal predictors likely agree with each other on most data points. As a result, model-providers are incentivized to select similar predictors, which decreases the overall predictive accuracy for users.

To study when non-monotonicity can occur, we first focus on a stylized setup that permits closed-form calculations of the social welfare at equilibrium (Section 3). Using this characterization, in three concrete binary classification setups, we show that the equilibrium social welfare can be non-monotonic in Bayes risk. In particular, we vary representations along three axes—the per-representation Bayes risks, the noise level of representations, and the dimension of the data representations—and exhibit non-monotonicity in each case (Figure 2).

Going beyond the stylized setup of Section 3, in Section 4 we consider linear function classes and demonstrate empirically that the social welfare can be non-monotonic in the data representation quality. We consider a binary image classification task on CIFAR-10 where data representations are obtained from the last-layer representations of AlexNet, VGG16, ResNet18, ResNet34, and

ResNet50, pretrained on ImageNet. Better representations (as measured by Bayes risk) can again perform worse under competition (Figure 4). We also consider synthetic data where we can vary representation quality more systematically, again finding ubiquitous non-monotonicities.

Altogether, our results demonstrate that the classical setting of a single model-provider can be a poor proxy for understanding multiple competing model-providers. This suggests that caution is needed when inferring that increased social welfare necessarily follows from the continuing trend towards improvements in predictive accuracy in machine learning models. Machine learning researchers and regulators should evaluate methods in environments with competing model-providers in order to reasonably assess the implications of raw performance improvements for social welfare.

## 1.1 Related work

Our work connects to research threads on the *welfare implications of algorithmic decisions* and *competition between data-driven platforms*.

**Welfare implications of algorithmic decisions.** Recent work investigates *algorithmic monoculture* [KR21; BCK<sup>+</sup>22], a setting in which multiple model-providers use the same predictor. In these works, monoculture is intrinsic to the decision-making pipeline: model-providers are given access to a shared algorithmic ranking [KR21] or shared components in the training pipeline [BCK<sup>+</sup>22]. In contrast, in our work, monoculture may arise endogenously from competition, as a result of scaling trends. Model-providers are always given access to the same function classes and data, but whether or not monoculture arises depends on the quality of data representations and its impact on the incentives of model-providers. Our work thus offers a new perspective on algorithmic monoculture, suggesting that it may arise naturally in competitive settings as a side effect of improvements in data representation quality.

More broadly, researchers have identified several sources of mismatch between predictive accuracy and downstream welfare metrics. This includes *narrowing* of a classifier under repeated interactions with users [HSN<sup>+</sup>18], *preference shaping* of users induced by a recommendation algorithm [CDR<sup>+</sup>22; DM22; CHR<sup>+</sup>22], *strategic adaptation* by users under a classifier [BKS12; HMP<sup>+</sup>16], and the *long-term impact of algorithmic decisions* [LDR<sup>+</sup>18; LWH<sup>+</sup>20].

**Competition between data-driven platforms.** Our work is also related to the literature on competing predictors. The model in our paper shares similarities with the work of Ben-Porat and Tennenholtz [BT17; BT19], who studied equilibria between competing predictors. Ben-Porat and Tennenholtz [BT17; BT19] show that empirical risk minimization is not an optimal strategy for a model-provider under competition and design algorithms that compute the best-responses; in contrast, our focus is on the equilibrium social welfare and how it changes with data representation quality. The specifics of our model also slightly differ from the specifics of Ben-Porat and Tennenholtz [BT17; BT19]. In their model, each user has an accuracy target that they wish to achieve and randomly chooses between model-providers that meet that accuracy target; in contrast, in our model, each user noisily chooses the model-provider that minimizes their loss and model-providers can have asymmetric market reputations.

Our work also relates to *bias-variance games* [FGH<sup>+</sup>19] between competing model-providers. However, Feng et al. [FGH<sup>+</sup>19] focus on the equilibrium strategies for the model-provider, but do not consider equilibrium social welfare for users; in contrast, our work focuses on the equilibrium social welfare. The model of Feng et al. [FGH<sup>+</sup>19] also differs from the model in our work. In Feng et al. [FGH<sup>+</sup>19], a model-provider action is modeled as choosing an error *distribution* for each user, where the randomness in the error is intended to capture randomness in the training data

samples and in the predictor; moreover, the action set includes error distributions with a range of different variances. In contrast, in our population-level setup with deterministic predictors, the error distribution for every user is always a point mass (variance 0). Thus, the equilibrium characterization of Feng et al. [FGH<sup>+</sup>19] does not translate to our model. The specifics of the model-provider utility in the work of Feng et al. [FGH<sup>+</sup>19] differs slightly from our model as well.

Other aspects studied in this research thread include competition between model-providers using *out-of-box* learning algorithms that do not directly optimize for market share [GZK<sup>+</sup>21; KGZ22; DCR<sup>+</sup>22], competition between model-providers selecting *regularization parameters* that tune model complexity [IK22], competition between *bandit algorithms* where data directly comes from users [AMS<sup>+</sup>20; JJH22], and competition between *algorithms dueling* for a user [IKL<sup>+</sup>11]. Our work also relates to *classical economic models of product differentiation* such as Hotelling’s model [Hot81; dGT79] (see Anderson, de Palma, and Thisse [AdPT92] for a textbook treatment), as well as the emerging area of *platform competition* [see, e.g., JS21; CP21].

## 2 Model

We focus on a binary classification setup with input space  $X \subseteq \mathbb{R}^d$  and output space  $Y = \{0, 1\}$ . Each user has an input  $x$  and a corresponding true output  $y$ , drawn from a distribution  $\mathcal{D}$  over  $X \times Y$ . Model providers choose predictors  $f$  from some model family  $\mathcal{F} \subseteq [0, 1]^X$ . A user’s loss given predictor  $f$  is  $\ell(f(x), y) = |y - f(x)|$ . In Section 3, we take  $\mathcal{F} = \{0, 1\}^X$  to be all binary functions, while in Section 4 we consider linear predictors of the form  $f(x) = \text{sigmoid}(\langle w, x \rangle + b)$ .

We study competition between  $m \geq 2$  model-providers for users, building on the model of Ben-Porat and Tennenholtz [BT17; BT19]. We index the model-providers by  $[m] := \{1, 2, \dots, m\}$ , and let  $f_j$  denote the predictor chosen by model provider  $j$ . After the model-providers choose predictors  $f_1, \dots, f_m$ , each user then chooses one of the  $m$  model-providers to link to, based on prediction accuracy. Model-providers aim to optimize the number of users that they win.

**User decisions.** Users noisily pick the model-provider offering the best predictions for them. That is, a user with representation  $x$  and true label  $y$  chooses a model-provider  $j^*(x, y)$  such that the loss  $|y - f_{j^*(x, y)}(x)|$  is the smallest across all model-providers  $j \in [m]$ , subject to noise in user decisions. More formally, we model user noise with the logit model [Tra02], also known as the Boltzmann rationality model:

$$\mathbb{P}[j^*(x, y) = j] = \frac{e^{-\ell(f_j(x), y)/c}}{\sum_{j'=1}^m e^{-\ell(f_{j'}(x), y)/c}}, \quad (1)$$

where  $c > 0$  denotes a noise parameter. We extend this model to account for uneven market reputations across decisions in Section 3.3, and defer further discussion of modeling choices for user decisions to Appendix A.

**Model provider incentives.** A model-provider’s utility is captured by the market share that they win. That is, model-provider  $j$ ’s utility is

$$u(f_j; \mathbf{f}_{-j}) := \mathbb{E}_{(x, y) \sim \mathcal{D}} [\mathbb{P}[j^*(x, y) = j]],$$

where  $\mathbf{f}_{-j}$  denotes the predictors chosen by the other model-providers and where the expectation is over  $(x, y)$  drawn from  $\mathcal{D}$ . Since the market shares always sum to one, this is a constant-sum game.

Each model-provider chooses a *best response* to the predictors of other model-providers. That is, model-provider  $j$  chooses a predictor  $f_j$  such that

$$f_j \in \arg \max_{f \in \mathcal{F}} u(f_j; \mathbf{f}_{-j}).$$

The best-response captures that model-providers optimize for market share. In practice, model-providers may do so via A/B testing to steer towards predictors that maximize profit, or by actively collecting data on market segments where competitors are performing poorly.

We study market outcomes  $\mathbf{f} = (f_1, f_2, \dots, f_m)$  that form a Nash equilibrium. Recall that  $(f_1, f_2, \dots, f_m)$  is a *pure strategy Nash equilibrium* if for every  $j \in [m]$ , model-provider  $j$ 's predictor is a best-response to  $\mathbf{f}_{-j}$ : that is,  $f_j \in \arg \max_{f \in \mathcal{F}} u(f_j; \mathbf{f}_{-j})$ . In well-behaved instances, pure-strategy equilibria exist (see theoretical results in Section 3.1 and simulation results in Section 4). However, for our results in Section 3.3, we must turn to mixed strategy equilibria where model-providers instead choose distributions  $\mu$  over  $\mathcal{F}$ .

**Quality of market outcome for users.** We are interested in studying the quality of a market outcome  $\mathbf{f} = (f_1, f_2, \dots, f_m)$  in terms of user utility. The quality of  $\mathbf{f}$  is determined by the overall *social loss* that it induces on the user population, after users choose between model-providers:

$$\text{SL}(f_1, \dots, f_m) := \mathbb{E}[\ell(f_{j^*(x,y)}(x), y)]. \quad (2)$$

When  $f_1, \dots, f_m$  is a Nash equilibrium, we refer to  $\text{SL}(f_1, \dots, f_m)$  as the *equilibrium social loss*.

