A coupled multipoint stress multipoint flux mixed finite element method for Biot poroelasticity model

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Abstract

In this work we present a mixed finite element method for five-field Biot's consolidation model that reduces to cell-centered finite differences for displacement and pressure on quadrilateral and simplicial grids. The method is guaranteed to perform robustly with discontinuous full tensor permeability coefficients and heterogeneous elasticity parameters, which is verified by the error analysis. Our approach is motivated by multipoint flux approximation (MPFA) method and multipoint stress approximation (MPSA) methods, while the approach we take is based on more recent multipoint flux mixed finite element (MFMFE) method and multipoint stress mixed finite element (MSMFE) method, for Darcy and linear elasticity models, respectively. Our scheme couples the latter two methods for the spatial discretization of the Biot's poroelasticity system, and is based on the lowest order Brezzi-Douglas-Marini mixed finite element spaces. The special quadrature rule is then employed, that allows for the local stress, rotation and velocity elimination and leads to a symmetric and positive-definite system for displacements and pressures. Theoretical and numerical studies indicate first-order accuracy in all variables in their natural norms.

1 Introduction

Geoscience applications such as environmental cleanup, petroleum production, solid waste disposal, and carbon sequestration are inherently coupled with field phenomena such as surface subsidence, uplift displacement, pore collapse, cavity generation, hydraulic fracturing, thermal fracturing, wellbore collapse, sand production, and fault activation. This coupled nature of fluid motion through porous media and solid deformation makes it challenging for numerical modeling and simulation.

In this work we use is the classical Biot consolidation system in poroelasticity [9, 37] under a quasi-static assumption as the mathematical model for such coupled fluid-solid system. The system of equations consists of an equilibrium equation for the solid and a mass balance equation for the fluid. The contribution of the fluid pressure to the total stress of the solid, and the divergence of the solid displacement represents an additional term in the fluid content. Numerical modeling of this coupled system is well studied in the literature. In [27, 28], Taylor-Hood finite elements are employed for a displacementpressure variational formulation. A least squares formulation that approximates directly the solid stress and the fluid velocity is studied in [23, 24]. Finite difference schemes on staggered grids designed to avoid nonphysical oscillations at early times have been developed in 1D in [16, 19]. The method in [16] can handle discontinuous coefficients through harmonic averaging. A formulation based on mixed finite element (MFE) methods for flow and continuous Galerkin (CG) for elasticity has been proposed in [31, 32]. The coupled multipoint flux mixed finite element method (MFMFE) for flow and CG method for elasticity has been studied in [41]. On the other hand, as the MFE methods for elasticity become more popular in the finite element community, the five-field MFE formulation for the Biot system was presented in [25]. The advantages of this approach is that the fluid and mechanics approximations are locally mass conservative and the fluid velocity and poroelastic stress are computed directly. Moreover, this approach guarantees robustness and locking-free properties with respect to physical parameters. In [20], a parallel domain decomposition method has been developed

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for coupling a time-dependent poroelastic model in a localized region with an elastic model in adjacent regions. Each model is discretized independently on nonmatching grids and the systems are coupled using DG jumps and mortars. Applications of the Biot system to the computational modeling of coupled reservoir flow and geomechanics can be found in [12, 17, 18, 36].

The focus of this paper is to develop a discretization method for the poroelasticity system in the mixed form that is suitable for irregular and rough grids, discontinuous full tensor permeabilities and Lamé coefficeints that are often encountered in modeling subsurface flows. To this end, we develop a formulation that couples multipoint flux mixed finite element (MFMFE) methods for flow with multipoint stress mixed finite element (MSMFE) methods for elasticity. The MFMFE method was developed for Darcy flow in [22, 42, 40]. It is locally conservative with continuous fluxes and can be viewed within a variational framework as a mixed finite element method with special approximating spaces and quadrature rules. The MFMFE method allows for an accurate and efficient treatment of irregular geometries and heterogeneities such as faults, layers, and pinchouts that require highly distorted grids and discontinuous coefficients. The resulting discretizations are cell-centered with convergent pressures and velocities on general hexahedral and simplicial grids. The reader is referred to [39] for the performance of the MFMFE method for flow on a benchmark test using rough 3D grids and anisotropic coefficients. Similarly, the MSMFE method was developed in [3, 4] and shares the same locality properties with continuous normal stresses. The method is also derived within an variational framework as a mixed finite element method for elasticity with weak symmetry using special approximating spaces and quadrature rules.

As the MFMFE method was motivated by the multipoint flux approximation (MPFA) methods [2, 1, 14, 15], the MSMFE method was motivated by the multipoint stress approxiamtion (MPSA) [29]. Both frameworks allow for local flux and stress elimination around grid vertices and reduction to a cell-centered pressure and displacement scheme, respectively. The coupled scheme based on MPSA and MPFA methods for the elasticity and flow parts of the Biot system was proposed in [30]. Similar elimination is achieved in the MFMFE and MSMFE variational framework, by employing appropriate finite element spaces and special quadrature rules. Both methods are based on the BDM1 [10] spaces with a trapezoidal quadrature rule applied on the reference element, [22, 42, 40]. Our goal in this paper is to emphasize the applicability of the MSMFE method for solid mechanics in the Biot system, which, together with the MFMFE method used for the flow part of the model will result in an efficient technique for solving a coupled saddle-point type problem.

In this paper, we develop convergence analysis for the MFMFE-MSMFE numerical approximation of the time-dependent poroelasticity system. We study the symmetric version of the MFMFE and MSMFE methods on simplicial grids in 2 and 3 dimensions and quadrilateral grids. Theoretical and numerical results demonstrate first-order convergence in time and space for the fluid pressure and velocity, as well as for the poroelastic stress, solid displacement and rotation.

The rest of the paper is organized as follows. The problem formulation and the numerical approximation are presented in Section 2. Well-posedness of the proposed coupled MFMFE-MSMFE method is studied in Section 3. Section 4 shows the reduction of the method to the cell-centered finite difference (CCFD) scheme. The convergence analysis for the continuous in time scheme is developed in Sections 5. Finally, Section 6 is devoted to computational experiments

2 Definition of the method.

2.1 Preliminaries.

In this section we recall the formulation of the elasticity system based on weak imposition of the symmetry of a stress tensor, and its discretization by a mixed finite element method. We then propose the modification of said method to obtain a multipoint stress mixed finite element method for linear elasticity with weak symmetry and further provide its stability and error analysis.

Let Ω be a simply connected bounded domain of $\mathbb{R}^d$, $d = 2, 3$ occupied by a linearly elastic porous body. We write $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{N}$ for the spaces of $d \times d$ matrices, symmetric matrices and skew-symmetric matrices, all over the field of real numbers, respectively. The material properties are described at each point $x \in \Omega$ by a compliance tensor $A = A(x)$, which is a symmetric, bounded and uniformly positive definite linear operator acting from $\mathbb{S} \to \mathbb{S}$. We also assume that an extension of A to an operator $\mathbb{M} \to \mathbb{M}$ still possesses the above properties. However, in a case of homogeneous and isotropic body,

$$A\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu + d\lambda}\operatorname{tr}(\sigma)I\right),$$

where I is a $d \times d$ identity matrix and $\mu > 0, \lambda \geq 0$ are Lamé coefficients. Conventionally K stands for the permeability tensor, c_0 - for mass storativity and α represents the Biot-Willis constant.

Throughout the paper the divergence operator is the usual divergence for vector fields, which produces vector field when applied to matrix field by taking the divergence of each row. We will also use the curl operator which is the usual curl when applied to vector fields in three dimension, and defined as

$$\operatorname{curl}\phi = (\partial_2\phi, -\partial_1\phi)$$

for a scalar function ϕ in two dimension. Similarly, for a vector field in two dimension or a matrix field in three dimension, curl operator produces a matrix field by acting row-wise.

