A coupled multipoint stress multipoint flux mixed finite element method for Biot poroelasticity model

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Abstract

In this work we present a mixed finite element method for five-field Biot's consolidation model that reduces to cell-centered finite differences for displacement and pressure on quadrilateral and simplicial grids. The method is guaranteed to perform robustly with discontinuous full tensor permeability coefficients and heterogeneous elasticity parameters, which is verified by the error analysis. Our approach is motivated by multipoint flux approximation (MPFA) method and multipoint stress approximation (MPSA) methods, while the approach we take is based on more recent multipoint flux mixed finite element (MFMFE) method and multipoint stress mixed finite element (MSMFE) method, for Darcy and linear elasticity models, respectively. Our scheme couples the latter two methods for the spatial discretization of the Biot's poroelasticity system, and is based on the lowest order Brezzi-Douglas-Marini mixed finite element spaces. The special quadrature rule is then employed, that allows for the local stress, rotation and velocity elimination and leads to a symmetric and positive-definite system for displacements and pressures. Theoretical and numerical studies indicate first-order accuracy in all variables in their natural norms.

1 Introduction

to be written

2 Definition of the method.

2.1 Preliminaries.

In this section we recall the formulation of the elasticity system based on weak imposition of the symmetry of a stress tensor, and its discretization by a mixed finite element method. We then propose the modification of said method to obtain a multipoint stress mixed finite element method for linear elasticity with weak symmetry and further provide its stability and error analysis.

Let Ω be a simply connected bounded domain of $\mathbb{R}^d$, $d = 2, 3$ occupied by a linearly elastic porous body. We write $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{N}$ for the spaces of $d \times d$ matrices, symmetric matrices and skew-symmetric matrices, all over the field of real numbers, respectively. The material properties are described at each point $x \in \Omega$ by a compliance tensor $A = A(x)$, which is a symmetric, bounded and uniformly positive definite linear operator acting from $\mathbb{S} \to \mathbb{S}$. We also assume that an extension of A to an operator $\mathbb{M} \to \mathbb{M}$ still possesses the above properties. However, in a case of homogeneous and isotropic body,

$$A\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu + d\lambda}\operatorname{tr}(\sigma)I\right),$$

where I is a $d \times d$ identity matrix and $\mu > 0, \lambda \geq 0$ are Lamé coefficients. Conventionally K stands for the permeability tensor, c_0 - for mass storativity and α represents the Biot-Willis constant.

Throughout the paper the divergence operator is the usual divergence for vector fields, which produces vector field when applied to matrix field by taking the divergence of each row. We will also use the curl operator which is the usual curl when applied to vector fields in three dimension, and defined as

$$\operatorname{curl}\phi = (\partial_2\phi, -\partial_1\phi)$$

for a scalar function ϕ in two dimension. Similarly, for the vector field in two dimension or the matrix field in three dimension, curl operator produces vector field or matrix field, respectively, by acting row-wise.

Therein for the rest of this paper, C denotes a generic positive constant that is independent of the discretization parameter h. We will also use the following standard notation. For a domain $G \subset \mathbb{R}^d$, the $L^2(G)$ inner product and

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norm for scalar and vector valued functions are denoted $(\cdot,\,\cdot)_G$ and $\|\cdot\|_G$, respectively. The norms and seminorms of the Sobolev spaces $W^{k,p}(G)$, $k \in \mathbb{R}, p > 0$ are denoted by $\|\cdot\|_{k,p,G}$ and $|\cdot|_{k,p,G}$, respectively. The norms and seminorms of the Hilbert spaces $H^k(G)$ are denoted by $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$, respectively. We omit G in the subscript if $G = \Omega$. For a section of the domain or element boundary $S \subset \mathbb{R}^{d-1}$ we write $\langle\cdot,\,\cdot\rangle_S$ and $\|\cdot\|_S$ for the $L^2(S)$ inner product (or duality pairing) and norm, respectively. We will also use the spaces

$$H(\mathrm{div};\Omega) = \{v \in L^2(\Omega,\mathbb{R}^d) : \operatorname{div} v \in L^2(\Omega)\},$$
$$H(\mathrm{div};\Omega,\mathbb{M}) = \{\tau \in L^2(\Omega,\mathbb{M}) : \operatorname{div}\tau \in L^2(\Omega,\mathbb{R}^d)\},$$

equipped with the norm

$$\|\tau\|_{\mathrm{div}} = \left(\|\tau\|^2 + \|\operatorname{div}\tau\|^2\right)^{1/2}.$$

Given a vector field f on Ω representing body forces, the quasi-static Biot system determines the displacement u, together with the Darcy velocity z and pressure p:

$$-\operatorname{div}\sigma(u) = f, \qquad \text{in } \Omega, \tag{2.1}$$
$$K^{-1}z + \nabla p = 0, \qquad \text{in } \Omega \tag{2.2}$$
$$\frac{\partial}{\partial t}(c_0 p + \alpha \operatorname{div} u) + \operatorname{div} z = q, \qquad \text{in } \Omega \tag{2.3}$$

where the poroelastic stress $\sigma(u)$ is such that:

$$\sigma(u) = \sigma_E(u) - \alpha p I,$$

where $\sigma_E(u) = 2\mu\epsilon(u) + \lambda \operatorname{div} u$ is the elastic stress. To close the system, the appropriate boundary conditions should also be prescribed

$$u = g_u \quad \text{on } \Gamma_D^{displ}, \tag{2.4}$$
$$p = g_p \quad \text{on } \Gamma_D^{pres}, \qquad z\cdot n = 0 \quad \text{on } \Gamma_N^{vel}, \tag{2.5}$$

where $\Gamma_D^{displ} = \Gamma_D^{pres} \cup \Gamma_N^{vel} = \partial\Omega$ are boundaries on which Dirichlet and Neumann data is specified for displacement, pressure and normal fluxes, respectively. We assume for simplicity that $\Gamma_D^* \neq \emptyset$, for $* = \{displ,\, pres\}$.

In order to simplify the notation we introduce operators S and Ξ as

$$\begin{aligned}
&\text{when } d=2 \quad S(\phi) = \phi \quad \text{for } \phi \in \mathbb{R}^d, && \Xi(\psi) = \begin{pmatrix} 0 & \psi \\ -\psi & 0 \end{pmatrix} \quad \text{for } \psi \in \mathbb{R},\\
&\text{when } d=3 \quad S(\phi) = \operatorname{tr}(\phi)I - \phi^T \quad \text{for } \phi \in \mathbb{M}, && \Xi(\psi) = \begin{pmatrix} 0 & -\psi_3 & \psi_2 \\ \psi_3 & 0 & -\psi_1 \\ -\psi_2 & \psi_1 & 0 \end{pmatrix} \quad \text{for } \psi \in \mathbb{R}^d.
\end{aligned} \tag{2.6}$$

Note that both of these operators are invertible, and it is shown in [?] and [?] that

$$\mathrm{Skew}(\mathrm{curl}\ \phi) = -\Xi(\mathrm{div}\ S(\phi)), \tag{2.7}$$

where $\mathrm{Skew}(\omega) = \left(\omega - \omega^T\right)$ is an operator $\mathbb{M} \to \mathbb{N}$. We also introduce the asymmetry operator as follows

$$\mathrm{as}\,(\omega) = \Xi^{-1}(\mathrm{Skew}(\omega)), \quad \text{for } d = 2, 3. \tag{2.8}$$

Another direct computation immediately yields that for all $\tau \in \mathbb{M}$ and $\xi \in \mathbb{N}$

$$(\tau,\,\xi) = \left(\mathrm{as}\,(\tau),\,\Xi^{-1}(\xi)\right), \quad \text{and} \quad \|\xi\| \sim \|\Xi^{-1}(\xi)\|. \tag{2.9}$$

We notice that due to the constitutive equation in a linear elasticity system, $A\sigma_E = \nabla u - \Xi(\gamma)$, we have

$$\operatorname{div} u = \operatorname{tr}(A\sigma_E)$$

Then the problem reads: find $(\sigma, u, \gamma, z, p)$ such that

$$(A\sigma,\,\tau) + (A\alpha p I,\,\tau) + (u,\,\operatorname{div}\tau) + (\gamma,\,\tau) = \langle g_u,\,\tau\,n\rangle \qquad \forall \tau \in \mathbb{X} \tag{2.10}$$
$$(\operatorname{div}\sigma,\,v) = -\,(f,\,v) \qquad \forall v \in V \tag{2.11}$$
$$(\sigma,\,\xi) = 0 \qquad \forall \xi \in \mathbb{W} \tag{2.12}$$
$$\left(K^{-1}z,\,q\right) - (p,\,\operatorname{div} q) = -\langle g_p,\,v\cdot n\rangle \qquad \forall q \in Z \tag{2.13}$$
$$c_0\left(\frac{\partial p}{\partial t},\,w\right) + \alpha\left(\frac{\partial}{\partial t}A\sigma,\,wI\right) + \alpha\left(\frac{\partial}{\partial t}\operatorname{tr}(A\alpha p I),\,w\right) + (\operatorname{div} z,\,w) = (g,\,w) \qquad \forall w \in W, \tag{2.14}$$
$$\sigma\,n = 0 \qquad \text{on } \Gamma_N^{stress}, \tag{2.15}$$
$$u\cdot n = 0 \qquad \text{on } \Gamma_N^{vel}. \tag{2.16}$$

where the spaces are

$$\mathbb{X} = \{\tau \in H(\mathrm{div};\Omega,\mathbb{M}) : \tau\, n = 0 \text{ on } \Gamma_N^{stress}\}, \quad V = L^2(\Omega,\mathbb{R}^d), \quad \mathbb{W} = L^2(\Omega,\mathbb{N}),$$
$$Z = \{v \in H(\mathrm{div};\Omega,\mathbb{R}^d) : v\cdot n = 0 \text{ on } \Gamma_N^{vel}\}, \quad W = L^2(\Omega).$$

We notice here that $\Gamma_N^{stress} \cup \Gamma_D^{displ} = \Gamma_N^{vel} \cup \Gamma_D^{pres} = \Gamma$. It was shown in [**?**] that (2.10)-(2.16) has a unique solution.

2.2 Finite element mappings.

We start with providing the necessary basic results that will be used in the later derivations of the multi-point stress-flux mixed finite element method for the problem.

Let $\mathcal{T}_h$ be a finite element partition of a polygonal domain $\Omega \in \mathbb{R}^d$, consisting of triangles and/or convex quadrilaterals in two dimensions and tetrahedra in three dimensions. Let $h = \max_{E\in\mathcal{T}_h} \mathrm{diam}(E)$ be the mesh characteristic size, representing the largest diameter of an element in the given partition. We also assume the partition $\mathcal{T}_h$ to be shape-regular and quasi-uniform [**?**]. For any element $E \in \mathcal{T}_h$ there exists a bijection mapping $F_E : \hat{E} \to E$, where $\hat{E}$ is a reference element. We denote the Jacobian matrix by DF_E and introduce $J_E = |\det(DF_E)|$. Let the inverse mapping be denoted by F_E^{-1}, its Jacobian matrix by DF_E^{-1}, and let $J_{F_E^{-1}} = |\det(DF_E^{-1})|$. We have that

$$DF_E^{-1}(x) = (DF_E)^{-1}(\hat{x}), \qquad J_{F_E^{-1}}(x) = \frac{1}{J_E(\hat{x})}.$$

In case of triangular meshes, $\hat{E}$ is the reference right triangle with vertices $\hat{\mathbf{r}}_1 = (0,0^T)$, $\hat{\mathbf{r}}_2 = (1,0)^T$ and $\hat{\mathbf{r}}_3 = (0,1)^T$. Let $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ be the corresponding vertices of E, oriented counterclockwise. In this case F_E is a linear mapping of the following form

$$F_E(\hat{\mathbf{r}}) = \mathbf{r}_1(1-\hat{x}-\hat{y}) + \mathbf{r}_2\hat{x} + \mathbf{r}_3\hat{y}, \tag{2.17}$$

with constant Jacobian matrix and determinant given by

$$DF_E = [\mathbf{r}_{21}, \mathbf{r}_{31}]^T \quad \text{and} \quad J_E = 2|E|, \tag{2.18}$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. The mapping for tetrahedra is described similarly.

In the case of convex quadrilaterals, $\hat{E}$ is the unit square with vertices $\hat{\mathbf{r}}_1 = (0,0)^T$, $\hat{\mathbf{r}}_2 = (1,0)^T$, $\hat{\mathbf{r}}_3 = (1,1)^T$ and $\hat{\mathbf{r}}_4 = (0,1)^T$. Denote by $\mathbf{r}_i = (x_i, y_i)^T$, $i = 1,\ldots,4$, the four corresponding vertices of element E. The outward unit normal vectors to the edges of E and $\hat{E}$ are denoted by n_i and $\hat{n}_i$, $i = 1,\ldots,4$, respectively. In this case F_E is the bilinear mapping given by

$$\begin{aligned} F_E(\hat{\mathbf{r}}) &= \mathbf{r}_1(1-\hat{x})(1-\hat{y}) + \mathbf{r}_2\hat{x}(1-\hat{y}) + \mathbf{r}_3\hat{x}\hat{y} + \mathbf{r}_4(1-\hat{x})\hat{y} \\ &= \mathbf{r}_1 + \mathbf{r}_{21}\hat{x} + \mathbf{r}_{41}\hat{y} + (\mathbf{r}_{34} - \mathbf{r}_{21})\hat{x}\hat{y}, \end{aligned} \tag{2.19}$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. It is easy to see that DF_E and J_E are linear functions of $\hat{x}$ and $\hat{y}$, i.e.

$$\begin{aligned} DF_E &= [(\mathbf{r}_{34} - \mathbf{r}_{21})\hat{y} + \mathbf{r}_{21}, \mathbf{r}_{41} - (\mathbf{r}_{34} - \mathbf{r}_{21})\hat{x}] \\ &= [\mathbf{r}_{21}, \mathbf{r}_{41}] + [(\mathbf{r}_{34} - \mathbf{r}_{21})\hat{y}, (\mathbf{r}_{34} - \mathbf{r}_{21})\hat{x}], \end{aligned} \tag{2.20}$$

$$J_E = 2|T_1| + 2(|T_2| - |T_1|)\hat{x} + 2(|T_4| - |T_1|)\hat{y}, \tag{2.21}$$

where $|T_i|$ is the area of a triangle enclosed by the two edges sharing $\mathbf{r}_i$. We notice that the Jacobian determinant J_E is uniformly positive, due to convexity of E.

Using the mapping definitions (2.17)-(2.21), a simple calculation verifies that for any edge (face) $e_i \subset \partial E$

$$n_i = \frac{1}{|e_i|} J_E (DF_E^{-1})^T \hat{n}_i. \tag{2.22}$$

Another direct computation using the mapping definitions together with shape-regularity and quasiuniformity of the grids, show that for all element types

$$\|DF_E\|_{0,\infty,\hat{E}} \sim h, \quad \|J_E\|_{0,\infty,\hat{E}} \sim h^d \quad \text{and} \quad \|J_{F_E^{-1}}\|_{0,\infty,\hat{E}} \sim h^{-d} \quad \forall E \in \mathcal{T}_h. \tag{2.23}$$

2.3 Mixed finite element spaces.

We consider $\mathbb{X}_h \times V_h \times \mathbb{W}_h$ to be the lowest order triple of the form $(\mathcal{BDM}_1)^d \times (\mathcal{P}_0)^d \times (\mathcal{P}_1^{cts})^{d\times d, skew}$ on simplicial elements, while in case of quadrilaterals, the triple is changed to $(\mathcal{BDM}_1)^d \times (\mathcal{Q}_0)^d \times (\mathcal{Q}_1^{cts})^{d\times d, skew}$. These triples were shown to be inf-sup stable for the mixed elasticity problem with weak symmetry in [**?**, **?**] for simplicial grids, and in [**?**] for the case of convex quadrilaterals. We also consider the lowest order $\mathcal{BDM}_1$ MFE spaces [**?**, **?**] for $Z_h \times W_h$.

On the reference simplex, these spaces are defined as

$$\hat{\mathbb{X}}_h(\hat{E}) = \left(\mathcal{P}_1(\hat{E})^d\right)^d, \qquad \hat{V}_h(\hat{E}) = \mathcal{P}_0(\hat{E})^d, \qquad \hat{\mathbb{W}}_h(\hat{E}) = \Xi(v),\, v \in \mathcal{P}_1(\hat{E}), \tag{2.24}$$

$$\hat{Z}_h(\hat{E}) = \mathcal{P}_1(\hat{E})^d, \qquad \hat{W}_h(\hat{E}) = \mathcal{P}_0(\hat{E}). \tag{2.25}$$

The definition of the said spaces on tetrahedra is obtained naturally from the one above.

On the reference unit square the stress and the velocity spaces are defined as

$$\begin{aligned}
\hat{\mathbb{X}}(\hat{E}) &= \left(\mathcal{P}_1(\hat{E})^2 + r\ \mathrm{curl}(\hat{x}^2\hat{y}) + s\ \mathrm{curl}(\hat{x}\hat{y}^2)\right)^2 \\
&= \begin{pmatrix} \alpha_1\hat{x} + \beta_1\hat{y} + \gamma_1 + r_1\hat{x}^2 + 2s_1\hat{x}\hat{y} & \alpha_2\hat{x} + \beta_2\hat{y} + \gamma_2 - 2r_1\hat{x}\hat{y} - s_1\hat{y}^2 \\ \alpha_3\hat{x} + \beta_3\hat{y} + \gamma_3 + r_2\hat{x}^2 + 2s_2\hat{x}\hat{y} & \alpha_4\hat{x} + \beta_4\hat{y} + \gamma_4 - 2r_2\hat{x}\hat{y} - s_2\hat{y}^2 \end{pmatrix} \\
\hat{V}_h(\hat{E}) &= \mathcal{P}_0(\hat{E})^d, \quad \hat{\mathbb{W}}_h(\hat{E}) = \Xi(v),\, v \in \mathcal{Q}_1(\hat{E}), \\
\hat{Z}(\hat{E}) &= \mathcal{P}_1(\hat{E})^2 + r\ \mathrm{curl}(\hat{x}^2\hat{y}) + s\ \mathrm{curl}(\hat{x}\hat{y}^2) \\
&= \begin{pmatrix} \alpha_5\hat{x} + \beta_5\hat{y} + \gamma_5 + r_3\hat{x}^2 + 2s_3\hat{x}\hat{y} \\ \alpha_6\hat{x} + \beta_6\hat{y} + \gamma_6 - 2r_3\hat{x}\hat{y} - s_3\hat{y}^2, \end{pmatrix} \\
\hat{W}_h(\hat{E}) &= \mathcal{P}_0(\hat{E}).
\end{aligned} \tag{2.26}$$

An important property these spaces possess is that

$$\widehat{\mathrm{div}}\hat{\mathbb{X}}(\hat{E}) = \hat{V}(\hat{E}), \quad \widehat{\mathrm{div}}\hat{Z}(\hat{E}) = \hat{W} \quad \text{and} \tag{2.27}$$

$$\forall \hat{\tau} \in \hat{\mathbb{X}}(\hat{E}),\, \hat{q} \in \hat{Z}(\hat{E}),\, \hat{e} \in \hat{E} \quad \hat{\tau}\,\hat{n}_{\hat{e}} \in \mathcal{P}_1(\hat{e})^d \text{ and } \hat{q}\cdot\hat{n}_{\hat{e}} \in \mathcal{P}_1(\hat{e}). \tag{2.28}$$

It is known [?], [?] that the degrees of freedom for $\mathcal{BDM}_1$ space can be chosen to be the values of normal fluxes at any two points on each edge $\hat{e}$ if $\hat{E}$ is a reference triangle, or any three points one each face $\hat{e}$ if $\hat{E}$ is a reference tetrahedron. This also applies to normal stresses in the case of $(\mathcal{BDM}_1)^d$. For this work we choose said points to be at the vertices of $\hat{e}$ for both the velocity and stress spaces. This choice is motivated by the use of quadrature rule introduced in the next section.

To define the above spaces on any physical element $E \in \mathcal{T}_h$ the following transformations are used

$$\tau \leftrightarrow \hat{\tau} : \tau = \frac{1}{J_E} DF_E \hat{\tau} \circ F_E^{-1}, \qquad v \leftrightarrow \hat{v} : v = \hat{v} \circ F_E^{-1},$$

$$\xi \leftrightarrow \hat{\xi} : \xi = \hat{\xi} \circ F_E^{-1}, \qquad \hat{q} \leftrightarrow \hat{q} : q = \frac{1}{J_E} DF_E \hat{q} \circ F_E^{-1},$$

$$w \leftrightarrow \hat{w} : w = \hat{w} \circ F_E^{-1},$$

here we consider $\tau \in \mathbb{X}$, $v \in V$, $\xi \in \mathbb{W}$, $q \in Z$ and $w \in W$.

