Biot MMMFE.

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1 Introduction

$\mathbb{X}$

2 Formulation of the methods

2.1 Mathematical formulation of model Problem.

Let $\Omega \subset \mathbb{R}^d$, $d = 2,3$ be a simply connected domain occupied by a linearly elastic porous body. We use the notation $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{N}$ for the spaces of $d \times d$ matrices, symmetric matrices and skew-symmetric matrices respectively, all over the field of real numbers, respectively. Throughout the paper the divergence operator is the usual divergence for vector fields, which produces vector field when applied to matrix field by taking the divergence of each row. We will also use the curl operator which is the usual curl when applied to vector fields in three dimension, and defined as

$$\operatorname{curl} \phi = (\partial_2 \phi, -\partial_1 \phi)$$

for a scalar function ϕ in two dimension. Similarly, for a vector field in two dimension or a matrix field in three dimension, curl operator produces a matrix field by acting row-wise. For the rest of this paper, C will denote a generic positive constant that is independent of the discretization parameter h.

We will make use of the following standard notations. For a set $G \subset \mathbb{R}^d$, the $L^2(G)$ inner product and norm are denoted by $(\cdot,\cdot)_G$ and $\|\cdot\|_G$ respectively, for scalar, vector and tensor valued functions. For a section of a subdomain boundary S we write $\langle\cdot,\cdot\rangle_S$ and $\|\cdot\|_S$ for the $L^2(S)$ inner product (or duality pairing) and norm, respectively. We omit subscript G if $G = \Omega$.

The material properties are described at each point $x \in \Omega$ by a compliance tensor $A = A(x)$, which is a symmetric, bounded and uniformly positive definite linear operator acting from $\mathbb{S} \to \mathbb{S}$. Therefore there exist constants $0 < \alpha_0 \le \alpha_1 < \infty$ such that

$$\alpha_0 \|\sigma\|^2 \le (A\sigma, \sigma) \le \alpha_1 \|\sigma\|^2. \tag{2.1}$$

We also assume that an extension of A to an operator $\mathbb{M} \to \mathbb{M}$ still possesses the above properties. In the special case of homogeneous and isotropic body operator A is given by,

$$A\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu + d\lambda}\operatorname{tr}(\sigma) I\right), \tag{2.2}$$

where I is a $d \times d$ identity matrix and $\mu > 0, \lambda \ge 0$ are Lame coefficients. Conventionally K stands for the permeability tensor, c_0 for mass storativity and α represents the Biot-Willis constant.

In this case

Given a vector field f on Ω representing body forces, the quasi-static Biot system determines the displacement u, together with the Darcy velocity z and pressure p:

$$-\operatorname{div}\sigma(u) = f, \qquad \text{in } \Omega, \tag{2.3}$$

$$K^{-1}z + \nabla p = 0, \qquad \text{in } \Omega, \tag{2.4}$$

$$\frac{\partial}{\partial t}(c_0 p + \alpha \operatorname{div} u) + \operatorname{div} z = q, \qquad \text{in } \Omega, \tag{2.5}$$

where the poroelastic stress $\sigma(u)$ is such that:

$$\sigma(u) = \sigma_E(u) - \alpha p I,$$

where $\sigma_E(u) = 2\mu\epsilon(u) + \lambda \operatorname{div} u\, I$ is the elastic stress and $\epsilon(u) := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$.

To close the system, the appropriate boundary conditions should also be prescribed

$$u = g_u \quad \text{on } \Gamma_D^{displ}, \qquad \sigma n = 0 \quad \text{on } \Gamma_N^{stress}, \tag{2.6}$$

$$p = g_p \quad \text{on } \Gamma_D^{pres}, \qquad z \cdot n = 0 \quad \text{on } \Gamma_N^{vel}, \tag{2.7}$$

where $\Gamma_D^{displ} \cup \Gamma_N^{stress} = \Gamma_D^{pres} \cup \Gamma_N^{vel} = \partial\Omega$ are boundaries on which Dirichlet and Neumann data is specified for displacement, pressure and normal fluxes, respectively. For simplicity, we assume non-zero measure for Γ_D^{displ} and Γ_D^{pres}.

