Understanding Sparse JL for Feature Hashing

Meena Jagadeesan (Harvard University)

mjagadeesan@college.harvard.edu

NeurIPS 2019

Dimensionality reduction (ℓ_2-to-ℓ_2)

A (randomized) map from $\mathbb{R}^n$ to $\mathbb{R}^m$ that "preserves geometry" of vectors.

Dimensionality reduction (ℓ_2-to-ℓ_2)

A (randomized) map from $\mathbb{R}^n$ to $\mathbb{R}^m$ that "preserves geometry" of vectors.

A pre-processing step in many applications:

Dimensionality reduction (ℓ_2-to-ℓ_2)

A (randomized) map from $\mathbb{R}^n$ to $\mathbb{R}^m$ that "preserves geometry" of vectors.

A pre-processing step in many applications:

- Document classification tasks (Weinberger et al. '09, etc)
- SVMs (Paul et al. '14)
- k-means/k-medians (Makarychev, Makarychev, Razenshteyn '18)
- Nearest neighbors (Ailon, Chazelle '09, Har-Peled et al. '14, Wei '19)
- Numerical linear algebra (Clarkson and Woodruff '12, Nelson and Nguyen '14, etc.)

Dimensionality reduction (ℓ_2-to-ℓ_2)

A (randomized) map from $\mathbb{R}^n$ to $\mathbb{R}^m$ that "preserves geometry" of vectors.

A pre-processing step in many applications:

- Document classification tasks (Weinberger et al. '09, etc)
- SVMs (Paul et al. '14)
- k-means/k-medians (Makarychev, Makarychev, Razenshteyn '18)
- Nearest neighbors (Ailon, Chazelle '09, Har-Peled et al. '14, Wei '19)
- Numerical linear algebra (Clarkson and Woodruff '12, Nelson and Nguyen '14, etc.)

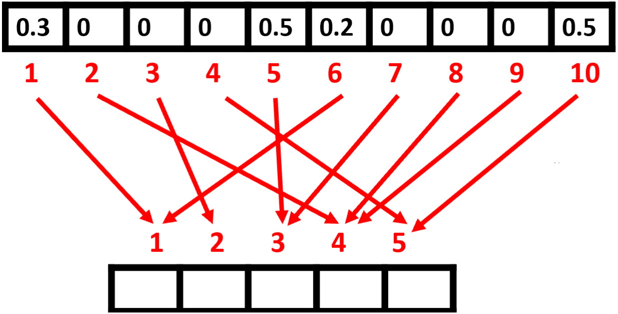
Our contribution: Theoretical analysis of a state-of-the-art dimensionality reduction scheme on feature vectors. Could inform how to optimally set parameters in practice.

Feature hashing (Weinberger et al. '09)

Use a **hash function** $h : \{1, \ldots, n\} \to \{1, \ldots, m\}$ on coordinates.

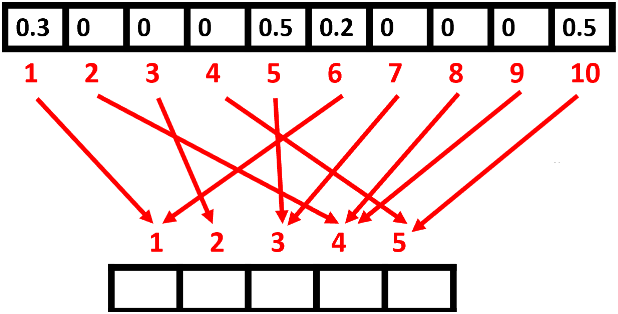
Feature hashing (Weinberger et al. '09)

Use a **hash function** $h : \{1, \ldots, n\} \to \{1, \ldots, m\}$ on coordinates.



Feature hashing (Weinberger et al. '09)

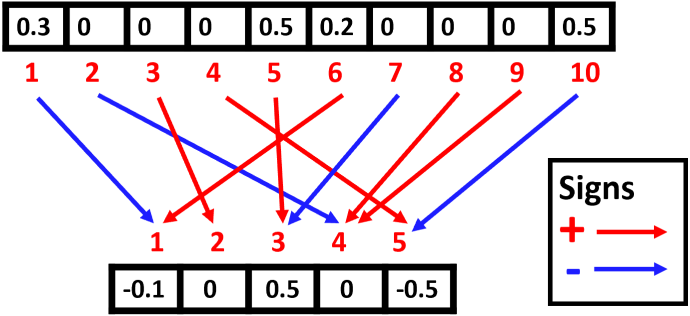
Use a **hash function** $h : \{1, \ldots, n\} \to \{1, \ldots, m\}$ on coordinates.