The first and the third transformations provided above are known as Piola transformation applied to tensor and vector valued functions, respectively. Its advantage is in preserving the normal components of the stress tensor and velocity vector on the edges (faces), and it satisfies the following properties

$$(\mathrm{div}\,\tau, v)_E = (\widehat{\mathrm{div}}\hat{\tau}, \hat{v})_{\hat{E}} \quad \text{and} \quad \langle \tau\, n_e, v\rangle_e = \langle \hat{\tau}\,\hat{n}_{\hat{e}}, \hat{v}\rangle_{\hat{e}}, \tag{2.29}$$

$$(\mathrm{div}\, q, w)_E = (\widehat{\mathrm{div}}\hat{q}, \hat{w})_{\hat{E}} \quad \text{and} \quad \langle q\cdot n_e, w\rangle_e = \langle \hat{q}\cdot\hat{n}_{\hat{e}}, \hat{w}\rangle_{\hat{e}}. \tag{2.30}$$

It also follows that for functions in stress and velocity spaces, there holds

$$\tau\, n_e = \frac{1}{J_E} DF_E \hat{\tau}\, \frac{1}{|e|} J_E (DF_E^{-1})^T \hat{n}_{\hat{e}} = \frac{1}{|e|}\hat{\tau}\,\hat{n}_{\hat{e}}, \tag{2.31}$$

$$q\cdot n_e = \frac{1}{J_E} DF_E \hat{q} \cdot \frac{1}{|e|} J_E (DF_E^{-1})^T \hat{n}_{\hat{e}} = \frac{1}{|e|}\hat{q}\cdot\hat{n}_{\hat{e}}. \tag{2.32}$$

First equation in (2.29) can be written as $(\mathrm{div}\,\tau, v)_E = (\widehat{\mathrm{div}\,\tau}, J_E\hat{v})_{\hat{E}}$ which leads to

$$\mathrm{div}\,\tau = \left(\frac{1}{J_E}\widehat{\mathrm{div}}\cdot\hat{\chi}\right) \circ F_E^{-1}(x), \tag{2.33}$$

showing that $\mathrm{div}\,\tau\big|_E$ is constant on simplicial elements. Similarly, one concludes that $\mathrm{div}\,q\big|_E$ is also constant on simplicial elements.

We now introduce the finite dimensional spaces for the method on a given partition of the domain $\mathcal{T}_h$:

$$\begin{aligned}
\mathbb{X}_h &= \{\tau \in \mathbb{X} : \quad \tau|_E \leftrightarrow \hat{\tau},\ \hat{\tau} \in \hat{\mathbb{X}}(\hat{E}) \quad \forall E \in \mathcal{T}_h\}, \\
V_h &= \{v \in V : \quad v|_E \leftrightarrow \hat{v},\ \hat{v} \in \hat{V}(\hat{E}) \quad \forall E \in \mathcal{T}_h\}, \\
\mathbb{W}_h &= \{\xi \in \mathbb{W} : \quad \xi|_E \leftrightarrow \hat{\xi},\ \hat{\xi} \in \hat{\mathbb{W}}(\hat{E}) \quad \forall E \in \mathcal{T}_h\}, \\
Z_h &= \{q \in Z : \quad q|_E \leftrightarrow \hat{q},\ \hat{q} \in \hat{Z}(\hat{E}) \quad \forall E \in \mathcal{T}_h\}, \\
W_h &= \{w \in W : \quad w|_E \leftrightarrow \hat{w},\ \hat{w} \in \hat{W}(\hat{E}) \quad \forall E \in \mathcal{T}_h\}.
\end{aligned} \tag{2.34}$$

We denote by Π a mixed projection operator acting on tensor valued functions, such that $\Pi : \mathbb{X} \cap H^1(\Omega, \mathbb{M}) \to \mathbb{X}_h$. We will also use the same notation for a projection operator acting on vector valued functions, so that in this case Π maps from $Z \cap H^1(\Omega, \mathbb{R}^d)$ onto Z_h. It was shown in [?],[?] and [?] that such projection operator exists and satisfies the following properties

$$\begin{aligned}
(\operatorname{div}(\Pi\tau - \tau), v) &= 0, \qquad \forall v \in V_h, \\
(\operatorname{div}(\Pi q - q), w) &= 0, \qquad \forall w \in W_h.
\end{aligned} \tag{2.35}$$

In both cases the operator Π is defined locally on each element E by

$$\Pi\tau \leftrightarrow \widehat{\Pi\tau}, \quad \widehat{\Pi\tau} = \hat{\Pi}\hat{\tau}, \tag{2.36}$$

$$\Pi q \leftrightarrow \widehat{\Pi q}, \quad \widehat{\Pi q} = \hat{\Pi}\hat{q}, \tag{2.37}$$

where $\hat{\Pi} : H^1(\hat{E}, \mathbb{M}) \to \hat{\mathbb{X}}_h(\hat{E})$ is the reference element projection operator satisfying

$$\forall \hat{e} \subset \partial\hat{E}, \qquad \langle (\hat{\Pi}\hat{\tau} - \hat{\tau})\hat{n}, \hat{\phi}_1 \rangle_{\hat{e}} = 0 \qquad \forall \hat{\phi}_1 \in (\mathcal{P}_1(\hat{e}))^d, \tag{2.38}$$

and similarly, $\hat{\Pi} : H^1(\hat{E}, \mathbb{R}^d) \to \hat{Z}_h(\hat{E})$ is an operator satisfying

$$\forall \hat{e} \subset \partial\hat{E}, \qquad \langle (\hat{\Pi}\hat{q} - \hat{q}) \cdot \hat{n}, \hat{\psi}_1 \rangle_{\hat{e}} = 0 \qquad \forall \hat{\psi}_1 \in \mathcal{P}_1(\hat{e}). \tag{2.39}$$

It is straightforward to see from (2.29), (2.36), (2.38) that $\tau\, n = 0$ on Γ_N^{stress} implies $\Pi\tau\, n = 0$ on Γ_N^{stress}. For this we note that for all $\phi \leftrightarrow \hat{\phi} \in (\mathcal{P}_1(\hat{e}))^d$,

$$\langle \Pi\tau\, n,\, \phi \rangle_e = \langle \widehat{\Pi\tau\, n},\, \hat{\phi} \rangle_{\hat{e}} = \langle \hat{\Pi}\hat{\tau}\, \hat{n},\, \hat{\phi} \rangle_{\hat{e}} = \langle \hat{\tau}\, \hat{n},\, \hat{\phi} \rangle = 0.$$

Similar argument using (2.30), (2.37), (2.39) shows that $q \cdot n = 0$ on Γ_N^{vel} implies $\Pi q \cdot n = 0$ on Γ_N^{vel}.

In addition to the mixed projection operator presented above, we will make use of a similar projection operator onto the lowest order Raviart-Thomas spaces [?, ?]. This additional construction is solely motivated by the purposes of error analysis on quadrilaterals, although for the uniformity of forthcoming proofs we would treat simplicial case in the same fashion. To deal with errors in stress and velocity variables we consider $\mathcal{RT}_0$ spaces of tensor and vector valued functions, respectively, where the former is obtained as 2 copies of the latter. Said spaces are defined on a unit square as follows

$$\hat{\mathbb{X}}^0(\hat{E}) = \begin{pmatrix} \alpha_1 + \beta_1\hat{x} & \alpha_2 + \beta_2\hat{y} \\ \alpha_3 + \beta_3\hat{x} & \alpha_4 + \beta_4\hat{y} \end{pmatrix}, \quad \hat{V}^0(\hat{E}) = \Big(Q_0(\hat{E})\Big)^2, \tag{2.40}$$

$$\hat{Z}^0(\hat{E}) = \begin{pmatrix} \alpha_5 + \beta_5\hat{x} \\ \alpha_6 + \beta_6\hat{y} \end{pmatrix}, \quad \hat{W}^0(\hat{E}) = Q_0(\hat{E}), \tag{2.41}$$

and on unit triangle as

$$\hat{\mathbb{X}}^0(\hat{E}) = \begin{pmatrix} \alpha_1 + \beta_1\hat{x} & \alpha_2 + \beta_1\hat{y} \\ \alpha_3 + \beta_2\hat{x} & \alpha_4 + \beta_2\hat{y} \end{pmatrix}, \quad \hat{V}^0(\hat{E}) = \Big(P_0(\hat{E})\Big)^2, \tag{2.42}$$

$$\hat{Z}^0(\hat{E}) = \begin{pmatrix} \alpha_5 + \beta_3\hat{x} \\ \alpha_6 + \beta_3\hat{y} \end{pmatrix}, \quad \hat{W}^0(\hat{E}) = P_0(\hat{E}). \tag{2.43}$$

In the case of unit tentrahedron $\hat{\mathbb{X}}^0(\hat{E})$ would have an additional row of components, while $\hat{Z}^0(\hat{E})$ - and additional entry in the vector. In all cases the

$$\begin{aligned}
&\operatorname{div} \hat{\mathbb{X}}^0(\hat{E}) = \hat{V}^0(\hat{e}) \text{ and } \hat{\tau}\,\hat{n} \in (\mathcal{P}_0(\hat{e}))^d, \\
&\operatorname{div} \hat{Z}^0(\hat{E}) = \hat{W}^0(\hat{e}) \text{ and } \hat{q} \cdot \hat{n} \in \mathcal{P}_0(\hat{e}).
\end{aligned}$$

The degrees of freedom of $\hat{\mathbb{X}}^0(\hat{E})$ are the values of normal stress $\hat{\tau}\,\hat{n}$ at the midpoints of all edges (faces) $\hat{e}$, similarly, the degrees of freedom of $\hat{Z}^0(\hat{E})$ are the values of normal fluxes $\hat{q}\cdot\hat{n}$ at the same points. The projection operator $\hat{\Pi}_0$ acting on tensor valued functions from $H^1(\Omega,\mathbb{M})$ onto $\hat{\mathbb{X}}^0(\hat{E})$; and acting on vector valued function so that $\hat{\Pi}_0 : H^1(\Omega,\mathbb{R}^d) \to \hat{Z}^0(\hat{E})$ satisfies

$$\forall \hat{e}\subset\partial\hat{E}, \qquad \langle(\hat{\Pi}_0\hat{\tau}-\hat{\tau})\hat{n}, \hat{\phi}_0\rangle_{\hat{e}} = 0 \qquad \forall\hat{\phi}_0\in(\mathcal{P}_0(\hat{e}))^d, \tag{2.44}$$

$$\forall \hat{e}\subset\partial\hat{E}, \qquad \langle(\hat{\Pi}_0\hat{q}-\hat{q})\cdot\hat{n}, \hat{\psi}_0\rangle_{\hat{e}} = 0 \qquad \forall\hat{\psi}_0\in\mathcal{P}_0(\hat{e}). \tag{2.45}$$

The spaces $\mathbb{X}_h^0$, V_h^0, Z_h^0 and W_h^0 on the entire partition $\mathcal{T}_h$ and the projection operator Π_0 for both tensor and vector valued functions are defined similarly to the case of $\mathcal{BDM}_1$ spaces. Notice also that $\mathbb{X}_h^0\subset\mathbb{X}_h$ and $Z_h^0\subset Z_h$, while the corresponding spaces V_h^0 and W_h^0 coincide with V_h and W_h, respectively. The definition of $\mathcal{RT}_0$ projector implies that

$$\operatorname{div}\tau = \operatorname{div}\Pi_0\tau \quad \text{and} \quad \|\Pi_0\tau\| \le C\|\tau\| \quad \forall\tau\in\mathbb{X}_h, \tag{2.46}$$

$$\operatorname{div}q = \operatorname{div}\Pi_0 q \quad \text{and} \quad \|\Pi_0 q\| \le C\|q\| \quad \forall q\in Z_h. \tag{2.47}$$

2.4 The $\mathcal{BDM}_1$ coupled mixed finite element method.

The lowest order coupled five field mixed finite element approximation of Biot's poroelasticity system of equations (2.10)-(2.16) reads as follows: Find $(\sigma_h, u_h, \gamma_h, z_h, p_h)\in\mathbb{X}_h\times V_h\times\mathbb{W}_h\times Z_h\times W_h$ such that:

$$(A\sigma_h,\,\tau) + (A\alpha p_h I,\,\tau) + (u_h,\,\operatorname{div}\tau) + (\gamma_h,\,\tau) = \langle g_u,\,\tau\,n\rangle_{\Gamma_D^{displ}} \qquad \forall\tau\in\mathbb{X}_h \tag{2.48}$$

$$(\operatorname{div}\sigma_h,\,v) = -\,(f,\,v) \qquad \forall v\in V_h \tag{2.49}$$

$$(\sigma_h,\,\xi) = 0 \qquad \forall\xi\in\mathbb{W}_h \tag{2.50}$$

$$\left(K^{-1}z_h,\,q\right) - (p_h,\,\operatorname{div}q) = -\langle g_p,\,v\cdot n\rangle_{\Gamma_D^{pres}} \qquad \forall q\in Z_h \tag{2.51}$$

$$c_0\left(\frac{\partial p_h}{\partial t},\,w\right) + \alpha\left(\frac{\partial}{\partial t}A\sigma_h,\,wI\right) + \alpha\left(\frac{\partial}{\partial t}\operatorname{tr}(A\alpha p_h I),\,w\right) + (\operatorname{div}z_h,\,w) = (g,\,w) \qquad \forall w\in W_h. \tag{2.52}$$

The method has a unique solution and is first order accurate for all of the variables in corresponding norms on simplicial and quadrilateral grids with our choices of elements [?]. While the method inherits all the advantages of a MFE method, its major drawback is in the resulting coupled algebraic system for five variables being of a saddle point type. Motivated by MFMFE and MSMFE methods, in the next sections we develop a quadrature rule that allows for local elimination of the stresses, rotations and fluxes which leads to a positive-definite cell-centered displacement-pressure system.

2.5 A quadrature rule.

For any pair of tensor or vector valued functions (ϕ,ψ) from $\mathbb{X}_h$ or Z_h, respectively, and for any linear uniformly bounded and positive-definite operator L we define the global quadrature rule

$$(L\phi,\psi)_Q \equiv \sum_{E\in\mathcal{T}_h}(L\phi,\psi)_{Q,E}.$$

The integration on any element E is performed by mapping to the reference element $\hat{E}$. The quadrature rule is defined on $\hat{E}$. Using the definition of the finite element spaces and omitting the subscript E, we get

$$\begin{aligned}\int_E L\phi\cdot\psi\,dx &= \int_{\hat{E}}\hat{L}\frac{1}{J}DF\hat{\phi}\cdot\frac{1}{J}DF\hat{\psi}\,J\,d\hat{x}\\ &= \int_{\hat{E}}\frac{1}{J}DF^T\hat{L}\,DF\hat{\phi}\cdot\hat{\psi}\,dx \equiv \int_{\hat{E}}\mathcal{L}\hat{\phi}\cdot\hat{\psi}\,d\hat{x},\end{aligned}$$

where $\cdot$ has a meaning of inner product for both tensor and vector valued functions, and

$$\mathcal{L}\phi = \frac{1}{J}DF^T\hat{L}\,DF\hat{\phi} \tag{2.53}$$

is also a symmetric and positive definite operator. Notice that due to (**??**),

$$\|\mathcal{L}\hat{\phi}\|_{\hat{E}} \sim h^{2-d}\|L\phi\|_E. \tag{2.54}$$

The quadrature rule on an element E is defined as

$$(L\phi,\psi)_{Q,E} \equiv (\mathcal{L}\hat{\phi},\hat{\psi})_{\hat{Q},\hat{E}} \equiv \frac{|\hat{E}|}{s}\sum_{i=1}^{s}\mathcal{L}\hat{\phi}(\hat{\mathbf{r}}_i):\hat{\psi}(\hat{\mathbf{r}}_i), \tag{2.55}$$

where $s = 3$ for the unit triangle and $s = 4$ for the unit tetrahedron or the unit square. This quadrature rule is often referred to as a vertex quadrature rule on unit simplices and as trapezoid rule on unit squares.

When applied to the elasticity and Darcy coercive terms in our coupled problem, the quadrature rule defined above guarantees the coupling of stress and velocity basis function only around vertices (see [**?**, **?**]), i.e., the coupled stress basis functions are only the ones associated with a corner, and same statement applies for the velocity basis functions.

We also construct the quadrature rule for the term involving stress with second variable being pressure or rotation. Given $\tau = \mathbb{X}_h$, $\zeta \in \mathbb{W}_h$ or $\zeta \in (W_h)^{d\times d}$ and any linear uniformly bounded positive-definite operator M we get:

$$\int_E M\tau : \zeta \, dx = \int_{\hat{E}} \frac{1}{J} \hat{M} DF \, \hat{\tau} : \hat{\zeta} \, J \, d\hat{x} = \int_{\hat{E}} \hat{M} DF \, \hat{\tau} : \hat{\zeta} \, d\hat{x} = \int_{\hat{E}} \mathcal{M}\hat{\tau} : \hat{\zeta} \, d\hat{x},$$

where $\mathcal{M}\hat{\tau} = \hat{M} DF \hat{\tau}$. For this case we also define

$$(\tau, \zeta)_{Q,E} \equiv \left(\mathcal{M}\hat{\tau}, \, \hat{\zeta}\right)_{\hat{\mathcal{Q}},\hat{E}} \equiv \frac{|\hat{E}|}{s} \sum_{i=1}^{s} \mathcal{M}\hat{\tau}(\hat{\mathbf{r}}_i) : \hat{\zeta}(\hat{\mathbf{r}}_i). \tag{2.56}$$

Remark 2.1. *The quadrature rules can be defined directly on an element E. It is easy to see from definitions* (2.55), (2.56) *that on simplicial elements, for $\phi, \psi \in \mathbb{X}_h$ or $\phi, \psi \in Z_h$, $\tau \in \mathbb{X}_h$ and $\zeta \in \mathbb{W}_h$ or $\zeta \in (W_h)^{d\times d}$*

$$(L\phi, \psi)_{Q,E} = \frac{|E|}{s} \sum_{i=1}^{s} L\phi(\mathbf{r}_i) \cdot \psi(\mathbf{r}_i), \quad (M\tau, \zeta)_{Q,E} = \frac{|E|}{s} \sum_{i=1}^{s} M\tau(\mathbf{r}_i) : \zeta(\mathbf{r}_i), \tag{2.57}$$

where L and M are any linear uniformly bounded and positive definite operators. On quadrilaterals the above definitions read as

$$(L\phi, \psi)_{Q,E} = \frac{1}{2} \sum_{i=1}^{4} |T_i| L\phi(\mathbf{r}_i) \cdot \psi(\mathbf{r}_i), \quad (M\tau, \zeta)_{Q,E} = \frac{1}{2} \sum_{i=1}^{4} |T_i| M\tau(\mathbf{r}_i) : \zeta(\mathbf{r}_i), \tag{2.58}$$

where $|T_i|$ is the area of a triangle formed by two edges sharing vertex $\mathbf{r}_i$.

*The above quadrature rules are closely related to some inner products arising in mimetic finite difference methods [**?**].*

For ϕ, $\psi \in \mathbb{X}_h$ or ϕ, $\psi \in Z_h$, $\tau \in \mathbb{X}_h$ and $\zeta \in \mathbb{W}_h$ or $\zeta \in (W_h)^{d\times d}$ denote the element quadrature errors by

$$\theta(L\phi, \psi) \equiv (L\phi, \psi)_E - (L\phi, \psi)_{Q,E} \tag{2.59}$$

$$\delta(M\tau, \zeta) \equiv (M\tau, \zeta)_E - (M\tau, \zeta)_{Q,E}, \tag{2.60}$$

and define the global quadrature errors by $\theta(L\phi, \psi)_E = \theta(L\phi, \psi)$, $\delta(M\tau, \zeta)_E = \delta(M\tau, \zeta)$. Similarly denote the quadrature errors on the reference element by

$$\hat{\theta}(\mathcal{L}\hat{\phi}, \hat{\psi}) \equiv (\mathcal{L}\hat{\phi}, \hat{\psi})_{\hat{E}} - (\mathcal{L}\hat{\phi}, \hat{\psi})_{Q,\hat{E}} \tag{2.61}$$

$$\hat{\delta}(\mathcal{M}\hat{\tau}, \hat{\zeta}) \equiv (\mathcal{M}\hat{\tau}, \hat{\zeta})_{\hat{E}} - (\mathcal{M}\hat{\tau}, \hat{\zeta})_{Q,\hat{E}}. \tag{2.62}$$

Lemma 2.1. *On simplicial elements, if $\chi \in \mathbb{X}_h(E)$ and $r \in Z_h(E)$, then*

$$\theta_E(\chi, \tau_0) = 0 \quad \textit{for all constant tensors } \tau_0,$$

$$\theta_E(r, v_0) = 0 \quad \textit{for all constant vectors } v_0.$$

Also, if $\zeta \in \mathbb{W}_h(E)$, then

$$\delta_E(\chi, \xi_0) = \delta_E(\tau_0, \zeta) = 0, \quad \textit{for all constant tensors } \xi_0 \textit{ and } \tau_0.$$

Proof. It is enough to consider τ_0 such that it has only one nonzero component, say, $(\tau_0)_{1,1} = 1$, the arguments for other cases are similar. Since the quadrature rule $(f)_E = \frac{|E|}{s} \sum_{i=1}^{s} f(\mathbf{r}_i)$ is exact for linear functions and using Remark 2.1 we have

$$(\chi, \, \tau_0)_{Q,E} = \frac{|E|}{s} \sum_{i=1}^{s} (\chi)_{1,1}(\mathbf{r}_i) = \int_E \chi : \tau_0 \, dx,$$

The same reasoning applies for the other two statements. □

Lemma 2.2. *On the reference square, for any $\hat{\chi} \in \hat{\mathbb{X}}_h(\hat{E})$ and $\hat{r} \in \hat{Z}_h(\hat{E})$,*

$$\left(\hat{\chi} - \hat{\Pi}_0 \hat{\chi}, \, \hat{\tau}_0\right)_{\hat{\mathcal{Q}},\hat{E}} = 0 \quad \textit{for all constant tensors } \hat{\tau}_0, \tag{2.63}$$

$$\left(\hat{r} - \hat{\Pi}_0 \hat{r}, \, \hat{v}_0\right)_{\hat{\mathcal{Q}},\hat{E}} = 0 \quad \textit{for all constant vectors } \hat{v}_0. \tag{2.64}$$

Proof. On any edge $\hat{e}$, if the degrees of freedom of $\hat{\chi}$ are $(\hat{\chi}_{\hat{e},11},\, \hat{\chi}_{\hat{e},12})^T$ and $(\hat{\chi}_{\hat{e},21}, \hat{\chi}_{\hat{e},22})^T$, then (2.44) and an application of trapezoid quadrature rule imply that

$$\hat{\Pi}_0\hat{\chi}\big|_E = \begin{pmatrix} \frac{1}{2}(\hat{\chi}_{\hat{e},11} + \hat{\chi}_{\hat{e},21}) \\ \frac{1}{2}(\hat{\chi}_{\hat{e},12} + \hat{\chi}_{\hat{e},22}) \end{pmatrix}.$$

Using (2.55) the simple calculation shows that the statement holds for the case of $\hat{\chi} \in \hat{\mathbb{X}}_h(\hat{E})$. Similar reasoning applied to the degrees of freedom of $\hat{r}$ shows that the statement is also valid for $\hat{r} \in \hat{Z}_h(\hat{E})$. $\square$

2.6 The coupled multipoint stress multipoint flux mixed finite element method.

We first introduce an L^2-orthogonal projection operator acting onto the space of piecewise constant scalar or vector valued function on the trace of $\mathcal{T}_h$ on $\partial\Omega$:

$$\mathcal{P}_0 : L^2(\partial\Omega, \mathbb{R}^d) \to \mathbb{X}_h^0\, n, \quad \text{such that } \forall \phi \in L^2(\Omega, \mathbb{R}^d), \quad \langle \phi - \mathcal{P}_0\phi,\, \tau\, n\rangle_{\partial\Omega} = 0, \quad \forall \tau \in \mathbb{X}_h^0, \tag{2.65}$$

$$\mathcal{P}_0 : L^2(\partial\Omega, \mathbb{R}) \to Z_h^0 \cdot n, \quad \text{such that } \forall \psi \in L^2(\Omega), \quad \langle \psi - \mathcal{P}_0\psi,\, q \cdot n\rangle_{\partial\Omega} = 0, \quad \forall q \in Z_h^0. \tag{2.66}$$

In the method proposed below, the Dirichlet boundary data for displacement and pressure variables is incorporated into the system via the projection operator defined above.

