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Robust Error Analysis of Coupled Mixed Methods for Biot's Consolidation Model

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Abstract We study the a priori error analysis of finite element methods for Biot's consolidation model. We consider a formulation which has the stress tensor, the fluid flux, the solid displacement, and the pore pressure as unknowns. Two mixed finite elements, one for linear elasticity and the other for mixed Poisson problems are coupled for spatial discretization, and we show that any pair of stable mixed finite elements is available. The novelty of our analysis is that the error estimates of all the unknowns are robust for material parameters. Specifically, the analysis does not need a uniformly positive storage coefficient, and the error estimates are robust for nearly incompressible materials. Numerical experiments illustrating our theoretical analysis are included.

Keywords Poroelasticity · Error analysis · Mixed finite elements

Mathematics Subject Classification 65N30 · 65N12

1 Introduction

Biot's consolidation model describes the deformation of an elastic porous medium and the viscous fluid flow inside it when the medium is saturated by the fluid [7]. This model has many applications in various engineering fields including geomechanics, petrolieum engineering, and biomedical engineering.

There are numerous studies of numerical schemes for Biot's consolidation model with finite element methods. In a series of papers [28–30], Murad et al. studied a formulation

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with the solid displacement and the pore pressure as unknowns using mixed finite elements for the Stokes equation. A discontinuous Galerkin method for the same formulation was also studied in [13]. A Galerkin least square method was proposed for a formulation with four unknowns, i.e., the solid displacement, a pseudo-stress tensor, the fluid flux, and the pore pressure [24]. A formulation with the solid displacement, the fluid flux, and the pore pressure as unknowns was studied with various couplings of continuous and discontinuous Galerkin methods, and mixed finite element methods [33–35]. A coupling of nonconforming and mixed finite element methods for the formulation was recently studied in [40]. For more information on previous studies we refer to [13,26,40] and the references therein.

In this paper we consider a formulation with four unknowns, i.e., the stress tensor, the solid displacement, the fluid flux, the pore pressure. This was considered recently in [41] using a combination of two mixed finite elements for the discretization of the problem, one for linear elasticity with symmetric stress tensors and the other for mixed Poisson problems. A numerical experiment in the paper shows that this approach can be advantageous to avoid non-physical pressure oscillations when the constrained storage coefficient of the problem vanishes. In this paper we also use a combination of two mixed finite elements for the discretization but we use mixed finite elements for linear elasticity with weakly symmetric stress because the elements with weakly symmetric stress can be advantageous with respect to efficient implementation and low computational costs. The main contribution of this paper is *a new error analysis providing the a priori error estimates that are robust for material parameters*. More specifically, we give error estimates of all the unknowns with the L^∞ norm in time and L^2 norm in space, and the estimates do not need strict positivity of the constrained storage coefficient s_0. Moreover, as in the Hellinger–Reissner formulation of linear elasticity [4], the error bounds are uniform for the parameter indicating incompressibility of the poroelastic medium, i.e., the error estimates are robust for nearly incompressible materials. To the best of our knowledge, an analytic proof of this robustness for incompressibility has not been addressed in literature.

The paper is organized as follows. In Sect. 2 we define notation and derive a variational formulation of the problem. In Sect. 3 we present finite element methods for the semidiscrete problem and show the a priori error analysis of semidiscrete solutions. We also prove robustness of the error estimate for nearly incompressible materials and show well-posedness of fully discrete solutions with the backward Euler discretization in time. In Sect. 4 numerical results illustrating our theoretical analysis are presented.

2 Biot Model and Variational Formulations

2.1 Notation

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with $n = 2$ or 3. Let $L^2(\Omega)$ be the set of square-integrable real-valued functions on Ω. The inner product of $L^2(\Omega)$ and the induced norm are denoted by $(\cdot, \cdot)$ and $\|\cdot\|_0$. For a finite-dimensional inner product space $\mathbb{X}$, let $L^2(\Omega; \mathbb{X})$ be the set of $\mathbb{X}$-valued functions such that each component of the functions is in $L^2(\Omega)$. The inner product of $L^2(\Omega; \mathbb{X})$ is naturally defined by the inner product of $\mathbb{X}$ and $L^2(\Omega)$, so we use the same notation $(\cdot, \cdot)$ and $\|\cdot\|_0$ to denote the inner product and norm on $L^2(\Omega; \mathbb{X})$. The inner product space $\mathbb{X}$ is the space of $\mathbb{R}^n$ vectors with standard inner product or a subspace of $n \times n$ matrices with the Frobenius inner product. For future reference we use $\mathbb{M}$, $\mathbb{S}$, $\mathbb{K}$ to denote the spaces of all, symmetric, skew-symmetric $n \times n$ matrices, respectively, and $\mathbb{V}$ to denote the space of column $\mathbb{R}^n$ vectors.

For a nonnegative integer m, $H^m(\Omega)$ denotes the standard Sobolev spaces of real-valued functions based on the L^2 norm, and $H^m(\Omega;\mathbb{X})$ is defined similarly based on $L^2(\Omega;\mathbb{X})$. For $m \geq 1$, we use $H_0^m(\Omega)$ to denote the subspace of $H^m(\Omega)$ with vanishing trace on $\partial\Omega$ [17], and $H_0^m(\Omega;\mathbb{X})$ is defined similarly. For simplicity we also use $\|\cdot\|_m$ to denote the H^m-norm for both of $H^m(\Omega)$ and $H^m(\Omega;\mathbb{X})$. We define $H(\mathrm{div}\,,\Omega)$ as

$$H(\mathrm{div}\,,\Omega) := \{v \in L^2(\Omega;\mathbb{R}^n) \ : \ \mathrm{div}\, v \in L^2(\Omega)\},$$

and the norm $(\|v\|_0^2 + \|\mathrm{div}\, v\|_0^2)^{1/2}$ is denoted by $\|v\|_{\mathrm{div}}$. We also define the space $H(\mathrm{div}\,,\Omega;\mathbb{M})$ and $H(\mathrm{div}\,,\Omega;\mathbb{S})$ as

$$\begin{aligned}
H(\mathrm{div}\,,\Omega;\mathbb{M}) &:= \{\tau \in L^2(\Omega;\mathbb{M}) \ : \ \mathrm{div}\,\tau \in L^2(\Omega;\mathbb{V})\},\\
H(\mathrm{div}\,,\Omega;\mathbb{S}) &:= H(\mathrm{div}\,,\Omega;\mathbb{M}) \cap L^2(\Omega;\mathbb{S}),
\end{aligned}$$

in which the divergence of τ is understood as the row-wise divergence of τ, and $\|\tau\|_{\mathrm{div}}$ is defined similarly as the norm of $H(\mathrm{div}\,)$.

Let $J = [0, T]$, $T > 0$ be an interval. For a reflexive Banach space $\mathcal{X}$, let $C^0(J;\mathcal{X})$ denote the set of functions $f : J \to \mathcal{X}$ which are continuous in $t \in J$. For an integer $m \geq 1$, we define

$$C^m(J;\mathcal{X}) = \{f \mid \partial^i f/\partial t^i \in C^0(J;\mathcal{X}),\ 0 \leq i \leq m\},$$

where $\partial^i f/\partial t^i$ is the ith time derivative in the sense of the Fréchet derivative in $\mathcal{X}$ (see e.g., [42]). For a function $f : J \to \mathcal{X}$, we define the space–time norm

$$\|f\|_{L^r(J;\mathcal{X})} = \begin{cases} \left(\int_J \|f\|_{\mathcal{X}}^r ds\right)^{1/r}, & 1 \leq r < \infty,\\ \operatorname{esssup}_{t\in J}\|f\|_{\mathcal{X}}, & r = \infty. \end{cases}$$

If the time interval is fixed, then we use $L^r\mathcal{X}$ instead of $L^r(J;\mathcal{X})$ for simplicity. We define the space–time Sobolev spaces $W^{k,r}(J;\mathcal{X})$ for a nonnegative integer k and $1 \leq r \leq \infty$ as the closure of $C^k(J;\mathcal{X})$ with the norm $\|f\|_{W^{k,r}\mathcal{X}} = \sum_{i=0}^k \|\partial^i f/\partial t^i\|_{L^r\mathcal{X}}$. We adopt a convention that $\|f, g\|_{\mathcal{X}} = \|f\|_{\mathcal{X}} + \|g\|_{\mathcal{X}}$ for the norm of a Banach space $\mathcal{X}$. For simplicity of notation, $\dot{f}$ will be used to denote $\partial f/\partial t$.

Finally, throughout this paper we use $X \lesssim Y$ to denote the inequality $X \leq cY$ with a generic constant $c > 0$ which is independent of mesh sizes. If needed, we will write c explicitly in inequalities but it can be different in each formula.

2.2 Biot's Consolidation Model

In this subsection we derive a formulation of Biot's consolidation model with four unknowns and establish a variational formulation of the problem.

Let Ω, a bounded Lipschitz domain in $\mathbb{R}^n$ with $n = 2$ or 3, be occupied by a fluid-saturated poroelastic body. Let $u : \Omega \to \mathbb{V}$ be the displacement of the poroelastic medium, $p : \Omega \to \mathbb{R}$ the pore pressure, $f : \Omega \to \mathbb{V}$ the body force, and $g : \Omega \to \mathbb{R}$ the source/sink density function of the fluid. The governing equations of Biot's consolidation model are

$$-\mathrm{div}\,\mathcal{C}\epsilon(u) + \alpha\nabla p = f \quad \text{in } \Omega, \tag{1}$$

$$s_0\dot{p} + \alpha\mathrm{div}\,\dot{u} - \mathrm{div}\,(\kappa\nabla p) = g \quad \text{in } \Omega, \tag{2}$$

where $\mathcal{C}$ is the elastic stiffness tensor, $\epsilon(u)$ is the linearized strain tensor, $s_0 \geq 0$ is the constrained specific storage coefficient, κ is the hydraulic conductivity tensor, and $\alpha > 0$ is the Biot–Willis constant which is close to 1. In order to understand the system (1)–(2) precisely, we need to explain the operators. First, by $\operatorname{grad} u$ we mean the $\mathbb{M}$-valued function such that each row is the gradient of each component of $u : \Omega \to \mathbb{V}$, and $\epsilon(u)$ is the symmetric matrix part of $\operatorname{grad} u$. There is no confusion of the divergence operator div for vector-valued functions but, when it is used for $\mathbb{M}$-valued functions, div will be (n-tuples of) the row-wise divergence of the $\mathbb{M}$-valued function which results in a $\mathbb{V}$-valued function. If q is a scalar function, then ∇q stands for the gradient of q as a column vector. With these conventions the equations in the system (1)–(2) are well-defined.

In general the elastic stiffness tensor $\mathcal{C}$ is a rank 4 tensor giving a symmetric positive definite linear map from $L^2(\Omega; \mathbb{S})$ into itself [21]. The coefficient $s_0 \geq 0$ is determined by material parameters such as the permeability (of the porous medium) and the bulk moduli of the solid and the fluid. The hydraulic conductivity tensor κ is defined by the permeability tensor of the solid divided by the fluid viscosity and it is positive definite. All the parameters $\mathcal{C}$, s_0, κ, and α are functions of $x \in \Omega$. For the derivation of these equations from physical modeling, we refer to standard porous media references, for example, [1,15].