How should collisions be handled?

Feature hashing (Weinberger et al. '09)

Use a **hash function** $h : \{1, \ldots, n\} \to \{1, \ldots, m\}$ on coordinates.



Use **random signs** to handle collisions (unbiased estimator of ℓ_2^2 norm).

Sparse Johnson-Lindenstrauss transform (KN '12)

Sparse Johnson-Lindenstrauss transform (KN '12)

Use many hash functions $h_1, h_2, \ldots, h_s : \{1, \ldots, n\} \to \{1, \ldots, m\}$.

Sparse Johnson-Lindenstrauss transform (KN '12)

Use many hash functions $h_1, h_2, \ldots, h_s : \{1, \ldots, n\} \to \{1, \ldots, m\}$.

- Anti-correlate hash functions so $h_j(i) \neq h_k(i)$.

Sparse Johnson-Lindenstrauss transform (KN '12)

Use many hash functions $h_1, h_2, \ldots, h_s : \{1, \ldots, n\} \to \{1, \ldots, m\}$.

- Anti-correlate hash functions so $h_j(i) \neq h_k(i)$.

Use random signs to deal with collisions.

Sparse Johnson-Lindenstrauss transform (KN '12)

Use many hash functions $h_1, h_2, \ldots, h_s : \{1, \ldots, n\} \to \{1, \ldots, m\}$.

- Anti-correlate hash functions so $h_j(i) \neq h_k(i)$.

Use random signs to deal with collisions.

Scale the resulting vector by $\frac{1}{\sqrt{s}}$.

Sparse Johnson-Lindenstrauss transform (KN '12)

Use many hash functions $h_1, h_2, \ldots, h_s : \{1, \ldots, n\} \to \{1, \ldots, m\}$.

- Anti-correlate hash functions so $h_j(i) \neq h_k(i)$.

Use random signs to deal with collisions.

Scale the resulting vector by $\frac{1}{\sqrt{s}}$.

Sparse JL is a state-of-the-art sparse dimensionality reduction.

Sparse Johnson-Lindenstrauss transform (KN '12)

Use many hash functions $h_1, h_2, \ldots, h_s : \{1, \ldots, n\} \to \{1, \ldots, m\}$.

- Anti-correlate hash functions so $h_j(i) \neq h_k(i)$.

Use random signs to deal with collisions.

Scale the resulting vector by $\frac{1}{\sqrt{s}}$.

Sparse JL is a state-of-the-art sparse dimensionality reduction.

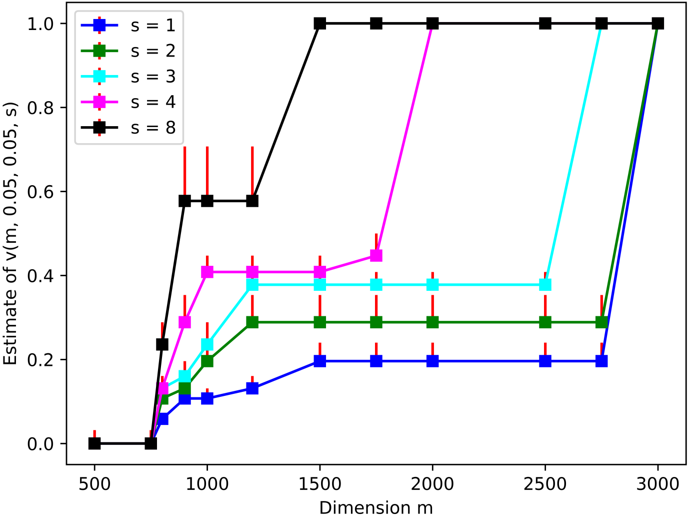
Central question

How should the # of hash functions s and dimension m be set?

Intuition for our contribution

Intuition for our contribution

The function v captures the performance of sparse JL on feature vectors.



Mathematical framework

Mathematical framework

Use a probability distribution $\mathcal{F}$ over maps $f : \mathbb{R}^n \to \mathbb{R}^m$.