Our method is defined as follows. We seek $(\sigma_h, u_h, \gamma_h, z_h, p_h) \in \mathbb{X}_h \times V_h \times \mathbb{W}_h \times Z_h \times W_h$ such that:

$$(A\sigma_h,\, \tau)_Q + (A\alpha p_h I,\, \tau) + (u_h,\, \operatorname{div}\tau) + (\gamma_h,\, \tau)_Q = \langle \mathcal{P}_0 g_u,\, \tau\, n\rangle_{\Gamma_D^{displ}} \qquad \forall \tau \in \mathbb{X}_h \tag{2.67}$$

$$(\operatorname{div}\sigma_h,\, v) = -\,(f,\, v) \qquad \forall v \in V_h \tag{2.68}$$

$$(\sigma_h,\, \xi)_Q = 0 \qquad \forall \xi \in \mathbb{W}_h \tag{2.69}$$

$$\left(K^{-1} z_h,\, q\right)_Q - (p_h,\, \operatorname{div} q) = -\langle \mathcal{P}_0 g_p,\, v \cdot n\rangle_{\Gamma_D^{pres}} \qquad \forall q \in Z_h \tag{2.70}$$

$$c_0\left(\frac{\partial p_h}{\partial t},\, w\right) + \alpha\left(\frac{\partial}{\partial t} A\sigma_h,\, wI\right)_Q + \alpha\left(\frac{\partial}{\partial t}\operatorname{tr}(A\alpha p_h I),\, w\right) + (\operatorname{div} z_h,\, w) = (g,\, w) \qquad \forall w \in W_h. \tag{2.71}$$

Before we prove well-posedness and stability of the method (2.67)-(2.71), we show several important results involving the quadrature rule (2.55).

Lemma 2.3. *If $E \in \mathcal{T}_h$ and $\phi \in L^2(E, \mathbb{M})$, $\phi \in L^2(E, \mathbb{R}^d)$ is a function mapped using Piola transformation, then*

$$\|\phi\|_E \sim h^{\frac{2-d}{d}} \|\phi\|_{\hat{E}}. \tag{2.72}$$

Proof. The statement follows from the bounds given in (**??**) and the following relations

$$\int_E \phi \cdot \phi\, dx = \int_{\hat{E}} \frac{1}{J} DF\hat{\phi} \cdot \frac{1}{J} DF\hat{\phi}\, d\hat{x},$$
$$\int_{\hat{E}} \hat{\phi} \cdot \hat{\phi}\, d\hat{x} = \int_E \frac{1}{J_{F^{-1}}} DF^{-1}\phi \cdot \frac{1}{J_{F^{-1}}} DF^{-1}\phi\, dx,$$

where $\cdot$ stands for the inner product when applied to tensor valued functions. $\square$

Lemma 2.4. *There exists a positive constant C independent of h, such that for any linear uniformly bounded and positive-definite operator L*

$$(L\phi,\, \phi)_Q \geq C\|\phi\|^2, \quad \forall \phi \in \mathbb{X}_h \text{ or } \forall \phi \in Z_h. \tag{2.73}$$

Proof. Let $\phi = \sum_{i=1}^s \sum_{j=1}^d \phi_{ij}\psi_{ij}$ on an element E where ψ_{ij} is a basis function. Using the definitions of the quadrature rule as in Remark 2.1 we obtain

$$(L\phi,\, \psi)_{Q,E} = \frac{|E|}{s}\sum_{i=1}^s L\phi(\mathbf{r}_i) \cdot \phi(\mathbf{r}_i) \geq C(l_0)\frac{|E|}{s}\sum_{i=1}^s \phi(\mathbf{r}_i) \cdot \phi(\mathbf{r}_i) \geq C(l_0)\frac{|E|}{s}\sum_{i=1}^s\sum_{j=1}^d \phi_{ij}^2,$$

where $C(l_0)$ involves the constant from the lower bound of the operator L. On the other hand

$$\|\phi\|_E^2 = \left(\sum_{i=1}^s\sum_{j=1}^d \phi_{ij}\psi_{ij},\, \sum_{k=1}^s\sum_{l=1}^d \phi_{kl}\psi_{kl}\right) \leq C|E|\sum_{i=1}^s\sum_{j=1}^d \phi_{ij}^2.$$

And the assertion of the lemma follows from the combination of the above two estimates. $\square$

The following corollary is a result of the above lemma. We present it without a proof, for details see [**?**, **?**].

Corollary 2.1. *The bilinear form $(L\phi,\, \psi)_Q$ is an inner product on $\mathbb{X}_h$ and Z_h, $(L\phi,\, \psi)_Q^{1/2}$ is also a norm in $\mathbb{X}_h$ and Z_h equivalent to $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{Z_h}$, respectively.*

3 Stability analysis in semidiscrete case

Let us assume for simplicity that $\Gamma_D^{displ} = \Gamma_D^{pres} = \partial\Omega$.

Step 1: L^2 in space estimates:

We differentiate (2.67) and choose $(\tau, v, \xi, q, w) = (\sigma_h, \partial_t u_h, \partial_t \gamma_h, z_h, p_h)$ int (2.67)-(2.71) to obtain the following system:

$$(A\partial_t\sigma_h,\, \sigma_h)_Q + (A\alpha\partial_t pI,\, \sigma_h)_Q + (\partial_t u_h,\, \operatorname{div}\sigma_h) + (\partial_t\gamma_h,\, \operatorname{as}\sigma_h)_Q = \langle \partial_t \mathcal{P}_0 g_u,\, \sigma_h\, n\rangle \tag{3.1}$$

$$(\operatorname{div}\sigma_h,\, \partial_t u_h) = -(f,\, \partial_t u_h) \tag{3.2}$$

$$(\operatorname{as}\sigma_h,\, \partial_t\gamma_h)_Q = 0 \tag{3.3}$$

$$\left(K^{-1}z_h,\, z_h\right)_Q - (p_h,\, \operatorname{div} z_h) = \langle \mathcal{P}_0 g_p,\, z_h\cdot n\rangle \tag{3.4}$$

$$c_0\,(\partial_t p_h,\, p_h) + \alpha\,(\partial_t \operatorname{tr}(A\sigma_h),\, p_h)_Q + \alpha\,(\partial_t\operatorname{tr}(A\alpha p_h I),\, p_h)_Q + (\operatorname{div} z_h,\, p_h) = (g,\, p_h) \tag{3.5}$$

Combining (3.1)-(3.5), we get

$$\begin{aligned}(A\partial_t\sigma_h,\, \sigma_h)_Q + (A\alpha\partial_t pI,\, \sigma_h)_Q + \left(K^{-1}z_h,\, z_h\right)_Q + c_0\,(\partial_t p_h,\, p_h) + \alpha\,(\partial_t\operatorname{tr}(A\sigma_h),\, p_h)_Q \\ + \alpha\,(\partial_t\operatorname{tr}(A\alpha p_h I),\, p_h)_Q = \langle\partial_t\mathcal{P}_0 g_u,\, \sigma_h\, n\rangle + (f,\, \partial_t u_h) + \langle\mathcal{P}_0 g_p,\, z_h\cdot n\rangle + (g,\, p_h)\,.\end{aligned} \tag{3.6}$$

Using the definition of quadrature rule (2.55) and product rule, we can write the first term on the left hand side of (3.6) as follows

$$\begin{aligned}(A\partial_t\sigma_h,\, \sigma_h)_Q &= \sum_{E\in\mathcal{T}_h}(A\partial_t\sigma_h,\, \sigma_h)_{E,Q} = \sum_{E\in\mathcal{T}_h}(\mathcal{A}\partial_t\hat\sigma_h,\, \hat\sigma_h)_{\hat E,Q} = \sum_{E\in\mathcal{T}_h}\frac{|\hat E|}{s}\sum_{i=1}^{s}\mathcal{A}\partial_t\hat\sigma_h(\hat{\mathbf{r}}_i) : \hat\sigma_h(\hat{\mathbf{r}}_i) \\ &= \sum_{E\in\mathcal{T}_h}\frac{|\hat E|}{s}\sum_{i=1}^{s}\partial_t\mathcal{A}^{1/2}\hat\sigma_h(\hat{\mathbf{r}}_i) : \mathcal{A}^{1/2}\hat\sigma_h(\hat{\mathbf{r}}_i) = \frac{1}{2}\sum_{E\in\mathcal{T}_h}\frac{|\hat E|}{s}\partial_t\sum_{i=1}^{s}\mathcal{A}^{1/2}\hat\sigma_h(\hat{\mathbf{r}}_i) : \mathcal{A}^{1/2}\hat\sigma_h(\hat{\mathbf{r}}_i) \\ &= \sum_{E\in\mathcal{T}_h}\frac{1}{2}\partial_t\left(A^{1/2}\sigma_h,\, A^{1/2}\sigma_h\right)_{E,Q} = \frac{1}{2}\partial_t\left(A^{1/2}\sigma_h,\, A^{1/2}\sigma_h\right)_Q\end{aligned}$$

and (3.6) becomes:

$$\begin{aligned}\frac{1}{2}\partial_t\left(A^{1/2}\sigma_h,\, A^{1/2}\sigma_h\right)_Q + (A\alpha\partial_t pI,\, \sigma_h)_Q + \alpha\,(\partial_t\operatorname{tr}(A\sigma_h),\, p_h)_Q + \alpha\,(\partial_t\operatorname{tr}(A\alpha p_h I),\, p_h)_Q \\ + \|K^{-1/2}z_h\|_Q^2 + \frac{c_0}{2}\partial_t\|p_h\|^2 = \langle\partial_t\mathcal{P}_0 g_u,\, \sigma_h\, n\rangle + (f,\, \partial_t u_h) + \langle\mathcal{P}_0 g_p,\, z_h\cdot n\rangle + (g,\, p_h)\,.\end{aligned} \tag{3.7}$$

We use the identity

$$\operatorname{tr}(\tau)w = t : (wI), \quad \forall \tau\in\mathbb{M},\ w\in\mathbb{R}$$

to combine the first four terms on the left-hand side of (3.7):

$$\begin{aligned}&\frac{1}{2}\partial_t\left(A^{1/2}\sigma_h,\, A^{1/2}\sigma_h\right)_Q + (A\alpha\partial_t pI,\, \sigma_h)_Q + \alpha\,(\partial_t\operatorname{tr}(A\sigma_h),\, p_h)_Q + \alpha\,(\partial_t\operatorname{tr}(A\alpha p_h I),\, p_h)_Q \\ &= \frac{1}{2}\partial_t\left(A^{1/2}\sigma_h,\, A^{1/2}\sigma_h\right)_Q + \alpha\left(A^{1/2}\partial_t p_h I,\, A^{1/2}\sigma_h\right)_Q + \alpha\left(\partial_t A^{1/2}\sigma_h,\, A^{1/2}p_h I\right)_Q + \frac{\alpha^2}{2}\left(\partial_t A^{1/2}p_h I,\, \partial_t A^{1/2}p_h I\right)_Q \\ &= \frac{1}{2}\partial_t\left(A^{1/2}(\sigma_h + \alpha p_h I),\, A^{1/2}(\sigma_h+\alpha p_h I)\right)_Q = \frac{1}{2}\partial_t\|A^{1/2}(\sigma_h+\alpha p_h I)\|_Q^2.\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) and using product rule, we obtain

$$\begin{aligned}\frac{1}{2}\partial_t\left[\|A^{1/2}(\sigma_h+\alpha p_h I)\|_Q^2 + c_0\|p_h\|^2\right] + \|K^{-1/2}z_h\|_Q^2 &= \langle\partial_t\mathcal{P}_0 g_u,\, \sigma_h\, n\rangle + (f,\, \partial_t u_h) + \langle\partial_t\mathcal{P}_0 g_p,\, z_h\cdot n\rangle + (g,\, p_h) \\ &= \langle\partial_t\mathcal{P}_0 g_u,\, \sigma_h\, n\rangle + \partial_t\,(f,\, u_h) - (\partial_t f,\, u_h) + \langle\mathcal{P}_0 g_p,\, z_h\cdot n\rangle + (g,\, p_h)\,.\end{aligned} \tag{3.9}$$

Next we integrate (3.9) in time from 0 to an arbitrary $t\in(0,T]$:

$$\begin{aligned}\frac{1}{2}\left[\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|_Q^2 + c_0\|p_h(t)\|^2\right] + \int_0^t \|K^{-1/2}z_h(s)\|_Q^2\,ds = \int_0^t ((g(s),\, p_h(s)) - (\partial_t f(s),\, u_h(s)))\,ds \\ + \int_0^t (\langle\partial_t\mathcal{P}_0 g_u(s),\, \sigma_h(s)\, n\rangle + \langle\mathcal{P}_0 g_p(s),\, z_h(s)\cdot n\rangle)\,ds + \frac{1}{2}\left[\|A^{1/2}(\sigma_h(0)+\alpha p_h I(0))\|_Q^2 + c_0\|p_h(0)\|^2\right] \\ + (f(t),\, u_h(t)) + (f(0),\, u_h(0))\end{aligned}$$

and apply Cauchy-Schwartz and Young inequalities:

$$\frac{1}{2}\left[\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|_Q^2 + c_0\|p_h(t)\|^2\right] + \int_0^t \|K^{-1/2}z_h(s)\|_Q^2\,ds$$
$$\le \epsilon\left(\|u_h(t)\|^2 + \int_0^t (\|p_h(s)\|^2 + \|u_h(s)\|^2)\,ds\right) + \tilde{\epsilon}\int_0^t (\|\sigma_h(s)\,n\|_{-\frac{1}{2}}^2 + \|z_h\cdot n\|_{-\frac{1}{2}}^2)\,ds$$
$$+\frac{C}{\epsilon}\left(\|f(t)\|^2 + \int_0^t (\|g(s)\|^2 + \|\partial_t f(s)\|^2)\,ds\right) + \frac{C}{\tilde{\epsilon}}\int_0^t (\|\partial_t \mathcal{P}_0 g_u(s)\|_{\frac{1}{2}}^2 + \|\mathcal{P}_0 g_p(s)\|_{\frac{1}{2}}^2)\,ds$$
$$+\frac{1}{2}\left[\|A^{1/2}(\sigma_h(0)+\alpha p_h I(0))\|_Q^2 + c_0\|p_h(0)\|^2 + \|u_h(0)\|^2 + \|f(0)\|^2\right]. \tag{3.10}$$

Using the inf-sup condition [REF TO MSMFE INF-SUP] and (2.67), we get

$$\|u_h\| + \|\gamma_h\| \le C \sup_{0\ne\tau\in\mathbb{X}_h} \frac{(u_h,\operatorname{div}\tau) + (\gamma_h,\operatorname{as}\tau)_Q}{\|\tau\|_{\mathrm{div}}} = C\sup_{0\ne\tau\in\mathbb{X}_h}\frac{-\left(A^{1/2}(\sigma_h+\alpha p_h I),\,A^{1/2}\tau\right)_Q + \langle \mathcal{P}_0 g_u,\,\tau\,n\rangle}{\|\tau\|_{\mathrm{div}}}$$
$$\le C\|A^{1/2}(\sigma_h+\alpha p_h I)\| + \|\mathcal{P}_0 g_u\|_{\frac{1}{2}}, \tag{3.11}$$

where in the last step we used [EQUIVALENCE OF NORMS].
Similarly, using the inf-sup condition [REF TO MFMFE INF-SUP] and (2.70), we have

$$\|p_h\| \le C\sup_{0\ne q\in Z_h}\frac{(p_h,\operatorname{div}q)}{\|q\|_{\mathrm{div}}} = C\sup_{0\ne q\in Z_h}\frac{\left(K^{-1}z_h,\,q\right)_Q + \langle \mathcal{P}_0 g_p,\,q\cdot n\rangle}{\|q\|_{\mathrm{div}}} \le C\|K^{-1/2}z_h\| + \|\mathcal{P}_0 g_p\|_{\frac{1}{2}}. \tag{3.12}$$

Combining (3.10)-(3.12) and using [EQUIVALENCE OF NORMS], we obtain

$$\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|^2 + \|u_h(t)\|^2 + \|\gamma_h(t)\|^2 + c_0\|p_h(t)\|^2 + \int_0^t (\|K^{-1/2}z_h(s)\|^2 + \|p_h(s)\|^2)\,ds$$
$$\le C\Bigg[\epsilon\left(\|u_h(t)\|^2 + \int_0^t (\|p_h(s)\|^2 + \|u_h(s)\|^2)\,ds\right) + \tilde{\epsilon}\int_0^t (\|\sigma_h(s)\,n\|_{-\frac{1}{2}} + \|z_h(s)\cdot n\|_{-\frac{1}{2}})\,ds$$
$$+\frac{C}{\epsilon}\left(\|f(t)\|^2 + \int_0^t (\|g(s)\|^2 + \|\partial_t f(s)\|^2)\,ds\right) + \frac{C}{\tilde{\epsilon}}\int_0^t (\|\partial_t \mathcal{P}_0 g_u(s)\|_{\frac{1}{2}}^2 + \|\mathcal{P}_0 g_p(s)\|_{\frac{1}{2}}^2)\,ds$$
$$+C\left[\|A^{1/2}(\sigma_h(0)+\alpha p_h I(0))\|_Q^2 + c_0\|p_h(0)\|^2 + \|u_h(0)\|^2 + \|f(0)\|^2\right] + \|\mathcal{P}_0 g_u(t)\|_{\frac{1}{2}}^2\Bigg].$$

Choosing ϵ small enough, we get

$$\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|^2 + \|u_h(t)\|^2 + \|\gamma_h(t)\|^2 + c_0\|p_h(t)\|^2 + \int_0^t (\|K^{-1/2}z_h(s)\|^2 + \|p_h(s)\|^2)\,ds$$
$$\le C\Bigg[\epsilon\int_0^t \|u_h(s)\|^2\,ds + \tilde{\epsilon}\int_0^t (\|\sigma_h(s)\,n\|_{-\frac{1}{2}}^2 + \|z_h(s)\cdot n\|_{-\frac{1}{2}}^2)\,ds$$
$$+\frac{C}{\tilde{\epsilon}}\int_0^t (\|\partial_t \mathcal{P}_0 g_u(s)\|_{\frac{1}{2}}^2 + \|\mathcal{P}_0 g_p(s)\|_{\frac{1}{2}}^2)\,ds + \left(\|f(t)\|^2 + \int_0^t (\|g(s)\|^2 + \|\partial_t f(s)\|^2)\,ds\right) + \|\mathcal{P}_0 g_u(t)\|_{\frac{1}{2}}^2$$
$$+\|A^{1/2}(\sigma_h(0)+\alpha p_h I(0))\|_Q^2 + c_0\|p_h(0)\|^2 + \|u_h(0)\|^2 + \|f(0)\|^2\Bigg]. \tag{3.13}$$

Let us denote the right hand side of (3.13) by RHS_1. We proceed with deriving estimates for $\operatorname{div}\sigma_h$ and $\operatorname{div} z_h$.