Then the variational formulation for (2.3) – (2.7) reads: find $(\sigma, u, \gamma, z, p) \in \mathbb{X} \times V \times \mathbb{W} \times Z \times W$ such that

$$\left(A\left(\sigma + \alpha p I\right), \tau\right)_\Omega + (u, \operatorname{div}\tau) + (\gamma, \tau) = \langle g_u, \tau\, n\rangle_{\Gamma_D^{displ}}, \qquad \forall \tau \in \mathbb{X}, \tag{2.8}$$

$$(\operatorname{div}\sigma, v) = -\left(f, v\right), \qquad \forall v \in V, \tag{2.9}$$

$$(\sigma, \xi) = 0, \qquad \forall \xi \in \mathbb{W}, \tag{2.10}$$

$$\left(K^{-1}z, q\right) - (p, \operatorname{div} q) = -\langle g_p, v \cdot n\rangle_{\Gamma_D^{pres}}, \qquad \forall q \in Z, \tag{2.11}$$

$$c_0\left(\frac{\partial p}{\partial t}, w\right) + \alpha\left(\frac{\partial}{\partial t}A\left(\sigma + \alpha p I\right), wI\right)_\Omega + (\operatorname{div} z, w) = (g, w), \qquad \forall w \in W, \tag{2.12}$$

where

$$\mathbb{X} = \left\{\tau \in H(\operatorname{div};\Omega,\mathbb{M}) : \tau\, n = 0 \text{ on } \Gamma_N^{stress}\right\}, \quad V = L^2(\Omega,\mathbb{R}^d), \quad \mathbb{W} = L^2(\Omega,\mathbb{N}),$$
$$Z = \left\{v \in H(\operatorname{div};\Omega,\mathbb{R}^d) : v \cdot n = 0 \text{ on } \Gamma_N^{vel}\right\}, \quad W = L^2(\Omega).$$

It was indeed shown in [13] that the coupled system (2.8)– (2.12) is well posed. As in the finite element literature, we proceed by converting the system into a finite dimensional problem which can be solve using a linear solver numerically.

In the finite dimensional setting using finite element spaces, the variational formulation corresponding to (2.8)– (2.12) reads as follows: find $(\sigma_h, u_h, \gamma_h, z_h, p_h) \in \mathbb{X}_h \times V_h \times \mathbb{W}_h \times Z_h \times W_h$ such that

$$\left(A\left(\sigma_h + \alpha p_h I\right), \tau\right)_\Omega + (u_h, \operatorname{div}\tau) + (\gamma_h, \tau) = \langle g_u, \tau\, n\rangle_{\Gamma_D^{displ}}, \qquad \forall \tau \in \mathbb{X}_h, \tag{2.13}$$

$$(\operatorname{div}\sigma_h, v) = -\left(f, v\right), \qquad \forall v \in V_h, \tag{2.14}$$

$$(\sigma_h, \xi) = 0, \qquad \forall \xi \in \mathbb{W}_h, \tag{2.15}$$

$$\left(K^{-1}z_h, q\right) - (p_h, \operatorname{div} q) = -\langle g_p, v \cdot n\rangle_{\Gamma_D^{pres}}, \qquad \forall q \in Z_h, \tag{2.16}$$

$$c_0\left(\frac{\partial p_h}{\partial t}, w\right) + \alpha\left(\frac{\partial}{\partial t}A\left(\sigma_h + \alpha p_h I\right), wI\right)_\Omega + (\operatorname{div} z_h, w) = (g, w), \qquad \forall w \in W_h, \tag{2.17}$$

where $\mathbb{X}_h \times V_h \times \mathbb{W}_h \times Z_h \times W_h \subset \mathbb{X} \times V \times \mathbb{W} \times Z \times W$ is a collection of FE spaces which are inf-sup stable.

For solving the mechanics part, $\mathbb{X}_h \times V_h \times \mathbb{W}_h$ could be one chosen from any of the stable triplets suitable for solving linear elasticity problem equations with weakly imposed symmetry. Examples of such triplets include Stenberg [15], Amara-Thomas [1], Arnold-Falk-Winther [4, 5, 6] and Cockburn-Gopalakrishnan-Guzman [8, 11] family of elements. In particular, as shown in (citation), these families satisfies the inf-sup condition,

for any $(u_h, \gamma_h) \in V_h \times \mathbb{W}_h$,

$$\|u_h\| + \|\gamma_h\| \leq C \sup_{0 \neq \tau \in \mathbb{X}_h} \frac{(u_h, \operatorname{div} \tau) + (\gamma_h, \tau)}{\|\tau\|_{H(\mathrm{div};\Omega,\mathbb{M})}}. \tag{2.18}$$