Therein for the rest of this paper, C denotes a generic positive constant that is independent of the discretization parameter h. We will also use the following standard notation. For a domain $G \subset \mathbb{R}^d$, the $L^2(G)$ inner product and norm for scalar and vector valued functions are denoted $(\cdot,\, \cdot)_G$ and $\|\cdot\|_G$, respectively. The norms and seminorms of the Sobolev spaces $W^{k,p}(G)$, $k \in \mathbb{R}, p > 0$ are denoted by $\|\cdot\|_{k,p,G}$ and $|\cdot|_{k,p,G}$, respectively. The norms and seminorms of the Hilbert spaces $H^k(G)$ are denoted by $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$, respectively. We omit G in the subscript if $G = \Omega$. For a section of the domain or element boundary $S \subset \mathbb{R}^{d-1}$ we write $\langle\cdot,\, \cdot\rangle_S$ and $\|\cdot\|_S$ for the $L^2(S)$ inner product (or duality pairing) and norm, respectively. We will also use the spaces

$$\begin{aligned}
H(\operatorname{div};\Omega) &= \{v \in L^2(\Omega,\mathbb{R}^d) : \operatorname{div} v \in L^2(\Omega)\},\\
H(\operatorname{div};\Omega,\mathbb{M}) &= \{\tau \in L^2(\Omega,\mathbb{M}) : \operatorname{div}\tau \in L^2(\Omega,\mathbb{R}^d)\},
\end{aligned}$$

equipped with the norm

$$\|\tau\|_{\operatorname{div}} = \left(\|\tau\|^2 + \|\operatorname{div}\tau\|^2\right)^{1/2}.$$

Given a vector field f on Ω representing body forces, the quasi-static Biot system determines the displacement u, together with the Darcy velocity z and pressure p:

$$-\operatorname{div}\sigma(u) = f, \qquad \text{in } \Omega, \tag{2.1}$$

$$K^{-1}z + \nabla p = 0, \qquad \text{in } \Omega \tag{2.2}$$

$$\frac{\partial}{\partial t}(c_0 p + \alpha \operatorname{div} u) + \operatorname{div} z = q, \qquad \text{in } \Omega \tag{2.3}$$

where the poroelastic stress $\sigma(u)$ is such that:

$$\sigma(u) = \sigma_E(u) - \alpha p I,$$

where $\sigma_E(u) = 2\mu\epsilon(u) + \lambda \operatorname{div} u\, I$ is the elastic stress. To close the system, the appropriate boundary conditions should also be prescribed

$$u = g_u \quad \text{on } \Gamma_D^{displ}, \qquad \sigma\, n = 0 \quad \text{on } \Gamma_N^{stress}, \tag{2.4}$$

$$p = g_p \quad \text{on } \Gamma_D^{pres}, \qquad z \cdot n = 0 \quad \text{on } \Gamma_N^{vel}, \tag{2.5}$$

where $\Gamma_D^{displ} \cup \Gamma_N^{stress} = \Gamma_D^{pres} \cup \Gamma_N^{vel} = \partial\Omega$ are boundaries on which Dirichlet and Neumann data is specified for displacement, pressure and normal fluxes, respectively. We assume for simplicity that $\Gamma_D^* \neq \emptyset$, for $* = \{displ,\, pres\}$.

We will also make use of the following notation. For a matrix τ, let

$$\operatorname{as}(\tau) = \tau_{12} - \tau_{21} \text{ in } 2d \text{ and } \operatorname{as}(\tau) = (\tau_{32} - \tau_{23}, \tau_{31} - \tau_{13}, \tau_{21} - \tau_{12})^T \text{ in } 3d,$$

and define the invertible Ξ as follows,

$$\Xi(\omega) = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \quad \text{for } \omega \in \mathbb{R}, \quad \Xi(\omega) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad \text{for } \omega \in \mathbb{R}^d. \tag{2.6}$$

It is easy to see then, that for all $\tau \in \mathbb{M}$ and $\xi \in \mathbb{N}$,

$$(\tau,\, \xi) = \left(\operatorname{as}(\tau),\, \Xi^{-1}(\xi)\right). \tag{2.7}$$

We notice that due to the constitutive equation in a linear elasticity system, $A\sigma_E = \nabla u - \Xi(\gamma)$, we have

$$\operatorname{div} u = \operatorname{tr}(A\sigma_E)$$

Then the problem reads: find $(\sigma, u, \gamma, z, p)$ such that

$$\begin{aligned}
&(A\sigma,\, \tau) + (A\alpha p I,\, \tau) + (u,\, \operatorname{div}\tau) + (\gamma,\, \tau) = \langle g_u,\, \tau\, n\rangle, && \forall \tau \in \mathbb{X}, && (2.8)\\
&(\operatorname{div}\sigma,\, v) = -\left(f,\, v\right), && \forall v \in V, && (2.9)\\
&(\sigma,\, \xi) = 0 && \forall \xi \in \mathbb{W}, && (2.10)\\
&\left(K^{-1}z,\, q\right) - (p,\, \operatorname{div} q) = -\langle g_p,\, v\cdot n\rangle, && \forall q \in Z, && (2.11)\\
&c_0\left(\frac{\partial p}{\partial t},\, w\right) + \alpha\left(\frac{\partial}{\partial t}A\sigma,\, wI\right) + \alpha\left(\frac{\partial}{\partial t}\operatorname{tr}(A\alpha p I),\, w\right) + (\operatorname{div} z,\, w) = (g,\, w)\,, && \forall w \in W, && (2.12)\\
&\sigma\, n = 0, && \text{on } \Gamma_N^{stress}, && (2.13)\\
&u\cdot n = 0, && \text{on } \Gamma_N^{vel}, && (2.14)
\end{aligned}$$

where the spaces are

$$\begin{aligned}
&\mathbb{X} = \left\{\tau \in H(\operatorname{div};\Omega,\mathbb{M}) : \tau\, n = 0 \text{ on } \Gamma_N^{stress}\right\}, \quad V = L^2(\Omega,\mathbb{R}^d), \quad \mathbb{W} = L^2(\Omega,\mathbb{N}),\\
&Z = \left\{v \in H(\operatorname{div};\Omega,\mathbb{R}^d) : v\cdot n = 0 \text{ on } \Gamma_N^{vel}\right\}, \quad W = L^2(\Omega).
\end{aligned}$$

It was shown in [25] that (2.8)-(2.14) has a unique solution.

2.2 Finite element mappings.

We start with providing the necessary basic results that will be used in the later derivations of the multi-point stress-flux mixed finite element method for the problem.

Let $\mathcal{T}_h$ be a finite element partition of a polygonal domain $\Omega \in \mathbb{R}^d$, consisting of triangles and/or convex quadrilaterals in two dimensions and tetrahedra in three dimensions. Let $h = \max_{E\in\mathcal{T}_h} \operatorname{diam}(E)$ be the mesh characteristic size, representing the largest diameter of an element in the given partition. We also assume the partition $\mathcal{T}_h$ to be shape-regular and quasi-uniform [13]. For any element $E \in \mathcal{T}_h$ there exists a bijection mapping $F_E : \hat{E} \to E$, where $\hat{E}$ is a reference element. We denote the Jacobian matrix by DF_E and introduce $J_E = |\det(DF_E)|$. Let the inverse mapping be denoted by F_E^{-1}, its Jacobian matrix by DF_E^{-1}, and let $J_{F_E^{-1}} = \left|\det(DF_E^{-1})\right|$. We have that

$$DF_E^{-1}(x) = (DF_E)^{-1}(\hat{x}), \qquad J_{F_E^{-1}}(x) = \frac{1}{J_E(\hat{x})}.$$

In case of triangular meshes, $\hat{E}$ is the reference right triangle with vertices $\hat{\mathbf{r}}_1 = (0, 0^T)$, $\hat{\mathbf{r}}_2 = (1,0)^T$ and $\hat{\mathbf{r}}_3 = (0,1)^T$. Let $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ be the corresponding vertices of E, oriented counterclockwise. In this case F_E is a linear mapping of the following form

$$F_E(\hat{\mathbf{r}}) = \mathbf{r}_1(1 - \hat{x} - \hat{y}) + \mathbf{r}_2\hat{x} + \mathbf{r}_3\hat{y}, \tag{2.15}$$

with constant Jacobian matrix and determinant given by

$$DF_E = [\mathbf{r}_{21}, \mathbf{r}_{31}]^T \quad \text{and} \quad J_E = 2|E|, \tag{2.16}$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. The mapping for tetrahedra is described similarly.