Step 2: $H(div)$ in space estimate for stress:

Testing (2.68) with $v = \operatorname{div}\sigma_h$, we immediately obtain a bound on divergence of stress:

$$\|\operatorname{div}\sigma_h\| \le \|f\|. \tag{3.14}$$

On the other hand setting $\tau = s_h$, $v = u_h$, $\xi = g_h$ in (2.67)-(2.69) and using equivalence of norms [EQUIVALENCE OF NORMS], we obtain

$$\|\sigma_h\|^2 \le C(\|p\|^2 + \|\mathcal{P}_0 g_u\|_{\frac{1}{2}}^2 + \|f\|^2) + \epsilon(\|\sigma_h\,n\|_{-\frac{1}{2}}^2 + \|u\|^2) \tag{3.15}$$

We combine (3.14)-(3.15) and integrate in time:

$$\int_0^t (\|\sigma_h(s)\|^2 + \|\operatorname{div}\sigma_h(s)\|^2)\,ds \le C\int_0^t \left((\|p(s)\|^2 + \|\mathcal{P}_0 g_u(s)\|_{\frac{1}{2}}^2 + \|f(s)\|^2) + \epsilon(\|\sigma_h(s)\,n\|_{-\frac{1}{2}}^2 + \|u(s)\|^2)\right)\,ds$$

Using (3.11), we obtain

$$
\begin{aligned}
\int_0^t (\|\sigma_h(s)\|_{\text{div}}^2 + \|u_h(s)\|^2 + \|\gamma_h(s)\|^2)\, ds &\le C \int_0^t (\|p(s)\|^2 + \|\mathcal{P}_0 g_u(s)\|_{\frac{1}{2}}^2 + \|f(s)\|^2)\, ds \\
&\le RHS_1 + \int_0^t (\|\mathcal{P}_0 g_u(s)\|_{\frac{1}{2}}^2 + \|f(s)\|^2)\, ds
\end{aligned} \tag{3.16}
$$

Step 3: $H(div)$ in space estimate for velocity:
It follows from equation (2.71) and [EQUIVALENCE OF NORMS] that

$$
\|\operatorname{div} z_h\| \le C \left(c_0 \|\partial_t p_h\| + \|A^{1/2} \partial_t (\sigma_h + \alpha p_h I)\| + \|g\| \right). \tag{3.17}
$$

To obtain control on first two terms on the right hand side of (3.17), we differentiate equations (2.67)-(2.70) and and combine (2.67)-(2.71), as it was done in (3.1)-(3.10), with the choice $(\tau, v, \xi, q, w) = (\partial_t \sigma_h, \partial_t u_h, \partial_t \gamma_h, z_h, \partial_t p_h)$:

$$
\begin{aligned}
&\int_0^t \left(\|A^{1/2} \partial_t (\sigma_h(s) + \alpha p_h I(s))\|_Q^2 + c_0 \|\partial_t p_h(s)\|^2 \right) ds + \frac{1}{2} \|K^{-1/2} z_h(t)\|_Q^2 \\
&\le \int_0^t \left(\|p_h(s)\| \|\partial_t g(s)\| + \|\partial_t u_h(s)\| \|\partial_t f(s)\| + \|\sigma_h\, n\|_{-\frac{1}{2}} \|\partial_t \mathcal{P}_0 g_u\|_{\frac{1}{2}} + \|z_h \cdot n\|_{-\frac{1}{2}} \|\partial_t \mathcal{P}_0 g_p\|_{\frac{1}{2}} \right) ds \\
&\quad + \|p_h(t)\| \|g(t)\| + \frac{1}{2} \|K^{-1/2} z_h(0)\|_Q^2 - \|p_h(0)\| \|g(0)\|.
\end{aligned} \tag{3.18}
$$

Using the inf-sup condition [REF TO MSMFE INF-SUP] and (2.67), differentiated in time, we get

$$
\|\partial_t u_h\| + \|\partial_t \gamma_h\| \le C \|A^{1/2} \partial_t (\sigma_h + \alpha p_h I)\| + \|\partial_t \mathcal{P}_0 g_u\|_{\frac{1}{2}}. \tag{3.19}
$$

Combining (3.12), (3.19) and (3.18), we get:

$$
\begin{aligned}
&\int_0^t \left(\|A^{1/2} \partial_t (\sigma_h(s) + \alpha p_h I(s))\|^2 + \|\partial_t u_h(s)\|^2 + \|\partial_t \gamma_h(s)\|^2 + c_0 \|\partial_t p_h(s)\|^2 \right) ds + \|K^{-1/2} z_h(t)\|^2 + \|p_h(t)\|^2 \\
&\le \epsilon \left(\int_0^t (\|p_h(s)\|^2 + \|\partial_t u_h(s)\|^2)\, ds + \|p_h(t)\|^2 \right) + \tilde{\epsilon} \int_0^t (\|\sigma_h(s)\, n\|_{-\frac{1}{2}}^2 + \|z_h(s) \cdot n\|_{-\frac{1}{2}}^2)\, ds \\
&\quad + \frac{C}{\epsilon} \left(\int_0^t (\|\partial_t g(s)\|^2 + \|\partial_t f(s)\|^2)\, ds + \|g(t)\|^2 \right) + \frac{C}{\tilde{\epsilon}} \int_0^t (\|\partial_t \mathcal{P}_0 g_u(s)\|_{\frac{1}{2}}^2 + \|\partial_t \mathcal{P}_0 g_p(s)\|_{\frac{1}{2}}^2)\, ds \\
&\quad + C(\|z_h(0)\|^2 + \|p_h(0)\|^2 + \|g(0)\|^2).
\end{aligned}
$$

Choosing ϵ small enough, we obtain

$$
\begin{aligned}
&\int_0^t \left(\|A^{1/2} \partial_t (\sigma_h(s) + \alpha p_h I(s))\|^2 + \|\partial_t u_h(s)\|^2 + \|\partial_t \gamma_h(s)\|^2 + c_0 \|\partial_t p_h(s)\|^2 \right) ds + \|K^{-1/2} z_h(t)\|^2 + \|p_h(t)\|^2 \\
&\le \tilde{\epsilon} \int_0^t (\|\sigma_h(s)\, n\|_{-\frac{1}{2}}^2 + \|z_h(s) \cdot n\|_{-\frac{1}{2}}^2)\, ds + \frac{C}{\tilde{\epsilon}} \int_0^t (\|\partial_t \mathcal{P}_0 g_u(s)\|_{\frac{1}{2}}^2 + \|\partial_t \mathcal{P}_0 g_p(s)\|_{\frac{1}{2}}^2)\, ds \\
&\quad + C \left(\int_0^t (\|\partial_t g(s)\|^2 + \|\partial_t f(s)\|^2)\, ds + \|g(t)\|^2 + \|z_h(0)\|^2 + \|p_h(0)\|^2 + \|g(0)\|^2 + RHS_1 \right).
\end{aligned} \tag{3.20}
$$

Integrating (3.17) in time and using (3.20), we get

$$
\begin{aligned}
&\int_0^t \|\operatorname{div} z_h(s)\|^2\, ds + \|K^{-1/2} z_h(t)\|^2 + \|p_h(t)\|^2 \\
&\le \tilde{\epsilon} \int_0^t (\|\sigma_h(s)\, n\|_{-\frac{1}{2}}^2 + \|z_h(s) \cdot n\|_{-\frac{1}{2}}^2)\, ds + \frac{C}{\tilde{\epsilon}} \int_0^t (\|\partial_t \mathcal{P}_0 g_u(s)\|_{\frac{1}{2}}^2 + \|\partial_t \mathcal{P}_0 g_p(s)\|_{\frac{1}{2}}^2)\, ds \\
&\quad + C \left(\int_0^t (\|g(s)\|^2 + \|\partial_t g(s)\|^2 + \|\partial_t f(s)\|^2)\, ds + \|g(t)\|^2 + \|z_h(0)\|^2 + \|p_h(0)\|^2 + \|g(0)\|^2 + RHS_1 \right).
\end{aligned} \tag{3.21}
$$

We note that initial condition for Darcy velocity can be computed as a suitable projection of $-K\nabla p(0)$, provided true initial condition is regular enough.

Step 4: combine everything and separate stress and pressure variables:
We combine (3.13), (3.16) and (3.21):

$$\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|^2+\|u_h(t)\|^2+\|\gamma_h(t)\|^2+\|z_h(t)\|^2+\|p_h(t)\|^2$$
$$+\int_0^t(\|\sigma_h(s)\|^2_{\mathrm{div}}+\|u_h(s)\|^2+\|\gamma_h(s)\|^2+\|z_h(s)\|^2_{\mathrm{div}}+\|p_h(s)\|^2)\,ds\le C\Big[\epsilon\int_0^t\|u_h(s)\|^2\,ds$$
$$+\int_0^t(\|\mathcal{P}_0 g_u(s)\|_{\frac12}+\|\partial_t\mathcal{P}_0 g_u(s)\|_{\frac12}+\|\mathcal{P}_0 g_p(s)\|_{\frac12}+\|\partial_t\mathcal{P}_0 g_p(s)\|_{\frac12}+\|g(s)\|^2+\|\partial_t g(s)\|^2+\|f(s)\|^2+\|\partial_t f(s)\|^2)\,ds$$
$$+\|f(t)\|^2+\|g(t)\|^2+\|\mathcal{P}_0 g_u(t)\|_{\frac12}+\|f(0)\|^2+\|g(0)\|^2$$
$$+\|A^{1/2}(\sigma_h(0)+\alpha p_h I(0))\|^2_Q+\|p_h(0)\|^2+\|u_h(0)\|^2+\|z_h(0)\|^2\Big]. \tag{3.22}$$

Finally, we note that we can obtain an estimate on $\|\sigma_h(t)\|$ as follows:

$$\begin{aligned}\|\sigma_h(t)\|\le C\|A^{1/2}\sigma_h(t)\|&\le C\left(\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|+\|A^{1/2}\alpha p_h I(t)\|\right)\\&\le C\left(\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|+\|p_h(t)\|\right)\end{aligned} \tag{3.23}$$

Then, (3.23) together with (3.14) yield

$$\|\sigma_h(t)\|_{\mathrm{div}}\le C\left(\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|+\|p_h(t)\|+\|f(t)\|\right) \tag{3.24}$$

Combination of (3.22)-(3.24) together with [REF TO STABILITY OF L2 PROJECTION] results in the following result.

Theorem 3.1. *Let $(\sigma_h,u_h,\gamma_h,z_h,p_h)\in\mathbb{X}_h\times V_h\times\Theta_h\times Z_h\times W_h$ be the solution of* (2.67)-(2.71). *Then the following stability estimate holds:*

$$\begin{aligned}&\|\sigma_h\|_{L^\infty(0,T;H(\mathrm{div},\Omega))}+\|u_h\|_{L^\infty(0,T;L^2(\Omega))}+\|\gamma_h\|_{L^\infty(0,T;L^2(\Omega))}+\|z_h\|_{L^\infty(0,T;L^2(\Omega))}+\|p_h\|_{L^\infty(0,T;L^2(\Omega))}\\&+\|\sigma_h\|_{L^2(0,T;H(\mathrm{div},\Omega))}+\|u_h\|_{L^2(0,T;L^2(\Omega))}+\|\gamma_h\|_{L^2(0,T;L^2(\Omega))}+\|z_h\|_{L^2(0,T;H(\mathrm{div},\Omega))}+\|p_h\|_{L^2(0,T;L^2(\Omega))}\\&\le C\Big[\|p_h(0)\|+\|\sigma_h(0)\|+\|u_h(0)\|+\|z_h(0)\|+\|f\|_{L^\infty(0,T;L^2(\Omega))}+\|f\|_{H^1(0,T;L^2(\Omega))}+\|g_p\|_{H^1(0,T;H^{1/2}(\partial\Omega))}\\&\quad+\|g\|_{L^\infty(0,T;L^2(\Omega))}+\|g\|_{H^1(0,T;L^2(\Omega))}+\|g_u\|_{L^\infty(0,T;H^{1/2}(\partial\Omega))}+\|g_u\|_{H^1(0,T;H^{1/2}(\partial\Omega))}\Big].\end{aligned} \tag{3.25}$$

4 Error analysis

4.1 Preliminaries

Due to the reduced approximation properties of the MFE spaces on general quadrilaterals [CITE [8] FROM MFMFE], we restrict the quadrilateral elements to be $O(h^2)$-perturbations of parallelograms:

$$\|\mathbf{r}_{34}-\mathbf{r}_{21}\|\le Ch^2.$$

In this case it is easy to verify that

$$|DF_E|_{1,\infty,\hat E}\le Ch^2\quad\text{and}\quad\left|\frac{1}{J_E}DF_E\right|_{j,\infty,\hat E}\le Ch^{j-1},\ j=1,2. \tag{4.1}$$

We introduce the L^2-projection operators $Q^0:L^2(\Omega)\to\mathcal{Q}_0$ and $Q^1:L^2(\Omega)\to\mathcal{Q}_1$ satisfying

$$(\phi-Q^0\phi,\psi_h)=0,\qquad\forall\psi_h\in\mathcal{Q}_0, \tag{4.2}$$
$$(\phi-Q^1\phi,\psi_h)=0,\qquad\forall\psi_h\in\mathcal{Q}_1. \tag{4.3}$$

We will use Q^0 operator for the approximation of displacement and pressure variables and Q^1 for the approximation of rotation.

Next lemma summarizes continuity and approximation properties of the projection operators.

Lemma 4.1. *There exists a constant $C > 0$ such that on simplices and h^2-parallelograms*

$$\|\phi - Q^0\phi\| \le C\|\phi\|_r h^r, \qquad \forall \phi \in H^r(\Omega), \qquad 0 \le r \le 1, \tag{4.4}$$
$$\|\phi - Q^1\phi\| \le C\|\phi\|_r h^r, \qquad \forall \phi \in H^r(\Omega), \qquad 0 \le r \le 1, \tag{4.5}$$
$$\|\psi - \Pi\psi\| \le C\|\psi\|_r h^r, \qquad \forall \psi \in H^r(\Omega), \qquad 1 \le r \le 2, \tag{4.6}$$
$$\|\psi - \Pi^0\psi\| \le C\|\psi\|_1 h, \qquad \forall \psi \in H^1(\Omega), \tag{4.7}$$
$$\|\operatorname{div}(\psi - \Pi\psi)\| + \|\operatorname{div}(\psi - \Pi^0\psi)\| \le C\|\operatorname{div}\psi\|_r h^r, \qquad \forall \psi \in H^{r+1}(\Omega), \qquad 0 \le r \le 1. \tag{4.8}$$

Moreover, for all elements $E \in \mathcal{T}_h$, there exists a constant $c > 0$, such that

$$\|\Pi\psi\|_{j,E} \le C\|\psi\|_j, \qquad \forall \psi \in H^j(\Omega), \quad j = 1, 2, \tag{4.9}$$
$$\|\Pi^0\psi\|_{1,E} \le C\|\psi\|_1, \qquad \forall \psi \in H^1(\Omega). \tag{4.10}$$

Proof. Proof of bounds for the L^2-projections (4.4)-(4.5) can be found in [CITE [18] from MFMFE]; and bounds (4.6)-(4.8) can be found in [CITE [15, 28] from MFMFE] for affine elements and [CITE [32, 8] from MFMFE] for h^2-parallelograms. Finally, for the proof of (4.9)-(4.10) was presented in [CITE MFMFE]. □

In the following lemma we state bounds for the terms arising from the use of quadrature rule.

Lemma 4.2. *If $K^{-1} \in W^{1,\infty}_{\mathcal{T}_h}$ and $A \in W^{1,\infty}_{\mathcal{T}_h}$, then there is a constant $C > 0$ such that*

$$|\theta\left(K^{-1}q,\, v\right)| \le C \sum_{E \in \mathcal{T}_h} h\|K^{-1}\|_{1,\infty,E}\|q\|_{1,E}\|v\|_E, \qquad \forall q \in V_h,\, v \in V_h^0, \tag{4.11}$$
$$|\theta\left(A\tau,\, \chi + wI\right)| \le C \sum_{E \in \mathcal{T}_h} h\|A\|_{1,\infty,E}\|\tau\|_{1,E}\|\chi + wI\|_E, \qquad \forall \tau \in \mathbb{X}_h,\, \chi \in \mathbb{X}_h^0, w \in W_h, \tag{4.12}$$
$$|\theta\left(Ap,\, w\right)| \le C \sum_{E \in \mathcal{T}_h} h\|A\|_{1,\infty,E}\|p\|_E\|w\|_E, \qquad \forall p, w \in W_h, \tag{4.13}$$
$$|\theta\left(\operatorname{as}\,\chi,\, \xi\right)| \le C \sum_{E \in \mathcal{T}_h} h\|\chi\|_{1,E}\|\xi\|_E, \qquad \forall \chi \in \mathbb{X}_h^0, \xi \in \Theta_h. \tag{4.14}$$

Moreover, on h^2-parallelograms, if $K^{-1} \in W^{1,\infty}_{\mathcal{T}_h}$ and $A \in W^{1,\infty}_{\mathcal{T}_h}$, there is a constant $c > 0$ such that

$$|\left(K^{-1}\Pi u,\, v - \Pi^0 v\right)_Q| \le ch\|q\|_1\|v\|, \qquad v \in V_h, \tag{4.15}$$
$$|\left(A(\Pi\sigma + Q^0 p),\, \chi - \Pi^0\chi\right)_Q| \le ch(\|\sigma\|_1 + \|p\|)\|\chi\|, \qquad \forall \chi \in \mathbb{X}_h, \tag{4.16}$$
$$|\left(\operatorname{as}(\chi - \Pi^0\chi),\, Q^1\gamma\right)_Q| \le ch\|\gamma\|_1\|\chi\|, \qquad \forall \chi \in \mathbb{X}_h. \tag{4.17}$$

Proof. The estimates (4.11) and (4.15) can be found in [CITE MFMFE], while (4.12), (4.14), (4.16) and (4.17) were proven in [CITE MSMFE] for $p = w = 0$.

Next we prove (4.12) for the case $w \neq 0$. We note that (4.13) can be obtained in the say way. We compute for any $E \in \mathcal{T}_h$

$$|\theta\left(A\tau,\, wI\right)_E| = |\theta\left(\hat{A}DF_E\hat{\tau},\, \hat{w}I\right)_{\hat{E}}| \le |\theta\left((\hat{A}DF_E - \overline{\hat{A}DF_E})\hat{\tau},\, \hat{w}I\right)_{\hat{E}}| + |\theta\left(\overline{\hat{A}DF_E}\hat{\tau},\, \hat{w}I\right)_{\hat{E}}|,$$

where the overline notation stands for the mean value. For the first term on the right hand side, we use Taylor expansion, (**??**) and (4.1):

$$\begin{aligned} |\theta\left((\hat{A}\,DF_E - \overline{\hat{A}\,DF_E})\hat{\tau},\, \hat{w}I\right)_{\hat{E}}| &\le C|\hat{A}\,DF_E|_{1,\infty,\hat{E}}\|\hat{\tau}\|_{\hat{E}}\|\hat{w}\|_{\hat{E}} \\ &\le C(|\hat{A}\,|_{1,\infty,\hat{E}}\|DF_E\|_{0,\infty,\hat{E}} + |DF_E|_{1,\infty,\hat{E}}\|\hat{A}\|_{0,\infty,\hat{E}})\|\hat{\tau}\|_{\hat{E}}\|\hat{w}\|_{\hat{E}} \\ &\le Ch\|A\|_{1,\infty,E}\|\tau\|_E\|w\|_E. \end{aligned} \tag{4.18}$$

For the second term we note that since the quadrature rule is exact for linears, $\theta\left(\overline{\hat{A}DF_E}\hat{\Pi}^0\hat{\tau},\, \hat{w}I\right)_{\hat{E}} = 0$. Therefore, using (**??**) and (4.7) we obtain

$$\begin{aligned} |\theta\left(\overline{\hat{A}DF_E}\hat{\tau},\, \hat{w}I\right)_{\hat{E}}| = |\theta\left(\overline{\hat{A}DF_E}(\hat{\tau} - \hat{\Pi}^0\hat{\tau}),\, \hat{w}I\right)_{\hat{E}}| &\le C\|\hat{A}\,DF_E\|_{0,\infty,\hat{E}}\|\hat{\tau} - \hat{\Pi}^0\hat{\tau}\|_{\hat{E}}\|\hat{w}\|_{\hat{E}} \\ &\le Ch\|A\|_{0,\infty,E}\|\tau\|_{1,E}\|w\|_E. \end{aligned} \tag{4.19}$$

Combining (4.18)-(4.19) and summing over all $E \in \mathcal{T}_h$, we get

$$|\theta\,(A\tau,\, wI)\,| \leq C \sum_{E\in\mathcal{T}_h} h\|A\|_{1,\infty,E}\|\tau\|_{1,E}\|w\|_E,$$

as desired. We use similar arguments to prove (4.16) with nonzero p. First, we write:

$$\begin{aligned}
|\,(A\,Q^0p,\, \chi - \Pi^0\chi)_{Q,E}\,| &= |\left(DF_E^T\,\hat{A}\,\widehat{Q^0p},\, \hat{\chi} - \hat{\Pi}^0\hat{\chi}\right)_{\hat{Q},\hat{E}}| \\
&\leq |\left(\overline{DF_E^T\,\hat{A}}\,\widehat{Q^0p},\, \hat{\chi} - \hat{\Pi}^0\hat{\chi}\right)_{\hat{Q},\hat{E}}| + |\left((DF_E^T\,\hat{A} - \overline{DF_E^T\,\hat{A}})\,\widehat{Q^0p},\, \hat{\chi} - \hat{\Pi}^0\hat{\chi}\right)_{\hat{Q},\hat{E}}|
\end{aligned}$$

The first term on right is equal to zero due to [REF TO LEMMA AS (2.40) IN MFMFE]. For the second term we use Taylor expansion, [REF TO EQUIVALENCE OF NORMS], (??), [REF TO L2 STABILITY OF Q0 AND PI0]:

$$\begin{aligned}
|\left((DF_E^T\,\hat{A} - \overline{DF_E^T\,\hat{A}})\,\widehat{Q^0p},\, \hat{\chi} - \hat{\Pi}^0\hat{\chi}\right)_{\hat{Q},\hat{E}}| &\leq C|DF_E^T\,\hat{A}|_{1,\infty,\hat{E}}\|\widehat{Q^0p}\|_{\hat{E}}\|\hat{\chi} - \hat{\Pi}^0\hat{\chi}\|_{\hat{E}} \\
&\leq Ch\|p\|_E\|\chi\|_E.
\end{aligned}$$