To derive a formulation with four unknowns we introduce the fluid flux $z = -\kappa \nabla p$ and the stress tensor $\sigma = \mathcal{C}\epsilon(u) - \alpha p \mathbb{I}$ as new unknowns, where $\mathbb{I}$ is the $n \times n$ identity matrix. By the definitions of σ and z, we have

$$\mathcal{A}^s(\sigma + \alpha p \mathbb{I}) - \epsilon(u) = 0, \tag{3}$$

$$\kappa^{-1} z + \nabla p = 0, \tag{4}$$

where $\mathcal{A}^s = \mathcal{C}^{-1}$, and we can rewrite (1) as

$$-\operatorname{div} \sigma = f. \tag{5}$$

In addition, observing that $\operatorname{div} u = \operatorname{tr} \epsilon(u) = \operatorname{tr} \mathcal{A}^s(\sigma + \alpha p \mathbb{I})$ where tr is the trace of matrices, we can rewrite (2) as

$$s_0 \dot{p} + \alpha \operatorname{tr} \mathcal{A}^s(\dot{\sigma} + \alpha \dot{p} \mathbb{I}) + \operatorname{div} z = g. \tag{6}$$

As a consequence, we obtain a system with four unknowns σ, u, z, p, and four equations (3)–(6). In order to be a well-posed problem, the equations (3)–(6) need appropriate boundary and initial conditions. We assume that there are two partitions of $\partial\Omega$,

$$\partial\Omega = \Gamma_p \cup \Gamma_f, \quad \partial\Omega = \Gamma_d \cup \Gamma_t,$$

with $|\Gamma_p|, |\Gamma_d| > 0$, i.e., the Lebesgue measures of Γ_p and Γ_d are positive. Let $\boldsymbol{n}$ be the outward unit normal vector field on $\partial\Omega$. Boundary conditions are given in general by

$$p(t) = p_0(t) \quad \text{on } \Gamma_p, \qquad z(t) \cdot \boldsymbol{n} = z_{\boldsymbol{n}}(t) \quad \text{on } \Gamma_f, \tag{7}$$

$$u(t) = u_0(t) \quad \text{on } \Gamma_d, \qquad \sigma(t)\boldsymbol{n} = \sigma_{\boldsymbol{n}}(t) \quad \text{on } \Gamma_t, \tag{8}$$

for all $t \in J$ with given

$$p_0 : J \times \Gamma_p \to \mathbb{R}, \quad z_{\boldsymbol{n}} : J \times \Gamma_f \to \mathbb{R}, \quad u_0 : J \times \Gamma_d \to \mathbb{V}, \quad \sigma_{\boldsymbol{n}} : J \times \Gamma_t \to \mathbb{V}.$$

For well-posedness of (3)–(6) with the above boundary conditions, we can adopt the argument in [37] used for well-posedness of the system (1)–(2) with same boundary conditions. Namely, we find expressions of σ and z in terms of p using (3)–(5), and apply these expressions to (6) to obtain a parabolic partial differential equation of p. Since well-posedness of the system is not our main interest, we will not pursue it further in this paper.

For spatial discretization we use two mixed finite element methods, one for linear elasticity of the Hellinger–Reissner formulation and the other for mixed Poisson problems. For discretization of linear elasticity we will use mixed finite elements for elasticity with weakly symmetric stress. Compared to mixed finite elements with symmetric stress tensors, the elements with weakly symmetric stress can be preferable because they usually require less computational costs and can be implemented with the Fraejis de Veubeke hybridization, which results in a system with reduced sizes [2,16].

In order to use mixed finite elements for elasticity with weakly symmetric stress, we introduce the skew-symmetric part of grad u, denoted by γ, as another unknown. This new unknown plays a role of a Lagrange multiplier for the symmetry of the stress tensor. To formulate it, we define $\mathcal{A}$ as an extension of $\mathcal{A}^s$ such that $\mathcal{A} = \mathcal{A}^s$ on $L^2(\Omega; \mathbb{S})$ and $\mathcal{A}$ is a positive constant multiple of the identity map on $L^2(\Omega; \mathbb{K})$, where $\mathbb{K}$ is the space of $n \times n$ skew-symmetric matrices. Then, recalling that γ is the skew-symmetric part of grad u, (3) can be written as

$$\mathcal{A}(\sigma + \alpha p\mathbb{I}) - \operatorname{grad} u + \gamma = 0, \tag{9}$$

and we have the symmetry constraint of σ,

$$(\sigma, \eta) = 0 \quad \forall \eta \in L^2(\Omega; \mathbb{K}). \tag{10}$$

Let us define the function spaces

$$\begin{aligned} &\Sigma = H(\operatorname{div}, \Omega; \mathbb{M}), \quad V = L^2(\Omega; \mathbb{V}), \quad \Gamma = L^2(\Omega; \mathbb{K}), \\ &W = H(\operatorname{div}, \Omega), \quad Q = L^2(\Omega), \end{aligned} \tag{11}$$

for unknowns $(\sigma, u, \gamma, z, p)$. Then, by integration by parts with vanishing boundary conditions, we can derive the following variational formulation: Find

$$(\sigma, p) \in C^1(J; \Sigma \times Q) \quad \text{and} \quad (u, \gamma, z) \in C^0(J; V \times \Gamma \times W),$$

such that

$$(\mathcal{A}(\sigma + \alpha p\mathbb{I}), \tau) + (u, \operatorname{div} \tau) + (\gamma, \tau) = 0, \qquad \forall \tau \in \Sigma, \tag{12}$$

$$(\operatorname{div} \sigma, v) + (\sigma, \eta) = -(f, v), \qquad \forall (v, \eta) \in V \times \Gamma, \tag{13}$$

$$-(\kappa^{-1} z, w) + (p, \operatorname{div} w) = 0, \qquad \forall w \in W, \tag{14}$$

$$(s_0 \dot{p}, q) + (\mathcal{A}(\dot{\sigma} + \alpha \dot{p}\mathbb{I}), \alpha q\mathbb{I}) + (\operatorname{div} z, q) = (g, q), \qquad \forall q \in Q. \tag{15}$$

In (15) we used $(\operatorname{tr} \mathcal{A}^s \xi, q) = (\operatorname{tr} \mathcal{A}\xi, q) = (\mathcal{A}\xi, q\mathbb{I})$ for a matrix ξ and a scalar q.

This variational formulation can be easily extended to problems with general boundary conditions. Suppose that boundary conditions are given as

$$p(t) = p_0(t) \quad \text{on } \Gamma_p, \qquad z(t) \cdot \boldsymbol{n} = z_{\boldsymbol{n}}(t) \quad \text{on } \Gamma_f, \tag{16}$$

$$u(t) = u_0(t) \quad \text{on } \Gamma_d, \qquad \sigma(t)\boldsymbol{n} = \sigma_{\boldsymbol{n}}(t) \quad \text{on } \Gamma_t, \tag{17}$$

with

$$p_0 : J \times \Gamma_p \to \mathbb{R}, \quad z_{\boldsymbol{n}} : J \times \Gamma_f \to \mathbb{R}, \quad u_0 : J \times \Gamma_d, \quad \sigma_{\boldsymbol{n}} : J \times \Gamma_t \to \mathbb{R}^n.$$

We define Σ_∂ and W_∂ as

$$\Sigma_\partial = \{\tau \in \Sigma \ : \ \tau\boldsymbol{n} = 0 \text{ on } \Gamma_t\}, \quad W_\partial = \{w \in W \ : \ w \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_f\}. \tag{18}$$

The boundary conditions $\sigma_{\boldsymbol{n}}$ and $z_{\boldsymbol{n}}$ are imposed as essential boundary conditions and we can obtain a variational formulation with function spaces Σ_∂ and W_∂ [9]. As a consequence, we have a variational formulation

$$(\mathcal{A}(\sigma + \alpha p \mathbb{I}), \tau) + (u, \operatorname{div} \tau) + (\gamma, \tau) = \langle u_0, \tau \boldsymbol{n} \rangle_{\Gamma_d}, \qquad \forall \tau \in \Sigma_\partial, \tag{19}$$

$$(\operatorname{div} \sigma, v) + (\sigma, \eta) = -(f, v), \qquad \forall (v, \eta) \in V \times \Gamma, \tag{20}$$

$$-(\kappa^{-1} z, w) + (p, \operatorname{div} w) = \langle p_0, w \cdot \boldsymbol{n} \rangle_{\Gamma_p}, \quad \forall w \in W_\partial, \tag{21}$$

$$(s_0 \dot{p}, q) + (\mathcal{A}(\dot{\sigma} + \alpha \dot{p} \mathbb{I}), \alpha q \mathbb{I}) + (\operatorname{div} z, q) = (g, q), \qquad \forall q \in Q, \tag{22}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_d}$ and $\langle \cdot, \cdot \rangle_{\Gamma_p}$ are the integrals of L^2 inner products on the indicated parts of boundary with the $(n-1)$-dimensional Lebesgue measure.

3 A Priori Error Analysis

In this section we consider semidiscretization of (12)–(15) and prove a priori error estimates. The discrete counterpart of (12)–(15) is to seek

$$(\sigma_h, p_h) \in C^1(J; \Sigma_h \times Q_h) \quad \text{and} \quad (u_h, \gamma_h, z_h) \in C^0(J; V_h \times \Gamma_h \times W_h),$$

such that

$$(\mathcal{A}(\sigma_h + \alpha p_h \mathbb{I}), \tau) + (u_h, \operatorname{div} \tau) + (\gamma_h, \tau) = 0 \qquad \forall \tau \in \Sigma_h, \tag{23}$$

$$(\operatorname{div} \sigma_h, v) + (\sigma_h, \eta) = -(f, v) \quad \forall (v, \eta) \in V_h \times \Gamma_h, \tag{24}$$

$$-(\kappa^{-1} z_h, w) + (p_h, \operatorname{div} w) = 0 \qquad \forall w \in W_h, \tag{25}$$

$$(s_0 \dot{p}_h, q) + (\mathcal{A}(\dot{\sigma}_h + \alpha \dot{p}_h \mathbb{I}), \alpha q \mathbb{I}) + (\operatorname{div} z_h, q) = (g, q) \qquad \forall q \in Q_h, \tag{26}$$

for appropriate finite element spaces $\Sigma_h, V_h, W_h, Q_h, \Gamma_h$ on a shape-regular mesh of Ω. Note that this is a differential algebraic equation because the time derivative is involved only in the last equation, so existence and uniqueness of its solutions are not straightforward. However, we do not discuss existence and uniqueness of solutions of this semidiscrete problem here because they can be established with standard techniques in the theory of differential algebraic equations [23]. Instead, we focus on illustrating details of the a priori error analysis of semidiscrete solutions. At the end of this section we will discuss existence and uniqueness of fully discrete solutions for the problem, which is sufficient for practical computation.