In the case of convex quadrilaterals, $\hat{E}$ is the unit square with vertices $\hat{\mathbf{r}}_1 = (0,0)^T$, $\hat{\mathbf{r}}_2 = (1,0)^T$, $\hat{\mathbf{r}}_3 = (1,1)^T$ and $\hat{\mathbf{r}}_4 = (0,1)^T$. Denote by $\mathbf{r}_i = (x_i, y_i)^T$, $i = 1, \ldots, 4$, the four corresponding vertices of element E. The outward unit normal vectors to the edges of E and $\hat{E}$ are denoted by n_i and $\hat{n}_i$, $i = 1, \ldots, 4$, respectively. In this case F_E is the bilinear mapping given by

$$\begin{aligned} F_E(\hat{\mathbf{r}}) &= \mathbf{r}_1(1-\hat{x})(1-\hat{y}) + \mathbf{r}_2\hat{x}(1-\hat{y}) + \mathbf{r}_3\hat{x}\hat{y} + \mathbf{r}_4(1-\hat{x})\hat{y} \\ &= \mathbf{r}_1 + \mathbf{r}_{21}\hat{x} + \mathbf{r}_{41}\hat{y} + (\mathbf{r}_{34} - \mathbf{r}_{21})\hat{x}\hat{y}, \end{aligned} \tag{2.17}$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. It is easy to see that DF_E and J_E are linear functions of $\hat{x}$ and $\hat{y}$, i.e.

$$\begin{aligned} DF_E &= [(\mathbf{r}_{34} - \mathbf{r}_{21})\hat{y} + \mathbf{r}_{21}, \mathbf{r}_{41} - (\mathbf{r}_{34} - \mathbf{r}_{21})\hat{x}] \\ &= [\mathbf{r}_{21}, \mathbf{r}_{41}] + [(\mathbf{r}_{34} - \mathbf{r}_{21})\hat{y}, (\mathbf{r}_{34} - \mathbf{r}_{21})\hat{x}], \end{aligned} \tag{2.18}$$

$$J_E = 2|T_1| + 2(|T_2| - |T_1|)\hat{x} + 2(|T_4| - |T_1|)\hat{y}, \tag{2.19}$$

where $|T_i|$ is the area of a triangle enclosed by the two edges sharing $\mathbf{r}_i$. We notice that the Jacobian determinant J_E is uniformly positive, due to convexity of E.

Using the mapping definitions (2.15)-(2.19), a simple calculation verifies that for any edge (face) $e_i \subset \partial E$

$$n_i = \frac{1}{|e_i|} J_E (DF_E^{-1})^T \hat{n}_i. \tag{2.20}$$

Another direct computation using the mapping definitions together with shape-regularity and quasi-uniformity of the grids, show that for all element types

$$\|DF_E\|_{0,\infty,\hat{E}} \sim h, \quad \|J_E\|_{0,\infty,\hat{E}} \sim h^d \quad \text{and} \quad \|J_{F_E^{-1}}\|_{0,\infty,\hat{E}} \sim h^{-d} \quad \forall E \in \mathcal{T}_h. \tag{2.21}$$

2.3 Mixed finite element spaces.

We consider $\mathbb{X}_h \times V_h \times \mathbb{W}_h$ to be the lowest order triple of the form $(\mathcal{BDM}_1)^d \times (\mathcal{P}_0)^d \times \left(\mathcal{P}_1^{cts}\right)^{d\times d, skew}$ on simplicial elements, while in case of quadrilaterals, the triple is changed to $(\mathcal{BDM}_1)^d \times (\mathcal{Q}_0)^d \times \left(\mathcal{Q}_1^{cts}\right)^{d\times d, skew}$. These triples were shown to be inf-sup stable for the mixed elasticity problem with weak symmetry in [6, 7] for simplicial grids, and in [4] for the case of convex quadrilaterals. We also consider the lowest order $\mathcal{BDM}_1$ MFE spaces [10, 11] for $Z_h \times W_h$.

On the reference simplex, these spaces are defined as

$$\hat{\mathbb{X}}_h(\hat{E}) = \left(\mathcal{P}_1(\hat{E})^d\right)^d, \qquad \hat{V}_h(\hat{E}) = \mathcal{P}_0(\hat{E})^d, \qquad \hat{\mathbb{W}}_h(\hat{E}) = \Xi(v),\, v \in \mathcal{P}_1(\hat{E}), \tag{2.22}$$

$$\hat{Z}_h(\hat{E}) = \mathcal{P}_1(\hat{E})^d, \qquad \hat{W}_h(\hat{E}) = \mathcal{P}_0(\hat{E}). \tag{2.23}$$

The definition of the said spaces on tetrahedra is obtained naturally from the one above.

On the reference unit square the stress and the velocity spaces are defined as

$$\begin{aligned} \hat{\mathbb{X}}(\hat{E}) &= \left(\mathcal{P}_1(\hat{E})^2 + r\ \mathrm{curl}(\hat{x}^2\hat{y}) + s\ \mathrm{curl}(\hat{x}\hat{y}^2)\right)^2 \\ &= \begin{pmatrix} \alpha_1\hat{x} + \beta_1\hat{y} + \gamma_1 + r_1\hat{x}^2 + 2s_1\hat{x}\hat{y} & \alpha_2\hat{x} + \beta_2\hat{y} + \gamma_2 - 2r_1\hat{x}\hat{y} - s_1\hat{y}^2 \\ \alpha_3\hat{x} + \beta_3\hat{y} + \gamma_3 + r_2\hat{x}^2 + 2s_2\hat{x}\hat{y} & \alpha_4\hat{x} + \beta_4\hat{y} + \gamma_4 - 2r_2\hat{x}\hat{y} - s_2\hat{y}^2 \end{pmatrix}, \\ \hat{V}_h(\hat{E}) &= \mathcal{P}_0(\hat{E})^d, \quad \hat{\mathbb{W}}_h(\hat{E}) = \Xi(v),\, v \in \mathcal{Q}_1(\hat{E}), \\ \hat{Z}(\hat{E}) &= \mathcal{P}_1(\hat{E})^2 + r\ \mathrm{curl}(\hat{x}^2\hat{y}) + s\ \mathrm{curl}(\hat{x}\hat{y}^2) \\ &= \begin{pmatrix} \alpha_5\hat{x} + \beta_5\hat{y} + \gamma_5 + r_3\hat{x}^2 + 2s_3\hat{x}\hat{y} \\ \alpha_6\hat{x} + \beta_6\hat{y} + \gamma_6 - 2r_3\hat{x}\hat{y} - s_3\hat{y}^2 \end{pmatrix}, \\ \hat{W}_h(\hat{E}) &= \mathcal{P}_0(\hat{E}). \end{aligned} \tag{2.24}$$

An important property these spaces possess is that

$$\widehat{\operatorname{div}}\hat{\mathbb{X}}(\hat{E}) = \hat{V}(\hat{E}), \quad \widehat{\operatorname{div}}\hat{Z}(\hat{E}) = \hat{W} \quad \text{and} \tag{2.25}$$

$$\forall \hat{\tau} \in \hat{\mathbb{X}}(\hat{E}),\ \hat{q} \in \hat{Z}(\hat{E}),\ \hat{e} \in \hat{E} \quad \hat{\tau}\,\hat{n}_{\hat{e}} \in \mathcal{P}_1(\hat{e})^d \text{ and } \hat{q} \cdot \hat{n}_{\hat{e}} \in \mathcal{P}_1(\hat{e}). \tag{2.26}$$

It is known [10, 11] that the degrees of freedom for $\mathcal{BDM}_1$ space can be chosen to be the values of normal fluxes at any two points on each edge $\hat{e}$ if $\hat{E}$ is a reference triangle, or any three points one each face $\hat{e}$ if $\hat{E}$ is a reference tetrahedron. This also applies to normal stresses in the case of $(\mathcal{BDM}_1)^d$. For this work we choose said points to be at the vertices of $\hat{e}$ for both the velocity and stress spaces. This choice is motivated by the use of quadrature rule introduced in the next section.