For the flow part, $Z_h \times W_h$ could be chosen from any of the stable pressure-velocity pair of MFE spaces like Raviart-Thomas or Brezzi-Douglas-Marini ($\mathcal{BDM}$) spaces . These families are presented in detail in [7] where its shown that they satisfy the following inf-sup condition,

for any $w_h \in W_h$,

$$||p_{h,i}^{*,n+1}(\lambda_h)||_{\Omega_i} \leq C \sup_{0 \neq q_h \in Z_h} \frac{(\nabla \cdot q_h, w_h)}{||q_h||_{H(\mathrm{div},\Omega)}}. \tag{2.19}$$

2.2 Formulation of monolithic MMMFE method using non-matching grids.

Let $\Omega = \cup_{i=1}^n \Omega_i$ be a union of non-overlapping shape regular polygonal subdomains. Let $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$, $\Gamma = \cup_{i,j=1}^n \Gamma_{i,j}$, and $\Gamma_i = \partial\Omega_i \cap \Gamma = \partial\Omega_i \setminus \partial\Omega$ denote the interior subdomain interfaces. The domain discretization technique we develop here is generalization of the techniques derived in [give reference to the other paper] to a general case where the sub-domains have non-matching grids with different scales of refinement. We make use of relatively coarser mortar finite elements of our choice satisfying certain conditions at the interface of the sub-domains. Let h_i be the diameter of maximal element in the mesh on Ω_i and $h = \max_i h_i$. Talk about the computationally cheap and non-matching grid advantages this method has over the previous paper in the introduction.

For $1 \leq i \leq n$, let $\mathbb{X}_{h,i} \times V_{h,i} \times \mathbb{W}_{h,i} \times Z_{h,i} \times W_{h,i}$ be a collection of MFE families defined on the subdomain Ω_i. We assume that $\mathbb{X}_{h,i}, V_{h,i}, \mathbb{W}_{h,i}, Z_{h,i}$ and W contain polynomials of degree less than or equal to $k \geq 1, l \geq 0, q \geq 0, r \geq 1$ and $s \geq 0$ respectively. Let us define global spaces as follows:

$$\mathbb{X}_h = \bigoplus_{1 \leq i \leq n} \mathbb{X}_{h,i}, \quad V_h = \bigoplus_{1 \leq i \leq n} V_{h,i}, \quad \mathbb{W}_{h,i} = \bigoplus_{1 \leq i \leq n} \mathbb{W}_{h,i} \quad Z_h = \bigoplus_{1 \leq i \leq n} Z_{h,i}, \quad W_h = \bigoplus_{1 \leq i \leq n} W_{h,i}.$$

Note that the definition of the global spaces $\mathbb{X}_h$ and Z_h do not impose continuity of normal components of stress or velocity across the sub-domain interfaces. This issue is taken care of using Lagrange multipliers defined on compatible mortar spaces on the interface. Let $\mathcal{T}_{h,i,j}$ be a shape regular quasi-uniform finite element partition of $\Gamma_{i,j}$ constructed using a simplical or quadrilateral mesh in dimension $d-1$ with maximal element diameter H. Define the global mortar fine element space on the union of sub-domain interfaces,Γ to be,

$$\Lambda_H = \bigoplus_{1 \leq i \leq n} \begin{pmatrix} \Lambda_{H,i,j}^u \\ \Lambda_{H,i,j}^p \end{pmatrix},$$

where $\Lambda_{H,i,j}^u \subset \left(L^2(\Gamma_{i,j})\right)^d$ and $\Lambda_{H,i,j}^p \subset L^2(\Gamma_{i,j})$ are mortar finite element spaces on $\Gamma_{i,j}$ representing the displacement and pressure Lagrange multipliers respectively. Aforementioned mortar spaces contain either continuous or discontinuous polynomials of degree degree up to $m \geq 0$. Conditions on the compatibility of the degrees and richness of the mortar spaces to have a well-posed and stable method will be discussed in the later sections.