Mathematical framework

Use a probability distribution $\mathcal{F}$ over maps $f : \mathbb{R}^n \to \mathbb{R}^m$.

What does it mean to "preserve geometry"?

Mathematical framework

Use a probability distribution $\mathcal{F}$ over maps $f : \mathbb{R}^n \to \mathbb{R}^m$.

What does it mean to "preserve geometry"?

For *each* $x, y \in \mathbb{R}^n$:

$$\mathbb{P}_{f \in \mathcal{F}}[(1 - \epsilon) \|x - y\|_2 \leq \|f(x) - f(y)\|_2 \leq (1 + \epsilon) \|x - y\|_2] > 1 - \delta,$$

for ϵ target error, δ target failure probability.

Mathematical framework

Use a probability distribution $\mathcal{F}$ over maps $f : \mathbb{R}^n \to \mathbb{R}^m$.

What does it mean to "preserve geometry"?

For *each* $x, y \in \mathbb{R}^n$:

$$\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\left\|x-y\right\|_2 \leq \left\|f(x)-f(y)\right\|_2 \leq (1+\epsilon)\left\|x-y\right\|_2] > 1-\delta,$$

for ϵ target error, δ target failure probability.

Focus on linear maps:

$$\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\left\|x\right\|_2 \leq \left\|f(x)\right\|_2 \leq (1+\epsilon)\left\|x\right\|_2] > 1-\delta.$$

Performance on feature vectors (Weinberger et al. '09)

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\left\|x\right\|_2 \leq \left\|f(x)\right\|_2 \leq (1+\epsilon)\left\|x\right\|_2] > 1-\delta.$

Performance on feature vectors (Weinberger et al. '09)

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\|x\|_2 \leq \|f(x)\|_2 \leq (1+\epsilon)\|x\|_2] > 1-\delta$.

Sometimes a much smaller m works on feature vectors in practice than traditional theory on $\mathbb{R}^n$ suggests...

Performance on feature vectors (Weinberger et al. '09)

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\|x\|_2 \leq \|f(x)\|_2 \leq (1+\epsilon)\|x\|_2] > 1-\delta$.

Sometimes a much smaller m works on feature vectors in practice than traditional theory on $\mathbb{R}^n$ suggests...

Consider vectors w/ small ℓ_∞-to-ℓ_2 norm ratio:

$$S_v = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq v\|x\|_2\}.$$

Performance on feature vectors (Weinberger et al. '09)

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\|x\|_2 \leq \|f(x)\|_2 \leq (1+\epsilon)\|x\|_2] > 1-\delta$.

Sometimes a much smaller m works on feature vectors in practice than traditional theory on $\mathbb{R}^n$ suggests...

Consider vectors w/ small ℓ_∞-to-ℓ_2 norm ratio:

$$S_v = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq v\|x\|_2\}.$$

$v(m, \epsilon, \delta, s) :=$ sup over $v \in [0,1]$ s.t. sparse JL meets ℓ_2 goal on $x \in S_v$.

Performance on feature vectors (Weinberger et al. '09)

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\|x\|_2 \leq \|f(x)\|_2 \leq (1+\epsilon)\|x\|_2] > 1-\delta$.

Sometimes a much smaller m works on feature vectors in practice than traditional theory on $\mathbb{R}^n$ suggests...

Consider vectors w/ small ℓ_∞-to-ℓ_2 norm ratio:

$$S_v = \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq v\|x\|_2\}.$$

$v(m, \epsilon, \delta, s) :=$ sup over $v \in [0,1]$ s.t. sparse JL meets ℓ_2 goal on $x \in S_v$.

We give a tight theoretical analysis of the function $v(m, \epsilon, \delta, s)$, that could inform how to optimally set s and m in practice.

Informal statement of main result

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\left\|x\right\|_2 \leq \left\|f(x)\right\|_2 \leq (1+\epsilon)\left\|x\right\|_2] > 1-\delta$..

$v(m, \epsilon, \delta, s) :=$ sup over $v \in [0,1]$ s.t. sparse JL meets ℓ_2 goal on $x \in S_v$.

Informal statement of main result

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\|x\|_2 \leq \|f(x)\|_2 \leq (1+\epsilon)\|x\|_2] > 1-\delta$..