To define the above spaces on any physical element $E \in \mathcal{T}_h$ the following transformations are used

$$\tau \leftrightarrow \hat{\tau} : \tau = \frac{1}{J_E} DF_E \hat{\tau} \circ F_E^{-1}, \qquad v \leftrightarrow \hat{v} : v = \hat{v} \circ F_E^{-1},$$

$$\xi \leftrightarrow \hat{\xi} : \xi = \hat{\xi} \circ F_E^{-1}, \qquad \hat{q} \leftrightarrow \hat{q} : q = \frac{1}{J_E} DF_E \hat{q} \circ F_E^{-1},$$

$$w \leftrightarrow \hat{w} : w = \hat{w} \circ F_E^{-1},$$

here we consider $\tau \in \mathbb{X}$, $v \in V$, $\xi \in \mathbb{W}$, $q \in Z$ and $w \in W$.

The first and the third transformations provided above are known as Piola transformation applied to tensor and vector valued functions, respectively. Its advantage is in preserving the normal components of the stress tensor and velocity vector on the edges (faces), and it satisfies the following properties

$$(\operatorname{div}\tau, v)_E = (\widehat{\operatorname{div}}\hat{\tau}, \hat{v})_{\hat{E}} \quad \text{and} \quad \langle \tau\, n_e, v\rangle_e = \langle \hat{\tau}\,\hat{n}_{\hat{e}}, \hat{v}\rangle_{\hat{e}}, \tag{2.27}$$

$$(\operatorname{div} q, w)_E = (\widehat{\operatorname{div}}\hat{q}, \hat{w})_{\hat{E}} \quad \text{and} \quad \langle q \cdot n_e, w\rangle_e = \langle \hat{q} \cdot \hat{n}_{\hat{e}}, \hat{w}\rangle_{\hat{e}}. \tag{2.28}$$

It also follows that for functions in stress and velocity spaces, there holds

$$\tau\, n_e = \frac{1}{J_E} DF_E \hat{\tau}\, \frac{1}{|e|} J_E (DF_E^{-1})^T \hat{n}_{\hat{e}} = \frac{1}{|e|} \hat{\tau}\,\hat{n}_{\hat{e}}, \tag{2.29}$$

$$q \cdot n_e = \frac{1}{J_E} DF_E \hat{q} \cdot \frac{1}{|e|} J_E (DF_E^{-1})^T \hat{n}_{\hat{e}} = \frac{1}{|e|} \hat{q} \cdot \hat{n}_{\hat{e}}. \tag{2.30}$$

First equation in (2.27) can be written as $(\operatorname{div}\tau, v)_E = (\widehat{\operatorname{div}\tau}, J_E\hat{v})_{\hat{E}}$ which leads to

$$\operatorname{div}\tau = \left(\frac{1}{J_E}\widehat{\operatorname{div}} \cdot \hat{\chi}\right) \circ F_E^{-1}(x), \tag{2.31}$$

showing that $\operatorname{div}\tau\big|_E$ is constant on simplicial elements. Similarly, one concludes that $\operatorname{div} q\big|_E$ is also constant on simplicial elements.

We now introduce the finite dimensional spaces for the method on a given partition of the domain $\mathcal{T}_h$:

$$\begin{aligned}
\mathbb{X}_h &= \{\tau \in \mathbb{X} : \quad \tau|_E \leftrightarrow \hat{\tau},\ \hat{\tau} \in \hat{\mathbb{X}}(\hat{E}) \quad \forall E \in \mathcal{T}_h\},\\
V_h &= \{v \in V : \quad v|_E \leftrightarrow \hat{v},\ \hat{v} \in \hat{V}(\hat{E}) \quad \forall E \in \mathcal{T}_h\},\\
\mathbb{W}_h &= \{\xi \in \mathbb{W} : \quad \xi|_E \leftrightarrow \hat{\xi},\ \hat{\xi} \in \hat{\mathbb{W}}(\hat{E}) \quad \forall E \in \mathcal{T}_h\},\\
Z_h &= \{q \in Z : \quad q|_E \leftrightarrow \hat{q},\ \hat{q} \in \hat{Z}(\hat{E}) \quad \forall E \in \mathcal{T}_h\},\\
W_h &= \{w \in W : \quad w|_E \leftrightarrow \hat{w},\ \hat{w} \in \hat{W}(\hat{E}) \quad \forall E \in \mathcal{T}_h\}.
\end{aligned} \tag{2.32}$$

We denote by Π a mixed projection operator acting on tensor valued functions, such that $\Pi : \mathbb{X} \cap H^1(\Omega, \mathbb{M}) \to \mathbb{X}_h$. We will also use the same notation for a projection operator acting on vector valued functions, so that in this case Π maps from $Z \cap H^1(\Omega, \mathbb{R}^d)$ onto Z_h. It was shown in [10, 11] and [38] that such projection operator exists and satisfies the following properties

$$\begin{aligned}
(\operatorname{div}(\Pi\tau - \tau), v) &= 0, \qquad \forall v \in V_h,\\
(\operatorname{div}(\Pi q - q), w) &= 0, \qquad \forall w \in W_h.
\end{aligned} \tag{2.33}$$

In both cases the operator Π is defined locally on each element E by

$$\Pi\tau \leftrightarrow \widehat{\Pi\tau}, \quad \widehat{\Pi\tau} = \hat{\Pi}\hat{\tau}, \tag{2.34}$$

$$\Pi q \leftrightarrow \widehat{\Pi q}, \quad \widehat{\Pi q} = \hat{\Pi}\hat{q}, \tag{2.35}$$

where $\hat{\Pi} : H^1(\hat{E}, \mathbb{M}) \to \hat{\mathbb{X}}_h(\hat{E})$ is the reference element projection operator satisfying

$$\forall \hat{e} \subset \partial\hat{E}, \qquad \langle(\hat{\Pi}\hat{\tau} - \hat{\tau})\hat{n}, \hat{\phi}_1\rangle_{\hat{e}} = 0 \qquad \forall \hat{\phi}_1 \in (\mathcal{P}_1(\hat{e}))^d, \tag{2.36}$$

and similarly, $\hat{\Pi} : H^1(\hat{E}, \mathbb{R}^d) \to \hat{Z}_h(\hat{E})$ is an operator satisfying

$$\forall \hat{e} \subset \partial\hat{E}, \qquad \langle(\hat{\Pi}\hat{q} - \hat{q}) \cdot \hat{n}, \hat{\psi}_1\rangle_{\hat{e}} = 0 \qquad \forall \hat{\psi}_1 \in \mathcal{P}_1(\hat{e}). \tag{2.37}$$

It is straightforward to see from (2.27), (2.34), (2.36) that $\tau\, n = 0$ on Γ_N^{stress} implies $\Pi\tau\, n = 0$ on Γ_N^{stress}. For this we note that for all $\phi \leftrightarrow \hat{\phi} \in (\mathcal{P}_1(\hat{e}))^d$,

$$\langle \Pi\tau\, n,\, \phi\rangle_e = \langle \widehat{\Pi\tau\, n},\, \hat{\phi}\rangle_{\hat{e}} = \langle \hat{\Pi}\hat{\tau}\, \hat{n},\, \hat{\phi}\rangle_{\hat{e}} = \langle \hat{\tau}\, \hat{n},\, \hat{\phi}\rangle = 0.$$

Similar argument using (2.28), (2.35), (2.37) shows that $q \cdot n = 0$ on Γ_N^{vel} implies $\Pi q \cdot n = 0$ on Γ_N^{vel}.