The MFE problem in the mortar DD setting give rise to a Dirichlet type sub-domain problem as follows: for $1 \leq i \leq n$, find $(\sigma_{h,i}, u_{h,i}, \gamma_{h,i}, z_{h,i}, p_{h,i}, \lambda_H) \in \mathbb{X}_{h,i} \times V_{h,i} \times \mathbb{W}_{h,i} \times Z_{h,i} \times W_{h,i} \times \Lambda_H$ such that:

$$
\begin{aligned}
&\left(A\left(\sigma_{h,i}+\alpha p_{h,i} I\right), \tau\right)_{\Omega_i}+\left(u_{h,i}, \operatorname{div} \tau\right)_{\Omega_i}+\left(\gamma_{h,i}, \tau\right)_{\Omega_i} \\
&\qquad\qquad = \left\langle g_u, \tau\, n\right\rangle_{\partial\Omega\cap\Gamma_D^{displ}} + \langle \lambda_h^u, \tau\, n\rangle_{\Gamma_i}, && \forall \tau \in \mathbb{X}_{h,i}, && (2.20)\\
&\left(\operatorname{div} \sigma_{h,i}, v\right)_{\Omega_i} = -\left(f, v\right)_{\Omega_i}, && \forall v \in V_{h,i}, && (2.21)\\
&\left(\sigma_{h,i}, \xi\right)_{\Omega_i} = 0, && \forall \xi \in \mathbb{W}_{h,i}, && (2.22)\\
&\left(K^{-1} z_{h,i}, q\right)_{\Omega_i} - \left(p_{h,i}, \operatorname{div} q\right)_{\Omega_i} = -\langle g_p, v\cdot n\rangle_{\partial\Omega\cap\Gamma_D^{pres}} - \langle \lambda_h^p, q\cdot n\rangle_{\Gamma_i}, && \forall q \in Z_{h,i}, && (2.23)\\
&c_0\left(\frac{\partial p_{h,i}}{\partial t}, w\right)_{\Omega_i} + \alpha\left(\frac{\partial}{\partial t} A\left(\sigma_{h,i}+\alpha p_{h,i} I\right), wI\right)_{\Omega_i} + \left(\operatorname{div} z_{h,i}, w\right)_{\Omega_i} \\
&\qquad\qquad = \left(g, w\right)_{\Omega_i}, && \forall w \in W_{h,i}, && (2.24)\\
&\sum_{i=1}^{n} \langle \sigma_{h,i}\, n_i, \mu^u\rangle_{\Gamma_i} = 0, && \forall \mu^u \in \Lambda_H^u, && (2.25)\\
&\sum_{i=1}^{n} \langle z_{h,i}\cdot n_i, \mu^p\rangle_{\Gamma_i} = 0, && \forall \mu^p \in \Lambda_H^p, && (2.26)
\end{aligned}
$$

where n_i is the outward unit normal vector field on Ω_i. Note that equations (2.25–2.26) enforces a notion of weak continuity of normal components of stress tensor and velocity vector across the sub-domain interfaces Γ.

For each subdomain Ω_i,we define a projection operators $\mathcal{Q}_{h,i}^u : \left(L^2(\Gamma_i)\right)^d \to \mathbb{X}_{h,i} n_i|_{\Gamma_i}$ and $\mathcal{Q}_{h,i}^p : L^2(\Gamma_i) \to V_{h,i}\cdot n_i|_{\Gamma_i}$ such that for any $(\phi_u, \phi_p) \in \left(L^2(\Gamma_i)\right)^d \times L^2(\Gamma_i)$,

$$
\begin{aligned}
&\left\langle \phi_u - \mathcal{Q}_{h,i}^u \phi_u, \tau n\right\rangle = 0, && \forall \tau \in \mathbb{X}_{h,i},\\
&\left\langle \phi_p - \mathcal{Q}_{h,i}^p \phi_p, v\cdot n\right\rangle = 0 && \forall v \in V_{h,i}.
\end{aligned}
$$

And finally we define $\mathcal{Q}_{h,i} : \left(L^2(\Gamma_i)\right)^d \times L^2(\Gamma_i) \to \mathbb{X}_{h,i} n_i|_{\Gamma_i} \times V_{h,i}\cdot n_i|_{\Gamma_i}$ as

$$
\mathcal{Q}_{h,i} = \begin{pmatrix} \mathcal{Q}_{h,i}^u \\ \mathcal{Q}_{h,i}^p \end{pmatrix}. \tag{2.27}
$$

.