In the rest of this paper we assume that $(\Sigma_h, V_h, \Gamma_h)$ is a stable mixed method for linear elasticity with weakly symmetric stress, and (W_h, Q_h) is a stable mixed method for the mixed Poisson problem. Before we describe assumptions **(S1)–(S4)** including the stability conditions of finite elements, we restrict our interest on specific combinations of $(\Sigma_h, V_h, \Gamma_h)$ and (W_h, Q_h) in order to have balanced convergence rates of unknowns. To state the conditions rigorously, let $\mathcal{O}(\Xi_h)$ be the best order of approximation of Ξ_h in the L^2-norm for a function space $\Xi \subset L^2(\Omega; \mathbb{X})$ and a finite element space $\Xi_h \subset \Xi$. For example, if $\Xi = L^2(\Omega)$ and Ξ_h is the space of piecewise discontinuous polynomials of degree $\leq r$ on a shape-regular mesh, then $\mathcal{O}(\Xi_h) = r + 1$. In this paper we always assume that $(\Sigma_h, V_h, \Gamma_h, W_h, Q_h)$ satisfies

$$\min\{\mathcal{O}(\Sigma_h), \mathcal{O}(\Gamma_h)\} = \mathcal{O}(V_h) = \mathcal{O}(W_h) = \mathcal{O}(Q_h) = r, \quad r \geq 1, \tag{27}$$

and

$$\mathcal{O}(V_h) = r', \quad (r' \geq 1) \quad r' = r \text{ or } r' = r - 1.$$

Here are the assumptions on finite elements:

(S1) $\operatorname{div}\Sigma_h = V_h$ and $\operatorname{div} W_h = Q_h$.

(S2) There exists bounded maps $\Pi_h^{\Sigma} : H^1(\Omega; \mathbb{M}) \to \Sigma_h$, $\Pi_h^W : H^1(\Omega; \mathbb{V}) \to Q_h$ such that

$$\operatorname{div}\Pi_h^{\Sigma}\tau = P_h^V \operatorname{div}\tau, \quad \operatorname{div}\Pi_h^W w = P_h^Q \operatorname{div} w,$$
$$\|\tau - \Pi_h^{\Sigma}\tau\|_0 \lesssim h^m\|\tau\|_m, \quad \|w - \Pi_h^W w\|_0 \lesssim h^m\|w\|_m, \quad m \geq 1,$$

where P_h^V and P_h^Q are the L^2 projections into V_h and Q_h, respectively.

(S3) For any $q \in Q_h$, there exists $w \in Z_h$ satisfying

$$\operatorname{div} w = q, \quad \|w\|_{\operatorname{div}} \lesssim \|q\|_0.$$

(S4) For any $(v, \eta) \in V_h \times \Gamma_h$, there exists $\tau \in \Sigma_h$ satisfying

$$\operatorname{div}\tau = v, \quad (\tau, \eta') = (\eta, \eta') \quad \forall \eta' \in \Gamma_h, \quad \|\tau\|_{\operatorname{div}} \lesssim \|v\|_0 + \|\eta\|_0.$$

There are a number of mixed finite elements satisfying these assumptions. For the mixed Poisson problems, all of the families of Raviart–Thomas, Brezzi–Douglas–Marini, Nédélec elements [11,31,32,36] fulfill these assumptions. However, we will only consider the Raviart–Thomas elements (2D) and the Nédélec 1st kind $H(\operatorname{div})$ (3D) elements for (W_h, Q_h) due to (27). For the mixed form of linear elasticity, there are many known elements satisfying these assumptions [3,5,14,20,38,39]. We refer to [10] and [25] for surveys of the elements for mixed Poisson and mixed elasticity problems, respectively.

For the a priori error analysis of the problem, we need interpolation operators into finite element spaces. As previously defined P_h^V and P_h^Q, we define $P_h^{\Gamma} : \Gamma \to \Gamma_h$ as the L^2 projection into Γ_h. For W_h let I_h^W be the canonical Raviart–Thomas–Nédélec interpolation. Then it is well-known that

$$\operatorname{div} I_h^W w = P_h^Q \operatorname{div} w, \tag{28}$$
$$\|w - I_h^W w\|_0 \lesssim h^m\|w\|_m, \quad 1 \leq m \leq r \quad \text{for } w \in H^m(\Omega; \mathbb{V}) \tag{29}$$

hold. For I_h^{Σ} we use the weakly symmetric elliptic projection introduced in [6]. To be self-contained, we describe its definition here. To define $I_h^{\Sigma} : \Sigma \to \Sigma_h$ we consider a problem seeking $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{\gamma}_h) \in \Sigma_h \times V_h \times \Gamma_h$ such that

$$(\tilde{\sigma}_h, \tau) + (\tilde{u}_h, \operatorname{div}\tau) + (\tilde{\gamma}_h, \tau) = (\sigma, \tau), \qquad \forall \tau \in \Sigma_h, \tag{30}$$
$$(\operatorname{div}\tilde{\sigma}_h, v) = (\operatorname{div}\sigma, v), \qquad \forall v \in V_h, \tag{31}$$
$$(\tilde{\sigma}_h, \eta) = (\sigma, \eta), \qquad \forall \eta \in \Gamma_h, \tag{32}$$

for $\sigma \in \Sigma$. This is the discrete counterpart of a mixed form of linear elasticity problem with weakly symmetric stress. Note that the continuous version of this linear elasticity problem has $(\sigma, 0, 0)$ as its solution. If we define $I_h^{\Sigma}\sigma$ by $\tilde{\sigma}_h$, then I_h^{Σ} is a bounded linear map from Σ to Σ_h. By (31), the fact $\operatorname{div}\Sigma_h = V_h$, and (32), we obtain

$$\operatorname{div} I_h^{\Sigma}\sigma = P_h^V \operatorname{div}\sigma \quad \text{and} \quad (I_h^{\Sigma}\sigma, \eta) = (\sigma, \eta) \quad \forall \eta \in \Gamma_h. \tag{33}$$

An improved error estimate of mixed elasticity problems yields [6]

$$\|I_h^{\Sigma}\tau - \tau\|_0 \lesssim h^m\|\tau\|_m, \quad 1 \leq m \leq r \quad \text{if } \tau \in H^m(\Omega; \mathbb{M}). \tag{34}$$

3.1 Error Analysis

Let $(\sigma, u, \gamma, z, p)$ and $(\sigma_h, u_h, \gamma_h, z_h, p_h)$ be solutions of (12)–(15) and (23)–(26), respectively. We define the difference of these two solutions as error terms, i.e., $e_\sigma = \sigma - \sigma_h$, and e_u, e_γ, e_z, e_p are defined similarly. For the error analysis we decompose these error terms into two components using interpolation operators. More precisely, we set

$$\begin{aligned} e_\sigma &= e_\sigma^I + e_\sigma^h := (\sigma - I_h^\Sigma \sigma) + (I_h^\Sigma \sigma - \sigma_h), \\ e_u &= e_u^I + e_u^h := (u - P_h^V u) + (P_h^V u - u_h), \\ e_\gamma &= e_\gamma^I + e_\gamma^h := (\gamma - P_h^\Gamma \gamma) + (P_h^\Gamma \gamma - \gamma_h), \\ e_z &= e_z^I + e_z^h := (z - I_h^W z) + (I_h^W z - z_h), \\ e_p &= e_p^I + e_p^h := (p - P_h^Q p) + (P_h^Q p - p_h). \end{aligned}$$

By well-known approximation properties of the interpolation operators P_h^V, P_h^Q, P_h^Γ, I_h^W and by (34), we have

$$\|e_\sigma^I\|_{L^\infty L^2} \lesssim h^m \|\sigma\|_{L^\infty H^m}, \quad 1 \le m \le r, \tag{35}$$

$$\|e_u^I\|_{L^\infty L^2} \lesssim h^m \|u\|_{L^\infty H^m}, \quad 1 \le m \le r', \tag{36}$$

$$\|e_z^I\|_{L^\infty L^2} \lesssim h^m \|z\|_{L^\infty H^m}, \quad 1 \le m \le r, \tag{37}$$

$$\|e_p^I\|_{L^\infty L^2} \lesssim h^m \|p\|_{L^\infty H^m}, \quad 1 \le m \le r, \tag{38}$$

$$\|e_\gamma^I\|_{L^\infty L^2} \lesssim h^m \|\gamma\|_{L^\infty H^m}, \quad 1 \le m \le r. \tag{39}$$

The following lemma will be useful in the error analysis.

Lemma 1 *Let $F, G, X : J \to \mathbb{R}$ be continuous, nonnegative functions. Suppose that $X(t)$ is continuously differentiable and satisfies*

$$X(t)^2 \le X(0)^2 + \int_0^t (F(s)X(s) + G(s))\, ds. \tag{40}$$

for all $t \in J$. Then for $t \in J$,

$$X(t) \le X(0) + \max\left\{ 2\int_0^t F(s)\, ds, \left(2\int_0^t G(s)\, ds\right)^{\frac{1}{2}} \right\}. \tag{41}$$

Proof To reduce the problem, let us define a statement:

(M) (41) holds when $X(t)$ is the maximum on the interval $[0, t]$

We first claim that, if **(M)** is true, then (41) holds for any $t \in J$. To see this, assume that **(M)** is true. For given $t_0 \in J$, if X attains its maximum on $[0, t_0]$ at t_0, then there is nothing to prove, so suppose that X attains its maximum value on the inverval $[0, t_0]$ at $\bar{t} \in [0, t_0]$, $\bar{t} < t_0$. Due to **(M)** and the nonnegativity of F and G, we have

$$\begin{aligned} X(t_0) <& X(\bar{t}) \le X(0) + \max\left\{ 2\int_0^{\bar{t}} F(s)\, ds, \left(2\int_0^{\bar{t}} G(s)\, ds\right)^{\frac{1}{2}} \right\} \\ \le& X(0) + \max\left\{ 2\int_0^{t_0} F(s)\, ds, \left(2\int_0^{t_0} G(s)\, ds\right)^{\frac{1}{2}} \right\}, \end{aligned}$$

so (41) holds for t_0 as well. Since $t_0 \in J$ is arbitrary, (41) for all $t \in J$.

The above argument implies that it is enough to prove **(M)**. Thus, we now show that (41) holds under the assumption that $X(t)$ is the maximum on the interval $[0, t]$. It is obvious that one of the following inequalities holds:

$$\int_0^t G(s)\,ds \le \int_0^t F(s)X(s)\,ds, \quad \int_0^t F(s)X(s)\,ds < \int_0^t G(s)\,ds. \tag{42}$$

Suppose that the first inequality in (42) holds. Then (40) gives

$$X(t)^2 \le X(0)^2 + 2\int_0^t F(s)X(s)\,ds.$$

If we divide both sides by $X(t)$, then we get

$$X(t) \le X(0) + 2\int_0^t F(s)\,ds,$$

because $X(t)$ is the maximum on $[0, t]$. Thus one case of (41) is proved.

To complete the proof, assume that the second inequality in (42) is true. Then (40) gives

$$X(t)^2 \le X(0)^2 + 2\int_0^t G(s)\,ds.$$

By taking square roots of both sides, and by the triangle inequality, we have

$$X(t) \le X(0) + \left(2\int_0^t G(s)\,ds\right)^{\frac{1}{2}},$$

so the other case of (41) is proved. □

We will use $\|\tau\|_{\mathcal{A}}$, $\|w\|_{\kappa^{-1}}$, $\|q\|_{s_0}$ to denote the quantities

$$(\mathcal{A}\tau, \tau)^{1/2}, \quad (\kappa^{-1}w, w)^{1/2}, \quad (s_0 q, q)^{1/2},$$

respectively. We also use $L^2_{\mathcal{A}}$ to denote the L^2 space with the norm $\|\cdot\|_{\mathcal{A}}$.

Here is the main result of this section.