In addition to the mixed projection operator presented above, we will make use of a similar projection operator onto the lowest order Raviart-Thomas spaces [34, 11]. This additional construction is solely motivated by the purposes of error analysis on quadrilaterals, although for the uniformity of forthcoming proofs we would treat simplicial case in the same fashion. To deal with errors in stress and velocity variables we consider $\mathcal{RT}_0$ spaces of tensor and vector valued functions, respectively, where the former is obtained as 2 copies of the latter. Said spaces are defined on a unit square as follows

$$\hat{\mathbb{X}}^0(\hat{E}) = \begin{pmatrix} \alpha_1 + \beta_1\hat{x} & \alpha_2 + \beta_2\hat{y} \\ \alpha_3 + \beta_3\hat{x} & \alpha_4 + \beta_4\hat{y} \end{pmatrix}, \quad \hat{V}^0(\hat{E}) = \Big(Q_0(\hat{E})\Big)^2, \tag{2.38}$$

$$\hat{Z}^0(\hat{E}) = \begin{pmatrix} \alpha_5 + \beta_5\hat{x} \\ \alpha_6 + \beta_6\hat{y} \end{pmatrix}, \quad \hat{W}^0(\hat{E}) = Q_0(\hat{E}), \tag{2.39}$$

and on unit triangle as

$$\hat{\mathbb{X}}^0(\hat{E}) = \begin{pmatrix} \alpha_1 + \beta_1\hat{x} & \alpha_2 + \beta_1\hat{y} \\ \alpha_3 + \beta_2\hat{x} & \alpha_4 + \beta_2\hat{y} \end{pmatrix}, \quad \hat{V}^0(\hat{E}) = \Big(P_0(\hat{E})\Big)^2, \tag{2.40}$$

$$\hat{Z}^0(\hat{E}) = \begin{pmatrix} \alpha_5 + \beta_3\hat{x} \\ \alpha_6 + \beta_3\hat{y} \end{pmatrix}, \quad \hat{W}^0(\hat{E}) = P_0(\hat{E}). \tag{2.41}$$

In the case of unit tentrahedron $\hat{\mathbb{X}}^0(\hat{E})$ would have an additional row of components, while $\hat{Z}^0(\hat{E})$ - and additional entry in the vector. In all cases the

$$\operatorname{div} \hat{\mathbb{X}}^0(\hat{E}) = \hat{V}^0(\hat{e}) \text{ and } \hat{\tau}\,\hat{n} \in (\mathcal{P}_0(\hat{e}))^d\,,$$
$$\operatorname{div} \hat{Z}^0(\hat{E}) = \hat{W}^0(\hat{e}) \text{ and } \hat{q}\cdot\hat{n} \in \mathcal{P}_0(\hat{e}).$$

The degrees of freedom of $\hat{\mathbb{X}}^0(\hat{E})$ are the values of normal stress $\hat{\tau}\,\hat{n}$ at the midpoints of all edges (faces) $\hat{e}$, similarly, the degrees of freedom of $\hat{Z}^0(\hat{E})$ are the values of normal fluxes $\hat{q}\cdot\hat{n}$ at the same points. The projection operator $\hat{\Pi}_0$ acting on tensor valued functions from $H^1(\Omega, \mathbb{M})$ onto $\hat{\mathbb{X}}^0(\hat{E})$; and acting on vector valued function so that $\hat{\Pi}_0 : H^1(\Omega, \mathbb{R}^d) \to \hat{Z}^0(\hat{E})$ satisfies

$$\begin{aligned} &\forall \hat{e} \subset \partial\hat{E}, \qquad \langle(\hat{\Pi}_0\hat{\tau} - \hat{\tau})\hat{n}, \hat{\phi}_0\rangle_{\hat{e}} = 0 \qquad \forall \hat{\phi}_0 \in (\mathcal{P}_0(\hat{e}))^d, \\ &\forall \hat{e} \subset \partial\hat{E}, \qquad \langle(\hat{\Pi}_0\hat{q} - \hat{q}) \cdot \hat{n}, \hat{\psi}_0\rangle_{\hat{e}} = 0 \qquad \forall \hat{\psi}_0 \in \mathcal{P}_0(\hat{e}). \end{aligned} \tag{2.42}$$

The spaces $\mathbb{X}_h^0$, V_h^0, Z_h^0 and W_h^0 on the entire partition $\mathcal{T}_h$ and the projection operator Π_0 for both tensor and vector valued functions are defined similarly to the case of $\mathcal{BDM}_1$ spaces. Notice also that $\mathbb{X}_h^0 \subset \mathbb{X}_h$ and $Z_h^0 \subset Z_h$, while the corresponding spaces V_h^0 and W_h^0 coincide with V_h and W_h, respectively. The definition of $\mathcal{RT}_0$ projector implies that

$$\begin{aligned} &\operatorname{div}\tau = \operatorname{div}\Pi_0\tau \quad \text{and} \quad \|\Pi_0\tau\| \le C\|\tau\| \quad \forall \tau \in \mathbb{X}_h, \\ &\operatorname{div} q = \operatorname{div}\Pi_0 q \quad \text{and} \quad \|\Pi_0 q\| \le C\|q\| \quad \forall q \in Z_h. \end{aligned} \tag{2.43}$$

2.4 The $\mathcal{BDM}_1$ coupled mixed finite element method.

The lowest order coupled five field mixed finite element approximation of Biot's poroelasticity system of equations (2.8)-(2.14) reads as follows: Find $(\sigma_h, u_h, \gamma_h, z_h, p_h) \in \mathbb{X}_h \times V_h \times \mathbb{W}_h \times Z_h \times W_h$ such that:

$$
\begin{aligned}
&(A\sigma_h,\, \tau) + (A\alpha p_h I,\, \tau) + (u_h,\, \operatorname{div}\tau) + (\gamma_h,\, \tau) = \langle g_u,\, \tau\, n\rangle_{\Gamma_D^{displ}} && \forall \tau \in \mathbb{X}_h && (2.44)\\
&(\operatorname{div}\sigma_h,\, v) = -\,(f,\, v) && \forall v \in V_h && (2.45)\\
&(\sigma_h,\, \xi) = 0 && \forall \xi \in \mathbb{W}_h && (2.46)\\
&\left(K^{-1} z_h,\, q\right) - (p_h,\, \operatorname{div} q) = -\langle g_p,\, v\cdot n\rangle_{\Gamma_D^{pres}} && \forall q \in Z_h && (2.47)\\
&c_0\left(\frac{\partial p_h}{\partial t},\, w\right) + \alpha\left(\frac{\partial}{\partial t} A\sigma_h,\, wI\right) + \alpha\left(\frac{\partial}{\partial t}\operatorname{tr}(A\alpha p_h I),\, w\right) + (\operatorname{div} z_h,\, w) = (g,\, w) && \forall w \in W_h. && (2.48)
\end{aligned}
$$

The method has a unique solution and is first order accurate for all of the variables in corresponding norms on simplicial and quadrilateral grids with our choices of elements [25]. While the method inherits all the advantages of a MFE method, its major drawback is in the resulting coupled algebraic system for five variables being of a saddle point type. Motivated by MFMFE and MSMFE methods, in the next sections we develop a quadrature rule that allows for local elimination of the stresses, rotations and fluxes which leads to a positive-definite cell-centered displacement-pressure system.

2.5 A quadrature rule.

For any pair of tensor or vector valued functions (ϕ, ψ) from $\mathbb{X}_h$ or Z_h, respectively, and for any linear uniformly bounded and positive-definite operator L we define the global quadrature rule

$$
(L\phi, \psi)_Q \equiv \sum_{E\in\mathcal{T}_h} (L\phi, \psi)_{Q,E}.
$$

The integration on any element E is performed by mapping to the reference element $\hat{E}$. The quadrature rule is defined on $\hat{E}$. Using the definition of the finite element spaces and omitting the subscript E, we get

$$
\begin{aligned}
\int_E L\phi\cdot\psi\, dx &= \int_{\hat{E}} \hat{L}\frac{1}{J}DF\hat{\phi}\cdot\frac{1}{J}DF\hat{\psi}\, J\, d\hat{x}\\
&= \int_{\hat{E}} \frac{1}{J}DF^T\hat{L}\, DF\hat{\phi}\cdot\hat{\psi}\, dx \equiv \int_{\hat{E}} \mathcal{L}\hat{\phi}\cdot\hat{\psi}\, d\hat{x},
\end{aligned}
$$

where $\cdot$ has a meaning of inner product for both tensor and vector valued functions, and

$$
\mathcal{L}\phi = \frac{1}{J}DF^T\hat{L}\, DF\hat{\phi} \qquad (2.49)
$$

is also a symmetric and positive definite operator. Notice that due to (2.21),

$$
\|\mathcal{L}\hat{\phi}\|_{\hat{E}} \sim h^{2-d}\|L\phi\|_E. \qquad (2.50)
$$

The quadrature rule on an element E is defined as

$$
(L\phi, \psi)_{Q,E} \equiv (\mathcal{L}\hat{\phi}, \hat{\psi})_{\hat{Q},\hat{E}} \equiv \frac{|\hat{E}|}{s}\sum_{i=1}^{s} \mathcal{L}\hat{\phi}(\hat{\mathbf{r}}_i) : \hat{\psi}(\hat{\mathbf{r}}_i), \qquad (2.51)
$$

where $s = 3$ for the unit triangle and $s = 4$ for the unit tetrahedron or the unit square. This quadrature rule is often referred to as a vertex quadrature rule on unit simplices and as trapezoid rule on unit squares.