□

4.2 Optimal convergence

We form the error system by subtracting the discrete problem (2.67)-(2.71) from the continuous one [REF TO CONTINUOUS EQUATIONS]:

$$\begin{aligned}
(A\sigma,\,\tau) - (A\sigma_h,\,\tau)_Q + (A\alpha pI,\,\tau) - (A\alpha p_hI,\,\tau)_Q + (u-u_h,\,\operatorname{div}\tau)& \\
+ (\gamma,\,\text{as}\,\tau) - (\gamma_h,\,\text{as}\,\tau)_Q = \langle g_u - Qg_u,\,\tau\,n\rangle,& \qquad \forall \tau \in \mathbb{X}_h, \qquad (4.20)\\
(\operatorname{div}\sigma - \operatorname{div}\sigma_h,\, v) = 0,& \qquad \forall v \in V_h, \qquad (4.21)\\
(\text{as}\,\sigma,\,\xi) - (\text{as}\,\sigma_h,\,\xi)_Q = 0,& \qquad \forall \xi \in \Theta_h, \qquad (4.22)\\
\left(K^{-1}z,\,q\right) - \left(K^{-1}z_h,\,q\right)_Q - (p-p_h,\,\operatorname{div}q) = \langle g_p - Qg_p,\,q\cdot n\rangle,& \qquad \forall q \in Z_h, \qquad (4.23)\\
c_0\,(\partial_t p - \partial_t p_h,\,w) + \alpha\,(\partial_t\operatorname{tr}(A\sigma),\,w) - \alpha\,(\partial_t\operatorname{tr}(A\sigma_h),\,w)_Q + \alpha\,(\partial_t\operatorname{tr}(A\alpha pI),\,w)& \\
-\alpha\,(\partial_t\operatorname{tr}(A\alpha p_hI),\,w)_Q + (\operatorname{div}z - \operatorname{div}z_h,\,w) = 0,& \qquad \forall w \in W_h. \qquad (4.24)
\end{aligned}$$

We split the errors into approximation and truncation errors as follows:

$$\begin{aligned}
e_s &= \sigma - \sigma_h = (\sigma - \Pi\sigma) + (\Pi\sigma - \sigma_h) := \psi_s + \phi_s,\\
e_u &= u - u_h = (u - Q^0u) + (Q^0u - u_h) := \psi_u + \phi_u,\\
e_\gamma &= \gamma - \gamma_h = (\gamma - Q^1\gamma) + (Q^1\gamma - \gamma_h) := \psi_\gamma + \phi_\gamma,\\
e_z &= z - z_h = (z - \Pi z) + (\Pi z - z_h) := \psi_z + \phi_z,\\
e_p &= p - p_h = (p - Q^0p) + (Q^0p - p_h) := \psi_p + \phi_p.
\end{aligned}$$

Step 1: L^2 in space estimates:
With these notations we can rewrite the first equation (4.20) in the error system in the following way:

$$\begin{aligned}
(A\phi_s,\,\tau)_Q + \alpha\,(A\phi_pI,\,\tau)_Q + (\phi_u,\,\operatorname{div}\tau) + (\phi_\gamma,\,\text{as}\,\tau)_Q &= (A\Pi\sigma,\,\tau)_Q - (A\sigma,\,\tau) + \alpha\left(AQ^0pI,\,\tau\right)_Q - \alpha\,(ApI,\,\tau)\\
&\quad + (\psi_u,\,\operatorname{div}\tau) + \left(Q^1\gamma,\,\text{as}\,\tau\right)_Q - (\gamma,\,\text{as}\,\tau) + \langle g_u - Qg_u,\,\tau\,n\rangle.
\end{aligned}$$

It follows from the definition of operator Q^0 (4.2) that $(\psi_u,\,\operatorname{div}\tau) = 0$. Combining the rest of the terms, we write

$$\begin{aligned}
(A\phi_s,\,\tau)_Q + \alpha\,(A\phi_pI,\,\tau)_Q + (\phi_u,\,\operatorname{div}\tau) + (\phi_\gamma,\,\text{as}\,\tau)_Q &= -\left(A(\sigma+\alpha pI),\,\tau - \Pi^0\tau\right) - \left(A(\psi_s + \alpha\psi_pI),\,\Pi^0\tau\right)\\
-\left(A(\Pi\sigma + \alpha Q^0pI),\,\Pi^0\tau\right) + \left(A(\Pi\sigma + \alpha Q^0pI),\,\Pi^0\tau\right)_Q &+ \left(A(\Pi\sigma + \alpha Q^0pI),\,\tau - \Pi^0\tau\right)_Q - \left(\gamma,\,\text{as}\,(\tau - \Pi^0\tau)\right)\\
-\left(\psi_\gamma,\,\text{as}\,\Pi^0\tau\right) - \left(Q^1\gamma,\,\text{as}\,\Pi^0\tau\right) + \left(Q^1\gamma,\,\text{as}\,\Pi^0\tau\right)_Q &+ \left(Q^1\gamma,\,\text{as}\,(\tau - \Pi^0\tau)\right)_Q + \langle g_u,\,(\tau - \Pi^0\tau)\,n\rangle, \qquad (4.25)
\end{aligned}$$

where we used [REF TO ORTHOGONALITY PROP OF L2 PROJECTION TO THE BOUNDARY SPACE]. Taking $\tau - \Pi^0\tau$ as a test function in REF TO CONTINUOUS FIRST EQ IN ELASTICITY, we obtain

$$\left(A(\sigma + \alpha pI),\,\tau - \Pi^0\tau\right) + \left(u,\,\operatorname{div}(\tau - \Pi^0\tau)\right) + \left(\gamma,\,\text{as}\,(\tau - \Pi^0\tau)\right) = \langle g_u,\,(\tau - \Pi^0\tau)\,n\rangle.$$

Hence, due to [REF TO PROPERTY OF PI AND PI* AS 2.22 in MFMFE3D],

$$-\left(A(\sigma+\alpha pI),\,\tau-\Pi^0\tau\right)-\left(\gamma,\,\mathrm{as}\,(\tau-\Pi^0\tau)\right)+\langle g_u,\,(\tau-\Pi^0\tau)\,n\rangle=0. \tag{4.26}$$

Combining (4.25)-(4.26) and rewriting terms, coming from the use of quadrature rule, we get

$$\begin{aligned}&(A\phi_s,\,\tau)_Q+\alpha\,(A\phi_pI,\,\tau)_Q+(\phi_u,\,\mathrm{div}\,\tau)+(\phi_\gamma,\,\mathrm{as}\,\tau)_Q=-\left(A(\psi_s+\alpha\psi_pI),\,\Pi^0\tau\right)-\left(\psi_\gamma,\,\mathrm{as}\,\Pi^0\tau\right)\\&-\theta\left(A\Pi\sigma,\,\Pi^0\tau\right)-\theta\left(A\alpha Q^0pI,\,\Pi^0\tau\right)-\theta\left(Q^1\gamma,\,\mathrm{as}\,\Pi^0\tau\right)+\left(A(\Pi\sigma+\alpha Q^0pI),\,\tau-\Pi^0\tau\right)_Q+\left(Q^1\gamma,\,\mathrm{as}\,(\tau-\Pi^0\tau)\right)_Q.\end{aligned} \tag{4.27}$$

Using [REF TO ORTHOGONALITY PROP OF PI OPERATOR] and (4.21) that

$$\mathrm{div}\,\phi_s=0. \tag{4.28}$$

From [REFERENCES TO THIRD CONTINUOUS AND DISCRETE EQS IN ELASTICICTY AND PROP OF PI(choose it to be weakly orthogonal to rotation)] we have:

$$(\mathrm{as}\,\phi_s,\,\xi)_Q=(\mathrm{as}\,\Pi\sigma,\,\xi)_Q-(\mathrm{as}\,\sigma_h,\,\xi)_Q=0, \tag{4.29}$$

where we used [REF TO PROP OF PI(choose it to be weakly orthogonal to rotation)] We rewrite (4.23) similarly to how it was done in (4.25)-(4.27):

$$\begin{aligned}\left(K^{-1}\phi_z,\,q\right)_q-(\phi_p,\,\mathrm{div}\,q)=(\psi_p,\,\mathrm{div}\,q)-\left(K^{-1}z,\,q-\Pi^0q\right)-\left(K^{-1}(z-\Pi z),\,\Pi^0q\right)-\left(K^{-1}\Pi z,\,\Pi^0q\right)\\+\left(K^{-1}\Pi z,\,\Pi_0q\right)_Q+\left(K^{-1}\Pi z,\,q-\Pi^0q\right)_Q-\langle g_p,\,(q-\Pi^0q)\cdot n\rangle.\end{aligned}$$

Using (4.2), we conclude that $(\psi_p,\,\mathrm{div}\,q)=0$. Moreover, testing REF TO CONTINUOUS FIRST EQ IN ELASTICITY with $q-\Pi^0q$, we also obtain

$$-\left(K^{-1}z,\,q-\Pi^0q\right)-\langle g_p,\,(q-\Pi^0q)\cdot n\rangle=0.$$

Hence, we have

$$\left(K^{-1}\phi_z,\,q\right)_Q-(\phi_p,\,\mathrm{div}\,q)=-\left(K^{-1}\psi_z,\,\Pi^0q\right)-\theta\left(K^{-1}\Pi z,\,\Pi^0q\right)+\left(K^{-1}\Pi z,\,q-\Pi^0q\right)_Q. \tag{4.30}$$

Finally, using (4.2) and [REF TO ORTHOGONALITY PROP OF PI OPERATOR], we rewrite the last equation, (4.24), in the error system as follows

$$\begin{aligned}c_0\,(\partial_t\phi_p,\,w)+\alpha\,(\partial_t\,\mathrm{tr}\,(A\phi_s),\,w)_Q+\alpha^2\,(\partial_t\,\mathrm{tr}\,(A\phi_p),\,w)_Q+(\mathrm{div}\,\phi_z,\,w)-\alpha\,(\partial_t\,\mathrm{tr}\,(A\psi_s),\,w)\\=-\alpha\theta\,(\partial_t\,\mathrm{tr}\,(A\Pi\sigma),\,w)-\alpha^2\,(\partial_t\,\mathrm{tr}\,(A\psi_pI),\,w)-\alpha^2\theta\,(\partial_t\,\mathrm{tr}\,(AQ^0pI),\,w)\,.\end{aligned} \tag{4.31}$$

Next we differentiate (4.27), set $\tau=\phi_s$, $\xi=\partial_t\phi_\gamma$, $q=\phi_z$, $w=\phi_p$ and combine (4.27)-(4.30):

$$\begin{aligned}&\frac{1}{2}\partial_t\left[\|A^{1/2}(\phi_s+\alpha\phi_pI)\|_Q^2+c_0\|\phi_p\|^2\right]+\left(K^{-1}\phi_z,\,\phi_z\right)_Q=\\&-\left(A\partial_t(\psi_s+\alpha\psi_pI),\,\Pi^0\phi_s\right)-\left(\partial_t\psi_\gamma,\,\mathrm{as}\,\Pi^0\phi_s\right)-\theta\left(A\partial_t\Pi\sigma,\,\Pi^0\phi_s+\alpha\phi_pI\right)-\theta\left(\partial_tQ^1\gamma,\,\mathrm{as}\,\Pi^0\phi_s\right)\\&+\left(A\partial_t(\Pi\sigma+\alpha Q^0pI),\,\phi_s-\Pi^0\phi_s\right)_Q+\left(\partial_tQ^1\gamma,\,\mathrm{as}\,(\phi_s-\Pi^0\phi_s)\right)_Q-\left(K^{-1}\psi_z,\,\Pi^0\phi_z\right)-\theta\left(K^{-1}\Pi z,\,\Pi^0\phi_z\right)\\&+\left(K^{-1}\Pi z,\,\phi_z-\Pi^0\phi_z\right)_Q-\alpha\,(\partial_t\,\mathrm{tr}\,(A\psi_s),\,\phi_p)-\alpha^2\,(\partial_t\,\mathrm{tr}\,(A\psi_pI),\,\phi_p)-\alpha\theta\left(\partial_tAQ^0pI,\,\Pi^0\phi_s+\alpha\phi_p\right).\end{aligned} \tag{4.32}$$

Using (4.4)-(4.6) and (4.10), we have

$$\begin{aligned}&\left|\left(A\partial_t(\psi_s+\alpha\psi_pI),\,\Pi^0\phi_s\right)+\left(\partial_t\psi_\gamma,\,\mathrm{as}\,\Pi^0\phi_s\right)+\left(K^{-1}\psi_z,\,\Pi^0\phi_z\right)+\alpha\,(\partial_t\,\mathrm{tr}\,(A\psi_s),\,\phi_p)-\alpha^2\,(\partial_t\,\mathrm{tr}\,(A\psi_pI),\,\phi_p)_Q\right|\\&\qquad\le Ch^2(\|\partial_t\sigma\|_1^2+\|\partial_tp\|_1^2+\|\partial_t\gamma\|_1^2+\|z\|_1^2)+\epsilon(\|\phi_s\|^2+\|\phi_p\|^2+\|\phi_z\|^2).\end{aligned} \tag{4.33}$$

Applying (4.11)-(4.14) and [REF TO CONTINUITY PROPERTY OF PI, Q AND PI0]

$$\begin{aligned}&\left|\theta\left(A\partial_t\Pi\sigma,\,\Pi^0\phi_s+\alpha\phi_pI\right)+\theta\left(K^{-1}\Pi z,\,\Pi^0\phi_z\right)-\alpha\theta\left(\partial_tAQ^0pI,\,\Pi^0\phi_s+\alpha\phi_p\right)-\theta\left(\partial_tQ^1\gamma,\,\mathrm{as}\,\Pi^0\phi_s\right)\right|\\&\qquad\le Ch^2(\|\partial_t\sigma\|_1^2+\|z\|_1^2+\|\partial_tp\|_0^2+\|\partial_t\gamma\|_0^2)+\epsilon(\|\phi_s\|^2+\|\phi_p\|^2+\|\phi_z\|^2).\end{aligned} \tag{4.34}$$

Due to (4.15) -(4.17) ,

$$\begin{aligned}&\left|\left(A\partial_t(\Pi\sigma+\alpha Q^0pI),\,\phi_s-\Pi^0\phi_s\right)_Q+\left(\partial_tQ\gamma,\,\mathrm{as}\,(\phi_s-\Pi^0\phi_s)\right)_Q+\left(K^{-1}\Pi z,\,\phi_z-\Pi^0\phi_z\right)_Q\right|\\&\qquad\le Ch^2(\|\partial_t\sigma\|_1^2+\|\partial_tp\|_1^2+\|\partial_t\gamma\|_1^2+\|z\|_1^2)+\epsilon(\|\phi_s\|^2+\|\phi_z\|^2).\end{aligned} \tag{4.35}$$

Next, we combine (4.32)-(4.35) and integrate the result in time from 0 to arbitrary $t \in (0, T]$:

$$\|A^{1/2}(\phi_s(t) + \alpha\phi_p I(t))\|_Q^2 + c_0\|\phi_p(t)\|^2 + \int_0^t \|K^{-1/2}\phi_z(s)\|_Q^2\, ds$$
$$\leq \epsilon \int_0^t (\|\phi_s(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_z(s)\|^2)\, ds + Ch^2 \int_0^t (\|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2 + \|z(s)\|_1^2)\, ds$$
$$+\|A^{1/2}(\phi_s(0) + \alpha\phi_p I(0))\|_Q^2 + c_0\|\phi_p(0)\|^2. \tag{4.36}$$

Choosing $\sigma_h(0) = \Pi\sigma(0)$ and $p_h(0) = Q^0 p(0)$, we obtain

$$\|A^{1/2}(\phi_s(0) + \alpha\phi_p I(0))\|_Q^2 + c_0\|\phi_p(0)\|^2 = 0. \tag{4.37}$$

Hence, we can write (4.36) as

$$\|A^{1/2}(\phi_s(t) + \alpha\phi_p I(t))\|_Q^2 + c_0\|\phi_p(t)\|^2 + \int_0^t \|K^{-1/2}\phi_z(s)\|_Q^2\, ds$$
$$\leq \epsilon \int_0^t (\|\phi_s(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_z(s)\|^2)\, ds + Ch^2 \int_0^t (\|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2 + \|z(s)\|_1^2)\, ds. \tag{4.38}$$

Using the inf-sup condition [REF TO MSMFE INF-SUP] and (4.20), we get

$$\|\phi_u\| + \|\phi_\gamma\| \leq C \sup_{0\neq\tau\in\mathbb{X}_h} \frac{(\phi_u, \operatorname{div}\tau) + (\phi_\gamma, \operatorname{as}\tau)_Q}{\|\tau\|_{\operatorname{div}}} = C \sup_{0\neq\tau\in\mathbb{X}_h} \Big(\frac{(A(\sigma_h + \alpha p_h I), \tau)_Q - (A(\sigma + \alpha p I), \tau)}{\|\tau\|_{\operatorname{div}}}$$
$$+ \frac{\left(Q^1\gamma, \operatorname{as}\tau\right) - (\gamma, \operatorname{as}\tau)_Q + \langle g_u - Q^0 g_u, \tau\, n\rangle}{\|\tau\|_{\operatorname{div}}} \Big). \tag{4.39}$$

Using the calculations in (4.25)-(4.27) and [REF TO ORTHOGONALITY PROP OF Q AND PI0 ON THE BOUNDARY] and Lemma (4.2) , we have

$$\begin{aligned}
&(A(\sigma_h + \alpha p_h I), \tau)_Q - (A(\sigma + \alpha p I), \tau) + \left(Q^1\gamma, \operatorname{as}\tau\right) - (\gamma, \operatorname{as}\tau)_Q + \langle g_u - Q^0 g_u, \tau\, n\rangle \\
&\quad = -\left(A(\phi_s + \alpha\phi_p I), \tau\right)_Q - \left(A(\psi_s + \alpha\psi_p I), \Pi^0\tau\right) - \left(\psi_\gamma, \operatorname{as}\Pi^0\tau\right) - \theta\left(A\Pi\sigma, \Pi^0\tau\right) \\
&\quad + \left(A(\Pi\sigma + \alpha Q^0 p I), \tau - \Pi^0\tau\right)_Q + \left(Q^1\gamma, \operatorname{as}(\tau - \Pi^0\tau)\right)_Q \\
&\quad \leq Ch(\|\sigma\|_1 + \|p\|_1 + \|\gamma\|_1)\|\tau\| + C\|A^{1/2}(\phi_s + \alpha\phi_p I)\|\|\tau\|
\end{aligned} \tag{4.40}$$

Combining (4.39) and (4.40) and using [REF TO ORTHOGONALITY PROP OF Q AND PI], we get

$$\|\phi_u\| + \|\phi_\gamma\| \leq Ch(\|\sigma\|_1 + \|p\|_1 + \|\gamma\|_1) + C\|A^{1/2}(\phi_s + \alpha\phi_p I)\|.$$

Thus, (4.38) becomes

$$\|A^{1/2}(\phi_s(t) + \alpha\phi_p I(t))\|^2 + \|\phi_u(t)\|^2 + \|\phi_\gamma(t)\|^2 + c_0\|\phi_p(t)\|^2 + \int_0^t \|\phi_z(s)\|^2\, ds$$
$$\leq \epsilon \int_0^t (\|\phi_s(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_z(s)\|^2)\, ds + Ch^2(\|\sigma(t)\|_1^2 + \|p(t)\|_1^2 + \|\gamma(t)\|_1^2),$$
$$+Ch^2 \int_0^t (\|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2 + \|z(s)\|_1^2)\, ds, \tag{4.41}$$

where we also used [REF TO EQUIVALENCE OF NORMS].