Theorem 1 *Let $(\sigma, u, \gamma, z, p)$ be an exact solution of (12)–(15) and assume that numerical initial data $\sigma_h(0)$, $p_h(0)$, $z_h(0)$ are given to satisfy*

$$\|\sigma(0) - \sigma_h(0), p(0) - p_h(0), z(0) - z_h(0)\|_0 \lesssim h^m, \quad 1 \le m \le r. \tag{43}$$

If $(\sigma_h, u_h, \gamma_h, z_h, p_h)$ is the solution of (23)–(26) with the numerical initial data, then, for $1 \le m \le r$, the following estimates hold:

$$\|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{L^\infty L^2_{\mathcal{A}}} \lesssim h^m \max\{\|\sigma, p, \gamma\|_{W^{1,1}H^m}, \|z\|_{L^2 H^m}\}, \tag{44}$$

$$\|e_\sigma^h\|_{L^\infty L^2} \lesssim h^m (\|\sigma, p, \gamma\|_{W^{1,2}H^m} + \|z\|_{W^{1,1}H^m}), \tag{45}$$

$$\|e_u^h, e_\gamma^h\|_{L^\infty L^2} \lesssim h^m \max\{\|\sigma, p, \gamma\|_{W^{1,1}H^m}, \|z\|_{L^2 H^m}\}, \tag{46}$$

$$\|e_z^h\|_{L^\infty L^2} \lesssim h^m (\|\sigma, p, \gamma\|_{W^{1,2}H^m} + \|z\|_{W^{1,1}H^m}), \tag{47}$$

$$\|e_p^h\|_{L^\infty L^2} \lesssim h^m (\|\sigma, p, \gamma\|_{W^{1,2}H^m} + \|z\|_{W^{1,1}H^m}). \tag{48}$$

Note that, for sufficiently regular solutions, $\|e_u^h\|_{L^\infty L^2} \lesssim h^r$ holds even though $r' = \mathcal{O}(V_h)$ is less than r.

Proof We begin the error analysis by taking differences of the variational form and the semidiscrete equations, which results in

$$(\mathcal{A}(e_\sigma + \alpha e_p \mathbb{I}), \tau) + (e_u, \operatorname{div} \tau) + (e_\gamma, \tau) = 0, \quad \forall \tau \in \Sigma_h, \tag{49}$$

$$(\operatorname{div} e_\sigma, v) + (e_\sigma, \eta) = 0, \quad \forall (v, \eta) \in V_h \times \Gamma_h, \tag{50}$$

$$(\kappa^{-1} e_z, w) + (e_p, \operatorname{div} w) = 0, \quad \forall w \in W_h, \tag{51}$$

$$(s_0 \dot{e}_p, q) + (\mathcal{A}(\dot{e}_\sigma + \alpha \dot{e}_p \mathbb{I}), \alpha q \mathbb{I}) - (\operatorname{div} e_z, q) = 0, \quad \forall q \in Q_h. \tag{52}$$

Note that **(S1)** and the definitions of P_h^Q, P_h^V yield

$$(e_p^I, \operatorname{div} w) = 0 \quad \forall w \in W_h \quad \text{and} \quad (e_u^I, \operatorname{div} \tau) = 0 \quad \forall \tau \in \Sigma_h.$$

In addition, the commuting diagram property of I_h^W in **(S2)** and the properties of I_h^Σ in (33) yield

$$(\operatorname{div} e_z^I, q) = 0 \quad \forall q \in Q_h \quad \text{and} \quad (\operatorname{div} e_\sigma^I, v) = (e_\sigma^I, \eta) = 0 \quad \forall (v, \eta) \in V_h \times \Gamma_h.$$

Considering these cancellations and using the error decompositions with e^I- and e^h-terms, the system (49)–(52) can be rewritten as

$$(\mathcal{A}(e_\sigma^h + \alpha e_p^h \mathbb{I}), \tau) + (e_u^h, \operatorname{div} \tau) + (e_\gamma^h, \tau) = -(\mathcal{A}(e_\sigma^I + \alpha e_p^I \mathbb{I}), \tau) - (e_\gamma^I, \tau), \tag{53}$$

$$(\operatorname{div} e_\sigma^h, v) + (e_\sigma^h, \eta) = 0, \tag{54}$$

$$(\kappa^{-1} e_z^h, w) + (e_p^h, \operatorname{div} w) = -(\kappa^{-1} e_z^I, w), \tag{55}$$

$$\begin{aligned}(s_0 \dot{e}_p^h, q) + (\mathcal{A}(\dot{e}_\sigma^h + \alpha \dot{e}_p^h \mathbb{I}), \alpha q \mathbb{I}) - (\operatorname{div} e_z^h, q) &= -(s_0 \dot{e}_p^I, q) \\ &\quad - (\mathcal{A}(\dot{e}_\sigma^I + \alpha \dot{e}_p^I \mathbb{I}), \alpha q \mathbb{I}),\end{aligned} \tag{56}$$

for any $(\tau, v, \eta, w, q) \in (\Sigma_h, V_h, \Gamma_h, W_h, Q_h)$.

We remark that (43) implies that $\|e_\sigma^h(0)\|_0 \lesssim h^m$ because

$$\begin{aligned}\|e_\sigma^h(0)\|_0 &= \|I_h^\Sigma \sigma(0) - \sigma_h(0)\|_0 \\ &\le \|I_h^\Sigma \sigma(0) - \sigma(0)\|_0 + \|\sigma(0) - \sigma_h(0)\|_0 \\ &\lesssim h^m \|\sigma(0)\|_m.\end{aligned}$$

We obtain $\|e_p^h(0), e_z^h(0)\|_0 \lesssim h^m$ with similar arguments.

Estimate of $\|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{\mathcal{A}}$: We first show

$$\|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{L^\infty L^2_{\mathcal{A}}} \lesssim h^m (\|\sigma(0), p(0)\|_m + \max\{\|\sigma, p, \gamma\|_{L^2 H^m}, \|z\|_{L^2 H^m}\}. \tag{57}$$

For its proof, taking the time derivative of (54), choosing $\tau = e_\sigma^h$, $v = \dot{e}_u^h$, $w = e_z^h$, $q = e_p^h$, $\eta = -\dot{e}_\gamma^h$, and adding all the equations (53)–(56), we have

$$\begin{aligned}&\frac{1}{2} \frac{d}{dt} (\|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{\mathcal{A}}^2 + \|e_p^h\|_{s_0}^2) + \|e_z^h\|_{\kappa^{-1}}^2 \\ &\quad = -(\kappa^{-1} e_z^I, e_z^h) - (\mathcal{A}(\dot{e}_\sigma^I + \alpha \dot{e}_p^I \mathbb{I}), e_\sigma^h + \alpha e_p^h \mathbb{I}) - (s_0 \dot{e}_p^I, e_p^h) - (\dot{e}_\gamma^I, e_\sigma^h).\end{aligned}$$

The Cauchy–Schwarz and arithmetic-geometric mean inequalities give

$$\begin{aligned}&\frac{1}{2} \frac{d}{dt} (\|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{\mathcal{A}}^2 + \|e_p^h\|_{s_0}^2) + \frac{1}{2} \|e_z^h\|_{\kappa^{-1}}^2 \\ &\quad \le \frac{1}{2} \|e_z^I\|_{\kappa^{-1}}^2 + \|\dot{e}_\sigma^I + \alpha \dot{e}_p^I \mathbb{I} + \dot{e}_\gamma^I\|_{\mathcal{A}} \|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{\mathcal{A}} + \|\dot{e}_p^I\|_{s_0} \|e_p^h\|_{s_0}.\end{aligned}$$

In this inequality we assumed that $(\mathcal{A}\tau, \dot{e}_\gamma^I) = (\tau, \dot{e}_\gamma^I)$ holds for simplicity but this equality holds with a positive constant in general, which is used in the extension of $\mathcal{A}^s$ to $\mathcal{A}$ for skew-symmetric tensors. Ignoring the nonnegative term $\|e_z^h\|_{\kappa^{-1}}^2/2$ in the above and applying Lemma 1 with

$$\begin{aligned} X &= \left(\|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{\mathcal{A}}\right)^2 + \|e_p^h\|_{s_0}^2)^{1/2}, \\ F &= \left(\|\dot{e}_\sigma^I + \alpha \dot{e}_p^I \mathbb{I} + \dot{e}_\gamma^I\|_{\mathcal{A}}^2 + \|\dot{e}_p^I\|_{s_0}^2\right)^{\frac{1}{2}}, \\ G &= \frac{1}{2}\|e_z^I\|_{\kappa^{-1}}^2, \end{aligned}$$

we can obtain

$$\begin{aligned} \|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{L^\infty L^2_{\mathcal{A}}} &\lesssim \|e_\sigma^h(0) + e_p^h(0)\mathbb{I}\|_0 + \max\{\|\dot{e}_\sigma^I, \dot{e}_p^I, \dot{e}_\gamma^I\|_{L^1L^2}, \|e_z^I\|_{L^2L^2}\} \\ &\lesssim h^m(\|\sigma(0), p(0)\|_m + \max\{\|\sigma, p, \gamma\|_{W^{1,1}H^m}, \|z\|_{L^2H^m}\}) \\ &\lesssim h^m \max\{\|\sigma, p, \gamma\|_{W^{1,1}H^m}, \|z\|_{L^2H^m}\}, \end{aligned}$$

where the last inequality is due to the Sobolev embedding $W^{1,1}H^m \subset L^\infty H^m$.

Estimates of $\|e_u^h, e_\gamma^h\|_{L^\infty L^2}$: By the inf-sup condition **(S4)** there exists $\tau \in \Sigma_h$ such that $\operatorname{div}\tau = e_u^h$, $(\tau, \eta) = (e_\gamma^h, \eta)$ for all $\eta \in \Gamma_h$, and $\|\tau\|_{\operatorname{div}}^2 \lesssim \|e_u^h\|_0^2 + \|e_\gamma^h\|_0^2$. Taking this τ in (53) and applying the triangle and Cauchy–Schwarz inequalities, we get

$$\begin{aligned} \|e_u^h\|_0^2 + \|e_\gamma^h\|_0^2 &= -(\mathcal{A}(e_\sigma + \alpha e_p \mathbb{I} + e_\gamma^I), \tau) \\ &\le (\|e_\sigma^I + \alpha e_p^I \mathbb{I} + e_\gamma^I\|_{\mathcal{A}} + \|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{\mathcal{A}})\|\tau\|_{\mathcal{A}} \\ &\lesssim (h^m\|\sigma, p, \gamma\|_{L^\infty H^m} + \|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{\mathcal{A}})(\|e_u^h\|_0^2 + \|e_\gamma^h\|_0^2)^{\frac{1}{2}}, \end{aligned}$$

where we used $\|\tau\|_{\mathcal{A}} \lesssim \|\tau\|_{\operatorname{div}} \lesssim \left(\|e_u^h\|_0^2 + \|e_\gamma^h\|_0^2\right)^{1/2}$ in the last inequality. Since $\|\sigma, p, \gamma\|_{L^\infty H^m} \lesssim \|\sigma, p, \gamma\|_{W^{1,1}H^m}$ by the Sobolev embedding, the above inequality and the estimate of $\|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{\mathcal{A}}$ yield

$$\|e_u^h, e_\gamma^h\|_{L^\infty L^2} \lesssim h^m \max\{\|\sigma, p, \gamma\|_{W^{1,1}H^m}, \|z\|_{L^2H^m}\}.$$