$v(m, \epsilon, \delta, s) :=$ sup over $v \in [0,1]$ s.t. sparse JL meets ℓ_2 goal on $x \in S_v$.

Theorem (Informal)

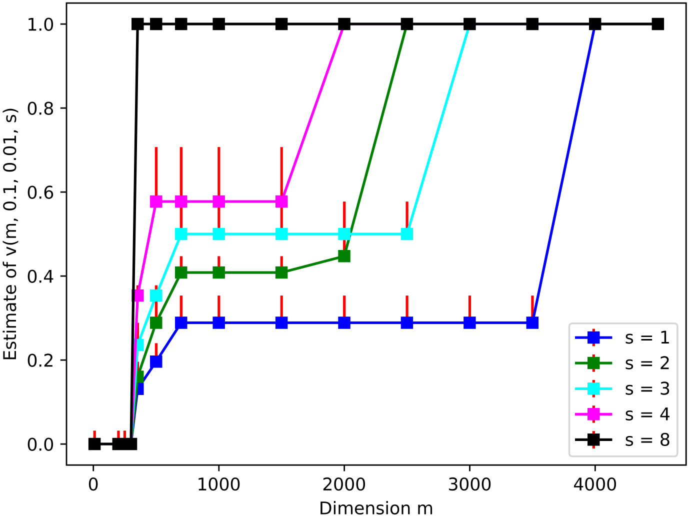
Sparse JL has **four regimes** *in terms of how it performs on norm preservation. For error ϵ and failure probability δ, sparse JL with projected dimension m and s hash functions has performance $v(m, \epsilon, \delta, s)$ equal to:*

$$\begin{cases} 1 \textit{ (full performance)} & \textit{High } m \\ \sqrt{s} B_1 \textit{ (partial performance)} & \textit{Middle } m \\ \sqrt{s} \min(B_1, B_2) \textit{ (partial performance)} & \textit{Middle } m \\ 0 \textit{ (poor performance)} & \textit{Small } m, \end{cases}$$

where B_1, B_2 are functions of m, ϵ, δ.

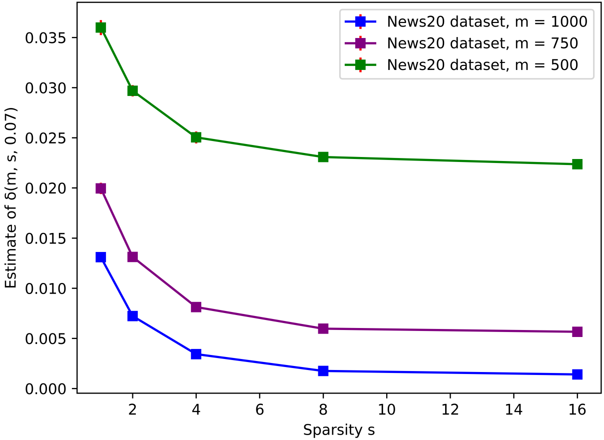
$v(m, \epsilon, \delta, s)$ on more synthetic data

$v(m, \epsilon, \delta, s)$ on more synthetic data

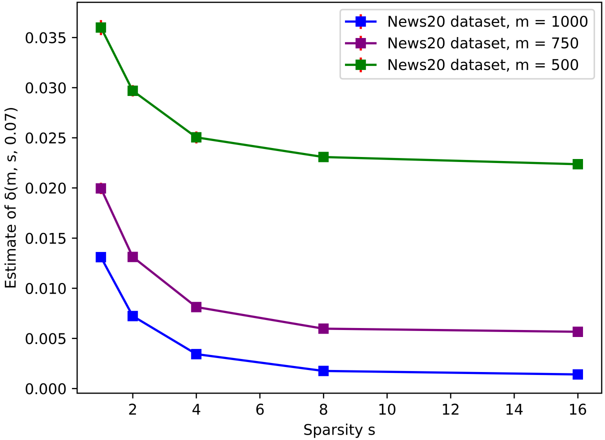


Sparse JL on News20 dataset

Sparse JL on News20 dataset



Sparse JL on News20 dataset



Sparse JL with ≥ 4 hash functions can perform much better than feature hashing in practice.