Our goal is to study how the equilibrium social loss changes when the representation quality (i.e., the quality of the input representations  $X$ ) improves. We formalize representation quality as the minimum risk  $\text{OPT}_{\text{single}}$  that a single model-provider could have achieved on the distribution  $\mathcal{D}$  with the model family  $\mathcal{F}$ . This means that  $\text{OPT}_{\text{single}}$  is equal to the Bayes risk:

$$\text{OPT}_{\text{single}} := \min_{f \in \mathcal{F}} \mathbb{E}[\ell(f(x), y)].$$

In the following sections, we show that the equilibrium social loss  $\text{SL}(f_1^*, \dots, f_m^*)$  can be non-monotonic in the representation quality (as measured by  $\text{OPT}_{\text{single}}$ ), when representations are varied along a variety of axes.

### 3 Non-monotonicity of Equilibrium Social Loss in a Stylized Setup

To understand when non-monotonicity can occur, we first consider a stylized setup that permits closed-form calculations of the social loss. We characterize the equilibrium social loss in this setup (Proposition 1), and apply this characterization to three concrete binary classification setups that vary representation quality along different axes (Section 3.2). We show that the equilibrium social loss can be non-monotonic in Bayes risk in all of these setups (Figures 2b-2c). Finally, we extend our model and results to model-providers with unequal market reputations (Section 3.3).

#### 3.1 Characterization of the equilibrium social loss

We assume the input space  $X$  is finite and let  $\mathcal{F} = \mathcal{F}_{\text{all}}$  contain all possible functions from  $X$  to  $\{0, 1\}$ . For simplicity, we also assume that users make noiseless decisions (i.e., we take  $c \rightarrow 0$ ), so a user's choice of model-provider  $j^*(x, y)$  is specified as follows:

$$\mathbb{P}[j^*(x, y) = j] = \begin{cases} 0 & \text{if } j \notin \arg \min_{j' \in [m]} |y - f_{j'}(x)| \\ \frac{1}{|\arg \min_{j' \in [m]} |y - f_{j'}(x)||} & \text{if } j \in \arg \min_{j' \in [m]} |y - f_{j'}(x)|. \end{cases} \quad (3)$$

In other words, users pick the model-provider with minimum loss, choosing randomly in case of ties.

Since  $\mathcal{F}_{\text{all}}$  lets model-providers make independent predictions about each representation  $x$ , the only source of error is noise in individual data points. To capture this, we define the *per-representation Bayes risk*  $\alpha(x)$  to be:

$$\alpha(x) := \min(\mathbb{P}(y = 1 \mid x), \mathbb{P}(y = 0 \mid x)). \quad (4)$$

The value  $\alpha(x)$  measures how random the label  $y$  is for a given representation  $x$ . As a result,  $\alpha(x)$  is the minimum error that a model-provider can hope to achieve on the given representation  $x$ . Increasing  $\alpha(x)$  increases the Bayes risk  $\text{OPT}_{\text{single}}$ : in particular,  $\text{OPT}_{\text{single}}$  is equal to the average value  $\mathbb{E}[\alpha(x)]$  across the population. The equilibrium social loss, however, depends on other aspects of  $\alpha(x)$ .

We characterize the equilibrium social loss in terms of the per-representation Bayes risks in the following proposition. Our characterization focuses on pure-strategy equilibria, which are guaranteed to exist in this setup (see Lemma 2 in Appendix C).

**Proposition 1.** *Let  $X$  be a finite set and let  $\mathcal{F} = \mathcal{F}_{\text{all}}$ . Suppose that user decisions are noiseless (i.e., user decisions are given by (3)). Suppose also that  $\alpha(x) \neq 1/m$  for all  $x \in X$ .<sup>1</sup> At any pure strategy Nash equilibrium  $f_1^*, \dots, f_m^*$ , the social loss  $SL(f_1^*, \dots, f_m^*)$  is equal to:*

$$SL(f_1^*, \dots, f_m^*) = \mathbb{E}_{(x,y) \sim \mathcal{D}} [\alpha(x) \cdot \mathbb{1}[\alpha(x) < 1/m]]. \quad (5)$$

The primary driver of Proposition 1 is that as the per-representation Bayes risk  $\alpha(x)$  decreases, the equilibrium predictions for  $x$  go from *heterogeneous* (different model-providers offer different predictions for  $x$ ) to *homogenous* (all model-providers offer the same prediction for  $x$ ). In particular, if  $\alpha(x)$  is below  $1/m$ , then all model-providers choose the Bayes optimal label  $y^* = \arg \max_{y'} \mathbb{P}[y' \mid x]$ , so predictions are homogeneous; on the other hand, if  $\alpha(x)$  is above  $1/m$ , then at least one model-provider will choose  $1 - y^*$ , so predictions are heterogeneous. When predictions are heterogeneous, each user is offered perfect predictive accuracy by some model-provider, which results in zero social loss. On the other hand, if predictions are homogeneous and all model-providers choose the Bayes optimal label, the social loss on  $x$  is the per-representation Bayes risk  $\alpha(x)$ . Putting this all together, the equilibrium social loss takes the value in (5). We defer a proof of Proposition 1 to Appendix C.

### 3.2 Non-monotonicity along several axes of varying representations

Using Proposition 1, we next vary representations along several axes and compute the equilibrium social loss, observing non-monotonicity in each case.

**Setting 1: Varying the per-representation Bayes risks.** Consider a population with a single value of  $x$  that has Bayes risk  $\alpha(x) = \alpha$ . We vary representation quality by varying  $\alpha$  from 0 to 0.5. Figure 2a depicts the result: by Proposition 1, the equilibrium social loss is zero if  $\alpha > 1/m$  and is  $\alpha$  if  $\alpha < 1/m$ , leading to non-monotonicity at  $\alpha = 1/m$ . When there are  $m \geq 3$  model-providers, the equilibrium social loss is thus non-monotonic in  $\alpha$ . (For  $m = 2$ , where  $\alpha = 1/2$  is the maximum possible per-representation Bayes risk, the equilibrium social loss is monotone in  $\alpha$ .) As the number of model-providers increases, the non-monotonicity occurs at a higher data representation quality (a lower Bayes risk).

<sup>1</sup>When  $\alpha(x) = 1/m$ , there turn out to be multiple pure-strategy equilibria with different social losses.



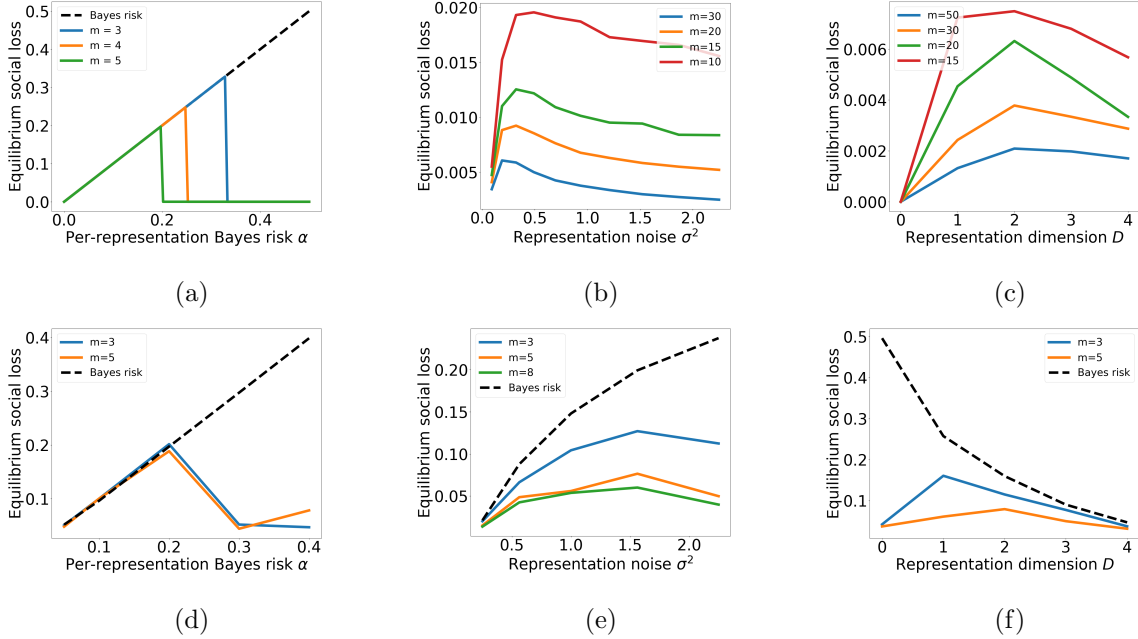


Figure 2: Equilibrium social loss (y-axis) versus data representation quality (x-axis) given  $m$  model providers, for different function classes  $\mathcal{F}$  (rows) and when representations are varied along different aspects (columns). Top row:  $\mathcal{F} = \mathcal{F}_{\text{all}}$ , with closed-form formula from Proposition 1. Bottom row: linear functions, computed via simulation (Section 4). We vary representations with respect to per-representation Bayes risk (a,d), noise level (b,e), and dimension (c,f). The dashed line indicates the Bayes risk (omitted if it is too high to fit on the axis). The Bayes risk is monotone, but the equilibrium social loss is non-monotone.

**Setting 2: Varying the representation noise.** Consider a one-dimensional population given by a mixture of two Gaussians (one for each class), where each Gaussian has variance  $\sigma^2$  (see Appendix B for the details of the setup). We vary the parameter  $\sigma$  to change the representation quality. Intuitively, a lower value of  $\sigma$  makes the Gaussians more well-separated, which improves representation quality (Bayes risk). By Proposition 1, the equilibrium social loss is  $\mathbb{E}[\alpha(x) \cdot \mathbb{1}[\alpha(x) < 1/m]]$ . For each value of  $\sigma$ , we estimate the equilibrium social loss by sampling representations  $x$  from the population and taking an average.<sup>2</sup> Figure 2b depicts the result: the equilibrium social loss is non-monotonic in  $\sigma$  (and thus the Bayes risk). Again, as the number of model-providers increases, the non-monotonicity occurs at a higher representation quality (a lower Bayes risk).