Estimate of $\|e_z^h\|_{L^\infty L^2}$: Taking time derivatives of (53)–(55), choosing $\tau = \dot{e}_\sigma^h$, $v = -\dot{e}_u^h$, $w = e_z^h$, $q = \dot{e}_p^h$, $\eta = -\dot{e}_\gamma^h$, and adding all the equations, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\kappa^{-1}e_z^h, e_z^h) + \|\dot{e}_\sigma^h + \alpha \dot{e}_p^h \mathbb{I}\|_{\mathcal{A}}^2 + \|\dot{e}_p^h\|_{s_0}^2 \\ &\quad = -(\kappa^{-1}\dot{e}_z^I, e_z^h) - (\mathcal{A}(\dot{e}_\sigma^I + \alpha \dot{e}_p^I \mathbb{I} + \dot{e}_\gamma^I), \dot{e}_\sigma^h + \alpha \dot{e}_p^h) - (s_0\dot{e}_p^I, \dot{e}_p^h). \end{aligned}$$

By the Cauchy–Schwarz and Young's inequalities we have

$$\frac{1}{2}\frac{d}{dt}\|e_z^h\|_{\kappa^{-1}}^2 \le \|\dot{e}_z^I\|_{\kappa^{-1}}\|e_z^h\|_{\kappa^{-1}} + \|\dot{e}_\sigma^I + \alpha \dot{e}_p^I \mathbb{I} + \dot{e}_\gamma^I\|_{\mathcal{A}}^2 + \|\dot{e}_p^I\|_{s_0}^2.$$

If we apply Lemma 1 to this inequality with

$$X = \|e_z^h\|_{\kappa^{-1}}, \quad F = \|\dot{e}_z^I\|_{\kappa^{-1}}, \quad G = \|\dot{e}_\sigma^I + \alpha \dot{e}_p^I \mathbb{I} + \dot{e}_\gamma^I\|_{\mathcal{A}}^2 + \|\dot{e}_p^I\|_{s_0}^2,$$

then the coercivity of κ^{-1}, boundedness of $\mathcal{A}$, and the Sobolev embedding will lead to

$$\begin{aligned} \|e_z^h\|_{L^\infty L^2} &\lesssim \|e_z^h(0)\|_0 + \|\dot{e}_z^I\|_{L^1L^2} + \|\dot{e}_\sigma^I, \dot{e}_p^I, \dot{e}_\gamma^I\|_{L^2L^2} \\ &\lesssim h^m(\|\sigma, p, \gamma\|_{W^{1,2}H^m} + \|z\|_{W^{1,1}H^m}). \end{aligned}$$

Estimate of $\|e_p^h\|_{L^\infty L^2}$: By the inf-sup condition in **(S3)** for (W_h, Q_h), there exists $w \in W_h$ such that $\operatorname{div} w = e_p^h$ and $\|w\|_{\operatorname{div}} \lesssim \|e_p^h\|_0$. Choosing this w in (55), we get

$$\|e_p^h\|_0^2 = (e_z^I + e_z^h, w)_{\kappa^{-1}} \lesssim (\|e_z^I\|_0 + \|e_z^h\|_0)\|e_p^h\|_0,$$

so the estimate of e_z^h and (37) give

$$\|e_p^h\|_{L^\infty L^2} \lesssim h^m(\|\sigma, p, \gamma\|_{W^{1,2}H^m} + \|z\|_{W^{1,1}H^m}).$$

Estimate of $\|e_\sigma^h\|_{L^\infty L^2}$: Combining the estimates of $\|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{\mathcal{A}}$ and $\|e_p^h\|_{L^\infty L^2}$, the triangle inequality, and coercivity of $\mathcal{A}$, we have

$$\|e_\sigma^h\|_{L^\infty L^2} \lesssim h^m(\|\sigma, p, \gamma\|_{W^{1,2}H^m} + \|z\|_{W^{1,1}H^m}).$$

The proof is completed. □

Remark 2 The quantity $(\sigma + \alpha\mathbb{I})$ has a physical meaning as the elastic stress tensor in the saturated poroelastic medium. The estimate of $\|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{L^\infty L^2_{\mathcal{A}}}$ plays a key role that any combinations of two mixed methods $(\Sigma_h, V_h, \Gamma_h)$ and (W_h, Q_h) are available because this estimate holds without any requirements on finite elements. In this estimate, an estimate of $\|e_p^h\|_{L^\infty L^2}$ can be obtained when s_0 is uniformly positive, and an estimate of $\|e_z^h\|_{L^2 L^2}$ can be also obtained.

Remark 3 The argument in the estimate of $\|e_z^h\|_{L^\infty L^2}$ can be also used for other formulations that the fluid flux and the pore pressure are present as unknowns [34,40].

If we combine (35)–(39) and the result of Theorem 1, then we have the following results.

Corollary 4 *Suppose that $(\sigma, u, \gamma, z, p)$ and $(\sigma_h, u_h, \gamma_h, z_h, p_h)$ are as in Theorem 1 with same assumptions. Then we have*

$$\begin{aligned}
\|e_\sigma + \alpha e_p \mathbb{I}\|_{L^\infty L^2_{\mathcal{A}}} &\lesssim h^m \max\{\|\sigma, p, \gamma\|_{W^{1,1}H^m}, \|z\|_{L^2 H^m}\}, \quad 1 \le m \le r,\\
\|e_\sigma\|_{L^\infty L^2} &\lesssim h^m(\|\sigma, p, \gamma\|_{W^{1,2}H^m} + \|z\|_{W^{1,1}H^m}), \quad 1 \le m \le r,\\
\|e_u\|_{L^\infty L^2} &\lesssim h^m \max\{\|\sigma, p, \gamma\|_{W^{1,1}H^m}, \|z\|_{L^2 H^m}\}, \quad 1 \le m \le r',\\
\|e_\gamma\|_{L^\infty L^2} &\lesssim h^m \max\{\|\sigma, p, \gamma\|_{W^{1,1}H^m}, \|z\|_{L^2 H^m}\}, \quad 1 \le m \le r,\\
\|e_z\|_{L^\infty L^2} &\lesssim h^m(\|\sigma, p, \gamma\|_{W^{1,2}H^m} + \|z\|_{W^{1,1}H^m}), \quad 1 \le m \le r,\\
\|e_p\|_{L^\infty L^2} &\lesssim h^m(\|\sigma, p, \gamma\|_{W^{1,2}H^m} + \|z\|_{W^{1,1}H^m}), \quad 1 \le m \le r.
\end{aligned}$$

3.2 Robustness of Error Estimates for Nearly Incompressible Materials

For isotropic elastic porous media, the elasticity tensor $\mathcal{C}$ has the form

$$\mathcal{C}\tau = 2\mu\tau + \lambda(\operatorname{tr}\tau)\mathbb{I}, \quad \tau \in L^2(\Omega; \mathbb{S}),$$

where the constants $\mu, \lambda > 0$ are Lamé coefficients, and

$$\mathcal{C}^{-1} = \mathcal{A}^s\tau = \frac{1}{2\mu}\left(\tau - \frac{\lambda}{2\mu + n\lambda}(\operatorname{tr}\tau)\mathbb{I}\right). \tag{58}$$

Throughout this subsection, we assume that $\mathcal{A}^s$ has this form and $\mathcal{A}$ is the extension of $\mathcal{A}^s$ to $L^2(\Omega; \mathbb{M})$ as before. One can see that the coercivity of $\mathcal{A}$ on $L^2(\Omega; \mathbb{M})$ is not uniform in λ. In other words,

$$c_\lambda\|\tau\|_0^2 \le \|\tau\|_{\mathcal{A}}^2, \quad \forall\tau \in L^2(\Omega; \mathbb{M})$$

holds with a constant $c_\lambda > 0$ but $c_\lambda \to 0$ as $\lambda \to +\infty$. It means that error estimates obtained using coercivity of $\mathcal{A}$ may have error bounds growing unboundedly as $\lambda \to +\infty$. The purpose of this subsection is to show that the error bounds in the previous subsection are uniform for arbitrarily large λ.

In the proof of Theorem 1 we can see that many error estimates rely on the estimate of $\|e_\sigma^h + \alpha e_p^h \mathbb{I}\|_{\mathcal{A}}$ and boundedness of the bilinear form $\mathcal{A}$. It is an important observation that estimates utilizing the $\mathcal{A}$-weighted norm and boundedness of $\mathcal{A}$ in (58) are uniform as $\lambda \to +\infty$. Thus, the only error estimate requiring coercivity of $\mathcal{A}$ is the error estimate of $\|e_\sigma^h\|_{L^\infty L^2}$.

Before we prove the main result, we need preliminary results. For a tensor τ we use τ^D to denote the deviatoric part of τ, i.e.,

$$\tau^D := \tau - \frac{1}{n}(\operatorname{tr} \tau)\mathbb{I}.$$

We define $H^1_\Gamma(\Omega)$ as

$$H^1_\Gamma(\Omega) = \{\phi \in H^1(\Omega; \mathbb{V}) \, : \, \phi|_{\Gamma_d} = 0\}, \tag{59}$$

where $\phi|_{\Gamma_d}$ is the trace of ϕ on $\Gamma_d \subset \partial\Omega$. Recall the definitions of Σ in (11) and Σ_∂ in (18) depending on boundary conditions $\Gamma_d = \partial\Omega$ and $\Gamma_d \neq \partial\Omega$.

Lemma 2 *Suppose that*

$$\tau \in \Sigma, \quad \int_\Omega \operatorname{tr} \tau \, dx = 0, \quad or \quad \tau \in \Sigma_\partial. \tag{60}$$

Then

$$\|\tau\|_0 \lesssim \|\tau^D\|_0 + \sup_{\phi \in H^1_\Gamma(\Omega)} \frac{(\operatorname{div} \tau, \phi)}{\|\phi\|_1}. \tag{61}$$

Proof Since $(\tau^D, (\operatorname{tr}\tau)\mathbb{I}) = 0$, $\|\tau\|_0 \lesssim \|\tau^D\|_0 + \|\operatorname{tr}\tau\|_0$ holds and it suffices to show that $\|\operatorname{tr}\tau\|_0$ is bounded by the right-hand side of (61). Due to the assumption (60), there exists $\phi \in H^1_\Gamma(\Omega)$ such that $\operatorname{div}^*\phi = \operatorname{tr}\tau$ and $\|\phi\|_1 \lesssim \|\operatorname{tr}\tau\|_0$, where div^* means the vertical divergence (cf. [19]). Then an algebraic identity and the integration by parts give

$$\begin{aligned} \|\operatorname{tr}\tau\|_0^2 &= (\operatorname{tr}\tau, \operatorname{div}^*\phi) = n(\tau, \operatorname{grad}\phi) - n(\tau^D, \operatorname{grad}\phi) \\ &= -n(\operatorname{div}\tau, \phi) - n(\tau^D, \operatorname{grad}\phi). \end{aligned}$$

We get the desired result from the Cauchy–Schwarz inequality, $\|\phi\|_1 \lesssim \|\operatorname{tr}\tau\|_0$, and dividing both sides by $\|\phi\|_1$. □

The proof of Lemma 2 is completely analogous to the proof of Lemma 3.1 in [4], with a simple modification for general boundary conditions. A similar result was proved in [12] with a Helmholtz decomposition.