Comparison to previous work

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\|x\|_2 \leq \|f(x)\|_2 \leq (1+\epsilon)\|x\|_2] > 1-\delta$..

$v(m, \epsilon, \delta, s) :=$ sup over $v \in [0,1]$ s.t. sparse JL meets ℓ_2 goal on $x \in S_v$.

Comparison to previous work

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\|x\|_2 \leq \|f(x)\|_2 \leq (1+\epsilon)\|x\|_2] > 1-\delta$..

$v(m, \epsilon, \delta, s) :=$ sup over $v \in [0,1]$ s.t. sparse JL meets ℓ_2 goal on $x \in S_v$.

Bounds on v:

- $v(m, \epsilon, \delta, 1)$ understood (Weinberger et al '09,..., Freksen et al. '18)
- $v(m, \epsilon, \delta, s)$ lower bound for *multiple hashing* (Weinberger et al '09)

Comparison to previous work

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\left\|x\right\|_2 \leq \left\|f(x)\right\|_2 \leq (1+\epsilon)\left\|x\right\|_2] > 1-\delta$..

$v(m, \epsilon, \delta, s) :=$ sup over $v \in [0,1]$ s.t. sparse JL meets ℓ_2 goal on $x \in S_v$.

Bounds on v:

- $v(m, \epsilon, \delta, 1)$ understood (Weinberger et al '09,..., Freksen et al. '18)
- $v(m, \epsilon, \delta, s)$ lower bound for *multiple hashing* (Weinberger et al '09)

Bounds for sparse JL on full space $\mathbb{R}^n$:

- Can set $m \approx \epsilon^{-2}\log(1/\delta)$, $s \approx \epsilon^{-1}\log(1/\delta)$ (Kane and Nelson '12)
- Can set $m \approx \min(2\epsilon^{-2}/\delta, \epsilon^{-2}\log(1/\delta)e^{\Theta(\epsilon^{-1}\log(1/\delta)/s)})$ (Cohen '16)

Comparison to previous work

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\|x\|_2 \leq \|f(x)\|_2 \leq (1+\epsilon)\|x\|_2] > 1-\delta.$.

$v(m, \epsilon, \delta, s) :=$ sup over $v \in [0,1]$ s.t. sparse JL meets ℓ_2 goal on $x \in S_v$.

Bounds on v:

- $v(m, \epsilon, \delta, 1)$ understood (Weinberger et al '09,..., Freksen et al. '18)
- $v(m, \epsilon, \delta, s)$ lower bound for *multiple hashing* (Weinberger et al '09)

Bounds for sparse JL on full space $\mathbb{R}^n$:

- Can set $m \approx \epsilon^{-2} \log(1/\delta)$, $s \approx \epsilon^{-1} \log(1/\delta)$ (Kane and Nelson '12)
- Can set $m \approx \min(2\epsilon^{-2}/\delta, \epsilon^{-2} \log(1/\delta) e^{\Theta(\epsilon^{-1} \log(1/\delta)/s)})$ (Cohen '16)

This work

Tight bounds on $v(m, \epsilon, \delta, s)$ **for a general** $s > 1$ *for sparse JL.*

Comparison to previous work

Goal: $\mathbb{P}_{f \in \mathcal{F}}[(1-\epsilon)\|x\|_2 \leq \|f(x)\|_2 \leq (1+\epsilon)\|x\|_2] > 1-\delta$..

$v(m, \epsilon, \delta, s) :=$ sup over $v \in [0,1]$ s.t. sparse JL meets ℓ_2 goal on $x \in S_v$.

Bounds on v:

- $v(m, \epsilon, \delta, 1)$ understood (Weinberger et al '09,..., Freksen et al. '18)
- $v(m, \epsilon, \delta, s)$ lower bound for *multiple hashing* (Weinberger et al '09)

Bounds for sparse JL on full space $\mathbb{R}^n$:

- Can set $m \approx \epsilon^{-2} \log(1/\delta)$, $s \approx \epsilon^{-1} \log(1/\delta)$ (Kane and Nelson '12)
- Can set $m \approx \min(2\epsilon^{-2}/\delta, \epsilon^{-2} \log(1/\delta) e^{\Theta(\epsilon^{-1} \log(1/\delta)/s)})$ (Cohen '16)

This work

Tight bounds on $v(m, \epsilon, \delta, s)$ **for a general** $s > 1$ *for sparse JL.*