Using the fact that $V_h^0 \times W_h$ is a stable Darcy pair, (4.23), [REF TO ORTHOGONALITY PROP OF Q ON THE BOUNDARY], [REF TO APPROX PROPERTY OF PI] and (4.11), we also obtain

$$\begin{aligned}
\|\phi_p\| &\leq C \sup_{0\neq q\in V_h^0} \frac{(\operatorname{div} q, \phi_p)}{\|q\|_{\operatorname{div}}} = C \sup_{0\neq q\in V_h^0} \frac{\left(K^{-1}z, q\right) - \left(K^{-1}z_h, q\right)_Q}{\|q\|_{\operatorname{div}}} \\
&= C \sup_{0\neq q\in V_h^0} \frac{\left(K^{-1}\phi_z, q\right)_Q - \left(K^{-1}\psi_z, q\right) + \theta\left(K^{-1}\Pi z, q\right)}{\|q\|_{\operatorname{div}}} \leq Ch\|z\|_1 + \|\phi_z\|.
\end{aligned} \tag{4.42}$$

Therefore, we have

$$\|A^{1/2}(\phi_s(t) + \alpha\phi_p I(t))\|^2 + \|\phi_u(t)\|^2 + \|\phi_\gamma(t)\|^2 + c_0\|\phi_p(t)\|^2 + \int_0^t (\|\phi_z(s)\|^2 + \|\phi_p(s)\|^2)\, ds$$
$$\leq \epsilon \int_0^t (\|\phi_s(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_z(s)\|^2)\, ds + Ch^2(\|\sigma(t)\|_1^2 + \|p(t)\|_1^2 + \|\gamma(t)\|_1^2),$$
$$+Ch^2 \int_0^t (\|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2 + \|z(s)\|_1^2). \tag{4.43}$$

Next, we choose $\tau = \phi_s$ in (4.27) and use (4.28)- (4.29) and (4.33)-(4.35):

$$C\|\phi_s\|^2 \leq -\alpha \left(A\phi_p I,\, \phi_s\right)_Q - \left(A(\psi_s + \alpha\psi_p I),\, \Pi^0\phi_s\right) - \left(\psi_\gamma,\, \text{as}\, \Pi^0\phi_s\right) - \theta\left(A\Pi\sigma,\, \Pi^0\phi_s\right)$$
$$- \theta\left(A\alpha Q^0 pI,\, \Pi^0\phi_s\right) - \theta\left(Q^1\gamma,\, \text{as}\, \Pi^0\phi_s\right) + \left(A(\Pi\sigma + \alpha Q^0 pI),\, \phi_s - \Pi^0\phi_s\right)_Q$$
$$+ \left(Q^1\gamma,\, \text{as}\,(\phi_s - \Pi^0\phi_s)\right)_Q \leq Ch^2(\|\sigma\|_1^2 + \|p\|_1^2 + \|\gamma\|_1^2) + C\|\phi_p\|^2 + \epsilon\|\phi_s\|^2,$$

where in the last step we used [REF TO APPROX PROPERTIES OF Q AND PI], [REF TO LEMMA THAT BOUNDS QUAD ERROR THETA] and [REF TO LEMMA THAT BOUNDS ANOTHER QUAD ERROR]. So, we have

$$\int_0^t \|\phi_s(s)\|^2\, ds \leq C\int_0^t h^2(\|\sigma(s)\|_1^2 + \|p(s)\|_1^2 + \|\gamma(s)\|_1^2)\, ds + C\int_0^t \|\phi_p(s)\|^2\, ds. \tag{4.44}$$

On the other hand, it follows from (4.39)-(4.40) and (4.44) that

$$\int_0^t (\|\phi_u(s)\| + \|\phi_\gamma(s)\|)\, ds \leq C\int_0^t (h(\|\sigma(s)\|_1 + \|p(s)\|_1 + \|\gamma(s)\|_1) + \|\phi_s(s)\| + \|\phi_p(s)\|)\, ds. \tag{4.45}$$

Combining (4.43)-(4.45), we obtain

$$\|A^{1/2}(\phi_s(t) + \alpha\phi_p I(t))\|^2 + \|\phi_u(t)\|^2 + \|\phi_\gamma(t)\|^2 + c_0\|\phi_p(t)\|^2$$
$$+ \int_0^t (\|\phi_z(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_s(s)\|^2 + \|\phi_u(s)\|^2 + \|\phi_\gamma(s)\|^2)\, ds$$
$$\leq \epsilon\int_0^t (\|\phi_s(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_z(s)\|^2)\, ds + Ch^2(\|\sigma(t)\|_1^2 + \|p(t)\|_1^2 + \|\gamma(t)\|_1^2),$$
$$+Ch^2\int_0^t (\|\sigma(s)\|_1^2 + \|\partial_t\sigma(s)\|_1^2 + \|p(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\gamma(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2 + \|z(s)\|_1^2). \tag{4.46}$$

Choosing ϵ small enough, we get

$$\|A^{1/2}(\phi_s(t) + \alpha\phi_p I(t))\|^2 + \|\phi_u(t)\|^2 + \|\phi_\gamma(t)\|^2 + c_0\|\phi_p(t)\|^2$$
$$+ \int_0^t (\|\phi_z(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_s(s)\|^2 + \|\phi_u(s)\|^2 + \|\phi_\gamma(s)\|^2)\, ds \leq Ch^2(\|\sigma(t)\|_1^2 + \|p(t)\|_1^2 + \|\gamma(t)\|_1^2),$$
$$+Ch^2\int_0^t (\|\sigma(s)\|_1^2 + \|\partial_t\sigma(s)\|_1^2 + \|p(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\gamma(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2 + \|z(s)\|_1^2). \tag{4.47}$$

Step 2: $H(div)$ in space estimate for stress and velocity:
Estimate for stress error follows immediately due to (4.28).
It follows from (4.31) that

$$\|\operatorname{div}\phi_z\| \leq c_0\|\partial_t\phi_p\| + \|\partial_t A^{1/2}(\phi_s + \alpha\phi_p I)\| + Ch(\|\sigma\|_1 + \|\partial_t\sigma\|_1). \tag{4.48}$$

Next we differentiate (4.27)-(4.30) , set $\tau = \partial_t\phi_s$, $\xi = \partial_t\phi_\gamma$, $q = \phi_z$, $w = \partial_t\phi_p$ and combine (4.27)-(4.31):

$$\frac{1}{2}\partial_t\|K^{-1/2}\phi_z\|_Q^2 + \|A^{1/2}\partial_t(\phi_s + \alpha\phi_p I)\|_Q^2 + c_0\|\partial_t\phi_p\|^2 = -\left(A\partial_t(\psi_s + \alpha\psi_p I),\, \Pi^0\partial_t\phi_s\right) - \left(\partial_t\psi_\gamma,\, \text{as}\, \Pi^0\partial_t\phi_s\right)$$
$$-\theta\left(A\partial_t\Pi\sigma,\, \Pi^0\partial_t\phi_s + \alpha\partial_t\phi_p I\right) + \left(A\partial_t(\Pi\sigma + \alpha Q^0 pI),\, \partial_t\phi_s - \Pi^0\partial_t\phi_s\right)_Q - \theta\left(\partial_t Q^1\gamma,\, \text{as}\, \partial_t\Pi^0\phi_s\right)$$
$$+ \left(\partial_t Q^1\gamma,\, \text{as}\,(\partial_t\phi_s - \Pi^0\partial_t\phi_s)\right)_Q - \left(K^{-1}\psi_z,\, \Pi^0\partial_t\phi_z\right) - \theta\left(K^{-1}\Pi z,\, \partial_t\Pi^0\phi_z\right) + \left(K^{-1}\Pi z,\, \partial_t\phi_z - \partial_t\Pi^0\phi_z\right)_Q$$
$$-\alpha\left(\partial_t\operatorname{tr}(A\psi_s),\, \partial_t\phi_p\right) - \alpha^2\left(\partial_t\operatorname{tr}(A\psi_p I),\, \partial_t\phi_p\right) - \alpha\theta\left(\partial_t AQ^0 p,\, \partial_t\Pi^0\phi_s + \alpha\partial_t\phi_p\right). \tag{4.49}$$

For all terms not corresponding to error in Darcy velocity, we repeat the arguments from (4.32)-(4.36), but group stress and pressure errors.

$$|-\theta\left(A\partial_t\Pi\sigma,\, \Pi^0\partial_t\phi_s + \alpha\partial_t\phi_p I\right) - \theta\left(\partial_t Q^1\gamma,\, \text{as}\, \partial_t\Pi^0\phi_s\right) - \alpha\theta\left(\partial_t AQ^0 p,\, \partial_t\Pi^0\phi_s + \alpha\partial_t\phi_p\right)|$$
$$= |\sum_{E\in\mathcal{T}_h}\left(\theta\left(A\partial_t\Pi\sigma,\, \Pi^0\partial_t(\phi_s + \alpha\phi_p I)\right)_E + \theta\left(\partial_t Q^1\gamma,\, \text{as}\, \Pi^0\partial_t(\phi_s + \alpha\phi_p I)\right)_E + \alpha\theta\left(\partial_t AQ^0 p,\, \Pi^0\partial_t(\phi_s + \alpha\phi_p I)\right)_E\right)|$$
$$\leq Ch^2(\|\partial_t\sigma\|_1^2 + \|\partial_t p\|_1^2 + \|\partial_t\gamma\|_1^2) + \epsilon\|\Pi^0\partial_t\phi_s + \alpha\partial_t\phi_p I\|^2, \tag{4.50}$$

where we used the fact that on every $E \in \mathcal{T}_h$, $\phi_p I|_E \in V_h^0(E) \times V_h^0(E)$ and also that as $\phi_p I = 0$. Similarly,

$$
\begin{aligned}
&| - \big(A\partial_t(\psi_s + \alpha\psi_p I), \Pi^0\partial_t\phi_s\big) - \big(\partial_t\psi_\gamma, \operatorname{as}\Pi^0\partial_t\phi_s\big) - \alpha\left(\partial_t \operatorname{tr}(A\psi_s), \partial_t\phi_p\right) - \alpha^2\left(\partial_t \operatorname{tr}(A\psi_p I), \partial_t\phi_p\right)| \\
&= | - \big(A\partial_t(\psi_s + \alpha\psi_p I), \partial_t(\Pi^0\phi_s + \alpha\phi_p)\big) - \big(\partial_t\psi_\gamma, \operatorname{as}\partial_t(\Pi^0\phi_s + \phi_p)\big)| \\
&= | \sum_{E\in\mathcal{T}_h} \Big(\big(A\partial_t(\psi_s + \alpha\psi_p I), \partial_t\Pi^0(\phi_s + \alpha\phi_p)\big)_E + \big(\partial_t\psi_\gamma, \operatorname{as}\partial_t\Pi^0(\phi_s + \phi_p)\big)_E\Big)| \\
&\le Ch^2(\|\partial_t\sigma\|_1^2 + \|\partial_t p\|_1^2 + \|\partial_t\gamma\|_1^2) + \epsilon\|\partial_t\phi_s + \alpha\partial_t\phi_p I\|^2,
\end{aligned}
\tag{4.51}
$$

and

$$
\begin{aligned}
&| \big(A\partial_t(\Pi\sigma + \alpha Q^0 pI), \partial_t\phi_s - \Pi^0\partial_t\phi_s\big)_Q + \big(\partial_t Q^1\gamma, \operatorname{as}(\partial_t\phi_s - \Pi^0\partial_t\phi_s)\big)_Q | \\
&= | \sum_{E\in\mathcal{T}_h} \Big(\big(A\partial_t(\Pi\sigma + \alpha Q^0 pI), \partial_t(\phi_s + \phi_p I) - \Pi^0\partial_t(\phi_s + \phi_p I)\big)_{Q,E} \\
&+ \big(\partial_t Q^1\gamma, \operatorname{as}(\partial_t(\phi_s + \phi_p I) - \Pi^0\partial_t(\phi_s + \phi_p I))\big)_{E,Q}\Big)| \le Ch^2(\|\partial_t\sigma\|_1^2 + \|\partial_t p\|_1^2 + \|\partial_t\gamma\|_1^2) + \epsilon\|\partial_t\phi_s + \alpha\partial_t\phi_p I\|^2.
\end{aligned}
\tag{4.52}
$$

Combining (4.49)-(4.52), we obtain

$$
\begin{aligned}
&\|K^{-1/2}\phi_z(t)\|_Q^2 + \int_0^t \Big(\|A^{1/2}\partial_t(\phi_s(s) + \alpha\phi_p I(s))\|_Q^2 + c_0\|\partial_t\phi_p(s)\|^2\Big)\, ds \le C\Big(\|K^{-1/2}\phi_z(0)\|_Q^2 \\
&+\epsilon\int_0^t \|\partial_t\phi_s(s) + \alpha\partial_t\phi_p(s)I\|^2\, ds + Ch^2\int_0^t (\|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2)\, ds \\
&+\int_0^t \Big(-\big(K^{-1}\psi_z(s), \Pi^0\partial_t\phi_z(s)\big) - \theta\left(K^{-1}\Pi z(s), \partial_t\Pi^0\phi_z(s)\right) + \big(K^{-1}\Pi z(s), \partial_t\phi_z(s) - \partial_t\Pi^0\phi_z(s)\big)_Q\Big)\, ds\Big).
\end{aligned}
\tag{4.53}
$$

We integrate by parts the terms involving error in Darcy velocity

$$
\begin{aligned}
&\int_0^t \Big(-\big(K^{-1}\psi_z(s), \Pi^0\partial_t\phi_z(s)\big) - \theta\left(K^{-1}\Pi z(s), \partial_t\Pi^0\phi_z(s)\right) + \big(K^{-1}\Pi z(s), \partial_t\phi_z(s) - \partial_t\Pi^0\phi_z(s)\big)_Q\Big)\, ds \\
&= -\big(K^{-1}\psi_z(t), \Pi^0\phi_z(t)\big) - \theta\left(K^{-1}\Pi z(t), \Pi^0\phi_z(t)\right) + \big(K^{-1}\Pi z(t), \phi_z(t) - \Pi^0\phi_z(t)\big)_Q \\
&+ \big(K^{-1}\psi_z(0), \Pi^0\phi_z(0)\big) + \theta\left(K^{-1}\Pi z(0), \Pi^0\phi_z(0)\right) + \big(K^{-1}\Pi z(0), \phi_z(0) - \Pi^0\phi_z(0)\big)_Q \\
&-\int_0^t \Big(-\big(K^{-1}\partial_t\psi_z(s), \Pi^0\phi_z(s)\big) - \theta\left(K^{-1}\partial_t\Pi z(s), \Pi^0\phi_z(s)\right) + \big(K^{-1}\partial_t\Pi z(s), \phi_z(s) - \Pi^0\phi_z(s)\big)_Q\Big)\, ds
\end{aligned}
$$

Choosing $z_h(0) = \Pi z(0)$, we obtain

$$
\big(K^{-1}\psi_z(0), \Pi^0\phi_z(0)\big) + \theta\left(K^{-1}\Pi z(0), \Pi^0\phi_z(0)\right) + \big(K^{-1}\Pi z(0), \phi_z(0) - \Pi^0\phi_z(0)\big)_Q = 0,
\tag{4.54}
$$

and for the rest of the terms we use [REF TO APPROX PROPERTIES OF PI], (4.11) and (4.15):

$$
\begin{aligned}
&-\big(K^{-1}\psi_z(t), \Pi^0\phi_z(t)\big) - \theta\left(K^{-1}\Pi z(t), \Pi^0\phi_z(t)\right) + \big(K^{-1}\Pi z(t), \phi_z(t) - \Pi^0\phi_z(t)\big)_Q \\
&-\int_0^t \Big(-\big(K^{-1}\partial_t\psi_z(s), \Pi^0\phi_z(s)\big) - \theta\left(K^{-1}\partial_t\Pi z(s), \Pi^0\phi_z(s)\right) + \big(K^{-1}\partial_t\Pi z(s), \phi_z(s) - \Pi^0\phi_z(s)\big)_Q\Big)\, ds \\
&\le C(h^2\|z(t)\|_1^2 + \epsilon\|\phi_z(t)\|^2) + \int_0^t (h^2\|\partial_t z(s)\|_1^2 + \epsilon\|\phi_z(s)\|^2)\, ds.
\end{aligned}
\tag{4.55}
$$

We combine (4.53)-(4.55):

$$
\begin{aligned}
&\|K^{-1/2}\phi_z(t)\|_Q^2 + \int_0^t \Big(\|A^{1/2}\partial_t(\phi_s(s) + \alpha\phi_p I(s))\|_Q^2 + c_0\|\partial_t\phi_p(s)\|^2\Big)\, ds \\
&\le C(h^2\|z(t)\|_1^2 + \epsilon\|\phi_z(t)\|^2) + C\int_0^t (h^2(\|\partial_t z(s)\|_1^2 + h^2\|\partial_t\sigma(s)\|_1^2 + \epsilon\|\phi_z(s)\|^2)\, ds.
\end{aligned}
\tag{4.56}
$$

Combining (4.56), (4.53),(4.42) and using [REF TO EQUIVALENCE OF NORMS], we obtain

$$
\begin{aligned}
&\|\phi_z(t)\|^2 + \|\phi_p(t)\|^2 + \int_0^t \left(\|\partial_t(\phi_s(s) + \alpha\phi_p I(s))\|^2 + c_0\|\partial_t\phi_p(s)\|^2\right)\, ds \\
&\le C\int_0^t h^2(\|\partial_t z(s)\|_1^2 + \|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2)\, ds \\
&+\epsilon\int_0^t (\|\phi_z(s)\|^2 + \|\partial_t(\phi_s(s) + \alpha\phi_p I(s))\|^2)\, ds + C(h^2\|z(t)\|_1^2 + \epsilon\|\phi_z(t)\|^2).
\end{aligned}
\tag{4.57}
$$

Hence, using (4.48) and (4.57), we get

$$\|\phi_z(t)\|^2 + \|\phi_p(t)\|^2 + \int_0^t \|\operatorname{div}\phi_z\|^2\,ds \le \epsilon \int_0^t \|\phi_z(s)\|^2\,ds$$
$$+C\left(\int_0^t h^2(\|\partial_t z(s)\|_1^2 + \|\sigma(s)\|_1^2 + \|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2)\,ds + \|z(t)\|_1^2\right). \tag{4.58}$$

Step 3: combine results:
We note that

$$\begin{aligned}\|\phi_s\| \le C\|A^{1/2}\phi_s\| &\le C\left(\|A^{1/2}(\phi_s+\alpha\phi_p I)\| + \|A^{1/2}\alpha\phi_p I\|\right)\\ &\le C\left(\|A^{1/2}(\phi_s+\alpha\phi_p I)\| + \|\phi_p\|\right).\end{aligned} \tag{4.59}$$

Therefore, combining (4.47), (4.58) and (4.59), we obtain the following result.

Theorem 4.1. *Let $(\sigma_h, u_h, \gamma_h, z_h, p_h) \in \mathbb{X}_h \times V_h \times \Theta_h \times Z_h \times W_h$ be the solution of (2.67)-(2.71) and let $(\sigma, u, \gamma, z, p) \in \mathbb{X}\times V\times\Theta\times Z\times W \cap H^1(0,T;(H^1(\Omega))^{2\times 2})\times H^1(0,T;(H^1(\Omega))^2)\times H^1(0,T;H^1(\Omega))\times H^1(0,T;(H^1(\Omega))^2)\times H^1(0,T;H^1(\Omega))$ be the solution of (2.10)-(2.14). Then the following error estimate holds:*

$$\begin{aligned}&\|\sigma-\sigma_h\|_{L^\infty(0,T;H(\operatorname{div},\Omega))} + \|u-u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\gamma-\gamma_h\|_{L^\infty(0,T;L^2(\Omega))} + \|z-z_h\|_{L^\infty(0,T;L^2(\Omega))}\\ &+\|p-p_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\sigma-\sigma_h\|_{L^2(0,T;H(\operatorname{div},\Omega))} + \|u-u_h\|_{L^2(0,T;L^2(\Omega))} + \|\gamma-\gamma_h\|_{L^2(0,T;L^2(\Omega))}\\ &+\|z-z_h\|_{L^2(0,T;H(\operatorname{div},\Omega))} + \|p-p_h\|_{L^2(0,T;L^2(\Omega))}\\ &\le Ch\Big(\|s\|_{H^1(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^1(\Omega))} + \|\gamma\|_{H^1(0,T;H^1(\Omega))} + \|z\|_{H^1(0,T;H^1(\Omega))} + \|p\|_{H^1(0,T;H^1(\Omega))}\\ &+\|\sigma\|_{L^\infty(0,T;H^1(\Omega))} + \|u\|_{L^\infty(0,T;L^2(\Omega))} + \|\gamma\|_{L^\infty(0,T;H^1(\Omega))} + \|z\|_{L^\infty(0,T;H^1(\Omega))} + \|p\|_{L^\infty(0,T;H^1(\Omega))}\Big).\end{aligned} \tag{4.60}$$

5 Numerical results

In this section we provide several numerical tests verifying the theoretically predicted convergence rates and illustrating the behavior of the method.

5.1 Test case 1

For the first test problem we take the poroelasticity system on a unit square $\Omega = [0,1]^2$ with Dirichlet pressure and displacement boundary conditions and the true solution

$$p = \exp(t)\sin(x+y), \quad u = \frac{1}{2}t\begin{pmatrix}x^2\\ y^2\end{pmatrix}.$$

The rest of the unknowns as well as the body forces can be determined from the equation using the values from Table 1. For the case of quadrilaterals, we perform convergence tests on three types of grids: unit square partitioned

Parameter	Symbol	Values
Lame coefficient	μ	79.3
Lame coefficient	λ	123.0
Permeability	K	$\mathbb{I}$
Mass storativity	c_0	1.0
Biot-Willis constant	α	1.0
Total time	T	10^{-2}
Time step	Δt	10^{-3}

Table 1: Parameters, test case 1.

into rectangles, h^2-parallelograms and the case when computational domain is obtained from the unit square via the following transformation

$$x = \hat{x} + 0.03\cos(3\pi\hat{x})\cos(3\pi\hat{y}), \quad y = \hat{y} - 0.04\cos(3\pi\hat{x})\cos(3\pi\hat{y}).$$

Convergence rates for all three cases are given in Tables 2-4.

cycle	# dofs	$\frac{\Vert p-p_h\Vert_{L^2(0,T;L^2(\Omega))}}{\Vert p\Vert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\Vert Q^0p-p_h\Vert_{L^2(0,T;L^2(\Omega))}}{\Vert p\Vert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\Vert p-p_h\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert p\Vert_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\Vert Q^0p-p_h\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert p\Vert_{L^\infty(0,T;L^2(\Omega))}}$	
0	313	9.198e-02	-	3.754e-03	-	9.205e-02	-	4.821e-03	-
1	1137	4.611e-02	1.00	9.593e-04	1.97	4.612e-02	1.00	1.147e-03	2.07
2	4321	2.306e-02	1.00	2.607e-04	1.88	2.307e-02	1.00	2.781e-04	2.04
3	16833	1.153e-02	1.00	7.035e-05	1.98	2.647e-03	1.03	2.373e-02	1.44
4	66433	5.766e-03	1.00	1.584e-05	2.07	5.766e-03	1.00	1.758e-05	2.00
5	263937	2.883e-03	1.00	3.599e-06	2.14	2.883e-03	1.00	4.116e-06	2.09
cycle	# dofs	$\frac{\Vert z-z_h\Vert_{L^2(0,T;L^2(\Omega))}}{\Vert z\Vert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\Vert \operatorname{div}(z-z_h)\Vert_{L^2(0,T;L^2(\Omega))}}{\Vert \operatorname{div} z\Vert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\Vert \sigma-\sigma_h\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert \sigma\Vert_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\Vert \operatorname{div}(\sigma-\sigma_h)\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert \operatorname{div}\sigma\Vert_{L^\infty(0,T;L^2(\Omega))}}$	
0	313	1.294e-01	-	2.049e-01	-	2.597e-02	-	2.463e-01	-
1	1137	6.145e-02	1.07	8.590e-02	1.25	1.155e-02	1.17	1.364e-01	0.85
2	4321	3.015e-02	1.03	3.289e-02	1.38	5.424e-03	1.09	6.421e-02	1.09
3	16833	1.500e-02	1.01	1.352e-02	1.28	2.647e-03	1.03	2.373e-02	1.44
4	66433	7.487e-03	1.00	6.096e-03	1.15	1.313e-03	1.01	8.135e-03	1.54
5	263937	3.742e-03	1.00	2.926e-03	1.06	6.549e-04	1.00	3.204e-03	1.34
cycle	# dofs	$\frac{\Vert u-u_h\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert u\Vert_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\Vert Q^0u-u_h\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert u\Vert_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\Vert Q^0u-u_h\Vert_{L^2(0,T;L^2(\Omega))}}{\Vert u\Vert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\Vert \gamma-\gamma_h\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert \gamma\Vert_{L^\infty(0,T;L^2(\Omega))}}$	
0	313	2.270e-01	-	1.982e-02	-	1.970e-02	-	9.508e-05	-
1	1137	1.140e-01	0.99	4.888e-03	2.02	4.913e-03	2.00	2.780e-05	1.77
2	4321	5.704e-02	1.00	1.222e-03	2.00	1.234e-03	1.99	7.890e-06	1.82
3	16833	2.852e-02	1.00	3.057e-04	2.00	3.092e-04	2.00	2.187e-06	1.85
4	66433	1.426e-02	1.00	7.650e-05	2.00	7.738e-05	2.00	5.957e-07	1.88
5	263937	7.132e-03	1.00	1.919e-05	2.00	1.937e-05	2.00	1.603e-07	1.89

Table 2: Computed numerical errors and convergence rates, rectangles.