When applied to the elasticity and Darcy coercive terms in our coupled problem, the quadrature rule defined above guarantees the coupling of stress and velocity basis function only around vertices (see [42, 3, 4]), i.e., the coupled stress basis functions are only the ones associated with a corner, and same statement applies for the velocity basis functions.

We also construct the quadrature rule for the term involving stress with second variable being pressure or rotation. Given $\tau = \mathbb{X}_h$, $\zeta \in \mathbb{W}_h$ or $\zeta \in (W_h)^{d\times d}$ and any linear uniformly bounded positive-definite operator M we get:

$$\int_E M\tau : \zeta \, dx = \int_{\hat{E}} \frac{1}{J} \hat{M} DF \,\hat{\tau} : \hat{\zeta} \, J \, d\hat{x} = \int_{\hat{E}} \hat{M} DF \,\hat{\tau} : \hat{\zeta} \, d\hat{x} = \int_{\hat{E}} \mathcal{M}\hat{\tau} : \hat{\zeta} \, d\hat{x},$$

where $\mathcal{M}\hat{\tau} = \hat{M}DF\hat{\tau}$. For this case we also define

$$(\tau, \zeta)_{Q,E} \equiv \left(\mathcal{M}\hat{\tau},\, \hat{\zeta}\right)_{\hat{\mathcal{Q}},\hat{E}} \equiv \frac{|\hat{E}|}{s} \sum_{i=1}^{s} \mathcal{M}\hat{\tau}(\hat{\mathbf{r}}_i) : \hat{\zeta}(\hat{\mathbf{r}}_i). \tag{2.52}$$

Remark 2.1. *The quadrature rules can be defined directly on an element E. It is easy to see from definitions* (2.51), (2.52) *that on simplicial elements, for $\phi, \psi \in \mathbb{X}_h$ or $\phi, \psi \in Z_h$, $\tau \in \mathbb{X}_h$ and $\zeta \in \mathbb{W}_h$ or $\zeta \in (W_h)^{d\times d}$*

$$(L\phi, \psi)_{Q,E} = \frac{|E|}{s} \sum_{i=1}^{s} L\phi(\mathbf{r}_i) \cdot \psi(\mathbf{r}_i), \quad (M\tau, \zeta)_{Q,E} = \frac{|E|}{s} \sum_{i=1}^{s} M\tau(\mathbf{r}_i) : \zeta(\mathbf{r}_i), \tag{2.53}$$

where L and M are any linear uniformly bounded and positive definite operators. On quadrilaterals the above definitions read as

$$(L\phi, \psi)_{Q,E} = \frac{1}{2} \sum_{i=1}^{4} |T_i| L\phi(\mathbf{r}_i) \cdot \psi(\mathbf{r}_i), \quad (M\tau, \zeta)_{Q,E} = \frac{1}{2} \sum_{i=1}^{4} |T_i| M\tau(\mathbf{r}_i) : \zeta(\mathbf{r}_i), \tag{2.54}$$

where $|T_i|$ is the area of a triangle formed by two edges sharing vertex $\mathbf{r}_i$.

The above quadrature rules are closely related to some inner products arising in mimetic finite difference methods [21].

For ϕ, $\psi \in \mathbb{X}_h$ or ϕ, $\psi \in Z_h$, $\tau \in \mathbb{X}_h$ and $\zeta \in \mathbb{W}_h$ or $\zeta \in (W_h)^{d\times d}$ denote the element quadrature errors by

$$\theta(L\phi, \psi) \equiv (L\phi, \psi)_E - (L\phi, \psi)_{Q,E} \tag{2.55}$$

$$\delta(M\tau, \zeta) \equiv (M\tau, \zeta)_E - (M\tau, \zeta)_{Q,E}, \tag{2.56}$$

and define the global quadrature errors by $\theta(L\phi, \psi)_E = \theta(L\phi, \psi)$, $\delta(M\tau, \zeta)_E = \delta(M\tau, \zeta)$. Similarly denote the quadrature errors on the reference element by

$$\hat{\theta}(\mathcal{L}\hat{\phi}, \hat{\psi}) \equiv (\mathcal{L}\hat{\phi}, \hat{\psi})_{\hat{E}} - (\mathcal{L}\hat{\phi}, \hat{\psi})_{Q,\hat{E}} \tag{2.57}$$

$$\hat{\delta}(\mathcal{M}\hat{\tau}, \hat{\zeta}) \equiv (\mathcal{M}\hat{\tau}, \hat{\zeta})_{\hat{E}} - (\mathcal{M}\hat{\tau}, \hat{\zeta})_{Q,\hat{E}}. \tag{2.58}$$

Lemma 2.1. *On simplicial elements, if $\chi \in \mathbb{X}_h(E)$ and $r \in Z_h(E)$, then*

$$\theta_E(\chi, \tau_0) = 0 \quad \textit{for all constant tensors } \tau_0,$$
$$\theta_E(r, v_0) = 0 \quad \textit{for all constant vectors } v_0.$$

Also, if $\zeta \in \mathbb{W}_h(E)$, then

$$\delta_E(\chi, \xi_0) = \delta_E(\tau_0, \zeta) = 0, \quad \textit{for all constant tensors } \xi_0 \textit{ and } \tau_0.$$

Proof. It is enough to consider τ_0 such that it has only one nonzero component, say, $(\tau_0)_{1,1} = 1$, the arguments for other cases are similar. Since the quadrature rule $(f)_E = \frac{|E|}{s} \sum_{i=1}^{s} f(\mathbf{r}_i)$ is exact for linear functions and using Remark 2.1 we have

$$(\chi,\, \tau_0)_{Q,E} = \frac{|E|}{s} \sum_{i=1}^{s} (\chi)_{1,1}(\mathbf{r}_i) = \int_E \chi : \tau_0 \, dx,$$

The same reasoning applies for the other two statements. □

Lemma 2.2. *On the reference square, for any $\hat{\chi} \in \hat{\mathbb{X}}_h(\hat{E})$ and $\hat{r} \in \hat{Z}_h(\hat{E})$,*

$$\left(\hat{\chi} - \hat{\Pi}_0\hat{\chi},\, \hat{\tau}_0\right)_{\hat{\mathcal{Q}},\hat{E}} = 0 \quad \textit{for all constant tensors } \hat{\tau}_0, \tag{2.59}$$

$$\left(\hat{r} - \hat{\Pi}_0\hat{r},\, \hat{v}_0\right)_{\hat{\mathcal{Q}},\hat{E}} = 0 \quad \textit{for all constant vectors } \hat{v}_0. \tag{2.60}$$

Proof. On any edge $\hat{e}$, if the degrees of freedom of $\hat{\chi}$ are $(\hat{\chi}_{\hat{e},11},\, \hat{\chi}_{\hat{e},12})^T$ and $(\hat{\chi}_{\hat{e},21}, \hat{\chi}_{\hat{e},22})^T$, then (2.42) and an application of trapezoid quadrature rule imply that

$$\hat{\Pi}_0\hat{\chi}\big|_E = \begin{pmatrix} \frac{1}{2}(\hat{\chi}_{\hat{e},11} + \hat{\chi}_{\hat{e},21}) \\ \frac{1}{2}(\hat{\chi}_{\hat{e},12} + \hat{\chi}_{\hat{e},22}) \end{pmatrix}.$$