**Setting 3: Varying the representation dimension.** We consider a four-dimensional population  $(X^{\text{all}}, Y)$ , and let the representation  $X$  consist of the first  $D$  coordinates of  $X^{\text{all}}$ , for  $D$  varying from 0 to 4 (see Appendix B for full details). Intuitively, a higher dimension  $D$  makes the representations more informative, thus improving representation quality (Bayes risk). As before, for each value of  $D$ , we estimate the equilibrium social loss by sampling representations  $x$  from the population and taking an average. Figure 2c depicts the result: the equilibrium social loss is once again non-monotonic in the representation dimension  $D$  (and thus the Bayes risk).

<sup>2</sup>Strictly speaking, we can't directly apply Proposition 1 to this setup since  $X$  is infinite. Informally speaking, this setting can be interpreted as taking a finite discretization of the data representation space.

**Discussion.** Settings 1-3 illustrate that equilibrium social loss can be non-monotonic in Bayes risk when representations are improved along many qualitatively different axes. The intuition is that varying representations along these axes can increase the values of  $\alpha(x)$  for inputs  $x$ ; by Proposition 1, these changes to  $\alpha(x)$  can lead to non-monotonicity in the equilibrium social loss. We will revisit Settings 1-3 for richer market structures (Section 3.3) and for linear predictors and noisy user decisions (Section 4.2).

### 3.3 Generalization to unequal market reputations

While we assumed above that users evenly break ties between model-providers, in reality, users might be more likely to choose model-providers with a higher market reputation (e.g., established, popular model-providers). This motivates us to incorporate market reputations into user decisions.

Formally, we assign to each model-provider  $j$  a *market reputation*  $w_j$ , and we replace the logit model in (1) with a weighted logit variant. When  $c \rightarrow 0$ , rather than breaking ties uniformly, they are instead broken proportionally to  $w_j$ :

$$\mathbb{P}[j^*(x, y) = j] = \begin{cases} 0 & \text{if } j \notin \arg \min_{j' \in [m]} |y - f_{j'}(x)| \\ \frac{w_j}{\sum_{j'' \in [m]} w_{j''} \cdot \mathbb{1}[j'' \in \arg \min_{j' \in [m]} |y - f_{j'}(x)|]} & \text{if } j \in \arg \min_{j' \in [m]} |y - f_{j'}(x)|. \end{cases} \quad (6)$$

See Appendix A for further discussion of this model. For simplicity, we assume that market reputations are normalized to sum to one.

Similarly to Proposition 1, we derive a closed-form formula for the equilibrium social loss, focusing on the case of  $m = 2$  model-providers for analytic tractability. We observe non-monotonicity as before, but with a more complex functional form.

**Proposition 2.** *Let  $X$  be a finite set and let  $\mathcal{F} = \mathcal{F}_{all}$ . Suppose there are  $m = 2$  model-providers with market reputations  $w_{min}$  and  $w_{max}$ , where  $w_{max} \geq w_{min}$  and  $w_{max} + w_{min} = 1$ . Suppose that user decisions are given by (6), and that  $\alpha(x) \neq w_{min}$  for all  $x \in X$ .<sup>3</sup> At any (mixed) Nash equilibrium  $(\mu_1, \mu_2)$ , the expected social loss is equal to:*

$$\mathbb{E}_{\substack{f_1 \sim \mu_1 \\ f_2 \sim \mu_2}} [SL(f_1, f_2)] = \mathbb{E}_{(x, y) \sim \mathcal{D}} \left[ \underbrace{\frac{(\alpha(x) - w_{min}) \cdot (w_{max} - \alpha(x))}{(1 - 2 \cdot w_{min})^2}}_{(A)} \cdot \mathbb{1}[\alpha(x) > w_{min}] + \underbrace{\alpha(x) \cdot \mathbb{1}[\alpha(x) < w_{min}]}_{(B)} \right]. \quad (7)$$

The high-level intuition for Proposition 2 is again that the equilibrium predictions go from heterogeneous to homogenous as  $\alpha(x)$  decreases. Term (A), which is realized for large  $\alpha(x)$ , captures the equilibrium social loss for heterogeneous predictions. Term (B), which is realized for small  $\alpha(x)$ , captures the equilibrium social loss for homogeneous predictions. We defer the proof of Proposition 2 to Appendix C.

The details of Proposition 2 differ from Proposition 1 in several ways. First, the transition point from heterogeneous to homogeneous predictions occurs at  $\alpha(x) = w_{min}$  as opposed to  $\alpha(x) = 1/2$ . In particular, the transition point depends on the market reputations rather than only the number of model-providers. Second, the equilibria have *mixed strategies* rather than pure strategies, because pure-strategy equilibria do not necessarily exist when market reputations are unequal (see

<sup>3</sup>As with Proposition 1, when  $\alpha(x)$  is equal to  $w_{min}$  for some value of  $x$ , there are multiple equilibria.



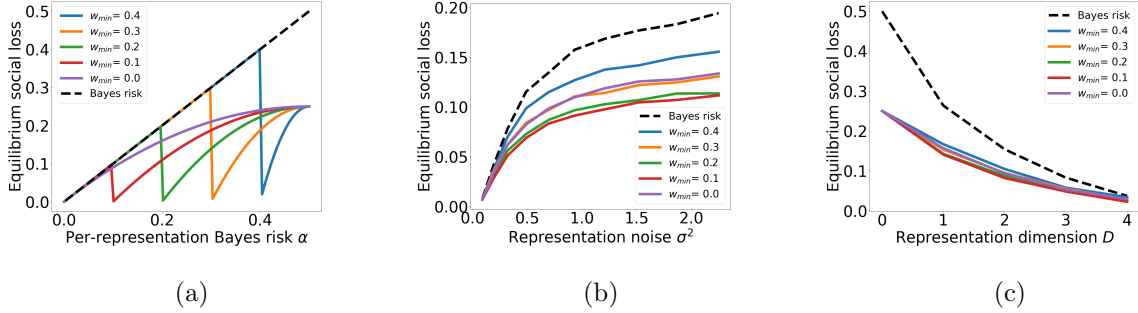


Figure 3: Equilibrium social loss (y-axis) versus data representation quality (x-axis) given two model-providers with market reputations  $[1 - w_{\min}, w_{\min}]$  when representations are varied along different aspects (columns). The equilibrium social loss is computed via the closed-form formula from Proposition 2. We vary representations with respect to per-representation Bayes risk (a), noise level (b), and dimension (c). The dashed line indicates the Bayes risk. The Bayes risk is monotone for all 3 axes of varying representations; on the other hand, the equilibrium social loss is non-monotone in the per-representation Bayes risk and monotone in noise level and dimension.

Lemma 4 in Appendix C). Third, the social loss at a representation  $x$  is no longer equal to zero for heterogeneous predictions—in particular, term (A) is now positive for all  $\alpha(x) > w_{\min}$  and increasing in  $\alpha(x)$ .

To better understand the implications of Proposition 2, we revisit Settings 1-3 from Section 3.2, considering the same three axes of varying representations with the same distributions over  $(x, y)$ . In contrast to Section 3.2, we consider 2 competing model-providers with unequal market positions rather than  $m$  competing model providers with equal market positions. Our results, described below, are depicted in Figure 3.

**Setting 1: Varying the per-representation Bayes risks.** Consider the same setup as Setting 1 in Section 3.2. Figure 3a depicts the non-monotonicity of the equilibrium social loss in the per-representation Bayes risk  $\alpha$  across different settings of market reputations for 2 competing model-providers. The discontinuity occurs at the smaller market reputation  $w_{\min}$ . Thus, as the market reputations of the 2 model-providers become closer together, the non-monotonicity occurs at a lower data representation quality (higher Bayes risk).

**Settings 2-3: Varying the representation noise or representation dimension.** Consider the setups from Settings 2-3 in Section 3.2. Figures 3b-3c depicts that the equilibrium social loss is *monotone* in data representation quality (Bayes risk) across different settings of market reputations for 2 competing model-providers.