2.3 Spaces of weakly continuous stress and velocity.

In this section, we introduce spaces of weakly continuous stress tensors and velocity vectors as follows:

$$
\mathbb{X}_{h,0} = \left\{ \tau \in \mathbb{X}_h : \sum_{i=1}^{n} \left(\tau n_i, \mu^u\right)_{\Gamma_i} = 0 \ \ \forall \mu^u \in \Lambda_H \right\}
$$

and

$$
Z_{h,0} = \left\{ q \in Z_h : \sum_{i=1}^{n} \left(q\cdot n_i, \mu^p\right)_{\Gamma_i} = 0 \ \ \forall \mu^p \in \Lambda_H \right\}.
$$

In order to analyze the system (2.20–2.26) using techniques developed for single domain system (2.13–2.17) in [give reference fo Ilona's and other relevant publications.], we assume that $\Gamma_D^{displ} = \Gamma_D^{pres} = \partial\Omega$ restate the DD subdomain problems using the weakly continuous spaces $\mathbb{X}_{h,0}$ and $Z_{h,0}$ as follows: find $(\sigma_h, u_h, \gamma_h, z_h, p_h) \in$

$\mathbb{X}_{h,0} \times V_h \times \mathbb{W}_h \times Z_{h,0} \times W_h$ such that

$$\left(A\left(\sigma_h + \alpha p_h I\right), \tau\right)_\Omega + (A, \tau)_\Omega + \sum_{i=1}^{n} (u_h, \operatorname{div} \tau)_{\Omega_i} + (\gamma_h, \tau)_\Omega = \langle g_u, \tau\, n\rangle_{\partial\Omega}, \qquad \forall \tau \in \mathbb{X}_{h,0}, \tag{2.28}$$

$$\sum_{i=1}^{n} (\operatorname{div} \sigma_h, v)_{\Omega_i} = -\left(f, v\right)_\Omega, \qquad \forall v \in V_h, \tag{2.29}$$

$$(\sigma_h, \xi)_\Omega = 0, \qquad \forall \xi \in \mathbb{W}_h, \tag{2.30}$$

$$\left(K^{-1} z_h, q\right)_\Omega - \sum_{i=1}^{n} (p_h, \operatorname{div} q)_{\Omega_i} = -\langle g_p, v \cdot n\rangle_{\partial\Omega}, \qquad \forall q \in Z_{h,0}, \tag{2.31}$$

$$c_0 \left(\frac{\partial p_h}{\partial t}, w\right)_\Omega + \alpha \left(\frac{\partial}{\partial t} A\left(\sigma_h + \alpha p_h I\right), wI\right)_\Omega + \sum_{i=1}^{n} \left(\operatorname{div} z_h, w\right)_{\Omega_i} = (g, w)_\Omega, \qquad \forall w \in W_h. \tag{2.32}$$

Note that we use the above formulation only for the sake of analysis and will present a reduction to interface problem approach to design the numerical algorithm.

3 Analysis of MMMFE for the monolithic system.

3.1 Well-posedness and stability analysis.

In this subsection, we show that the system (2.28)–(2.32) is well-posed and stable under the following assumption:

Assumption 1. *The mortar space Λ_H is chosen so that there exists a constant C independent of H and h such that the following inequality holds,*

$$\|\mu\|_{\Gamma_{i,j}} \le C\left(\|\mathcal{Q}_{h,i}\mu\|_{\Gamma_{i,j}} + \|\mathcal{Q}_{h,j}\mu\|_{\Gamma_{i,j}}\right), \quad \forall \mu \in \Lambda_H, \quad 1 \le i < j \le n. \tag{3.1}$$

Remark 1. Note that assumption (3.1) implies that the space Λ_H cannot be too rich compared to subdomain stress-velocity FE spaces(similar approach to [3]) and this in turn make sure that Λ_H^u and Λ_H^p are well controlled by their projection on to the normal traces of velocity and stress sub-domain spaces respectively. In practice, this condition can be easily obtained by taking a coarser mortar mesh satisfying $h < H \le 1$(see [3, 2, 14]).

Under the assumption (3.1), we present a inf-sup stability of the form (2.19) and (2.18) with respect to the newly defined weakly continuous velocity and stress tensor function as in sub-section 2.3.

Lemma 2. *Under assumption (3.1), there exists a linear operator $\Pi_0^E : H^{\frac{1}{2}+\epsilon}(\Omega, \mathbb{M}) \cap \mathbb{X} \to \mathbb{X}_{h,0}$ such that for any $\tau \in H^{\frac{1}{2}+\epsilon}(\Omega, \mathbb{M}) \cap \mathbb{X}$,*

$$\begin{aligned} &\sum_{i=1}^{n} \left(div\left(\Pi_0^E \tau - \tau\right), v\right)_{\Omega_i} = 0, && \forall v \in V_{h,i}, \\ &\left(\Pi_0^E \tau - \tau, \xi\right) = 0, && \forall \xi \in \mathbb{W}_h, \\ &\|\Pi_0^E \tau\| \le C\left(\|\tau\|_{\frac{1}{2}+\epsilon} + \|div \tau\|\right). \end{aligned}$$