Using (2.51) the simple calculation shows that the statement holds for the case of $\hat{\chi} \in \hat{\mathbb{X}}_h(\hat{E})$. Similar reasoning applied to the degrees of freedom of $\hat{r}$ shows that the statement is also valid for $\hat{r} \in \hat{Z}_h(\hat{E})$. □

2.6 The coupled multipoint stress multipoint flux mixed finite element method.

We first introduce an L^2-orthogonal projection operator acting onto the space of piecewise constant scalar or vector valued function on the trace of $\mathcal{T}_h$ on $\partial\Omega$:

$$\mathcal{P}_0 : L^2(\partial\Omega, \mathbb{R}^d) \to \mathbb{X}_h^0\, n, \quad \text{such that } \forall \phi \in L^2(\Omega, \mathbb{R}^d), \qquad \langle \phi - \mathcal{P}_0\phi,\, \tau\, n\rangle_{\partial\Omega} = 0, \quad \forall \tau \in \mathbb{X}_h^0, \tag{2.61}$$

$$\mathcal{P}_0 : L^2(\partial\Omega, \mathbb{R}) \to Z_h^0 \cdot n, \quad \text{such that } \forall \psi \in L^2(\Omega), \qquad \langle \psi - \mathcal{P}_0\psi,\, q \cdot n\rangle_{\partial\Omega} = 0, \quad \forall q \in Z_h^0. \tag{2.62}$$

In the method proposed below, the Dirichlet boundary data for displacement and pressure variables is incorporated into the system via the projection operator defined above.

Our method is defined as follows. We seek $(\sigma_h, u_h, \gamma_h, z_h, p_h) \in \mathbb{X}_h \times V_h \times \mathbb{W}_h \times Z_h \times W_h$ such that:

$$(A\sigma_h,\, \tau)_Q + (A\alpha p_h I,\, \tau)_Q + (u_h,\, \operatorname{div}\tau) + (\gamma_h,\, \tau)_Q = \langle \mathcal{P}_0 g_u,\, \tau\, n\rangle_{\Gamma_D^{displ}}, \qquad \forall \tau \in \mathbb{X}_h, \tag{2.63}$$

$$(\operatorname{div}\sigma_h,\, v) = -\,(f,\, v)\,, \qquad \forall v \in V_h, \tag{2.64}$$

$$(\sigma_h,\, \xi)_Q = 0, \qquad \forall \xi \in \mathbb{W}_h, \tag{2.65}$$

$$\left(K^{-1} z_h,\, q\right)_Q - (p_h,\, \operatorname{div} q) = -\langle \mathcal{P}_0 g_p,\, v\cdot n\rangle_{\Gamma_D^{pres}}, \qquad \forall q \in Z_h, \tag{2.66}$$

$$c_0\left(\frac{\partial p_h}{\partial t},\, w\right) + \alpha\left(\frac{\partial}{\partial t} A\sigma_h,\, wI\right)_Q + \alpha\left(\frac{\partial}{\partial t}\operatorname{tr}(A\alpha p_h I),\, w\right) + (\operatorname{div} z_h,\, w) = (g,\, w)\,, \quad \forall w \in W_h. \tag{2.67}$$

Before we prove well-posedness and stability of the method (2.63)-(2.67), we show several important results involving the quadrature rule (2.51).

Lemma 2.3. *If $E \in \mathcal{T}_h$ and $\phi \in L^2(E, \mathbb{M})$, $\phi \in L^2(E, \mathbb{R}^d)$ is a function mapped using Piola transformation, then*

$$\|\phi\|_E \sim h^{\frac{2-d}{d}}\|\phi\|_{\hat{E}}. \tag{2.68}$$

Proof. The statement follows from the bounds given in (2.21) and the following relations

$$\int_E \phi\cdot\phi\, dx = \int_{\hat{E}} \frac{1}{J} DF\hat{\phi} \cdot \frac{1}{J} DF\hat{\phi}\, d\hat{x},$$

$$\int_{\hat{E}} \hat{\phi}\cdot\hat{\phi}\, d\hat{x} = \int_E \frac{1}{J_{F^{-1}}} DF^{-1}\phi \cdot \frac{1}{J_{F^{-1}}} DF^{-1}\phi\, dx,$$

where $\cdot$ stands for the inner product when applied to tensor valued functions. □

Lemma 2.4. *There exists a positive constant C independent of h, such that for any linear uniformly bounded and positive-definite operator L*

$$(L\phi,\, \phi)_Q \geq C\|\phi\|^2, \quad \forall \phi \in \mathbb{X}_h \text{ or } \forall \phi \in Z_h. \tag{2.69}$$

Proof. Let $\phi = \sum_{i=1}^{s}\sum_{j=1}^{d} \phi_{ij}\psi_{ij}$ on an element E where ψ_{ij} is a basis function. Using the definitions of the quadrature rule as in Remark 2.1 we obtain

$$(L\phi,\, \psi)_{Q,E} = \frac{|E|}{s}\sum_{i=1}^{s} L\phi(\mathbf{r}_i)\cdot\phi(\mathbf{r}_i) \geq C(l_0)\frac{|E|}{s}\sum_{i=1}^{s}\phi(\mathbf{r}_i)\cdot\phi(\mathbf{r}_i) \geq C(l_0)\frac{|E|}{s}\sum_{i=1}^{s}\sum_{j=1}^{d}\phi_{ij}^2,$$

where $C(l_0)$ involves the constant from the lower bound of the operator L. On the other hand

$$\|\phi\|_E^2 = \left(\sum_{i=1}^{s}\sum_{j=1}^{d}\phi_{ij}\psi_{ij},\, \sum_{k=1}^{s}\sum_{l=1}^{d}\phi_{kl}\psi_{kl}\right) \leq C|E|\sum_{i=1}^{s}\sum_{j=1}^{d}\phi_{ij}^2.$$

And the assertion of the lemma follows from the combination of the above two estimates. □

The following corollary is a result of the above lemma. We present it without a proof, for details see [42, 3, 4].

Corollary 2.1. *The bilinear form $(L\phi,\, \psi)_Q$ is an inner product on $\mathbb{X}_h$ and Z_h, $(L\phi,\, \psi)_Q^{1/2}$ is also a norm in $\mathbb{X}_h$ and Z_h equivalent to $\|\cdot\|_{\mathbb{X}}$ and $\|\cdot\|_{Z_h}$, respectively.*

3 Stability analysis in semidiscrete case

In this section we show that the coupled multipoint stress multipoint flux system for the Biot model (2.63)-(2.67) is well-posed. Throughout this section we assume for simplicity that $\Gamma_D^{displ} = \Gamma_D^{pres} = \partial\Omega$.

Step 1: L^2 in space estimates:

We differentiate (2.63) and choose $(\tau, v, \xi, q, w) = (\sigma_h, \partial_t u_h, \partial_t\gamma_h, z_h, p_h)$ in equations (2.63)-(2.67) to obtain the following system:

$$(A\partial_t\sigma_h,\, \sigma_h)_Q + (A\alpha\partial_t pI,\, \sigma_h)_Q + (\partial_t u_h,\, \operatorname{div}\sigma_h) + (\partial_t\gamma_h,\, \operatorname{as}\sigma_h)_Q = \langle\partial_t\mathcal{P}_0 g_u,\, \sigma_h\, n\rangle, \tag{3.1}$$

$$(\operatorname{div}\sigma_h,\, \partial_t u_h) = -\left(f,\, \partial_t u_h\right), \tag{3.2}$$

$$(\operatorname{as}\sigma_h,\, \partial_t\gamma_h)_Q = 0, \tag{3.3}$$

$$\left(K^{-1}z_h,\, z_h\right)_Q - (p_h,\, \operatorname{div} z_h) = \langle\mathcal{P}_0 g_p,\, z_h\cdot n\rangle, \tag{3.4}$$

$$c_0\left(\partial_t p_h,\, p_h\right) + \alpha\left(\partial_t \operatorname{tr}(A\sigma_h),\, p_h\right)_Q + \alpha\left(\partial_t \operatorname{tr}(A\alpha p_h I),\, p_h\right)_Q + (\operatorname{div} z_h,\, p_h) = (g,\, p_h)\,. \tag{3.5}$$