Theorem 5 *Suppose that $(\sigma, u, \gamma, z, p)$ and $(\sigma_h, u_h, \gamma_h, z_h, p_h)$ are exact and discrete solutions of (12)–(15) and (23)–(26) with the assumptions as in Theorem 1. We also assume that $\mathcal{A}$ is the extension of (58), and α is piecewise constant on each mesh element. Then*

$$\|\sigma - \sigma_h\|_0 \le ch^m, \quad 1 \le m \le r,$$

holds with a constant c, which is uniformly bounded for arbitrarily large λ. Moreover, the same result holds for a solution of (19)–(22) (with $\Gamma_d \neq \partial\Omega$) and its discrete counterpart.

Proof Since I_h^Σ is independent of λ, $\|e_\sigma^I\|_0 \lesssim h^m$ holds with a constant independent of λ. By the triangle inequality, it is enough to have an estimate $\|e_\sigma^h\|_0$ with an upper bound uniformly bounded for arbitrarily large λ.

We first consider the case with $\Gamma_d = \partial\Omega$. Since Σ_h does not have any essential boundary condition on a part of $\partial\Omega$, we can take $\tau = \mathbb{I} \in \Sigma_h$ as a test function in error analysis. By taking $\tau = \mathbb{I}$ in (30), we also have

$$\int_\Omega \operatorname{tr}(\sigma - I_h^\Sigma\sigma)\,dx = \int_\Omega \operatorname{tr} e_\sigma^I\,dx = 0. \tag{62}$$

Taking $\tau = \mathbb{I}$ in (49), we have

$$(\mathcal{A}(e_\sigma + \alpha e_p\mathbb{I}), \mathbb{I}) = \frac{1}{2\mu + n\lambda}\int_\Omega (\operatorname{tr} e_\sigma + \alpha n e_p)dx = 0. \tag{63}$$

Let $|\Omega|$ be the n-dimensional Lebesgue measure of Ω and define

$$\beta := \frac{1}{n|\Omega|}\int_\Omega \operatorname{tr} e_\sigma\,dx.$$

Taking $\tau = \mathbb{I}$ in (49), we have

$$n|\Omega|\beta = \int_\Omega \operatorname{tr} e_\sigma\,dx = \int_\Omega \operatorname{tr} e_\sigma^h\,dx = -n\int_\Omega \alpha e_p\,dx = -n\int_\Omega \alpha e_p^h\,dx, \tag{64}$$

where the second, third, and fourth equalities are due to (62), (63), and the facts that α is piecewise constant and e_p^I is mean-value zero on each mesh element. Letting $\tilde{e}_\sigma^h = e_\sigma^h - \beta\mathbb{I}$ we have

$$\int_\Omega \operatorname{tr} \tilde{e}_\sigma^h\,dx = 0. \tag{65}$$

The triangle inequality gives

$$\|e_\sigma^h\|_0 \le \|\tilde{e}_\sigma^h\|_0 + |\beta|\|\mathbb{I}\|_0 = \|\tilde{e}_\sigma^h\|_0 + \sqrt{n}|\beta||\Omega|^{\frac{1}{2}}.$$

By (64) and Hölder inequality, we have

$$\sqrt{n}|\beta||\Omega|^{\frac{1}{2}} = \sqrt{n}|\Omega|^{-\frac{1}{2}}\left|\int_\Omega \alpha e_p^h dx\right| \lesssim \|e_p^h\|_0.$$

Recall that the estimate of $\|e_p^h\|_0$ is uniform in λ, so the above two inequalities imply that, to have a λ-uniform estimate of $\|e_\sigma^h\|_0$, we only need a λ-uniform estimate of $\|\tilde{e}_\sigma^h\|_0$. If we use (65) and the fact $\operatorname{div} e_\sigma^h = \operatorname{div} \tilde{e}_\sigma^h = 0$ by definition of $\tilde{e}_\sigma^h$, then (61) gives $\|\tilde{e}_\sigma^h\|_0 \lesssim \|(\tilde{e}_\sigma^h)^D\|_0$. In addition, it is easy to see $\|(\tilde{e}_\sigma^h)^D\|_0 \lesssim \|\tilde{e}_\sigma^h\|_\mathcal{A}$ from the definition of $\mathcal{A}$ (cf. [4, Lemma 3.2]), so it is sufficient to estimate $\|\tilde{e}_\sigma^h\|_\mathcal{A}$. A direct computation using the form of $\mathcal{A}$ in (58) gives

$$\begin{aligned}
\|\tilde{e}_\sigma^h\|_\mathcal{A}^2 &= \left(\mathcal{A}(e_\sigma^h - \beta\mathbb{I}), e_\sigma^h - \beta\mathbb{I}\right) \\
&= (\mathcal{A}e_\sigma^h, e_\sigma^h) - 2\beta(\mathcal{A}\mathbb{I}, e_\sigma^h) + \beta^2(\mathcal{A}\mathbb{I}, \mathbb{I}) \\
&= \|e_\sigma^h\|_\mathcal{A}^2 - \frac{2\beta}{2\mu + n\lambda}\int_\Omega \operatorname{tr} e_\sigma^h\,dx + \frac{n\beta^2}{2\mu + n\lambda}|\Omega| \quad \text{(by (58))} \\
&= \|e_\sigma^h\|_\mathcal{A}^2 - \frac{n\beta^2}{2\mu + n\lambda}|\Omega| \quad \text{(by (64))} \\
&\le \|e_\sigma^h\|_\mathcal{A}^2.
\end{aligned}$$

As we already have an estimate of $\|e_\sigma^h\|_{\mathcal{A}}$ which is uniform in λ, we obtain an estimate of $\|\tilde{e}_\sigma^h\|_0$ uniform in λ as well.

If $\Gamma_d \neq \partial\Omega$, then we can apply (61) directly to e_σ and have

$$\|e_\sigma\|_0 \lesssim \|(e_\sigma)^D\|_0 + \sup_{\phi\in H_\Gamma^1(\Omega)} \frac{(\operatorname{div} e_\sigma, \phi)}{\|\phi\|_1}. \tag{66}$$

Then it is enough to show that the first and second terms in the above are bounded by $\|e_\sigma^I\|_0$ and $\|e_\sigma^h\|_{\mathcal{A}}$. The first term $\|(e_\sigma)^D\|_0$ is easily estimated by

$$\|(e_\sigma)^D\|_0 \le \|(e_\sigma^I)^D\|_0 + \|(e_\sigma^h)^D\|_0 \lesssim \|e_\sigma^I\|_0 + \|e_\sigma^h\|_{\mathcal{A}}.$$

To estimate the second term, note that (50) with $\eta = 0$ implies that $\operatorname{div} e_\sigma = \operatorname{div}\sigma - P_h^V \operatorname{div}\sigma$, so

$$(\operatorname{div} e_\sigma, \phi) = (\operatorname{div}(\sigma - I_h^\Sigma\sigma), \phi) = -(\sigma - I_h^\Sigma\sigma, \operatorname{grad}\phi) = -(e_\sigma^I, \operatorname{grad}\phi).$$

Thus, the second term in (66) is bounded by $\|e_\sigma^I\|_0$. □

3.3 Superconvergence and a Local Post-Processing

We show that superconvergence of the displacement error is available for certain choices of $(\Sigma_h, V_h, \Gamma_h, W_h, Q_h)$. We also show that a new numerical displacement with higher order accuracy can be obtained from a local postprocessing of the superconvergent numerical displacement.

Suppose that $(\Sigma_h, V_h, \Gamma_h, W_h, Q_h)$ are elements such that

$$\mathcal{O}(\Sigma_h) = \mathcal{O}(\Gamma_h) = \mathcal{O}(W_h) = \mathcal{O}(Q_h) = r, \quad \mathcal{O}(V_h) = r - 1$$

with $r \ge 2$. Then we have obtained

$$\|e_\sigma^h, e_u^h, e_z^h, e_p^h, e_\gamma^h\|_0 \lesssim h^r \tag{67}$$

in Theorem 1 assuming that exact solutions are sufficiently regular. In particular, (67) implies that the convergence rate of $\|e_u^h\|_0$ is higher than the best approximation order of V_h. This allows us to use a local post-processing to obtain u_h^*, a numerical solution of u with better accuracy.

We describe the local post-processing here. Let V_h^* be the space of polynomials with 1 degree higher than V_h, $V_{h,0} \subset V_h^*$ be the space of piecewise constants, and $V_{h,0}^\perp$ be the L^2 orthogonal complement of $V_{h,0}$ in V_h^*. The post-processed solution u_h^* is defined as the solution of the system

$$(\operatorname{grad}_h u_h^*(t), \operatorname{grad}_h v) = (\mathcal{A}(\sigma_h(t) + \alpha p_h(t)\mathbb{I}) + \gamma_h(t), \operatorname{grad}_h v), \tag{68}$$

$$(u_h^*, v') = (u_h(t), v'), \tag{69}$$

for all $v \in V_{h,0}^\perp$ and $v' \in V_{h,0}$, where $\operatorname{grad}_h v$ is the element-wise gradient of v, which is an $n \times n$ matrix-valued function. It is easy to see that u_h^* is well-defined by checking that (68)–(69) is a nonsingular linear system. The error analysis of this post-processing is almost the same as the one in [6] but we present a detailed proof to be self-contained.

Theorem 6 *Suppose that the assumptions of Theorem 1 hold and $\|u\|_{W^{1,1}H^r}$ is finite. Assume also that $(\sigma_h, u_h, \gamma_h, z_h, p_h)$ is a semidiscrete solution of (23)–(26) and $r' = r - 1$. If we define $u_h^*(t) \in V_h^*$ by (68)–(69), then $u_h^*(t)$ satisfies*

$$\|u(t) - u_h^*(t)\|_0 \lesssim h^r. \tag{70}$$

Proof For simplicity we will omit the time variable t in the proof. Let P_h^0, P_h^*, $P_h^\perp$ be the L^2 projections into $V_{h,0}$, V_h^*, $V_{h,0}^\perp$, respectively, and $P_h^* = P_h^0 + P_h^\perp$ by definition. To prove (70), it suffices to show

$$\| P_h^* u - u_h^* \|_0 \lesssim h^r \tag{71}$$

because $\|u - P_h^* u\|_0 \lesssim h^r$ holds by the Bramble–Hilbert lemma. To estimate $\| P_h^* u - u_h^* \|_0$, consider the orthogonal decomposition

$$P_h^* u - u_h^* = P_h^0 (u - u_h^*) + P_h^\perp (u - u_h^*) \in V_{h,0} \oplus V_{h,0}^\perp.$$

From the facts $P_h^0 P_h^V u = P_h^0 u$ and $P_h^0 u_h^* = P_h^0 u_h$ (by the definition of u_h^*), we have $P_h^0 (u - u_h^*) = P_h^0 (P_h^V u - u_h)$. Recall that $\| P_h^V u - u_h \|_0 \lesssim h^r$ is already obtained in Theorem 1, so we only need to show $\| P_h^\perp (u - u_h^*) \|_0 \lesssim h^r$ in order to prove (71). Since

$$\operatorname{grad}_h u = \operatorname{grad} u = \mathcal{A}(\sigma + \alpha p \mathbb{I}) + \gamma,$$

we have

$$(\operatorname{grad}_h u, \operatorname{grad}_h w) = (\mathcal{A}(\sigma + \alpha p \mathbb{I}) + \gamma, \operatorname{grad}_h w)$$

for $w \in V_{h,0}^\perp$. By subtracting the equation (68) from this equation, we get, for $w \in V_{h,0}^\perp$,

$$(\operatorname{grad}_h (u - u_h^*), \operatorname{grad}_h w) = (\mathcal{A}(e_\sigma + \alpha e_p \mathbb{I}) + e_\gamma, \operatorname{grad}_h w). \tag{72}$$