$\implies$ *Characterization of sparse JL performance in terms of* ϵ, δ, *and* ℓ_∞*-to-*ℓ_2 *norm ratio for a general # of hash functions* s

Main result

Theorem

Under mild conditions, $v(m, \epsilon, \delta, s)$ is equal to $f'(m, \epsilon, \ln(1/\delta), s)$, where $f'(m, \epsilon, p, s)$ is defined to be:

$$\begin{cases} 1 & \text{if } m \geq \min\left(2\epsilon^{-2}e^{p}, \epsilon^{-2}pe^{\Theta\left(\max\left(1, \frac{p\epsilon^{-1}}{s}\right)\right)}\right) \\ \Theta\left(\sqrt{\epsilon s}\frac{\sqrt{\ln(\frac{m\epsilon^2}{p})}}{\sqrt{p}}\right) & \text{else, if } m \geq \max\left(\Theta(\epsilon^{-2}p), s \cdot e^{\Theta\left(\max\left(1, \frac{p\epsilon^{-1}}{s}\right)\right)}\right) \\ & \text{and } m \leq \epsilon^{-2}e^{\Theta(p)} \\ \Theta\left(\sqrt{\epsilon s}\min\left(\frac{\ln(\frac{m\epsilon}{p})}{p}, \frac{\sqrt{\ln(\frac{m\epsilon^2}{p})}}{\sqrt{p}}\right)\right) & \text{else, if } m \geq \Theta(\epsilon^{-2}p) \\ & \text{and } m \leq \min\left(\epsilon^{-2}e^{\Theta(p)}, s \cdot e^{\Theta\left(\max\left(1, \frac{p\epsilon^{-1}}{s}\right)\right)}\right) \\ 0 & \text{if } m \leq \Theta(\epsilon^{-2}p). \end{cases}$$

Conclusion

Tight analysis of $v(m, \epsilon, \delta, s)$ for uniform sparse JL for a general s. Could inform how to optimally set parameters in practice.

Characterization of sparse JL performance in terms of ϵ, δ, and ℓ_∞-to-ℓ_2 norm ratio for a general # of hash functions s.

Evaluation on real-world and synthetic data (sparse JL can perform much better than feature hashing).

Thank you!

PROOF OF MAIN RESULT

Sparse JL as a sparse random projection (KN '12)

Sparse JL as a sparse random projection (KN '12)

$\mathcal{A}_{s,m,n}$ a distribution over $m \times n$ matrices w/ s nonzero entries per column

Sparse JL as a sparse random projection (KN '12)

$\mathcal{A}_{s,m,n}$ a distribution over $m \times n$ matrices w/ s nonzero entries per column

Uniform: Mildly correlate hash functions so $h_j(i) \neq h_k(i)$.

Example (Uniform Sparse JL)

Uniformly choose s nonzero entries in each column; i.i.d signs for nonzero entries.

Block: Take $h_i : \{1, \ldots, n\} \to \{(m/s)(i-1)+1, \ldots, (m/s)(i)\}$

Example (Block Sparse JL)

Choose one nonzero coordinate per m/s-length block per column; i.i.d signs for nonzero entries.

Sparse JL as a sparse random projection (KN '12)

$\mathcal{A}_{s,m,n}$ a distribution over $m \times n$ matrices w/ s nonzero entries per column

Uniform: Mildly correlate hash functions so $h_j(i) \neq h_k(i)$.

Example (Uniform Sparse JL)

Uniformly choose s nonzero entries in each column; i.i.d signs for nonzero entries.

Block: Take $h_i : \{1, \ldots, n\} \to \{(m/s)(i-1)+1, \ldots, (m/s)(i)\}$

Example (Block Sparse JL)

Choose one nonzero coordinate per m/s-length block per column; i.i.d signs for nonzero entries.

Sparse JL distributions are state-of-the-art sparse random projections.

