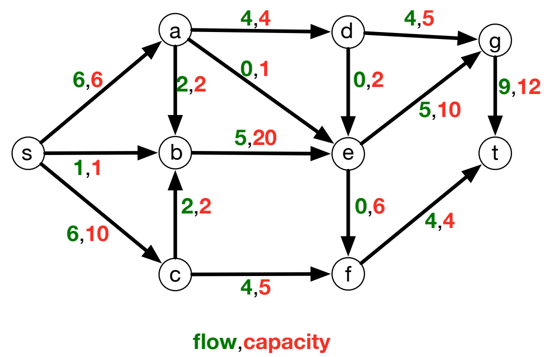
Solutions to Flow Network Practice Problems

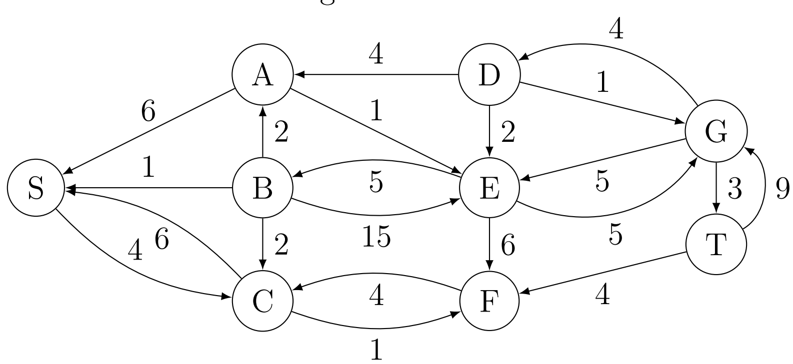
Practice problems:

1. [DPV] Problem 7.10 (max-flow = min-cut example)

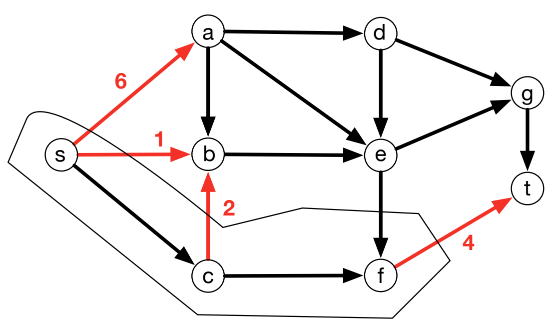
Here is a max flow in the given flow network:



The residual network G^f is the following:



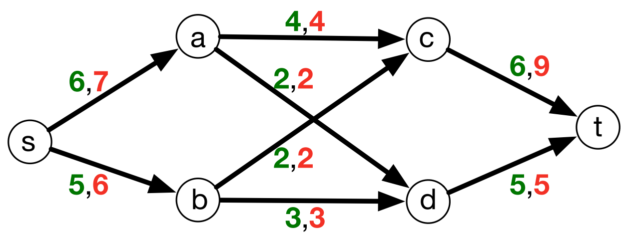
Looking at the residual network G^f, the set L of vertices reachable from s in G^f is $L = \{s, c, f\}$. This set L has capacity $13 = 6 + 1 + 2 + 4$. Note the capacity of the cut is determined by the original capacities, it does not depend on the flow. The capacity of this st-cut matches the size of the flow f and hence f is a max-flow and L defines a min-st-cut. Here is an illustration of this min-st-cut:



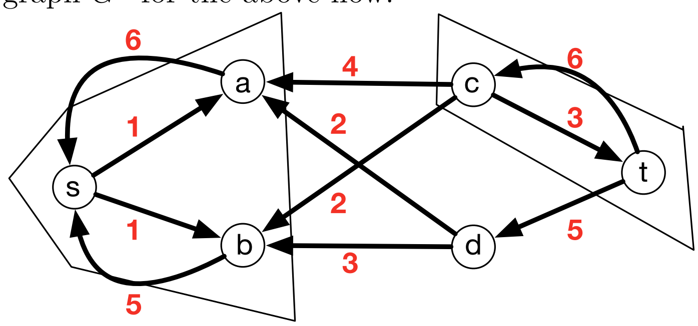
2. [DPV] Problem 7.17 (bottleneck edges)

Parts (a) and (b):

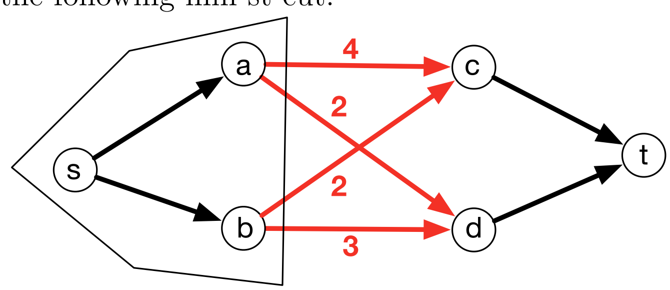
Here is the max flow in the given flow network:



Here is the residual graph G^f for the above flow:



In G^f the set of vertices reachable from s is $\{s, a, b\}$ and the set of vertices that can reach t is $\{c, t\}$. This gives the following min-st-cut:



Notice that the set $\{s, a, b\}$ has capacity $11 = 4 + 2 + 2 + 3$ which matches the size of the above flow.