**Discussion.** To interpret these results, observe that for 2 model-providers with equal market reputations ( $w_{\min} = 0.5$ ), the equilibrium social loss is always equal to the Bayes risk by Propositions 1-2, which trivially implies monotonicity. In contrast, Figure 3 shows that for unequal market positions ( $w_{\min} < 0.5$ ), the equilibrium social loss is non-monotonic in Bayes risk for Setting 1, though it is still monotonic in Bayes risk for Settings 2 and 3. (For comparison, recall from Figures 2a-2c that for  $m \gg 2$  model-providers with equal market reputations, non-monotonicity was exhibited for all three settings.) An interesting open question is identify other axes of varying

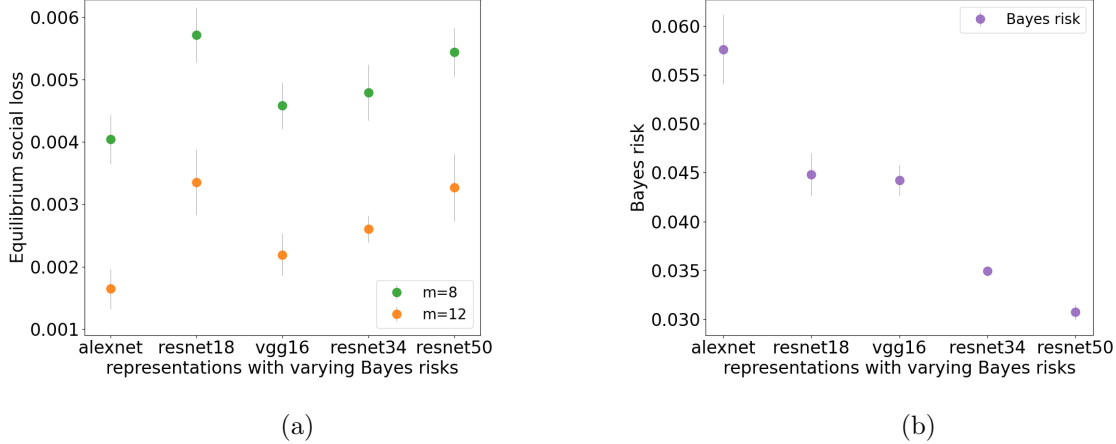


Figure 4: Equilibrium social loss (left) and Bayes risk (right) on a binary classification task on CIFAR-10 (Section 4.3). Representations are generated from different networks pre-trained on ImageNet. The points show the equilibrium social loss when  $m$  model-providers compete with each other (left) and the Bayes risk of a single model-provider in isolation (right). While Bayes risk is decreasing in this representation ordering, the equilibrium social loss is non-decreasing in this ordering. The equilibrium social loss is thus non-monotonic in representation quality as measured by Bayes risk.

representations, beyond Setting 1, which lead to non-monotonicity for 2 model-providers with unequal market reputations.

## 4 Empirical Analysis of Non-monotonicity for Linear Predictors

We next turn to linear predictors and demonstrate empirically that the social welfare can be non-monotonic in data representation quality in this setup as well.<sup>4</sup> We take  $X = \mathbb{R}^D$  and let  $\mathcal{F}_{\text{linear}}$  be the family of linear predictors  $f_{w,b} = \text{sigmoid}(\langle w, x \rangle + b)$  where  $w \in \mathbb{R}^D$  and  $b \in \mathbb{R}$ . Since this setting no longer admits closed-form formulae, we numerically estimate the equilibria using a variant of *best-response dynamics*, where model-providers repeatedly best-respond to the other predictors.

We first show on low-dimensional synthetic data that the insights from Section 3.2 readily generalize to linear predictors (see Figures 2d-2f). We then turn to natural data, considering a binary image classification task for CIFAR-10 and using pretrained networks—AlexNet, VGG16, and various ResNets—to generate high-dimensional representations (ranging from 2048 to 4096). In this setting we again find that the equilibrium social loss can be non-monotonic in the Bayes risk (see Figure 4).

### 4.1 Best-response dynamics implementation

To enable efficient computation, we assume the distribution  $\mathcal{D}$  corresponds to a finite dataset with  $N$  data points. We calculate equilibria using an approximation of best-response dynamics. Model-providers (players) iteratively (and approximately) best-respond to the other players' actions. We implement the approximate best-response as running several steps of gradient descent.

<sup>4</sup>The code can be found at <https://github.com/mjagadeesan/competition-nonmonotonicity>.

In more detail, for each  $j \in [m]$ , we initialize the model parameters  $w_j \in \mathbb{R}^d, b_j \in \mathbb{R}$  as mean zero Gaussians with standard deviation  $\sigma$ . Our algorithm then proceeds in stages. At a given stage, we iterate through the model-providers in the order  $1, \dots, m$ . When  $j$  is chosen, first we decide whether to reinitialize: if the risk  $\mathbb{E}_{(x,y) \sim \mathcal{D}}[\ell(f_{w_j, b_j}(x), y)]$  exceeds a threshold  $\rho$ , we re-initialize  $w_j$  and  $b_j$  (sampling from mean zero Gaussians as before); otherwise, we do not reinitialize. Then we run gradient descent on  $u(\cdot; \mathbf{f}_{-j})$  (computing the gradient on the full dataset of  $N$  points) with learning rate  $\eta$  for  $I$  iterations, updating the parameters  $w_j$  and  $b_j$ . At the end of a stage, the stopping condition is that for every  $j \in [m]$ , model-provider  $j$ 's utility  $u(f_j, \mathbf{f}_{-j})$  has changed by at most  $\epsilon$  relative to the previous stage. If the stopping condition is not met, we proceed to the next stage.

## 4.2 Simulations on synthetic data

We first revisit Settings 1-3 from Section 3.2, considering the same three axes of varying representations with the same distributions over  $(x, y)$ . In contrast to Section 3.2, we restrict the model family to linear predictors  $\mathcal{F}_{\text{linear}}$  instead of allowing all predictors  $\mathcal{F}_{\text{all}}$ . We also set the noise parameter  $c$  in user decisions (1) to 0.3. Our goal is to examine if the findings from Section 3 generalize to this new setting.

We compute the equilibria for each of the following (continuous) distributions as follows. First, we let  $\mathcal{D}$  be the empirical distribution over  $N = 10,000$  samples from the continuous distribution. Then we run the best-response dynamics described in Section 4.1 with  $\rho = 0.3$ ,  $I = 5000$ ,  $\eta = 0.1$ ,  $\epsilon = 0.01$ , and  $\sigma = 0.1$ . We then compute the equilibrium social loss according to (2). We also compute the Bayes optimal predictor with gradient descent. See Appendix B for full details.

Our results, described below, are depicted in Figures 2d-2f (row 2). We compare these results with Figures 2a-2c (row 1), which shows the analogous results for  $\mathcal{F}_{\text{all}}$  from Section 3.2.

**Setting 1: Varying the per-representation Bayes risks.** Consider the same single  $x$  setup as in Setting 1 in Section 3.2. The only parameter of the predictor is the bias  $b \in \mathbb{R}$  (i.e., we treat  $x$  as zero-dimensional). Figure 2d shows that the equilibrium social loss is non-monotonic in  $\alpha$ , which mirrors the non-monotonicity in Figure 2a.

**Setting 2: Varying the representation noise.** Consider the same one-dimensional mixture-of-Gaussians distribution as in Setting 2 in Section 3.2. (The weight  $w$  is one-dimensional.) We again vary the noise  $\sigma$  to change the representation quality. Figure 2e shows that the equilibrium social loss is non-monotonic in the noise  $\sigma$ , which again mirrors the non-monotonicity in Figure 2b.

**Setting 3: Varying the representation dimension.** Consider the same four-dimensional population as in Setting 3 in Section 3.2. We vary the representation dimension  $D$  from 0 to 4 to change the representation quality. Figure 2f shows that the equilibrium social loss is non-monotonic in the dimension  $D$ , which once again mirrors the non-monotonicity in Figure 2c.

**Discussion.** In summary, in Figure 2, rows 1 and 2 exhibit similar non-monotonicities. This illustrates that the insights from Section 3.1 translate to linear predictors and noisy user decisions.

## 4.3 Simulations on CIFAR-10

Finally, we turn to experiments with natural data. While we have directly varied the informativeness of data representations thus far, representations in practice are frequently generated by pretrained

models. The choice of the pretrained model implicitly influences representation quality, as measured by Bayes risk on the downstream task. In this section, we consider how the equilibrium social loss changes with representations generated from pretrained models of varying quality. We restrict the model family to linear predictors  $\mathcal{F}_{\text{linear}}$  and set the noise parameter  $c$  in user decisions (1) to 0.1.

We consider a binary image classification task on CIFAR-10 [Kri09] with 10,000 images. Class 0 is defined to be {airplane, bird, automobile, ship, horse, truck} and the class 1 is defined to be {cat, deer, dog, frog}. We treat the set of 10,000 images and labels as the population of users, meaning that it is both the training set and the validation set.<sup>5</sup> Representations are generated from five models—AlexNet [KSH12], VGG16 [SZ15], ResNet18, ResNet34, and ResNet50 [HZR<sup>+</sup>16]—pretrained on ImageNet [DDS<sup>+</sup>09]. The representation dimension is 4096 for AlexNet and VGG16, 512 for ResNet18 and ResNet34, and 2048 for ResNet50.

We compute the equilibria as follows. First, we let  $\mathcal{D}$  be the distribution described above with  $N = 10,000$  data points. Then we run the best-response dynamics described in Section 4.1 for  $m = 8$  and  $m = 12$  model-providers with  $\rho = 0.3$ ,  $I = 5000$ ,  $\epsilon = 0.001$ ,  $\sigma = 0.1$ , and a learning rate schedule that starts at  $\eta = 0.1$ . We then compute the equilibrium social loss according to (2). We also compute the Bayes risk using gradient descent. For full experimental details, see Appendix B.

Figure 4 shows that the equilibrium social loss can be non-monotone in the Bayes risk. For example, for  $m = 12$ , AlexNet outperforms ResNet18, ResNet34, and ResNet50 with respect to equilibrium social loss, even though the Bayes risk of AlexNet is substantially higher than the Bayes risk ResNet18. Similarly, VGG16 also outperforms ResNet50 despite having a higher Bayes risk. Even within the family of ResNets, the equilibrium social loss is non-monotonic the Bayes risk: e.g., ResNet34 outperforms ResNet50, even though the Bayes risk of ResNet34 is higher than that of ResNet50. Similar non-monotonicities are also exhibited for  $m = 8$ .

## 5 Discussion

We showed that the monotonicity of scaling trends can be violated under competition. In particular, we demonstrated that when multiple model-providers compete with each for users, improving data representation quality (as measured by Bayes risk) can *increase* the overall loss at equilibrium. We exhibited the non-monotonicity of the equilibrium social loss in the Bayes risk when representations are varied along several axes (per-representation Bayes risk, noise, dimension, and pre-trained model used to generate the representations).

