Restoring a Blurry Image

MTH 410, F18

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Contents

1 Background

Tikhonov regularization is a process for approximating $\mathbf{x}$ as a solution to the least squares optimization problem

$$\min_{\widetilde{\mathbf{x}} \in \mathbb{R}^n} f(\widetilde{\mathbf{x}}), \quad \text{where} \quad f(\widetilde{\mathbf{x}}) \stackrel{def}{=} \|\widetilde{\mathbf{d}} - \mathbf{A}\widetilde{\mathbf{x}}\|^2 + \lambda^2 \|\widetilde{\mathbf{x}}\|^2 \tag{1}$$

with $\widetilde{\mathbf{x}}$ being found as the solution to the linear system

$$(\mathbf{A}^T\mathbf{A} + \lambda^2 \mathbf{I})\widetilde{\mathbf{x}}_\lambda = \mathbf{A}^T \widetilde{\mathbf{d}} \tag{2}$$

The following algorithm produces the desired $\widetilde{\mathbf{x}}$:

Given $\mathbf{A}$, $\widetilde{\mathbf{D}}$, λ;
  $for\ i = 1 : m$
    $\widetilde{\mathbf{d}} = \widetilde{\mathbf{D}}(:, i)$ % column i of the noisy data matrix $\widetilde{\mathbf{D}}$
    evaluate $\widetilde{\mathbf{x}}_i$ by solving (2)
  end

Compiling $\widetilde{\mathbf{X}}_\lambda = [\widetilde{\mathbf{x}}_1\ \widetilde{\mathbf{x}}_2 \ldots \widetilde{\mathbf{x}}_m]$ yields the solution matrix $\widetilde{\mathbf{X}}_\lambda \approx \mathbf{X}$, where $\mathbf{X}$ is the original, undistorted matrix.

2 Application

One area of use for Tikhonov regularization is image restoration. Observe the following blurred image of a US dollar bill:

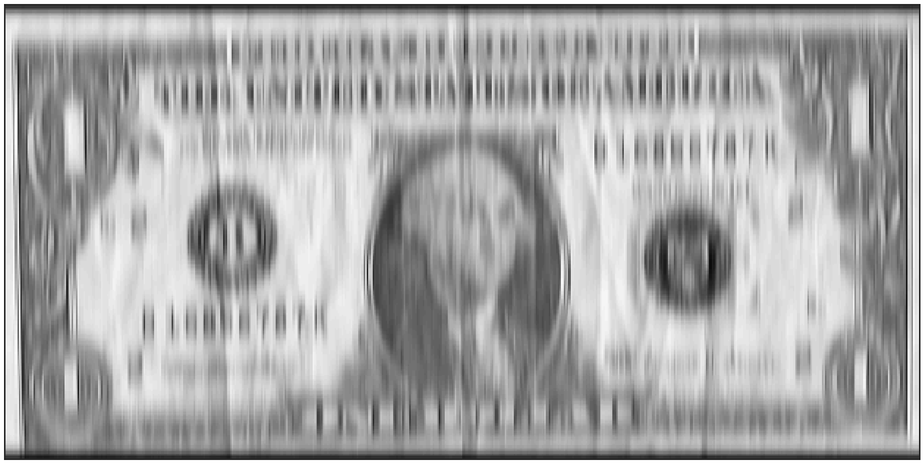

Figure 1: Blurred dollar bill, original

In order to decipher the serial number on the bill, we apply Tikhonov regularization (the code for which can be found in section 3) to the image data matrix with various values of λ until we have a reasonably clear result. Hansen [1] found that small – 10^{-1} to 10^{-5} – values of λ were appropriate for minimizing (1).

For $\lambda = 1$, we have

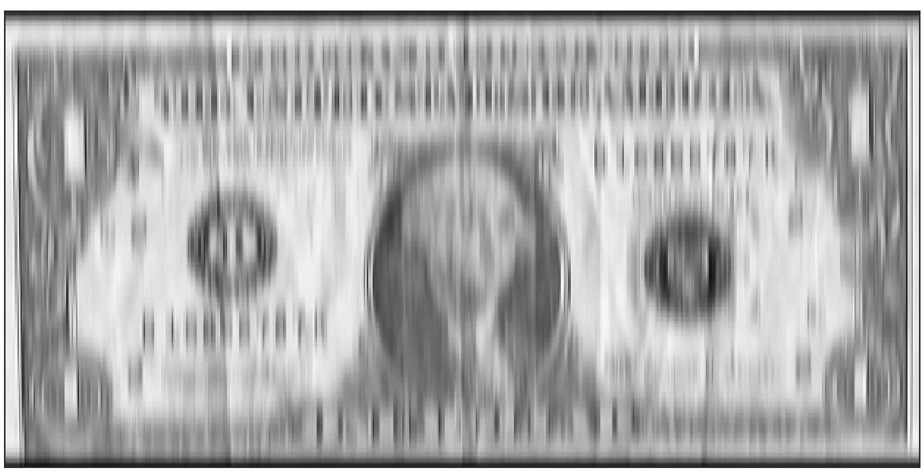

Figure 2: Blurred dollar bill, regularized, $\lambda = 1$

The resolution of which is perhaps slightly worse than the original. $\lambda = 10^{-1}$ yields

Figure 3: Blurred dollar bill, regularized, $\lambda = 10^{-1}$

This is significantly clearer. The next several figures show the results of further values of λ.

Figure 4: Blurred dollar bill, regularized, $\lambda = 10^{-2}$

Figure 5: Blurred dollar bill, regularized, $\lambda = 10^{-3}$

Figure 6: Blurred dollar bill, regularized, $\lambda = 10^{-4}$

Figure 7: Blurred dollar bill, regularized, $\lambda = 10^{-5}$

Figure 8: Blurred dollar bill, regularized, $\lambda = 10^{-6}$

We see that figures 6 and 7 – $\lambda = 10^{-4}$ and 10^{-5}, respectively – produce the clearest serial numbers, and that as we progress into figure 8, $\lambda = 10^{-6}$, we introduce more noise into the solution norm, obscuring the picture again.

Based on the results, the serial number appears to be D16856787K, meaning it was manufactured in Cleveland [2]. However, due to the lack of total clarity, it is possible for the D to be a B and for it to have been manufactured in New York.

3 Code

```
function Xapp = tikhonov(lambda,Dapp)

% This function uses Tikhonov regularization to approximate an original
% matrix from a given noisy matrix.

% lambda   - regularization parameter
% Dapp     - noisy data matrix

[Dm,Dn] = size(Dapp); % size constraints for function
L = 0.45; % given parameter

% create B, A, I matrices

B = zeros(Dm,Dm);

for i = 1:Dm
    B(i,i) = 1 - 2*L;
    if (i <= Dm-1)
        B(i,i+1) = L;
        B(i+1,i) = L;
    end
end

A = B^25; % non-singular transformation matrix
I = eye(size(A)); % Dm x Dm identity matrix

% solve approximate X original matrix

for j = 1:Dn
    d = Dapp(:,j);
    x = (A'*A + lambda^2*I)\A'*d;
    Xapp(:,j) = x;
end

end
```

References

[1] Hansen, P.C. (2000). "The L-curve and its use in the numerical treatment of inverse problems." 3. https://www.sintef.no/globalassets/project/evitameeting/2005/lcurve.pdf (accessed November 21, 2018).

[2] U.S. Bureau of Engraving and Printing. "Federal Reserve Bank and Serial Number Relationship Table." MoneyFactory.gov. https://www.moneyfactory.gov/frbsntable.html (accessed November 21, 2018).