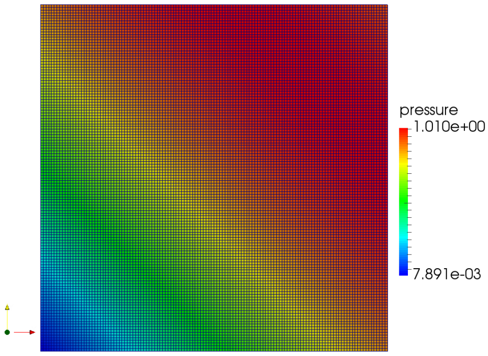
cycle	# dofs	$\frac{\Vert p-p_h\Vert_{L^2(0,T;L^2(\Omega))}}{\Vert p\Vert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\Vert Q^0p-p_h\Vert_{L^2(0,T;L^2(\Omega))}}{\Vert p\Vert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\Vert p-p_h\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert p\Vert_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\Vert Q^0p-p_h\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert p\Vert_{L^\infty(0,T;L^2(\Omega))}}$	
0	221	1.229e-01	-	1.878e-02	-	1.231e-01	-	1.943e-02	-
1	825	6.205e-02	0.99	4.448e-03	2.08	6.210e-02	0.99	4.822e-03	2.01
2	3185	3.109e-02	1.00	1.119e-03	1.99	3.109e-02	1.00	1.176e-03	2.04
3	12513	1.555e-02	1.00	2.862e-04	1.97	1.555e-02	1.00	2.934e-04	2.00
4	49601	7.775e-03	1.00	7.187e-05	1.99	7.775e-03	1.00	7.373e-05	1.99
5	197505	3.888e-03	1.00	1.749e-05	2.04	3.888e-03	1.00	1.835e-05	2.01
cycle	# dofs	$\frac{\Vert z-z_h\Vert_{L^2(0,T;L^2(\Omega))}}{\Vert z\Vert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\Vert \operatorname{div}(z-z_h)\Vert_{L^2(0,T;L^2(\Omega))}}{\Vert \operatorname{div} z\Vert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\Vert \sigma-\sigma_h\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert \sigma\Vert_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\Vert \operatorname{div}(\sigma-\sigma_h)\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert \operatorname{div}\sigma\Vert_{L^\infty(0,T;L^2(\Omega))}}$	
0	221	3.278e-01	-	4.499e-01	-	2.606e-01	-	5.335e-01	-
1	825	1.762e-01	0.90	2.050e-01	1.13	1.300e-01	1.00	3.026e-01	0.82
2	3185	9.128e-02	0.95	8.625e-02	1.25	6.594e-02	0.98	1.510e-01	1.00
3	12513	4.649e-02	0.97	3.811e-02	1.18	3.340e-02	0.98	5.944e-02	1.34
4	49601	2.348e-02	0.99	1.784e-02	1.09	1.685e-02	0.99	2.205e-02	1.43
5	197505	1.180e-02	0.99	8.691e-03	1.04	8.467e-03	0.99	9.267e-03	1.25
cycle	# dofs	$\frac{\Vert u-u_h\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert u\Vert_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\Vert Q^0u-u_h\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert u\Vert_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\Vert Q^0u-u_h\Vert_{L^2(0,T;L^2(\Omega))}}{\Vert u\Vert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\Vert \gamma-\gamma_h\Vert_{L^\infty(0,T;L^2(\Omega))}}{\Vert \gamma\Vert_{L^\infty(0,T;L^2(\Omega))}}$	
0	221	2.928e-01	-	4.694e-02	-	5.069e-02	-	6.095e-04	-
1	825	1.458e-01	1.01	1.219e-02	1.95	1.366e-02	1.89	2.941e-04	1.05
2	3185	7.272e-02	1.00	3.154e-03	1.95	3.548e-03	1.94	1.156e-04	1.35
3	12513	3.634e-02	1.00	8.074e-04	1.97	9.047e-04	1.97	4.301e-05	1.43
4	49601	1.817e-02	1.00	2.040e-04	1.98	2.277e-04	1.99	1.565e-05	1.46
5	197505	9.083e-03	1.00	5.118e-05	1.99	5.700e-05	2.00	5.630e-06	1.48

Table 3: Computed numerical errors and convergence rates, h^2-parallelograms.

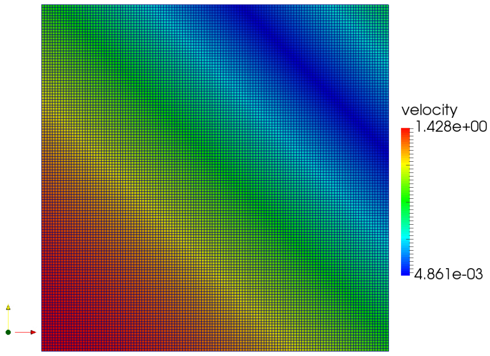
cycle	# dofs	$\frac{\lVert p-p_h\rVert_{L^2(0,T;L^2(\Omega))}}{\lVert p\rVert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\lVert Q^0 p-p_h\rVert_{L^2(0,T;L^2(\Omega))}}{\lVert p\rVert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\lVert p-p_h\rVert_{L^\infty(0,T;L^2(\Omega))}}{\lVert p\rVert_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\lVert Q^0 p-p_h\rVert_{L^\infty(0,T;L^2(\Omega))}}{\lVert p\rVert_{L^\infty(0,T;L^2(\Omega))}}$	
0	313	9.209e-02	-	3.935e-03	-	9.217e-02	-	4.963e-03	-
1	1137	4.616e-02	1.00	1.012e-03	1.96	4.617e-02	1.00	1.184e-03	2.07
2	4321	2.309e-02	1.00	2.758e-04	1.88	2.309e-02	1.00	2.879e-04	2.04
3	16833	1.155e-02	1.00	7.033e-05	1.97	1.155e-02	1.00	7.308e-05	1.98
4	66433	5.773e-03	1.00	1.689e-05	2.06	5.773e-03	1.00	1.832e-05	2.00
5	263937	2.886e-03	1.00	3.851e-06	2.13	2.886e-03	1.00	4.316e-06	2.09
cycle	# dofs	$\frac{\lVert z-z_h\rVert_{L^2(0,T;L^2(\Omega))}}{\lVert z\rVert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\lVert \operatorname{div}(z-z_h)\rVert_{L^2(0,T;L^2(\Omega))}}{\lVert \operatorname{div} z\rVert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\lVert \sigma-\sigma_h\rVert_{L^\infty(0,T;L^2(\Omega))}}{\lVert \sigma\rVert_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\lVert \operatorname{div}(\sigma-\sigma_h)\rVert_{L^\infty(0,T;L^2(\Omega))}}{\lVert \operatorname{div}\sigma\rVert_{L^\infty(0,T;L^2(\Omega))}}$	
0	313	1.314e-01	-	2.359e-01	-	7.852e-02	-	2.803e-01	-
1	1137	6.295e-02	1.06	1.052e-01	1.16	4.724e-02	0.73	1.519e-01	0.88
2	4321	3.098e-02	1.02	4.512e-02	1.22	2.539e-02	0.90	7.239e-02	1.07
3	16833	1.543e-02	1.01	2.033e-02	1.15	1.315e-02	0.95	2.852e-02	1.34
4	66433	7.710e-03	1.00	9.663e-03	1.07	6.686e-03	0.98	1.111e-02	1.36
5	263937	3.855e-03	1.00	4.745e-03	1.03	3.370e-03	0.99	4.912e-03	1.18
cycle	# dofs	$\frac{\lVert u-u_h\rVert_{L^\infty(0,T;L^2(\Omega))}}{\lVert u\rVert_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\lVert Q^0 u-u_h\rVert_{L^\infty(0,T;L^2(\Omega))}}{\lVert u\rVert_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\lVert Q^0 u-u_h\rVert_{L^2(0,T;L^2(\Omega))}}{\lVert u\rVert_{L^2(0,T;L^2(\Omega))}}$		$\frac{\lVert \gamma-\gamma_h\rVert_{L^\infty(0,T;L^2(\Omega))}}{\lVert \gamma\rVert_{L^\infty(0,T;L^2(\Omega))}}$	
0	313	2.316e-01	-	2.280e-02	-	2.395e-02	-	3.994e-04	-
1	1137	1.161e-01	1.00	6.747e-03	1.76	7.074e-03	1.76	2.146e-04	0.90
2	4321	5.803e-02	1.00	2.056e-03	1.71	2.139e-03	1.73	9.298e-05	1.21
3	16833	2.901e-02	1.00	5.662e-04	1.86	5.867e-04	1.87	3.713e-05	1.32
4	66433	1.450e-02	1.00	1.467e-04	1.95	1.517e-04	1.95	1.404e-05	1.40
5	263937	7.251e-03	1.00	3.715e-05	1.98	3.838e-05	1.98	5.156e-06	1.45

Table 4: Computed numerical errors and convergence rates, transformed mesh.

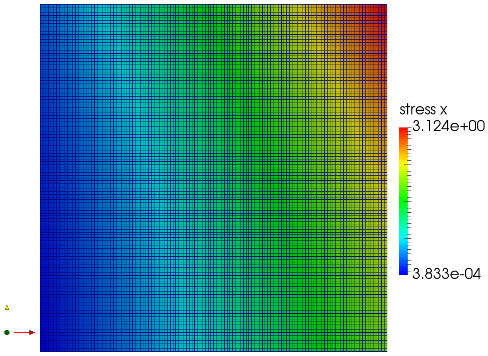
Figures 1-3 present the solution obtained on fifth level of refinement and the last time step.



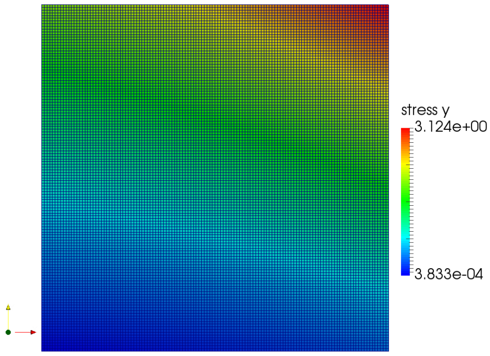
(a) Darcy pressure



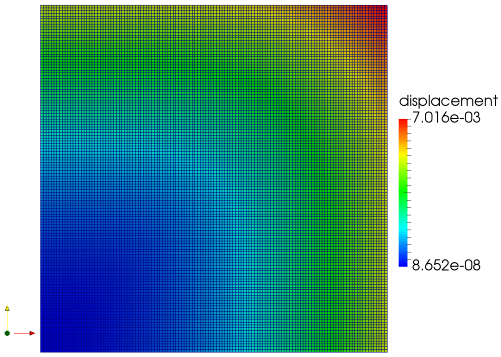
(b) Darcy velocity



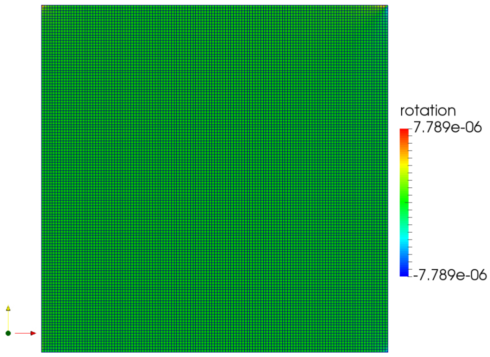
(c) Elastic stress, x-component



(d) Elastic stress, y-component

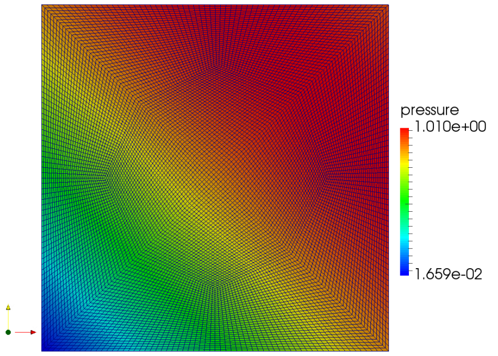


(e) Displacement

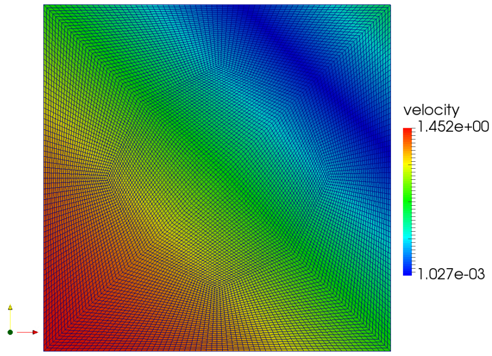


(f) Rotation

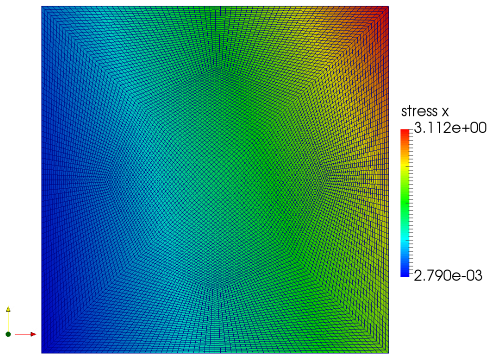
Figure 1: Solution on rectangular grid.



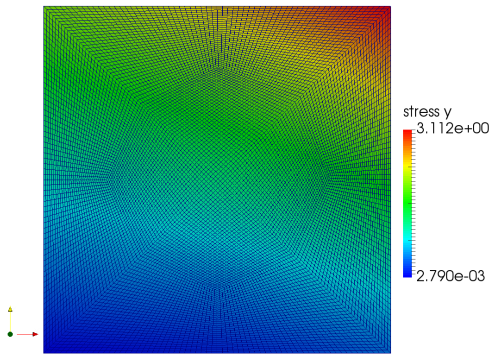
(a) Darcy pressure



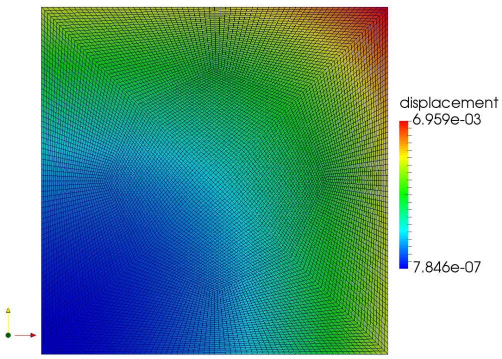
(b) Darcy velocity



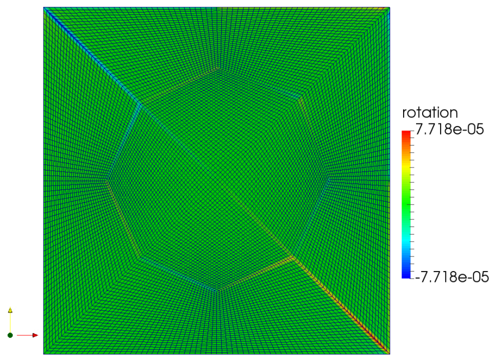
(c) Elastic stress, x-component



(d) Elastic stress, y-component

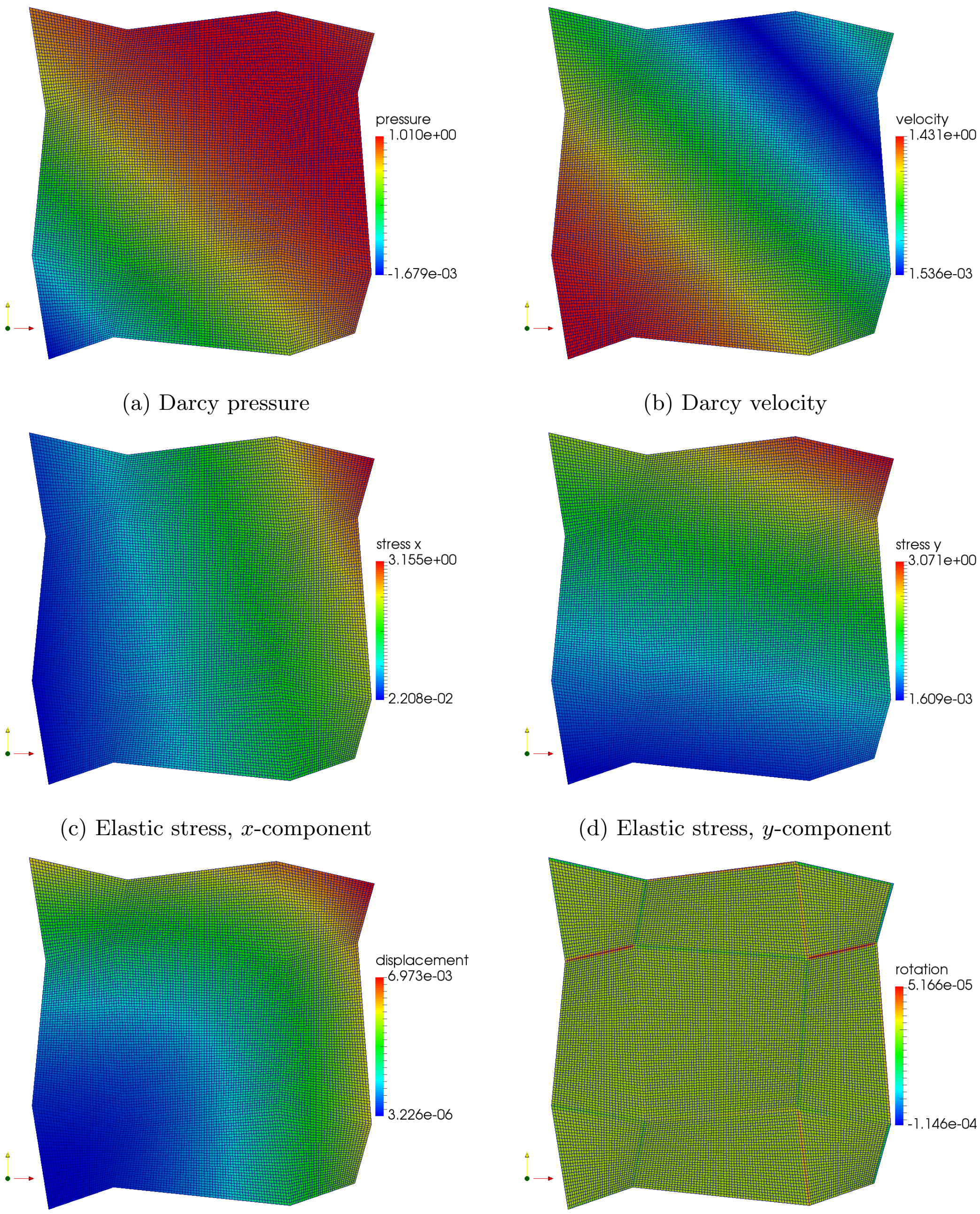


(e) Displacement



(f) Rotation

Figure 2: Solution on h^2-parallelograms.



(a) Darcy pressure

(b) Darcy velocity

(c) Elastic stress, x-component

(d) Elastic stress, y-component

(e) Displacement

(f) Rotation

Figure 3: Solution on transformed domain.