An interesting direction for future work is to further characterize the regimes when the equilibrium social loss is monotonic versus non-monotonic in data representation quality as measured by Bayes risk. For example, an interesting open question is to generalize our theoretical results from Section 3 to more general function classes and distributions of market reputations. Moreover, another interesting direction would be generalize our empirical findings from Section 4 to other axes of varying data representations and to non-linear classes of predictors. Finally, while we have focused on binary classification tasks, it would be interesting to generalize our findings to regression tasks with continuous outputs or to generative AI tasks with text-based or image-based outputs.

More broadly, the non-monotonicity of equilibrium social welfare in scale under competition establishes a disconnect between scaling trends in the single model-provider setting and in the competitive setting. In particular, typical scaling trends (e.g. [KMH<sup>+</sup>20; SK20; BDK<sup>+</sup>21; HBM<sup>+</sup>22; HKH<sup>+</sup>21])—which show increasing scale reliably increases predictive accuracy for a single model-provider in isolation—may not translate to competitive settings such as digital marketplaces. Thus,

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<sup>5</sup>We make this choice to be consistent with the rest of the paper, where we focus on population-level behavior and thus do not consider generalization error.

understanding the downstream impact of scale on user welfare in digital marketplaces will likely require understanding how scaling trends behave under competition. We hope that our work serves as a starting point for analyzing and eventually characterizing the scaling trends of learning systems in competitive settings.

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## A Discussion of Model of User Decisions

The user decision given by (1) is the standard logit model for discrete choice decisions [Tra02] which is also known as the Boltzmann rationality model. In the limit as  $c \rightarrow 0$ , a user with representation  $x$  and label  $y$  select from the set of model-providers  $\arg \min_{j \in [m]} \ell(f_j(x), y)$  that achieve the minimum loss; in particular, the user chooses a model-provider from this set with probability proportional to the model-provider’s market reputation. For  $c > 0$ , the specification in equation (1) captures that users evaluate a model-provider based on a noisy perception of the loss.

Recall that in Section 3.3, we extend this model to account for uneven market reputations across decisions. Given market reputations  $w_1, \dots, w_m$ , users choose a predictor according to:

$$\mathbb{P}[j^*(x, y) = j] = \frac{w_j \cdot e^{-\ell(f_j(x), y)/c}}{\sum_{j'=1}^m w_{j'} \cdot e^{-\ell(f_{j'}(x), y)/c}}. \quad (8)$$

When the market reputations are all equal ( $w_1 = \dots = w_m$ ), equation (8) exactly corresponds to (1). When the market reputations  $w_j$  are not equal, equation (8) captures that users place a higher weight on model-providers with a higher market reputation. This captures that users are more likely to choose a popular model-provider than a very small model-provider without much reputation.

Implicit in this model is asymmetric information between the model-providers and users. While the only information that a model-provider has about users is their representations, a user can make decisions based on noisy perceptions of their own loss (which can depend on their label). This captures that, even if users are unlikely to know their own labels, users can experiment with multiple model-providers to (noisily) determine which one maximizes their utility. The inclusion of market reputations reflects that users are more likely to experiment with and ultimately choose popular model-providers than less popular model-providers.

## B Additional Details of Simulations

**Hyperparameters.** We introduce a temperature parameter  $\tau$  within our loss function, defining the loss  $\ell(f_{w,b}(x), y)$  to be  $|\text{sigmoid}((\langle w, x \rangle + b)/\tau) - 1|$ . This reparameterizes, but does not change, the model family.

When we run the best-response dynamics, we always initialize the model parameters as mean-zero Gaussians with standard deviation  $\sigma = 0.1$ . The threshold risk  $\rho$  is set to be 0.3, and when we reinitialize model parameters, we again initialize them as mean-zero Gaussians with standard deviation 0.1. The number of iterations  $I$  of gradient descent is taken to be 5000. For Section 4.2, we set  $\tau = 0.1$ ,  $\epsilon = 0.01$ , and  $\eta = 0.1$ . For Section 4.3, we set  $\tau = 1.0$ ,  $\epsilon = 0.001$ , and  $\eta$  with the following learning rate schedule to expedite convergence:  $\eta = 0.1$  if the risk  $\mathbb{E}_{(x,y) \sim \mathcal{D}}[\ell(f_{w_j, b_j}(x), y)]$  is at least 0.3,  $\eta = 0.5$  if the risk is in  $[0.2, 0.3)$ ,  $\eta = 1$  if the risk is in  $[0.15, 0.2)$ ,  $\eta = 2$  if the risk is in  $[0.1, 0.15)$ , and  $\eta = 5$  if the risk is in  $[0.07, 0.1)$ , and  $\eta = 10$  if the risk is less than 0.07. We ran over ten trials for each data point in Figure 4b and error bars show two standard errors from the mean.

In addition to computing the equilibria, we also approximate the optimal Bayes risk. For Section 4.2, we run gradient descent for 10,000 iterations with learning rate equal to one and parameters initialized to independent Gaussians with zero mean and standard deviation 0.1. For Section 4.3, we run gradient descent for 50,000 iterations with learning rate equal to 10 and parameters initialized to independent Gaussians with zero mean and standard deviation 0.1.

**Generation of the synthetic dataset.** In Setting 1 (Figures 2a, 3a, and 2d), we consider a zero-dimensional population where  $Y | X$  is distributed as a Bernoulli with probability  $\alpha$ . In Figure 2d, the meaning of a zero-dimensional representation is that the only parameter is the bias.

In Setting 2 (Figures 2b, 3b, and 2e), we consider a one-dimensional population given by a mixture of Gaussians. In particular, the Gaussian  $X | Y = 0$  is distributed as  $N(-\mu, \sigma^2)$  and the Gaussian  $X | Y = 1$  is distributed as  $N(\mu, \sigma^2)$ . The distribution of the labels is given by  $\mathbb{P}[Y = 1] = 0.4$  and  $\mathbb{P}[Y = 0] = 0.6$ .

In Setting 3 (Figures 2c, 3c, and 2f), let  $D_{\text{base}} = 4$ . The distribution over  $(X^{\text{all}}, Y)$  consists of  $D_{\text{base}}$  subpopulations. We define the distribution of  $(X^{\text{all}}, Y)$  as follows: each subpopulation  $1 \leq i \leq D_{\text{base}}$  has a different mean vector  $\mu_i \in \mathbb{R}^{D_{\text{base}}}$  and is distributed as  $X^{\text{all}} \sim Y = 0 \sim N(-\mu_i, \sigma^2)$ , let  $X^{\text{all}} \sim Y = 1 \sim N(\mu_i, \sigma^2)$ , and let  $\mathbb{P}[Y = 0] = \mathbb{P}[Y = 1] = 1/2$ . We define  $(\mu_i)_d = 0$  for  $1 \leq d \leq i - 1$  and  $(\mu_i)_d = 1$  for  $i \leq d \leq D_{\text{base}}$ . If the representation dimension is  $D$ , then we define  $X$  to consist of the first  $D$  coordinates of  $X^{\text{all}}$ . When  $D = 0$ , the model-provider is not given representations and thus must assign all users to the same output. (Our setup captures that the dimension  $D$  must be at least  $i$  to see any nontrivial features about subpopulation  $i$ .)

In each case, we draw 10,000 samples and take the resulting empirical distribution to be  $\mathcal{D}$ .

**Generation of the CIFAR-10 task.** We consider a binary classification task consisting of the first 10,000 images in the training set of CIFAR-10. The class 0 is defined to be {airplane, bird, automobile, horse, truck} and class 1 is defined to be {cat, deer, dog, frog}. To generate representations, we use the pretrained models from the Pytorch `torchvision.models` package; these models were pretrained on ImageNet.

**Compute details.** We run our simulations on a single A100 GPU.

## C Additional Results and Proofs for Section 3

### C.1 Proofs for Section 3.1

We first show that we can decompose model-provider actions into independent decisions about each representation  $x$ . To formalize this, let  $\mathcal{D}$  be the data distribution, and let  $\mathcal{D}_x$  be the conditional distribution over  $(X, Y) | X = x$  where  $(X, Y) \sim \mathcal{D}$ . Let  $\mathcal{F}_{\text{all}}^x := \{f^0, f^1\}$  be the class of the (two) functions from a single representation  $x$  to  $\{0, 1\}$ , where  $f^0(x) = 0$  and  $f^1(x) = 1$ .

**Lemma 1.** *Let  $X$  be a finite set of representations, let  $\mathcal{F} = \mathcal{F}_{\text{all}}$ , and let  $\mathcal{D}$  be the distribution over  $(X, Y)$ . For each  $x \in X$ , let  $\mathcal{D}_x$  be the conditional distribution over  $(X, Y) | X = x$  where  $(X, Y) \sim \mathcal{D}$ , and let  $\mathcal{F}_{\text{all}}^x := \{f^0, f^1\}$  be the class of the (two) functions from a single representation  $x$  to  $\{0, 1\}$ , where  $f^0(x) = 0$  and  $f^1(x) = 1$ . Suppose that user decisions are noiseless (i.e.,  $c \rightarrow 0$ , so user decisions are given by (3)). A market outcome  $f_1, \dots, f_m$  is a pure-strategy equilibrium if and only if for every  $x \in X$ , the market outcome  $(f^{f_1(x)}, \dots, f^{f_m(x)})$  is a pure-strategy equilibrium for  $\mathcal{F}_{\text{all}}^x$  with data distribution  $\mathcal{D}_x$ .*

The intuition is that since  $\mathcal{F}_{\text{all}}$  is all possible functions, model-providers make independent decisions for each data representation.