Proof. The proof depends on the construction of a correction term to force the solutions to (2.20)–(2.24) to be in the weakly continuous spaces and also making use of (2.18) for each sub-domain Ω_i. Complete proof is given in [12, Lemma 3.4]. □

Lemma 3. *Under assumption (3.1), there exists a linear operator* $\Pi_0^D : H_0^1(\Omega)^n \to Z_{h,0}$ *such that for any* $\forall q \in Z_{h,0}$,

$$\left(div\left(\Pi_0^D q - q\right), w\right)_{\Omega_i} = 0, \qquad \forall w \in W_h,$$
$$\|\Pi_0^D q\|_{\mathbb{X}} \le C|q|_{H^1(\Omega)}.$$

Proof. This Lemma is proved similar to Lemma 2. Details can be found in the proof of [10, Lemma 3.6]. □

Lemma 2 and Lemma 3 along with a simple variant of Fortin's Lemma [9, 7] gives the following theorem (see [10, Theorem 3.1] for details) which gives inf-sup stability bounds with respect to the weakly continuous spaces of stress and velocity.

Theorem 4. *Under assumption (3.1), there exists positive constants* C_E *and* C_D *independent of the discretization parameters* h *and* H *such that*

for any $(u_h, \gamma_h) \in V_h \times \mathbb{W}_h$,

$$\|u_h\| + \|\gamma_h\| \le C_E \sup_{0 \ne \tau \in \mathbb{X}_{h,0}} \frac{\sum_{i=1}^n (u_h, \operatorname{div} \tau)_{\Omega_i} + (\gamma_h, \tau)_\Omega}{\|\tau\|_{H(\operatorname{div};\Omega,\mathbb{M})}}, \tag{3.2}$$

and for any $w_h \in W_h$,

$$||p_{h,i}^{*,n+1}(\lambda_h)||_{\Omega_i} \le C_D \sup_{0 \ne q_h \in Z_{h,0}} \frac{\sum_{i=1}^n (\nabla \cdot q_h, w_h)_{\Omega_i}}{||q_h||_{H(\operatorname{div},\Omega)}}. \tag{3.3}$$

Using stability analysis similar to that in give reference to theorem and section in Ilona's paper and inf-sup stability bounds (3.2) and (3.3) for the modified stress and velocity spaces, we obtain the following stability bound for monolithic MMMFE method.

Theorem 5. *Let* $(\sigma_h(t), u_h(t), \gamma_h(t), z_h(t), p_h(t)) \in \mathbb{X}_{h,0} \times V_h \times \mathbb{W}_h \times Z_{h,0} \times W_h$ *be the solution to the system of equations (2.28)–(2.32) for* $t \in [0, T]$, *then the following stability bound holds:*

$$\begin{aligned}
&\sum_{i=1}^n \|\sigma_h\|_{L^\infty(0,T;H(div,\Omega_i))} + \|u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\gamma_h\|_{L^\infty(0,T;L^2(\Omega))} + \|z_h\|_{L^\infty(0,T;L^2(\Omega))} + \|p_h\|_{L^\infty(0,T;L^2(\Omega))} \\
&+ \sum_{i=1}^n \|\sigma_h\|_{L^2(0,T;H(div,\Omega_i))} + \|u_h\|_{L^2(0,T;L^2(\Omega))} + \|\gamma_h\|_{L^2(0,T;L^2(\Omega))} + \|z_h\|_{L^2(0,T;L^2(\Omega))} + \|p_h\|_{L^2(0,T;L^2(\Omega))} \\
&\le C[\|\sigma_h(0)\| + \|u_h(0)\| + \|z_h(0)\| + \|p_h(0)\| + \|f\|_{L^\infty(0,T;L^2(\Omega))} + \|f\|_{H^1(0,T;L^2(\Omega))} + \|g\|_{H^1(0,T;L^2(\Omega))} \\
&+ \|g\|_{L^\infty(0,T;L^2(\Omega))} + \|g_u\|_{H^1(0,T;H^{\frac{1}{2}}(\partial\Omega))} + \|g_u\|_{L^\infty(0,T;H^{\frac{1}{2}}(\partial\Omega))} + \|g_p\|_{H^1(0,T;H^{\frac{1}{2}}(\partial\Omega))}].
\end{aligned}$$

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