Using the equality

$$u - u_h^* = (u - P_h^* u) + (P_h^* u - u_h^*) = (u - P_h^* u) + P_h^\perp (u - u_h^*) + P_h^0 (u - u_h^*)$$

to replace $u - u_h^*$ in (72), a direct computation gives

$$\begin{aligned} &(\operatorname{grad}_h (P_h^\perp (u - u_h^*)), \operatorname{grad}_h w) \\ &\quad = -(\operatorname{grad}_h (u - P_h^* u) - (\mathcal{A}(e_\sigma + \alpha e_p \mathbb{I}) + e_\gamma, \operatorname{grad}_h w) \end{aligned}$$

because $\operatorname{grad}_h P_h^0 (u - u_h^*) = 0$. Taking $w = P_h^\perp (u - u_h^*)$ in this equation and multiplying h, we have

$$h \| \operatorname{grad}_h P_h^\perp (u - u_h^*) \|_0 \lesssim h (\| \operatorname{grad}_h (u - P_h^* u) \|_0 + \| \mathcal{A}(e_\sigma + \alpha e_p \mathbb{I}) + e_\gamma \|_0). \tag{73}$$

Using the estimate $\| P_h^\perp (u - u_h) \|_0 \lesssim h \| \operatorname{grad}_h P_h^\perp (u - u_h) \|_0$, we get

$$\| P_h^\perp (u - u_h^*) \|_0 \lesssim h (\| \operatorname{grad}_h (u - P_h^* u) \|_0 + \| \mathcal{A}(e_\sigma + \alpha e_p \mathbb{I}) + e_\gamma \|_0).$$

Now we only need to prove that the two terms on the right-hand side of the above inequality are bounded by ch^r. For the first term, one can see

$$h \| \operatorname{grad}_h (u - P_h^* u) \|_0 \lesssim h^r \|u\|_r$$

by the Bramble–Hilbert lemma. For the second term we use the a priori error estimates of σ, p, γ in Corollary 4, and the triangle inequality. □

3.4 Well-Posedness of Fully Discrete Solutions

In this section we discuss the a priori error analysis of fully discrete solutions. Let N be a positive integer and $\Delta t = T/N$ be the time-step size. Define $t_i = i\Delta t$ for $i = 0, 1, \ldots, N$ and

$$f^i = f(t_i), \quad \partial_t f^i = \frac{f^i - f^{i-1}}{\Delta t}. \tag{74}$$

Denoting the ith time step solution by $(\sigma_h^i, u_h^i, \gamma_h^i, z_h^i, p_h^i)$, the full discretization of (23)–(26) with the backward Euler scheme is

$$(\mathcal{A}(\sigma_h^i + \alpha p_h^i \mathbb{I}), \tau) + (u_h^i, \operatorname{div} \tau) + (\gamma_h^i, \tau) = 0, \tag{75}$$

$$(\operatorname{div} \sigma_h^i, v) + (\sigma_h^i, \eta) = -(f^i, v), \tag{76}$$

$$(\kappa^{-1} z_h^i, w) + (p_h^i, \operatorname{div} w) = 0, \tag{77}$$

$$\left(s_0 \partial_t p_h^i, q\right) + \left(\mathcal{A}\left(\partial_t \sigma_h^i + \alpha \partial_t p_h^i \mathbb{I}\right), \alpha q \mathbb{I}\right) - (\operatorname{div} z_h^i, q) = (g^i, q), \tag{78}$$

for $i \geq 1$ and any $(\tau, v, \eta, w, q) \in \Sigma_h \times V_h \times \Gamma_h \times W_h \times Q_h$. This is a system of linear equations with the same number of equations and unknowns, so it is a nonsingular linear system if it has a unique solution. Suppose that f^i, g^i, and the $(i-1)$th time step solution vanish in the above system. Then we show that the ith step solution must vanish. To show it, we multiply Δt to (78) and get

$$(\mathcal{A}(\sigma_h^i + \alpha p_h^i \mathbb{I}), \tau) + (u_h^i, \operatorname{div} \tau) + (\gamma_h^i, \tau) = 0, \quad \forall \tau \in \Sigma_h, \tag{79}$$

$$(\operatorname{div} \sigma_h^i, v) + (\sigma_h^i, \eta) = 0, \quad \forall (v, \eta) \in V_h \times \Gamma_h, \tag{80}$$

$$(\kappa^{-1} z_h^i, w) + (p_h^i, \operatorname{div} w) = 0, \quad \forall w \in W_h, \tag{81}$$

$$\left(s_0 p_h^i, q\right) + \left(\mathcal{A}\left(\sigma_h^i + \alpha p_h^i \mathbb{I}\right), \alpha q \mathbb{I}\right) - \Delta t (\operatorname{div} z_h^i, q) = 0, \quad \forall q \in Q_h. \tag{82}$$

Taking $\tau = \sigma_h^i$, $v = u_h^i$, $w = \Delta t z_h^i$, $q = p_h^i$, $\eta = -\gamma_h^i$ and adding all the equations, we have

$$\|\sigma_h^i + \alpha p_h^i \mathbb{I}\|_{\mathcal{A}}^2 + (s_0 p_h^i, p_h^i) + \Delta t (\kappa^{-1} z_h^i, z_h^i) = 0,$$

so $\sigma_h^i + \alpha p_h^i \mathbb{I}$ and z_h^i vanish. Note that p_h^i does not necessarily vanish because we do not assume that s_0 is strictly positive on the whole Ω. However, the inf-sup condition **(S3)**, the fact $z_h^i = 0$, and (81) can conclude that $p_h^i = 0$, and therefore $\sigma_h^i = 0$ as well because $\sigma_h^i + \alpha p_h^i = 0$. Now $u_h^i = 0$ and $\gamma_h^i = 0$ can be obtained using the inf-sup condition **(S4)** and (79).

The a priori error analysis of differential algebraic equation with implicit time discretization is well-studied [23], so here we state the result and will not show the detailed proof of the error estimates of the fully discrete solutions.

Theorem 7 *Suppose that $(\sigma, u, \gamma, z, p)$ and $(\sigma_h^i, u_h^i, \gamma_h^i, z_h^i, p_h^i)$ are solutions of (12)–(15) and (75)–(78), respectively. Then the following estimate holds:*

$$\sup_{0 \leq i \leq N} \|\sigma^i - \sigma_h^i, u^i - u_h^i, \gamma^i - \gamma_h^i, z^i - z_h^i, p^i - p_h^i\|_0 \leq c(\Delta t + h^m), \quad 1 \leq m \leq r,$$

with a constant c depending on regularity of the exact solution.

Let us give a remark on other time discretizations. We may use other implicit time discretization schemes with higher order time discretization errors but they may need numerical initial data compatible with algebraic equations of the problem. For example, if the Crank–Nicolson method is used and numerical initial data do not satisfy the algebraic equations of the problem, then the numerical solution does not satisfy the algebraic equations at all time steps due to the time stepping algorithm of the Crank–Nicolson method. In order to avoid it, we may choose numerical initial data as a numerical solution of the algebraic equations. An alternative way is to take only one time step with the backward Euler method with very small time-step size and to continue time stepping with the Crank–Nicolson method.

4 Numerical Results

We present numerical experiments in this section. For simplicity of presentation, we assume that the poroelastic medium is homogeneous and isotropic, i.e.,

$$\begin{aligned} \mathcal{C}\tau &= 2\mu\tau + \lambda \operatorname{tr}\tau\mathbb{I}, \quad \tau \in L^2(\Omega;\mathbb{S}), \\ \mathcal{A}^s\tau &= \frac{1}{2\mu}\left(\tau - \frac{\lambda}{2\mu + n\lambda}\operatorname{tr}\tau\mathbb{I}\right), \quad \tau \in L^2(\Omega;\mathbb{S}) \end{aligned} \tag{83}$$

with positive constants μ and λ.

We set $\Omega = [0, 1]\times[0, 1]$, and displacement boundary conditions will be given in all examples. In our experiments, we used two combinations of mixed finite elements for elasticity and for mixed Poisson problems. As the first combination we use the lowest order Arnold–Falk–Winther (AFW_1) element [5] for linear elasticity and the lowest order Raviart–Thomas (RT_1) element for mixed Poisson problem. Note that we use the indices of Raviart–Thomas elements in the FEniCS package (cf. [27]), which may be different from other literature. The lowest order AFW element has the lowest order Brezzi–Douglas–Marini element [11] in each row of Σ_h, $\mathbb{V}$-valued discontinuous piecewise constant polynomials as V_h, and $\mathbb{K}$-valued discontinuous piecewise constant polynomials as Γ_h. Another combination is the lowest order Taylor–Hood based (THB_1) element for linear elasticity and the second lowest order RT element (RT_2) for mixed Poisson problem. The lowest order Taylor–Hood based element [25] has the lowest order BDM element in each row of Σ_h, $\mathbb{V}$-valued piecewise constant polynomials as V_h, and $\mathbb{K}$-valued continuous piecewise linear polynomials as Γ_h. For the stability analysis of the AFW_1 and THB_1 elements, we refer to [5,8,18,22,25].

In our numerical experiments the mesh is structured with mesh size h and we take the backward Euler time discretization with time-step size $0 < \Delta t < 1$. We set $\Delta t = h^2$ or $\Delta t = h^3$ in order to make the convergence rate of the time discretization errors higher than the one of the spatial discretization errors, so we can compare the convergence rates of the spatial discretization errors ignoring influences of time discretization errors. The expected convergence rates of the errors from the theoretical analysis are summarized in Table 2. All numerical experiments are implemented using Dolfin, the Python interface of the FEniCS package [27] (Table 1).