*Proof.* Let  $\mathcal{D}^R$  be the marginal distribution of  $X$  with respect to the distribution  $(X, Y) \sim \mathcal{D}$ . First, we write model-provider  $j$ 's utility as:

$$u(f_j; \mathbf{f}_{-j}) = \mathbb{E}_{(x,y) \sim \mathcal{D}} [\mathbb{P}[j^*(x, y) = j]] = \mathbb{E}_{x' \sim \mathcal{D}^R} \left[ \mathbb{E}_{(x,y) \sim \mathcal{D}_{x'}} [\mathbb{P}[j^*(x, y) = j]] \right], \quad (9)$$

where  $\mathbf{f}_{-j}$  denotes the predictors chosen by the other model-providers. The key intuition for the proof will be that the predictions  $[f_1(x''), \dots, f_m(x'')]$  affect  $\mathbb{E}_{(x,y) \sim \mathcal{D}_{x'}} [\mathbb{P}[j^*(x, y) = j]]$  if and only if  $x' = x''$ .

First we show that if  $f_1, \dots, f_m$  is a pure-strategy equilibrium, then  $(f^{f_1(x')}, \dots, f^{f_m(x')})$  is a pure-strategy equilibrium for  $\mathcal{F}_{\text{all}}^{x'}$  with data distribution  $\mathcal{D}_{x'}$ . Assume for sake of contradiction that  $(f^{f_1(x')}, \dots, f^{f_m(x')})$  is not an equilibrium. Then there exists  $j' \in [m]$  such that model-provider  $j'$  would achieve higher utility if they switched from  $f^{f_{j'}(x')}$  to  $f^{1-f_{j'}(x')}$ . Let  $f'_{j'}$  be the predictor given by  $f'_{j'}(x) = f_{j'}(x)$  if  $x \neq x'$  and  $f'_{j'}(x') = 1 - f_{j'}(x')$ . By equation (9), this would mean that  $u(f'_{j'}; \mathbf{f}_{-j'})$  is strictly higher than  $u(f_{j'}; \mathbf{f}_{-j'})$  which is a contradiction.

Next, we show that if  $(f^{f_1(x')}, \dots, f^{f_m(x')})$  is a pure-strategy equilibrium for  $\mathcal{F}_{\text{all}}^{x'}$  with data distribution  $\mathcal{D}_{x'}$  for all  $x' \in X$  then  $f_1, \dots, f_m$  is a pure-strategy equilibrium. Assume for sake of contradiction that there exists  $j'$  such that  $u(f'_{j'}; \mathbf{f}_{-j'}) > u(f_{j'}; \mathbf{f}_{-j'})$ . By equation (9), there must exist  $x'$  such that  $\mathbb{E}_{(x,y) \sim \mathcal{D}_{x'}} [\mathbb{P}[j^*(x, y) = j']]$  is higher for  $f'_{j'}$  than  $f_{j'}$ . This means that  $(f^{f_1(x')}, \dots, f^{f_m(x')})$  is not an equilibrium (since  $f^{1-f_{j'}(x')}$  would be a better response for model-provider  $j'$ ) which is a contradiction.  $\square$

We next show that a pure-strategy equilibrium exists by applying standard equilibrium existence results.

**Lemma 2.** *Let  $X$  be a finite set of representations, let  $\mathcal{F} = \mathcal{F}_{\text{all}}$ , and let  $\mathcal{D}$  be the distribution over  $(X, Y)$ . For any  $m \geq 2$ , there exists a pure strategy equilibrium.*

*Proof.* By Lemma 1, it suffices to show that there exists a pure-strategy equilibrium whenever there is a single data representation  $X = \{x\}$ . In this case, the function class  $\mathcal{F}_{\text{all}}$  consists of two predictors  $f^0$  given by  $f^0(x) = 0$  and  $f^1$  given by  $f^1(x) = 1$ . The game between model-providers is thus a 2-action game with symmetric utility functions. Thus, it must possess a (possibly asymmetric) pure Nash equilibrium [CRV<sup>+</sup>04].  $\square$

We next prove Proposition 1. The high-level intuition of the proof is as follows. By Lemma 1, we can focus on one data representation  $x$  at a time. Let  $y^* = \arg \max_y \mathbb{P}[y \mid x]$  be the Bayes optimal label of  $x$ . The proof boils down to characterizing when the market outcome,  $f_j(x) = y^*$  for  $j \in [m]$ , is an equilibrium, and the equilibrium social loss is determined by whether this market outcome is an equilibrium or not.

*Proof of Proposition 1.* Let  $\mathcal{D}^R$  be the marginal distribution of  $X$  with respect to the distribution  $(X, Y) \sim \mathcal{D}$ . Let  $f_1, \dots, f_m$  be a pure-strategy equilibrium. The social loss is equal to:

$$\begin{aligned} \text{SL}(f_1, \dots, f_m) &= \mathbb{E}[\ell(f_{j^*(x,y)}(x), y)] \\ &= \mathbb{E}_{x' \sim \mathcal{D}^R} \left[ \mathbb{E}_{(x,y) \sim \mathcal{D}} [\ell(f_{j^*(x,y)}(x), y) \mid x = x'] \right] \\ &= \mathbb{E}_{x' \sim \mathcal{D}^R} \left[ \mathbb{E}_{(x,y) \sim \mathcal{D}_{x'}} [\ell(f_{j^*(x,y)}(x), y)] \right], \end{aligned}$$

where  $\mathcal{D}_{x'}$  denotes the conditional distribution  $(X, Y) \mid X = x'$  where  $(X, Y) \sim \mathcal{D}$ . Thus, to analyze the overall social loss, we can separately analyze the social loss on each distribution  $\mathcal{D}_{x'}$  and then average across distributions. It suffices to show that  $\mathbb{E}_{\mathcal{D}_{x'}} [\ell(f_{j^*(x,y)}(x), y)] = \alpha(x')$  if  $\alpha(x') < 1/m$  and zero if  $\alpha(x') > 1/m$ .

To compute the social loss on  $\mathcal{D}_{x'}$ , we first apply Lemma 1. This means that  $(f_1(x'), \dots, f_m(x'))$  is pure-strategy equilibrium with  $\mathcal{D}_{x'}$ . We characterize the equilibrium structure for  $\mathcal{D}_{x'}$  and use this characterization to compute the equilibrium social loss.

**Equilibrium structure for  $\mathcal{D}_{x'}$ .** For notational convenience, let  $y_i := f_i(x')$  denote the label chosen by model-provider  $i$  and let  $y^* = \arg \max_y \mathbb{P}[y \mid x']$  be the Bayes optimal label for  $x'$ . We also abuse notation slightly and let  $u(y_1; y_{-j})$  be model-provider 1's utility if they choose the label  $y_1$  for  $x'$  and the other model-provider's choose  $y_{-j}$ .

We first show that all model-providers choosing  $y^*$  is an equilibrium if and only if  $\alpha(x') \leq 1/m$ . Let's fix  $y_j = y^*$  for all  $j \geq 2$  and look at model-provider 1's utility. We see that  $u(y^*; y_{-j}) = 1/m$  and  $u(1 - y^*; y_{-j}) = \alpha(x')$ . This means that  $y^*$  is a best-response (i.e.,  $y^* \in \arg \max_y u(y; y_{-j})$ ) if and only if  $\alpha(x') \leq 1/m$ .

We next show that if  $\alpha(x') < 1/m$ , then the market outcome  $y_i = y^*$  for all  $i \in [m]$  is the only pure-strategy equilibrium. Let  $y_1, \dots, y_m$  be a pure-strategy equilibrium. It suffices to show that  $y^*$  is the unique best response to  $y_{-j}$ ; that is, that  $\{y^*\} = \arg \max_y u(y; y_{-j})$ . To show this, let  $m'$  denote the size of the set  $\{2 \leq i \leq m \mid y_i = y^*\}$ . First, if  $m' = 0$ , then we have that

$$u(y^*; y_{-j}) = 1 - \alpha(x') > 1/m = u(1 - y^*; y_{-j}),$$

where  $1 - \alpha(x') > 1/m$  follows from the fact that  $1 - \alpha(x') \geq 1/2 \geq 1/m$  along with our assumption that  $\alpha(x') \neq 1/m$ . This demonstrates that  $y^*$  is indeed the unique best response. If  $m' = m - 1$ , then we have that:

$$u(y^*; y_{-j}) = 1/m > \alpha(x') = u(1 - y^*; y_{-j}),$$

as desired. Finally, if  $1 \leq m' \leq m - 2$ , then:

$$u(y^*; y_{-j}) = \frac{1 - \alpha(x')}{m' + 1} \geq \frac{1 - \alpha(x')}{m - 1} > \frac{1}{m} > \alpha(x') > \frac{\alpha(x')}{m - m'} = u(1 - y^*; y_{-j}),$$

as desired.

Finally, we show that all model-providers choosing  $1 - y^*$  is never an equilibrium. Let's fix  $y_j = 1 - y^*$  and look at model-provider 1's utility. We see that:

$$u(y^*; y_{-j}) = 1 - \alpha(x') > \frac{\alpha(x')}{m} = u(1 - y^*; y_{-j}),$$

which shows that  $y^*$  is the unique best response as desired.

**Characterization of equilibrium social loss.** It follows from (5) that the equilibrium social loss  $\mathbb{E}_{(x,y) \sim \mathcal{D}_{x'}}[\ell(f_{j^*}(x,y)(x), y)]$  is  $\alpha(x')$  if all of the model-providers choose  $y_i = y^*$ , it is zero if a nonzero number of model-providers choose  $y^*$  and a nonzero number of model-providers choose  $1 - y^*$ , and it is  $1 - \alpha(x')$  if all of the model-providers choose  $1 - y^*$ .