5.2 Test case 2

The second test case is similar, but the parameters know vary over the domain. We still consider a unit square $\Omega = [0,1]^2$ with Dirichlet pressure and displacement boundary conditions. The true solution is now given as

$$p = p = 5\sin(t)(x^3y^4 + x^2 + \sin(xy)\cos(y)), \quad u = 5\cos(t)\begin{pmatrix} x^4 + y^3 + \sin(xy) * cos(x) \\ x^3 + y^4 + \cos(xy) * sin(y) \end{pmatrix}.$$

The permeability tensor is

$$K = \begin{pmatrix} (x+1)^2 + y^2 & \sin(xy) \\ \sin(xy) & (x+1)^2 \end{pmatrix},$$

while the Lame coefficients and the compliance tensor are computed using the following values of Poisson's ratio and Young's modulus:

$$E = \sin(3\Pi x)\sin(3\Pi y) + 5, \quad \nu = 0.2,$$

using the relations

$$\lambda = \frac{E\nu}{(1-\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+2\nu)}.$$

Convergence rates for all three cases are given in Tables 5-7.

cycle	# dofs	$\frac{\|p-p_h\|_{L^2(0,T;L^2(\Omega))}}{\|p\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|Q^0p-p_h\|_{L^2(0,T;L^2(\Omega))}}{\|p\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|p-p_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|p\|_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\|Q^0p-p_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|p\|_{L^\infty(0,T;L^2(\Omega))}}$	
0	313	1.690e+01	-	1.761e+01	-	1.724e+01	-	1.797e+01	-
1	1137	5.503e+00	1.62	5.556e+00	1.66	7.301e+00	1.24	7.378e+00	1.28
2	4321	1.466e+00	1.91	1.466e+00	1.92	2.053e+00	1.83	2.058e+00	1.84
3	16833	3.748e-01	1.97	3.721e-01	1.98	5.269e-01	1.96	5.272e-01	1.96
4	66433	9.647e-02	1.96	9.380e-02	1.99	1.326e-01	1.99	1.326e-01	1.99
5	263937	2.612e-02	1.88	2.394e-02	1.97	3.322e-02	2.00	3.321e-02	2.00
cycle	# dofs	$\frac{\|z-z_h\|_{L^2(0,T;L^2(\Omega))}}{\|z\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|\operatorname{div}(z-z_h)\|_{L^2(0,T;L^2(\Omega))}}{\|\operatorname{div} z\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|\sigma-\sigma_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\|\operatorname{div}(\sigma-\sigma_h)\|_{L^\infty(0,T;L^2(\Omega))}}{\|\operatorname{div}\sigma\|_{L^\infty(0,T;L^2(\Omega))}}$	
0	313	3.198e+01	-	1.259e+02	-	8.959e-02	-	1.602e+02	-
1	1137	1.555e+01	1.04	7.050e+01	0.84	3.296e-02	1.44	1.239e+02	0.37
2	4321	4.634e+00	1.75	2.179e+01	1.69	1.042e-02	1.66	4.420e+01	1.49
3	16833	1.212e+00	1.93	5.873e+00	1.89	3.703e-03	1.49	1.284e+01	1.78
4	66433	3.072e-01	1.98	1.524e+00	1.95	1.587e-03	1.22	3.483e+00	1.88
5	263937	7.776e-02	1.98	3.914e-01	1.96	7.555e-04	1.07	9.214e-01	1.92
cycle	# dofs	$\frac{\|u-u_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|u\|_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\|Q^0u-u_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|u\|_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\|Q^0u-u_h\|_{L^2(0,T;L^2(\Omega))}}{\|u\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|\gamma-\gamma_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|\gamma\|_{L^\infty(0,T;L^2(\Omega))}}$	
0	313	2.257e-01	-	4.961e-02	-	4.939e-02	-	1.230e-01	-
1	1137	1.103e-01	1.03	1.207e-02	2.04	1.199e-02	2.04	4.260e-02	1.53
2	4321	5.480e-02	1.01	3.000e-03	2.01	2.977e-03	2.01	1.429e-02	1.58
3	16833	2.736e-02	1.00	7.486e-04	2.00	7.429e-04	2.00	4.701e-03	1.60
4	66433	1.367e-02	1.00	1.870e-04	2.00	1.856e-04	2.00	1.563e-03	1.59
5	263937	6.836e-03	1.00	4.673e-05	2.00	4.637e-05	2.00	5.292e-04	1.56

Table 5: Computed numerical errors and convergence rates, rectangles.

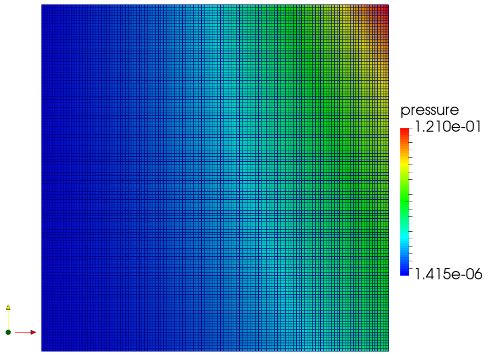
cycle	# dofs	$\frac{\|p-p_h\|_{L^2(0,T;L^2(\Omega))}}{\|p\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|Q^0p-p_h\|_{L^2(0,T;L^2(\Omega))}}{\|p\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|p-p_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|p\|_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\|Q^0p-p_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|p\|_{L^\infty(0,T;L^2(\Omega))}}$	
0	221	3.207e+01	-	3.600e+01	-	2.397e+01	-	2.691e+01	-
1	825	9.376e+00	1.77	9.668e+00	1.90	1.198e+01	1.00	1.235e+01	1.12
2	3185	2.857e+00	1.71	2.877e+00	1.75	3.845e+00	1.64	3.875e+00	1.67
3	12513	7.453e-01	1.94	7.442e-01	1.95	1.007e+00	1.93	1.009e+00	1.94
4	49601	1.899e-01	1.97	1.876e-01	1.99	2.535e-01	1.99	2.536e-01	1.99
5	197505	4.960e-02	1.94	4.744e-02	1.98	6.349e-02	2.00	6.349e-02	2.00
cycle	# dofs	$\frac{\|z-z_h\|_{L^2(0,T;L^2(\Omega))}}{\|z\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|\operatorname{div}(z-z_h)\|_{L^2(0,T;L^2(\Omega))}}{\|\operatorname{div} z\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|\sigma-\sigma_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\|\operatorname{div}(\sigma-\sigma_h)\|_{L^\infty(0,T;L^2(\Omega))}}{\|\operatorname{div}\sigma\|_{L^\infty(0,T;L^2(\Omega))}}$	
0	221	3.412e+01	-	9.925e+01	-	2.876e-01	-	8.825e+01	-
1	825	2.420e+01	0.50	1.101e+02	-0.15	1.274e-01	1.17	1.842e+02	-1.06
2	3185	8.480e+00	1.51	4.174e+01	1.40	6.393e-02	0.99	8.916e+01	1.05
3	12513	2.303e+00	1.88	1.228e+01	1.77	3.245e-02	0.98	3.044e+01	1.55
4	49601	5.892e-01	1.97	3.459e+00	1.83	1.642e-02	0.98	9.686e+00	1.65
5	197505	1.506e-01	1.97	9.597e-01	1.85	8.274e-03	0.99	2.937e+00	1.72
cycle	# dofs	$\frac{\|u-u_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|u\|_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\|Q^0u-u_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|u\|_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\|Q^0u-u_h\|_{L^2(0,T;L^2(\Omega))}}{\|u\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|\gamma-\gamma_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|\gamma\|_{L^\infty(0,T;L^2(\Omega))}}$	
0	221	3.161e-01	-	1.340e-01	-	1.336e-01	-	5.789e-01	-
1	825	1.393e-01	1.18	2.071e-02	2.69	2.040e-02	2.71	2.702e-01	1.10
2	3185	6.853e-02	1.02	4.970e-03	2.06	4.904e-03	2.06	1.108e-01	1.29
3	12513	3.415e-02	1.00	1.266e-03	1.97	1.254e-03	1.97	4.287e-02	1.37
4	49601	1.706e-02	1.00	3.209e-04	1.98	3.186e-04	1.98	1.610e-02	1.41
5	197505	8.529e-03	1.00	8.073e-05	1.99	8.021e-05	1.99	5.934e-03	1.44

Table 6: Computed numerical errors and convergence rates, h^2-parallelograms.

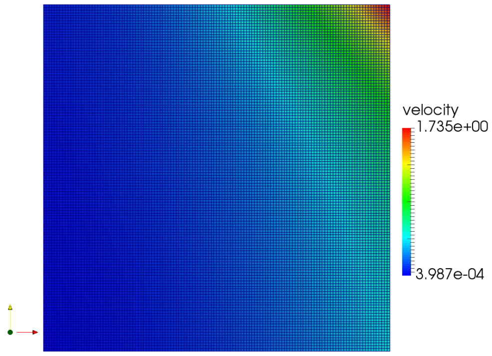
cycle	# dofs	$\frac{\|p-p_h\|_{L^2(0,T;L^2(\Omega))}}{\|p\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|Q^0p-p_h\|_{L^2(0,T;L^2(\Omega))}}{\|p\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|p-p_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|p\|_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\|Q^0p-p_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|p\|_{L^\infty(0,T;L^2(\Omega))}}$	
0	313	1.524e+01	-	1.580e+01	-	1.774e+01	-	1.841e+01	-
1	1137	5.381e+00	1.50	5.425e+00	1.54	7.372e+00	1.27	7.441e+00	1.31
2	4321	1.470e+00	1.87	1.470e+00	1.88	2.102e+00	1.81	2.107e+00	1.82
3	16833	3.785e-01	1.96	3.757e-01	1.97	5.425e-01	1.95	5.428e-01	1.96
4	66433	9.768e-02	1.95	9.504e-02	1.98	1.368e-01	1.99	1.368e-01	1.99
5	263937	2.650e-02	1.88	2.435e-02	1.96	3.431e-02	2.00	3.430e-02	2.00
cycle	# dofs	$\frac{\|z-z_h\|_{L^2(0,T;L^2(\Omega))}}{\|z\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|\operatorname{div}(z-z_h)\|_{L^2(0,T;L^2(\Omega))}}{\|\operatorname{div} z\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|\sigma-\sigma_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\|\operatorname{div}(\sigma-\sigma_h)\|_{L^\infty(0,T;L^2(\Omega))}}{\|\operatorname{div}\sigma\|_{L^\infty(0,T;L^2(\Omega))}}$	
0	313	3.650e+01	-	1.516e+02	-	1.135e-01	-	2.239e+02	-
1	1137	1.613e+01	1.18	7.713e+01	0.97	4.990e-02	1.19	1.512e+02	0.57
2	4321	4.826e+00	1.74	2.392e+01	1.69	2.247e-02	1.15	5.303e+01	1.51
3	16833	1.270e+00	1.93	6.541e+00	1.87	1.092e-02	1.04	1.568e+01	1.76
4	66433	3.227e-01	1.98	1.726e+00	1.92	5.453e-03	1.00	4.377e+00	1.84
5	263937	8.179e-02	1.98	4.504e-01	1.94	2.735e-03	1.00	1.191e+00	1.88
cycle	# dofs	$\frac{\|u-u_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|u\|_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\|Q^0u-u_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|u\|_{L^\infty(0,T;L^2(\Omega))}}$		$\frac{\|Q^0u-u_h\|_{L^2(0,T;L^2(\Omega))}}{\|u\|_{L^2(0,T;L^2(\Omega))}}$		$\frac{\|\gamma-\gamma_h\|_{L^\infty(0,T;L^2(\Omega))}}{\|\gamma\|_{L^\infty(0,T;L^2(\Omega))}}$	
0	313	2.267e-01	-	4.634e-02	-	4.627e-02	-	2.456e-01	-
1	1137	1.114e-01	1.03	1.131e-02	2.03	1.127e-02	2.04	1.258e-01	0.97
2	4321	5.544e-02	1.01	2.931e-03	1.95	2.918e-03	1.95	5.508e-02	1.19
3	16833	2.769e-02	1.00	7.504e-04	1.97	7.470e-04	1.97	2.178e-02	1.34
4	66433	1.384e-02	1.00	1.892e-04	1.99	1.883e-04	1.99	8.151e-03	1.42
5	263937	6.920e-03	1.00	4.739e-05	2.00	4.718e-05	2.00	2.974e-03	1.45

Table 7: Computed numerical errors and convergence rates, transformed mesh.

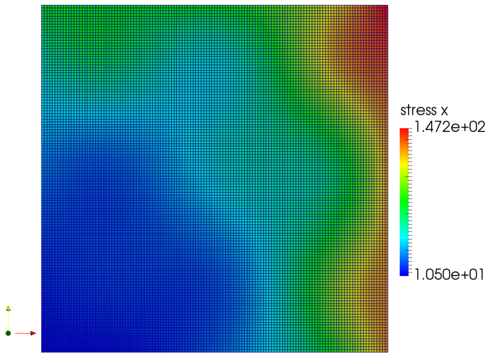
Figures 4-6 present the solution obtained on fifth level of refinement and the last time step.



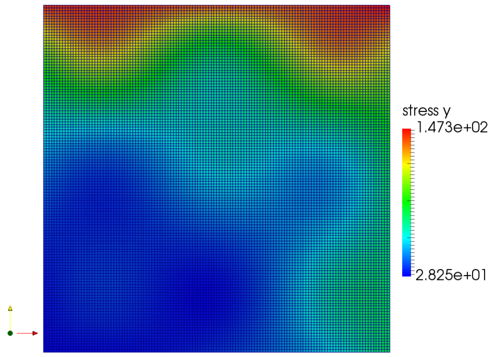
(a) Darcy pressure



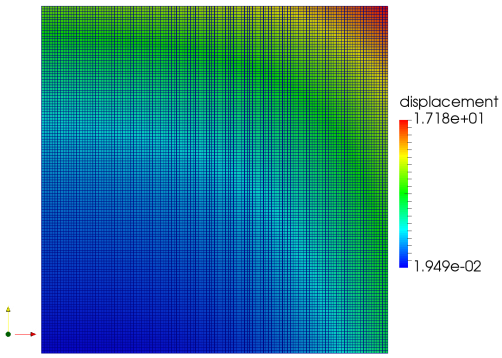
(b) Darcy velocity



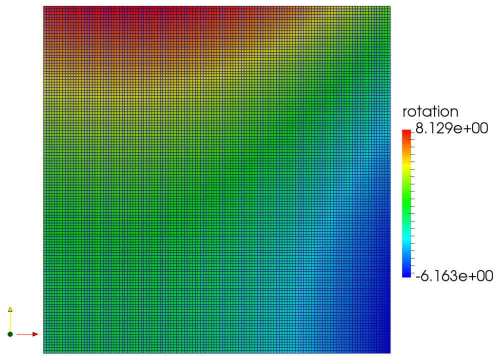
(c) Elastic stress, x-component



(d) Elastic stress, y-component

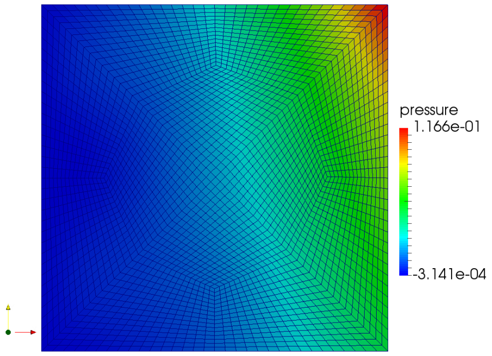


(e) Displacement

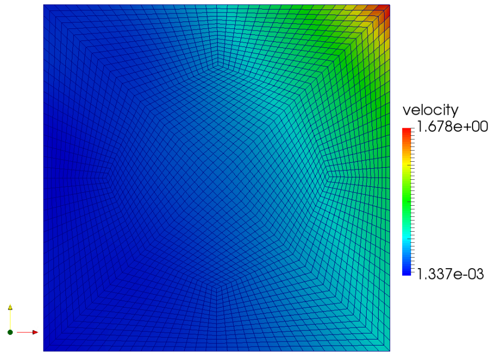


(f) Rotation

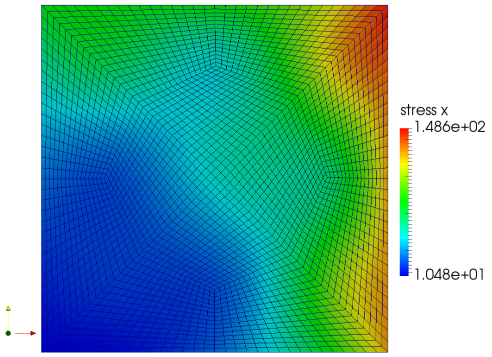
Figure 4: Solution on rectangular grid.



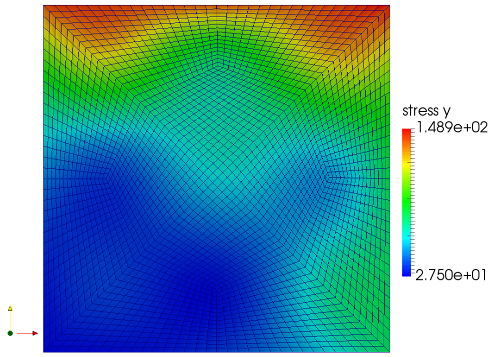
(a) Darcy pressure



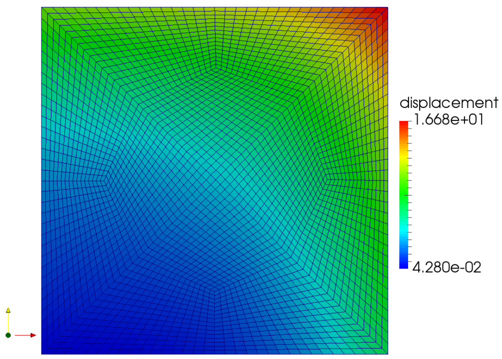
(b) Darcy velocity



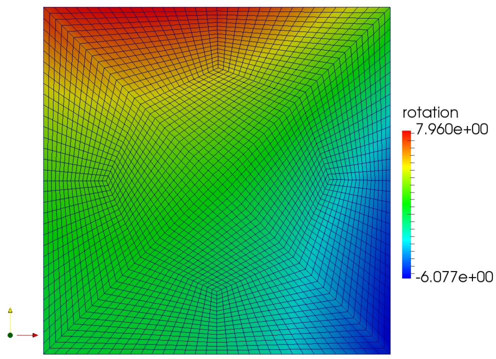
(c) Elastic stress, x-component



(d) Elastic stress, y-component

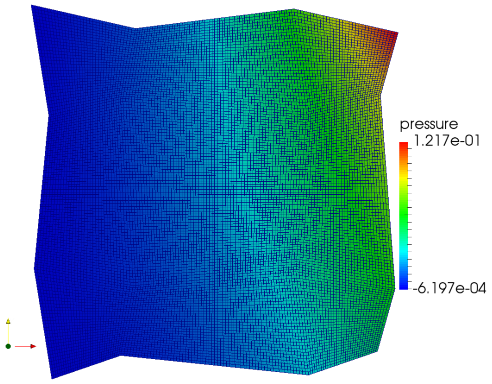


(e) Displacement

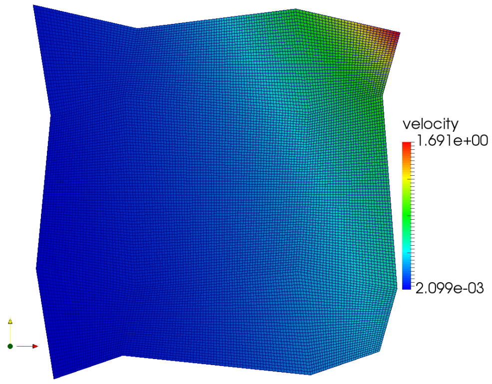


(f) Rotation

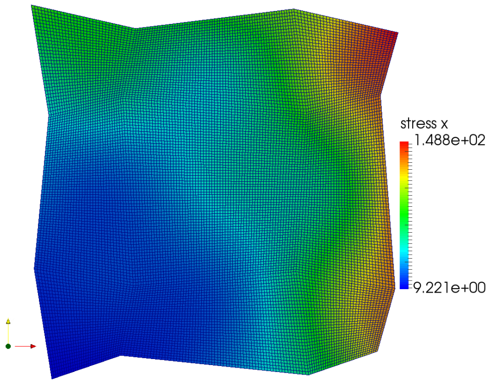
Figure 5: Solution on h^2-parallelograms.



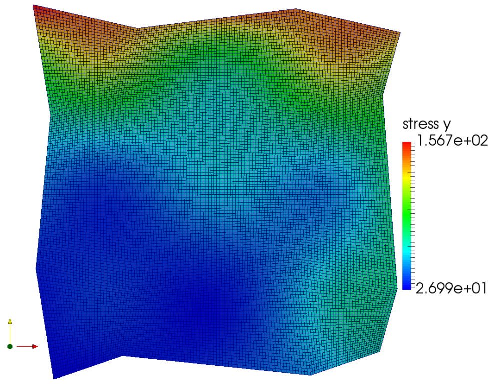
(a) Darcy pressure



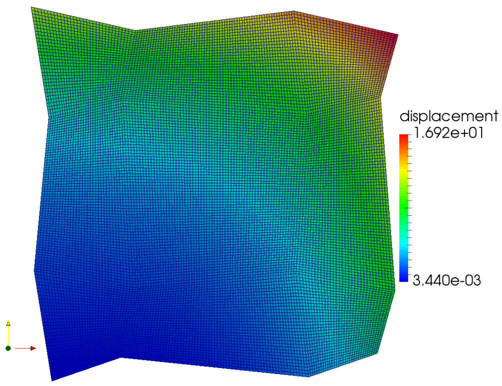
(b) Darcy velocity



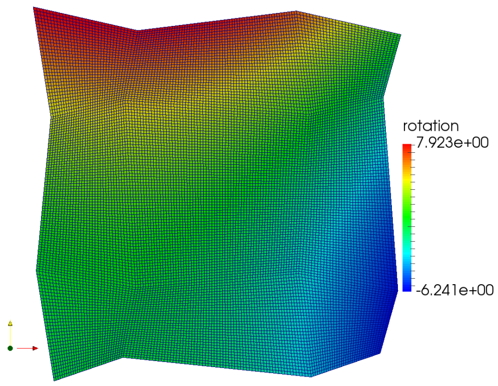
(c) Elastic stress, x-component



(d) Elastic stress, y-component



(e) Displacement



(f) Rotation

Figure 6: Solution on transformed domain.