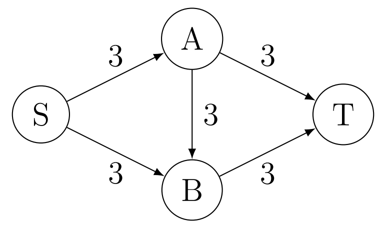
Part (c):

An edge $\overrightarrow{uv}$ in the original flow network G is a *bottleneck edge* if increasing its capacity results in an increase in the size of the maximum flow.

There are two bottleneck edges in the above network, they are the edges $\overrightarrow{ac}$ and $\overrightarrow{bc}$.

Part (d):

Here is an example of a flow network with 4 vertices and no bottleneck edges:



Alternatively, in the flow network from question 7.17, if the capacity of the edge $\overrightarrow{ct}$ was reduced from 9 to 6 then there will be no bottleneck edges in this flow network.

Part (e):

(a) Algorithm:

Our algorithm for finding bottleneck edges is as follows:

(1) Find a maximum flow f on G.

(2) Using f^*, create the residual graph G^f

(3) Run Explore from s in G^f. Let S be the set of vertices reachable from s in G^f.

(4) Create the reverse of G^f, G^r

(5) Run Explore from t in the reverse graph G^r. Let T be the set of vertices reachable from t in G^r; note the set T are those vertices which can reach t in G^f.

(6) For each $\overrightarrow{vw} \in E(G)$, output $\overrightarrow{vw}$ as a bottleneck edge if $v \in S$ and $w \in T$.

(b) Correctness:

We start by finding a maximum flow f for the flow network G. Consider an edge $\overrightarrow{vw}$ in the flow network G. Increasing the capacity of $\overrightarrow{vw}$ results in an increase in maximum flow value if and only if there exists a path from s to v and a path from w to t in G^f. This is because if there exists these two paths then more flow can be sent from s to v, then along the edge $\overrightarrow{vw}$, and finally from v to t.

Note that this algorithm looks for a path $s \to v$ and $w \to t$. What if these two paths share one or more edges? Then, the joined path will have one or more cycles. So, we can drop that cycle (or cycles) and get a shorter path from $s \to t$, but will this path still go through (v, w)? If one of the cycles contains edge $e = (v, w)$, then we have an augmenting path in G^f not using e, which would mean f is not a max flow. Hence, e cannot be in any of the cycles, so our algorithm works.

(c) Run time Analysis:

Since steps 2 through 6 each take $O(n + m)$ time, the running time is dominated by the running time of the maximum flow algorithm used in step 1 (either $O(mC)$ for Ford-Fulkerson or $O(m^2n)$ for Edmonds-Karp)

4. [DPV] Problem 7.19 (verifying max-flow)

Given a flow network $G = (V, E)$ and a flow f, we need to verify if f is a **valid max-flow** in linear time.

(a) Algorithm:

First, we check whether f is valid. That is,

- we check if flow f violates edge capacities, i.e., $0 \leq f_e \leq c_e$ for all $e \in E$. W
- we check if flow f is conserved, i.e., for any node $u \in V \backslash \{s, t\}$,

  $\sum_{(w,u)\in E} f_{wu} = \sum_{(u,z)\in E} f_{uz}$.

Next, if f is valid (otherwise we return false), we check if f is a maximum flow. To verify that f is of maximum size, we construct the residual graph G^f. We then run Explore from s on G^f to check if there is a path from s to t. If t is reachable from s then there is an augmenting path and hence f is not of maximum size. On the other hand if t is not reachable from s in G^f then we know that f is of maximum size.

(b) Correctness:

We know that f is a maximum flow if and only if there is no augmenting path from s to t in the residual graph.

(c) Run time Analysis:

Validating the flow takes $O(n+m)$ to check edge capacities and if flow is conserved. A single round of Explore on the flow network is also $O(n + m)$, thus the algorithm runs in linear time.