Let's combine this with our equilibrium characterization results. If  $\alpha(x') < 1/m$ , then the unique equilibrium is at  $y_i = y^*$  so the equilibrium social loss is  $\alpha(x)$  as desired. If  $\alpha(x') > 1/m$ , then neither  $y_i = y^*$  for all  $i \in [m]$  nor  $y_i = 1 - y^*$  for all  $i \in [m]$  is an equilibrium. Since there exists a pure strategy equilibrium by Lemma 2, there must be a pure strategy equilibrium where a nonzero number of model-providers choose  $y^*$  and a nonzero number of model-providers choose  $1 - y^*$ . The equilibrium social loss is thus zero.

Note when  $\alpha(x') = 1 - 1/m$ , there is actually an equilibrium where all of the model-providers choose  $y_i = y^*$ , 0 and an equilibrium where a nonzero number of model-providers choose  $y^*$  and a nonzero number of model-providers choose  $1 - y^*$ ; thus, the equilibrium social loss can be zero or  $1/m$ .

□



	$y_2 = 1 - y^*$	$y_2 = y^*$
$y_1 = 1 - y^*$	$(w_{\max}, w_{\min})$	$(\alpha(x), 1 - \alpha(x))$
$y_1 = y^*$	$(1 - \alpha(x), \alpha(x))$	$(w_{\max}, w_{\min})$

Table 1: Let  $X = \{x\}$ ,  $\mathcal{F} = \mathcal{F}_{\text{all}}$ , user decisions are noiseless, and user decisions are noiseless (i.e.,  $c \rightarrow 0$ , so user decisions are given by (6)). Suppose that there are  $m = 2$  model-providers with market reputations  $w_{\min}$  and  $w_{\max}$ , where  $w_{\max} \geq w_{\min}$  and  $w_{\max} + w_{\min} = 1$ . Let let  $y^* = \arg \max_y \mathbb{P}[y | x]$  be the Bayes optimal label for  $x'$ . The table shows the game matrix when model-provider 1 chooses the label  $y_1$  and model provider 2 chooses the label  $y_2$ .

## C.2 Proofs for Section 3.3

A useful lemma is the following calculation of the game matrix when there is a single representation  $X = \{x\}$ .

**Lemma 3.** *Let  $X = \{x\}$ , and let  $\mathcal{F} = \mathcal{F}_{\text{all}}$ . Suppose that there are  $m = 2$  model-providers with market reputations  $w_{\min}$  and  $w_{\max}$ , where  $w_{\max} \geq w_{\min}$  and  $w_{\max} + w_{\min} = 1$ . Suppose that user decisions are noiseless (i.e.,  $c \rightarrow 0$ , so user decisions are given by (6)). Then the game matrix is specified by Table 1.*

*Proof.* This follows from applying (6) and using the fact that  $\ell(y, y') = \mathbb{1}[y \neq y']$ .  $\square$

We show that pure strategy equilibria are no longer guaranteed to exist when model-providers have unequal market reputations, even when there is a single representation  $X = \{x\}$ .

**Lemma 4.** *Let  $X = \{x\}$  let  $\mathcal{F} = \mathcal{F}_{\text{all}}$ . Suppose that there are  $m = 2$  model-providers with market reputations  $w_{\min}$  and  $w_{\max}$ , where  $w_{\max} \geq w_{\min}$  and  $w_{\max} + w_{\min} = 1$ . Suppose that user decisions are noiseless (i.e.,  $c \rightarrow 0$ , so user decisions are given by (6)). If  $\alpha(x) > w_{\min}$ , then a pure strategy equilibrium does not exist.*

*Proof.* For notational convenience, let  $y_i := f_i(x')$  denote the label chosen by model-provider  $i$  and let  $y^* = \arg \max_y \mathbb{P}[y | x']$  be the Bayes optimal label for  $x'$ . We also abuse notation slightly and let  $u_i(y; y')$  be model-provider  $i$ 's utility if they choose the label  $y$  for  $x$  and the other model-providers choose  $y'$ . The proof follows from the game matrix show in Table 1 (Lemma 3). Using the fact that model-provider 1 must best-respond to model-provider 2's action, this leaves  $y_1 = 1 - y^*$ ,  $y_2 = 1 - y^*$  and  $y_1 = y^*$ ,  $y_2 = y^*$ . However, neither of these market outcomes captures a best-response for model-provider 2: if  $y_1 = 1 - y^*$ , then model-provider 2's unique best response is  $y^*$ ; if  $y_1 = y^*$ , then model-provider 2's unique best response is  $1 - y^*$ . This rules out the existence of a symmetric or asymmetric pure strategy equilibrium.  $\square$

Given the lack of existence of pure strategy equilibria, we must turn to mixed strategies. A mixed strategy equilibrium is guaranteed to exist since the game has finitely many actions  $\mathcal{F}_{\text{all}}$  and finitely many players  $m$ . Let  $(\mu_1, \mu_2, \dots, \mu_m)$  denote a mixed strategy profile over  $\mathcal{F}_{\text{all}}$ . We show the following analogue of Lemma 1 that allows us to again decompose model-provider actions into independent decisions about each representation  $x$ . To formalize this, let  $\mathcal{D}$  be the data distribution, and again let  $\mathcal{D}_x$  be the conditional distribution of  $(X, Y)$  when  $X = x$ , where  $(X, Y) \sim \mathcal{D}$ . Again, let  $\mathcal{F}_{\text{all}}^x := \{f_0, f_1\}$  be the class of the (two) functions from a single representation  $x$  to  $\{0, 1\}$ , where  $f_0(x) = 0$  and  $f_1(x) = 1$ . Given a mixed strategy profile  $\mu$  and a representation  $x$ , we define the

conditional mixed strategy  $\mu^x$  over  $\mathcal{F}_{\text{all}}^x := \{f_0, f_1\}$  to be defined so  $\mathbb{P}_{\mu^x}[f_i] := \mathbb{P}_{f \sim \mu}[f(x) = i]$  for  $i \in \{0, 1\}$ .

**Lemma 5.** *Let  $X$  be a finite set of representations, let  $\mathcal{F} = \mathcal{F}_{\text{all}}$ , and let  $\mathcal{D}$  be the distribution over  $(X, Y)$ . For each  $x \in X$ , let  $\mathcal{D}_x$  be the conditional distribution of  $(X, Y)$  given  $X = x$ , where  $(X, Y) \sim \mathcal{D}$ , and let  $\mathcal{F}_{\text{all}}^x := \{f_0, f_1\}$  be the class of the (two) functions from a single representation  $x$  to  $\{0, 1\}$ , where  $f_0(x) = 0$  and  $f_1(x) = 1$ . Suppose that user decisions are noiseless (i.e.,  $c \rightarrow 0$ , so user decisions are given by (3)). A strategy profile  $(\mu_1, \mu_2, \dots, \mu_m)$  is an equilibrium if and only if for every  $x \in X$ , the market outcome  $(\mu_1^x, \mu_2^x, \dots, \mu_m^x)$  (where  $\mu_1^x, \dots, \mu_m^x$  are the conditional mixed strategies defined above) is an equilibrium for  $\mathcal{F}_{\text{all}}^x$  with data distribution  $\mathcal{D}_x$ .*

*Proof.* The proof follows similarly to the proof of Lemma 5, but some minor generalizations to account for mixed strategy equilibria. Let  $\mathcal{D}^R$  be the marginal distribution of  $X$  with respect to the distribution  $(X, Y) \sim \mathcal{D}$ . Let  $\mathcal{D}^R$  be the marginal distribution of  $X$  with respect to the distribution  $(X, Y) \sim \mathcal{D}$ . First, we write model-provider  $j$ 's utility as:

$$\mathbb{E}_{\substack{f_j \sim \mu_j \\ \mathbf{f}_{-j} \sim \mu_{-j}}} [u(f_j; \mathbf{f}_{-j})] = \mathbb{E}_{\substack{f_j \sim \mu_j \\ \mathbf{f}_{-j} \sim \mu_{-j}}} \left[ \mathbb{E}_{(x, y) \sim \mathcal{D}} [\mathbb{P}[j^*(x, y) = j]] \right] = \mathbb{E}_{x' \sim \mathcal{D}^R} \left[ \mathbb{E}_{\substack{f_j \sim \mu_j^{x'} \\ \mathbf{f}_{-j} \sim \mu_{-j}^{x'}}} \left[ \mathbb{E}_{(x, y) \sim \mathcal{D}_{x'}} [\mathbb{P}[j^*(x, y) = j]] \right] \right]. \quad (10)$$

where  $\mu_{-j}$  denotes the mixed strategies chosen by the other model-providers.

First we show that if  $\mu_1, \mu_2, \dots, \mu_m$  is an equilibrium, then  $(\mu_1^{x'}, \dots, \mu_m^{x'})$  is an equilibrium for  $\mathcal{F}_{\text{all}}^{x'}$  with data distribution  $\mathcal{D}_{x'}$ . Let  $f_j$  be in  $\text{supp}(\mu_{j'})$ . Assume for sake of contradiction that  $(\mu_1^{x'}, \dots, \mu_m^{x'})$  is not an equilibrium. Then there exists  $j' \in [m]$  such that model-provider  $j'$  would achieve higher utility on  $f^{1-f_{j'}(x')}$  than  $f^{f_{j'}(x')}$ . Let  $f'_{j'}$  be the predictor given by  $f'_{j'}(x) = f_{j'}(x)$  if  $x \neq x'$  and  $f'_{j'}(x') = 1 - f_{j'}(x')$ . By equation (10), this would mean that  $u(f'_{j'}; \mu_{-j'})$  is strictly higher than  $u(f_{j'}; \mu_{-j'})$  which is a contradiction.