High-level approach of our analysis

(r, i)th coordinate is $\eta_{r,i}\sigma_{r,i}/\sqrt{s}$, where $\eta_{r,i} \in \{0, 1\}$, $\sigma_{r,i} \in \{-1, 1\}$

High-level approach of our analysis

(r, i)th coordinate is $\eta_{r,i}\sigma_{r,i}/\sqrt{s}$, where $\eta_{r,i} \in \{0, 1\}$, $\sigma_{r,i} \in \{-1, 1\}$

Analyze moments of "error" rv $\|Ax\|_2^2 - 1$ for $\|x\|_2 = 1$:

High-level approach of our analysis

(r, i)th coordinate is $\eta_{r,i}\sigma_{r,i}/\sqrt{s}$, where $\eta_{r,i} \in \{0, 1\}$, $\sigma_{r,i} \in \{-1, 1\}$

Analyze moments of "error" rv $\|Ax\|_2^2 - 1$ for $\|x\|_2 = 1$:

$$R(x_1, \dots, x_n) = \frac{1}{s} \sum_{1 \leq i \neq j \leq n} \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}x_i x_j$$

This random variable has been repeatedly analyzed in the literature.

High-level approach of our analysis

(r, i)th coordinate is $\eta_{r,i}\sigma_{r,i}/\sqrt{s}$, where $\eta_{r,i} \in \{0, 1\}$, $\sigma_{r,i} \in \{-1, 1\}$

Analyze moments of "error" rv $\|Ax\|_2^2 - 1$ for $\|x\|_2 = 1$:

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{1 \leq i \neq j \leq n} \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}x_i x_j$$

This random variable has been repeatedly analyzed in the literature.

But... existing bounds are limited to $s = 1$ (Freksen et al., etc.)

High-level approach of our analysis

(r, i)th coordinate is $\eta_{r,i}\sigma_{r,i}/\sqrt{s}$, where $\eta_{r,i} \in \{0, 1\}$, $\sigma_{r,i} \in \{-1, 1\}$

Analyze moments of "error" rv $\|Ax\|_2^2 - 1$ for $\|x\|_2 = 1$:

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{1 \leq i \neq j \leq n} \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}x_i x_j$$

This random variable has been repeatedly analyzed in the literature.

But... existing bounds are limited to $s = 1$ (Freksen et al., etc.) or limited to $v = 1$ (Kane and Nelson '12, Cohen et al. '18, etc.).

High-level approach of our analysis

(r, i)th coordinate is $\eta_{r,i}\sigma_{r,i}/\sqrt{s}$, where $\eta_{r,i} \in \{0, 1\}$, $\sigma_{r,i} \in \{-1, 1\}$

Analyze moments of "error" rv $\|Ax\|_2^2 - 1$ for $\|x\|_2 = 1$:

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{1 \leq i \neq j \leq n} \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}x_i x_j$$

This random variable has been repeatedly analyzed in the literature.

But... existing bounds are limited to $s = 1$ (Freksen et al., etc.) or limited to $v = 1$ (Kane and Nelson '12, Cohen et al. '18, etc.).

Need tight bounds on $\mathbb{E}[R(x_1, \ldots, x_n)^p]$ on S_v at every threshold v.

Bounding moments of $R(x_1, \ldots, x_n)$ at every threshold v

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{1 \le i \ne j \le n} \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}x_i x_j$$

Bounding moments of $R(x_1, \ldots, x_n)$ at every threshold v

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{1 \le i \neq j \le n} \sum_{r=1}^{m} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} x_i x_j$$

Complexities: $\eta_{r,i}$ are correlated,

Bounding moments of $R(x_1, \ldots, x_n)$ at every threshold v

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{1 \leq i \neq j \leq n} \sum_{r=1}^{m} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} x_i x_j$$

Complexities: $\eta_{r,i}$ are correlated, sum has $\Theta(mn^2)$ terms

Bounding moments of $R(x_1, \ldots, x_n)$ at every threshold v

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{1 \leq i \neq j \leq n} \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}x_i x_j$$

Complexities: $\eta_{r,i}$ are correlated, sum has $\Theta(mn^2)$ terms

Issues with existing approaches:

Bounding moments of $R(x_1, \ldots, x_n)$ at every threshold v

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{1 \leq i \neq j \leq n} \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}x_i x_j$$

Complexities: $\eta_{r,i}$ are correlated, sum has $\Theta(mn^2)$ terms

Issues with existing approaches:

1. Not clear how to generalize combinatorics of (Kane and Nelson '12, Freksen et al. '18, etc.)

Bounding moments of $R(x_1, \ldots, x_n)$ at every threshold v

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{1 \le i \neq j \le n} \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}x_i x_j$$

Complexities: $\eta_{r,i}$ are correlated, sum has $\Theta(mn^2)$ terms

Issues with existing approaches:

1. Not clear how to generalize combinatorics of (Kane and Nelson '12, Freksen et al. '18, etc.)
2. Existing non-combinatorial approaches not sufficiently tight (Cohen et. al '18, Cohen '16, etc.)