Example 8 For the displacement and pressure

$$u(t, x, y) = \begin{pmatrix} x\cos(t) \\ (1 + y^2)\cos(t + 1)\sin(\pi x) \end{pmatrix} \quad \text{and} \quad p(t, x, y) = x^2 y \cos(t^2), \tag{84}$$

the stiffness tensor (83) and given parameters $\mu, \lambda, \kappa, \alpha$, one can compute σ, z, γ, f, and g, using the equations (3)–(6) and the definition of γ. We compute a numerical solution of this exact solution with inhomogeneous displacement boundary conditions using the formulation (12)–(15). The numerical results with Element 1 (AFW_1 and RT_1) and Element 2 (THB_1

Table 1 Finite element spaces for unknowns (BDM_k: the kth lowest order Brezzi–Douglas–Marini element, RT_k: the kth lowest order Raviart–Thomas element, CG_k: the Lagrange finite element with polynomials of degree $\leq k$, DG_k: the finite element with discontinuous polynomials of degree $\leq k$)

	Σ_h	V_h	Γ_h	W_h	Q_h
Element 1	BDM_1	DG_0	DG_0	RT_1	DG_0
Element 2	BDM_1	DG_0	CG_1	RT_2	DG_1

Table 2 Expected (spatial) convergence rates

Error Norm	$\sigma - \sigma_h$ $L^\infty L^2$	$u - u_h$ $L^\infty L^2$	$u - u_h^*$ $L^\infty L^2$	$\gamma - \gamma_h$ $L^\infty L^2$	$z - z_h$ $L^\infty L^2$	$p - p_h$ $L^\infty L^2$
Element 1	1	1	1	1	1	1
Element 2	2	1	2	2	2	2

Table 3 Errors and convergence rates of unknowns at $t = 1.0$ with Element 1 for the exact solution with the displacement (84) ($\mu = \lambda = 10$, $s_0 = 1$, $\Delta t = h^2$)

$\frac{1}{h}$	$\|\sigma - \sigma_h\|$		$\|u - u_h\|$		$\|u - u_h^*\|$		$\|\gamma - \gamma_h\|$		$\|z - z_h\|$		$\|p - p_h\|$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
4	5.93e+0	–	1.92e−1	–	1.10e−1	–	2.50e−1	–	1.51e−1	–	6.24e−2	–
8	2.37e+0	1.32	8.94e−2	1.10	3.54e−2	1.64	1.16e−1	1.11	7.78e−2	0.95	3.19e−2	0.97
16	1.10e+0	1.11	4.26e−2	1.07	1.03e−2	1.78	5.53e−2	1.06	3.90e−2	1.00	1.55e−2	1.05
32	5.40e−1	1.03	2.09e−2	1.03	2.77e−3	1.90	2.72e−2	1.02	1.95e−2	1.00	7.59e−3	1.03
64	2.69e−1	1.00	1.04e−2	1.01	7.15e−4	1.95	1.35e−2	1.01	9.75e−3	1.00	3.77e−3	1.01

Table 4 Errors and convergence rates of unknowns at $t = 1.0$ with Element 2 for the exact solution with the displacement (84) ($\mu = \lambda = 10$, $s_0 = 1$, $\Delta t = h^3$)

$\frac{1}{h}$	$\|\sigma - \sigma_h\|$		$\|u - u_h\|$		$\|u - u_h^*\|$		$\|\gamma - \gamma_h\|$		$\|z - z_h\|$		$\|p - p_h\|$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
4	3.19e+0	–	1.62e−1	–	3.63e−2	–	2.23e−1	–	2.61e−2	–	1.09e−2	–
8	7.97e−1	2.00	8.26e−2	0.98	8.13e−3	2.16	4.20e−2	2.41	7.08e−3	1.88	2.84e−3	1.93
16	2.03e−1	1.97	4.14e−2	1.00	2.17e−3	1.91	9.63e−3	2.13	1.87e−3	1.92	7.61e−4	1.90
32	5.14e−2	1.98	2.07e−2	1.00	5.55e−4	1.96	2.34e−3	2.04	4.77e−4	1.97	2.00e−4	1.93
64	1.29e−2	1.99	1.04e−2	1.00	1.40e−4	1.99	5.79e−4	2.01	1.20e−4	1.99	5.13e−5	1.96

and RT_2) are given through Tables 3, 4, 5 and 6. The parameter values, mesh and time-step sizes are explained in the tables.

In Tables 3 and 5, with different values of s_0, we carried out the local post-processing in the previous section. The post-processed solutions show second order convergence but this superconvergence is not covered in the error analysis. Numerical experiments of the same exact solutions with Element 2 are presented in Tables 4 and 6 and all the convergence rates

Table 5 Errors and convergence rates of unknowns at $t = 1.0$ with Element 1 for the exact solution with the displacement (84) ($\mu = \lambda = 10$, $s_0 = 10^{-3}$, $\Delta t = h^2$)

$\frac{1}{h}$	$\|\sigma - \sigma_h\|$		$\|u - u_h\|$		$\|u - u_h^*\|$		$\|\gamma - \gamma_h\|$		$\|z - z_h\|$		$\|p - p_h\|$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
4	5.93e+00	–	1.92e−1	–	1.10e−1	–	2.50e−1	–	1.67e−1	–	7.82e−2	–
8	2.37e+00	1.32	8.94e−2	1.10	3.54e−2	1.64	1.16e−1	1.11	8.12e−2	1.05	3.55e−2	1.14
16	1.10e+00	1.11	4.26e−2	1.07	1.03e−2	1.78	5.53e−2	1.06	3.95e−2	1.04	1.60e−2	1.15
32	5.40e−01	1.03	2.09e−2	1.03	2.77e−3	1.90	2.72e−2	1.02	1.96e−2	1.01	7.66e−3	1.06
64	2.69e−01	1.00	1.04e−2	1.01	7.14e−4	1.95	1.35e−2	1.01	9.76e−3	1.00	3.78e−3	1.02

Table 6 Errors and convergence rates of unknowns at $t = 1.0$ with Element 2 for the exact solution with the displacement (84) ($\mu = \lambda = 10$, $s_0 = 10^{-3}$, $\Delta t = h^3$)

$\frac{1}{h}$	$\|\sigma - \sigma_h\|$		$\|u - u_h\|$		$\|u - u_h^*\|$		$\|\gamma - \gamma_h\|$		$\|z - z_h\|$		$\|p - p_h\|$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
4	3.19e+0	–	1.62e−1	–	3.63e−2	–	2.23e−1	–	2.65e−2	–	1.24e−2	–
8	7.97e−1	2.00	8.26e−2	0.98	8.13e−3	2.16	4.20e−2	2.41	6.68e−3	1.99	2.97e−3	2.06
16	2.03e−1	1.97	4.14e−2	1.00	2.17e−3	1.91	9.63e−3	2.13	1.75e−3	1.93	7.37e−4	2.01
32	5.14e−2	1.98	2.07e−2	1.00	5.55e−4	1.96	2.34e−3	2.04	4.46e−4	1.98	1.85e−4	2.00
64	1.29e−2	1.99	1.04e−2	1.00	1.40e−4	1.99	5.79e−4	2.01	1.12e−4	1.99	4.63e−5	2.00

Table 7 Errors and convergence rates of unknowns at $t = 1.0$ with Element 1 for the exact solution with the displacement (84) ($\mu = \lambda = 10$, $s_0 = 0$, $\Delta t = h^3$)

$\frac{1}{h}$	$\|\sigma - \sigma_h\|$		$\|u - u_h\|$		$\|u - u_h^*\|$		$\|\gamma - \gamma_h\|$		$\|z - z_h\|$		$\|p - p_h\|$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
4	5.93e+0	–	1.92e−1	–	1.10e−1	–	2.50e−1	–	1.67e−1	–	7.82e−2	–
8	2.37e+0	1.32	8.93e−2	1.10	3.54e−2	1.64	1.15e−1	1.11	8.12e−2	1.05	3.55e−2	1.14
16	1.10e+0	1.11	4.26e−2	1.07	1.03e−2	1.78	5.53e−2	1.06	3.95e−2	1.04	1.60e−2	1.15
32	5.40e−1	1.03	2.09e−2	1.03	2.77e−3	1.90	2.72e−2	1.02	1.96e−2	1.01	7.66e−3	1.06
64	2.69e−1	1.00	1.04e−2	1.01	7.14e−4	1.95	1.35e−2	1.01	9.76e−3	1.00	3.78e−3	1.02

are in agreement with the expected convergence rates. In Tables 7 and 8, we carried out numerical experiments for an exact solution with $s_0 = 0$, and can see that convergence rates are not influenced by this vanishing s_0.

Example 9 In order to illustrate that our methods are robust for nearly incompressible materials, we consider a problem on $\Omega = [0, 1] \times [0, 1]$ with

$$f = \begin{pmatrix} xy \\ \sin t \end{pmatrix}, \quad \mu = 10, \quad \kappa = 1, \quad s_0 = 10^{-3},$$

and boundary conditions

Table 8 Errors and convergence rates of unknowns at $t = 1.0$ with Element 2 for the exact solution with the displacement (84) ($\mu = \lambda = 10$, $s_0 = 0$, $\Delta t = h^3$)

$\frac{1}{h}$	$\|\sigma - \sigma_h\|$		$\|u - u_h\|$		$\|u - u_h^*\|$		$\|\gamma - \gamma_h\|$		$\|z - z_h\|$		$\|p - p_h\|$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
4	3.19e+0	–	1.62e−1	–	3.63e−2	–	2.23e−1	–	2.65e−2	–	1.24e−2	–
8	7.97e−1	2.00	8.26e−2	0.98	8.13e−3	2.16	4.20e−2	2.41	6.68e−3	1.99	2.97e−3	2.07
16	2.03e−1	1.97	4.14e−2	1.00	2.17e−3	1.91	9.63e−3	2.13	1.75e−3	1.93	7.37e−4	2.01
32	5.14e−2	1.98	2.07e−2	1.00	5.55e−4	1.96	2.34e−3	2.04	4.46e−4	1.98	1.85e−4	2.00
64	1.29e−2	1.99	1.04e−2	1.00	1.40e−4	1.99	5.79e−4	2.01	1.12e−4	1.99	4.63e−5	2.00

Table 9 Relative L^2 errors and convergence rates of σ, u, and z for large λ values

λ	$\frac{1}{h}$	$\|\sigma - \sigma_h\|/\|\sigma\|$		$\|u - u_h\|/\|u\|$		$\|z - z_h\|/\|z\|$	
		Error	Rate	Error	Rate	Error	Rate
10^1	4	4.51e−01	–	3.46e−01	–	4.09e−01	–
	8	2.94e−01	0.62	1.53e−01	1.17	2.24e−01	0.87
	16	1.80e−01	0.71	5.94e−02	1.37	1.14e−01	0.98
	32	1.04e−01	0.78	2.11e−02	1.49	5.52e−02	1.04
10^4	4	6.23e−01	–	5.88e−01	–	4.82e−01	–
	8	4.62e−01	0.43	3.19e−01	0.88	2.54e−01	0.92
	16	3.15e−01	0.55	1.50e−01	1.09	1.26e−01	1.01
	32	1.98e−01	0.67	6.12e−02	1.29	5.98e−02	1.07
10^7	4	6.24e−01	–	5.89e−01	–	4.82e−01	–
	8	4.63e−01	0.43	3.20e−01	0.88	2.55e−01	0.92
	16	3.16e−01	0.55	1.50e−01	1.09	1.26e−01	1.01
	32	1.98e−01	0.67	6.14e−02	1.29	5.98e−02	1.07
10^{10}	4	6.24e−01	–	5.89e−01	–	4.82e−01	–
	8	4.63e−01	0.43	3.20e−01	0.88	2.55e−01	0.92
	16	3.16e−01	0.55	1.50e−01	1.09	1.26e−01	1.01
	32	1.98e−01	0.67	6.14e−02	1.29	5.98e−02	1.07

$$\sigma \boldsymbol{n} = 0, \quad z \cdot \boldsymbol{n} = 0, \quad \text{on } \Gamma = \{(x, y) \in \mathbb{R}^2 \,:\, y < 1\}.$$

Since we do not know the exact solution, we compute a numerical solution with the mesh of Ω bisecting 128×128 rectangles, and use it to compute the errors of other numerical solutions with coarser meshes. For simplicity, we use Element 1 in Table 2. We present relative L^2 errors and convergence rates of σ, u, z for different λ values and mesh refinements in Table 9. Due to the limit of computational resources, our numerical experiments did not reach the asymptotic regime of convergence rates but they clearly show that relative L^2 errors of σ, u, z, are not influenced by large λ values.

5 Conclusion

In the paper, we propose a new finite element method for Biot's consolidation model and show the a priori error estimates of semidiscrete problems. In particular, our error estimates do not require strictly positive s_0, and they are robust for nearly incompressible materials. We illustrate the validity of our analysis by numerical experiments.

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