Next, we show that if  $(\mu_1^{x'}, \dots, \mu_m^{x'})$  is an equilibrium for  $\mathcal{F}_{\text{all}}^{x'}$  with data distribution  $\mathcal{D}_{x'}$  for all  $x' \in X$  then  $\mu_1, \dots, \mu_m$  is an equilibrium. Let  $f_j$  be in  $\text{supp}(\mu_{j'})$ . Assume for sake of contradiction that there exists  $j'$  such that  $u(f'_{j'}; \mu_{-j'}) > u(f_j; \mu_{-j'})$ . By equation (9), there must exist  $x'$  such that  $\mathbb{E}_{\mathbf{f}_{-j'} \sim \mu_{-j'}^{x'}} [\mathbb{E}_{(x, y) \sim \mathcal{D}_{x'}} [\mathbb{P}[j^*(x, y) = j']]]$  is higher for  $f'_{j'}$  than  $f_{j'}$ . This means that  $(\mu_1^{x'}, \dots, \mu_m^{x'})$  is not an equilibrium, which is a contradiction.  $\square$

We now prove Proposition 2.

*Proof of Proposition 2.* Let  $\mathcal{D}^R$  be the marginal distribution of  $x$  with respect to the distribution

$(x, y) \sim \mathcal{D}$ . Let  $\mu_1, \mu_2$  be a mixed strategy equilibrium. The social loss is equal to:

$$\begin{aligned}
\mathbb{E}_{\substack{f_1 \sim \mu_1 \\ f_2 \sim \mu_2}} [\text{SL}(f_1, f_2)] &= \mathbb{E}[\ell(f_{j^*(x,y)}(x), y)] \\
&= \mathbb{E}_{\substack{f_1 \sim \mu_1 \\ f_2 \sim \mu_2}} \left[ \mathbb{E}_{x' \sim \mathcal{D}^R} \left[ \mathbb{E}_{(x,y) \sim \mathcal{D}} [\ell(f_{j^*(x,y)}(x), y) \mid x = x'] \right] \right] \\
&= \mathbb{E}_{\substack{f_1 \sim \mu_1 \\ f_2 \sim \mu_2}} \left[ \mathbb{E}_{x' \sim \mathcal{D}^R} \left[ \mathbb{E}_{(x,y) \sim \mathcal{D}_{x'}} [\ell(f_{j^*(x,y)}(x), y)] \right] \right] \\
&= \mathbb{E}_{x' \sim \mathcal{D}^R} \left[ \mathbb{E}_{\substack{f_1 \sim \mu_1^* \\ f_2 \sim \mu_2^*}} \left[ \mathbb{E}_{(x,y) \sim \mathcal{D}_{x'}} [\ell(f_{j^*(x,y)}(x), y)] \right] \right] \\
&= \mathbb{E}_{x' \sim \mathcal{D}_X} \left[ \mathbb{E}_{\substack{f_1 \sim \mu_1^{x'} \\ f_2 \sim \mu_2^{x'}}} \left[ \mathbb{E}_{(x,y) \sim \mathcal{D}_{x'}} [\ell(f_{j^*(x,y)}(x), y)] \right] \right]
\end{aligned}$$

where  $\mathcal{D}_{x'}$  denotes the conditional distribution  $(X, Y) \mid X = x'$  where  $(X, Y) \sim \mathcal{D}$  and where  $\mu^x$  denotes the conditional mixed strategy  $\mathcal{F}_{\text{all}}^x := \{f^0, f^1\}$  to be defined so  $\mathbb{P}_{\mu^x}[f^i] := \mathbb{P}_{f \sim \mu}[f(x) = i]$  for  $i \in \{0, 1\}$ . Thus, to analyze the overall social loss, we can separately analyze the social loss on each distribution  $\mathcal{D}_{x'}$  and then average across distributions. It suffices to show that:

$$\mathbb{E}_{\substack{f_1 \sim \mu_1^{x'} \\ f_2 \sim \mu_2^{x'}}} \left[ \mathbb{E}_{(x,y) \sim \mathcal{D}_{x'}} [\ell(f_{j^*(x,y)}(x), y)] \right] = \begin{cases} \alpha(x') & \text{if } \alpha(x') < w_{\min} \\ \frac{2(\alpha(x') - w_{\min}) \cdot (w_{\max} - \alpha(x'))}{(1 - 2 \cdot w_{\min})^2} & \text{if } \alpha(x') > w_{\min}. \end{cases}$$

To compute the social loss on  $\mathcal{D}_{x'}$ , we first apply Lemma 5. This means that  $(\mu_1^{x'}, \mu_2^{x'})$  is mixed-strategy equilibrium with  $\mathcal{D}_{x'}$ . We characterize the equilibrium structure for  $\mathcal{D}_{x'}$  and use this characterization to compute the equilibrium social loss.

Our main technical ingredient is the game matrix in Table 1 (Lemma 3). We will slightly abuse notation and view choosing the label  $y$  as the strategy of the model-provider. Accordingly, we view a mixed strategy as a distribution over  $\{0, 1\}$ . For notational convenience, let  $y_i := f_i(x')$  denote the label chosen by model-provider  $i$  and let  $y^* = \arg \max_y \mathbb{P}[y \mid x']$  be the Bayes optimal label for  $x'$ . We split into two cases:  $\alpha(x') < w_{\min}$  and  $\alpha(x') > w_{\min}$ .

**Case 1:**  $\alpha(x') < w_{\min}$ . We claim that the unique equilibrium is a pure strategy equilibrium where  $y_1 = y_2 = y^*$ . First, if  $\alpha(x) < w_{\min}$ , we show that choosing  $y^*$  is a strictly dominant strategy for model-provider 1. This follows from the fact that  $1 - \alpha(x) > w_{\max}$  and  $w_{\max} \geq w_{\min} > \alpha(x)$ . Thus, model-provider 1 must play a pure strategy where they always choose  $y_1 = y^*$ . When model-provider 1 chooses  $y^*$ , then the unique best response for model-provider 2 is also to choose  $y^*$  since  $\alpha(x') < w_{\min}$ . This establishes that  $y_1 = y_2 = y^*$  is the unique equilibrium. This also implies that the equilibrium social loss satisfies:

$$\mathbb{E}_{\substack{f_1 \sim \mu_1^{x'} \\ f_2 \sim \mu_2^{x'}}} \left[ \mathbb{E}_{(x,y) \sim \mathcal{D}_{x'}} [\ell(f_{j^*(x,y)}(x), y)] \right] = \alpha(x')$$

as desired.

**Case 2:**  $\alpha(x') > w_{\min}$ . Let  $p_1 = \mathbb{P}_{\mu_1^{x'}}[y_1 = y^*]$  and let  $p_2 = \mathbb{P}_{\mu_2^{x'}}[y_2 = y^*]$ . By Lemma 4, a pure strategy equilibrium does not exist. Thus, we consider mixed strategies. Since pure strategy equilibria do not exist, at least one of  $p_1$  and  $p_2$  must be strictly between zero and one. We compute  $p_1$  and  $p_2$ , splitting into two cases: (1)  $p_1 > 0$  and (2)  $p_2 > 0$ .

If  $p_1 > 0$ , then we know that model-provider 1 must be indifferent between choosing  $y^*$  and  $1 - y^*$ . This means that:

$$p_2 \alpha(x') + (1 - p_2) w_{\max} = (1 - p_2)(1 - \alpha(x')) + p_2 w_{\max}.$$

Solving for  $p_2$ , we obtain:

$$p_2 = \frac{w_{\max} - (1 - \alpha(x'))}{2w_{\max} - 1} = \frac{\alpha(x') - w_{\min}}{1 - 2w_{\min}} > 0.$$

If  $p_2 > 0$ , then we know that model-provider 2 must be indifferent between choosing  $y^*$  and  $1 - y^*$ . This means that:

$$p_1 \alpha(x') + (1 - p_1) w_{\min} = (1 - p_1)(1 - \alpha(x')) + p_1 w_{\min}.$$

Solving for  $p_1$ , we obtain:

$$p_1 = \frac{(1 - \alpha(x')) - w_{\min}}{1 - 2w_{\min}} = \frac{w_{\max} - \alpha(x)}{1 - 2w_{\min}} > 0.$$

Putting this all together, we see that:

$$\begin{aligned} p_1 &= \frac{w_{\max} - \alpha(x')}{1 - 2w_{\min}} \\ p_2 &= \frac{\alpha(x') - w_{\min}}{1 - 2w_{\min}}, \end{aligned}$$

and in fact  $p_1 + p_2 = 1$ .

Using this characterization of  $p_1$  and  $p_2$ , we see that the equilibrium social loss is equal to:

$$\begin{aligned} \mathbb{E}_{\substack{f_1 \sim \mu_1^{x'} \\ f_2 \sim \mu_2^{x'}}} \left[ \mathbb{E}_{(x,y) \sim \mathcal{D}_{x'}} [\ell(f_{j^*(x,y)}(x), y)] \right] &= \alpha(x') \mathbb{P}[y_1 = y^*] \mathbb{P}[y_2 = y^*] + (1 - \alpha(x')) \mathbb{P}[y_1 = 1 - y^*] \mathbb{P}[y_2 = 1 - y^*] \\ &= \alpha(x') p_1 p_2 + (1 - \alpha(x')) (1 - p_1)(1 - p_2) \\ &= \alpha(x') p_1 p_2 + (1 - \alpha(x')) p_1 p_2 \\ &= p_1 p_2 \\ &= \frac{(\alpha(x') - w_{\min}) \cdot (w_{\max} - \alpha(x))}{(1 - 2 \cdot w_{\min})^2}, \end{aligned}$$

as desired. □