Bounding moments of $R(x_1, \ldots, x_n)$ at every threshold v

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{1 \leq i \neq j \leq n} \sum_{r=1}^{m} \eta_{r,i}\eta_{r,j}\sigma_{r,i}\sigma_{r,j}x_i x_j$$

Complexities: $\eta_{r,i}$ are correlated, sum has $\Theta(mn^2)$ terms

Issues with existing approaches:

1. Not clear how to generalize combinatorics of (Kane and Nelson '12, Freksen et al. '18, etc.)
2. Existing non-combinatorial approaches not sufficiently tight (Cohen et. al '18, Cohen '16, etc.)

We use a non-combinatorial approach with Rademacher-specific bounds.

Overview of our approach

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{m} Z_r(x_1, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{m} \left(\sum_{1 \le i \neq j \le n} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} x_i x_j \right).$$

Overview of our approach

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{m} Z_r(x_1, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{m} \left(\sum_{1 \leq i \neq j \leq n} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} x_i x_j \right).$$

Lower bound: $\mathbb{E}[Z_r(x_1, \ldots, x_n)^q] = \mathbb{E}_\eta[\mathbb{E}_\sigma[Z_r(x_1, \ldots, x_n)^q]]$

Overview of our approach

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{m} Z_r(x_1, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{m} \left(\sum_{1 \le i \ne j \le n} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} x_i x_j \right).$$

Lower bound: $\mathbb{E}[Z_r(x_1, \ldots, x_n)^q] = \mathbb{E}_\eta[\mathbb{E}_\sigma[Z_r(x_1, \ldots, x_n)^q]]$

1. Suffices to pick "worst" vector in each S_v
2. View $Z_r(v, \ldots, v, 0, \ldots, 0)$ as a quadratic form of ± 1 rvs
   Use known quadratic form moments bounds (Latała '99)
3. Take expectation over $\eta_{r,i}$; carefully combine over $r \in [m]$

Overview of our approach

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{m} Z_r(x_1, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{m} \left(\sum_{1 \le i \neq j \le n} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} x_i x_j \right).$$

Lower bound: $\mathbb{E}[Z_r(x_1, \ldots, x_n)^q] = \mathbb{E}_\eta[\mathbb{E}_\sigma[Z_r(x_1, \ldots, x_n)^q]]$

1. Suffices to pick "worst" vector in each S_v
2. View $Z_r(v, \ldots, v, 0, \ldots, 0)$ as a quadratic form of ± 1 rvs
   Use known quadratic form moments bounds (Latała '99)
3. Take expectation over $\eta_{r,i}$; carefully combine over $r \in [m]$

Upper bound: need to consider $R(x_1, \ldots, x_n)$ for every $x \in S_v$

Overview of our approach

$$R(x_1, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{m} Z_r(x_1, \ldots, x_n) = \frac{1}{s} \sum_{r=1}^{m} \left(\sum_{1 \le i \neq j \le n} \eta_{r,i} \eta_{r,j} \sigma_{r,i} \sigma_{r,j} x_i x_j \right).$$

Lower bound: $\mathbb{E}[Z_r(x_1, \ldots, x_n)^q] = \mathbb{E}_\eta[\mathbb{E}_\sigma[Z_r(x_1, \ldots, x_n)^q]]$

1. Suffices to pick "worst" vector in each S_v
2. View $Z_r(v, \ldots, v, 0, \ldots, 0)$ as a quadratic form of ± 1 rvs
   Use known quadratic form moments bounds (Latała '99)
3. Take expectation over $\eta_{r,i}$; carefully combine over $r \in [m]$

Upper bound: need to consider $R(x_1, \ldots, x_n)$ for every $x \in S_v$

1. Create tractable versions of estimates in (Latała '97, '99)
   Structure of $Z_r(x_1, \ldots, x_n)$ is helpful
2. Combine over $r \in [m]$ using (Latała '97)