Combining (3.1)-(3.5), we get

$$\begin{aligned}(A\partial_t\sigma_h,\, \sigma_h)_Q &+ (A\alpha\partial_t pI,\, \sigma_h)_Q + \left(K^{-1}z_h,\, z_h\right)_Q + c_0\left(\partial_t p_h,\, p_h\right) + \alpha\left(\partial_t\operatorname{tr}(A\sigma_h),\, p_h\right)_Q \\ &+ \alpha\left(\partial_t\operatorname{tr}(A\alpha p_h I),\, p_h\right)_Q = \langle\partial_t\mathcal{P}_0 g_u,\, \sigma_h\, n\rangle + (f,\, \partial_t u_h) + \langle\mathcal{P}_0 g_p,\, z_h\cdot n\rangle + (g,\, p_h)\,. \end{aligned}\tag{3.6}$$

Using the definition of the quadrature rule (2.51) and the product rule, we can write the first term on the left hand side of (3.6) as follows

$$\begin{aligned}(A\partial_t\sigma_h,\, \sigma_h)_Q &= \sum_{E\in\mathcal{T}_h}(A\partial_t\sigma_h,\, \sigma_h)_{E,Q} = \sum_{E\in\mathcal{T}_h}(\mathcal{A}\partial_t\hat{\sigma}_h,\, \hat{\sigma}_h)_{\hat{E},Q} = \sum_{E\in\mathcal{T}_h}\frac{|\hat{E}|}{s}\sum_{i=1}^{s}\mathcal{A}\partial_t\hat{\sigma}_h(\hat{\mathbf{r}}_i) : \hat{\sigma}_h(\hat{\mathbf{r}}_i) \\ &= \sum_{E\in\mathcal{T}_h}\frac{|\hat{E}|}{s}\sum_{i=1}^{s}\partial_t\mathcal{A}^{1/2}\hat{\sigma}_h(\hat{\mathbf{r}}_i) : \mathcal{A}^{1/2}\hat{\sigma}_h(\hat{\mathbf{r}}_i) = \frac{1}{2}\sum_{E\in\mathcal{T}_h}\frac{|\hat{E}|}{s}\partial_t\sum_{i=1}^{s}\mathcal{A}^{1/2}\hat{\sigma}_h(\hat{\mathbf{r}}_i) : \mathcal{A}^{1/2}\hat{\sigma}_h(\hat{\mathbf{r}}_i) \\ &= \sum_{E\in\mathcal{T}_h}\frac{1}{2}\partial_t\left(A^{1/2}\sigma_h,\, A^{1/2}\sigma_h\right)_{E,Q} = \frac{1}{2}\partial_t\left(A^{1/2}\sigma_h,\, A^{1/2}\sigma_h\right)_Q\end{aligned}$$

and (3.6) becomes:

$$\begin{aligned}\frac{1}{2}\partial_t\left(A^{1/2}\sigma_h,\,A^{1/2}\sigma_h\right)_Q &+ (A\alpha\partial_t pI,\,\sigma_h)_Q + \alpha\,(\partial_t\,\mathrm{tr}\,(A\sigma_h),\,p_h)_Q \\ &+ \alpha\,(\partial_t\,\mathrm{tr}\,(A\alpha p_h I),\,p_h)_Q + \|K^{-1/2}z_h\|_Q^2 + \frac{c_0}{2}\partial_t\|p_h\|^2 \\ &= \langle\partial_t\mathcal{P}_0 g_u,\,\sigma_h\,n\rangle + (f,\,\partial_t u_h) + \langle\mathcal{P}_0 g_p,\,z_h\cdot n\rangle + (g,\,p_h)\,. \end{aligned} \tag{3.7}$$

Using the identity

$$\mathrm{tr}\,(\tau)w = t:(wI),\quad \forall\tau\in\mathbb{M},\,w\in\mathbb{R},$$

we combine the first four terms on the left-hand side of (3.7):

$$\begin{aligned}&\frac{1}{2}\partial_t\left(A^{1/2}\sigma_h,\,A^{1/2}\sigma_h\right)_Q + (A\alpha\partial_t pI,\,\sigma_h)_Q + \alpha\,(\partial_t\,\mathrm{tr}\,(A\sigma_h),\,p_h)_Q + \alpha\,(\partial_t\,\mathrm{tr}\,(A\alpha p_h I),\,p_h)_Q \\ &= \frac{1}{2}\partial_t\left(A^{1/2}\sigma_h,\,A^{1/2}\sigma_h\right)_Q + \alpha\left(A^{1/2}\partial_t p_h I,\,A^{1/2}\sigma_h\right)_Q \\ &\quad + \alpha\left(\partial_t A^{1/2}\sigma_h,\,A^{1/2}p_h I\right)_Q + \frac{\alpha^2}{2}\left(\partial_t A^{1/2}p_h I,\,\partial_t A^{1/2}p_h I\right)_Q \\ &= \frac{1}{2}\partial_t\left(A^{1/2}(\sigma_h+\alpha p_h I),\,A^{1/2}(\sigma_h+\alpha p_h I)\right)_Q = \frac{1}{2}\partial_t\|A^{1/2}(\sigma_h+\alpha p_h I)\|_Q^2. \end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8) and using the product rule, one gets

$$\begin{aligned}\frac{1}{2}\partial_t&\left[\|A^{1/2}(\sigma_h+\alpha p_h I)\|_Q^2 + c_0\|p_h\|^2\right] + \|K^{-1/2}z_h\|_Q^2 \\ &= \langle\partial_t\mathcal{P}_0 g_u,\,\sigma_h\,n\rangle + (f,\,\partial_t u_h) + \langle\partial_t\mathcal{P}_0 g_p,\,z_h\cdot n\rangle + (g,\,p_h) \\ &= \langle\partial_t\mathcal{P}_0 g_u,\,\sigma_h\,n\rangle + \partial_t\,(f,\,u_h) - (\partial_t f,\,u_h) + \langle\mathcal{P}_0 g_p,\,z_h\cdot n\rangle + (g,\,p_h)\,. \end{aligned} \tag{3.9}$$

Next, integrating (3.9) in time from 0 to an arbitrary $t\in(0,T]$:

$$\begin{aligned}&\frac{1}{2}\left[\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|_Q^2 + c_0\|p_h(t)\|^2\right] + \int_0^t \|K^{-1/2}z_h(s)\|_Q^2\,ds \\ &= \int_0^t ((g(s),\,p_h(s)) - (\partial_t f(s),\,u_h(s)))\;ds + \int_0^t (\langle\partial_t\mathcal{P}_0 g_u(s),\,\sigma_h(s)\,n\rangle + \langle\mathcal{P}_0 g_p(s),\,z_h(s)\cdot n\rangle)\,ds \\ &\quad + \frac{1}{2}\left[\|A^{1/2}(\sigma_h(0)+\alpha p_h I(0))\|_Q^2 + c_0\|p_h(0)\|^2\right] + (f(t),\,u_h(t)) + (f(0),\,u_h(0))\end{aligned}$$

and applying Cauchy-Schwartz and Young inequalities one gets:

$$\begin{aligned}&\frac{1}{2}\left[\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|_Q^2 + c_0\|p_h(t)\|^2\right] + \int_0^t \|K^{-1/2}z_h(s)\|_Q^2\,ds \\ &\le \epsilon\left(\|u_h(t)\|^2 + \int_0^t(\|p_h(s)\|^2 + \|u_h(s)\|^2)\,ds\right) + \tilde{\epsilon}\int_0^t(\|\sigma_h(s)\,n\|_{-1/2}^2 + \|z_h\cdot n\|_{-1/2}^2)\,ds \\ &\quad + \frac{C}{\epsilon}\left(\|f(t)\|^2 + \int_0^t(\|g(s)\|^2 + \|\partial_t f(s)\|^2)\,ds\right) + \frac{C}{\tilde{\epsilon}}\int_0^t(\|\partial_t\mathcal{P}_0 g_u(s)\|_{1/2}^2 + \|\mathcal{P}_0 g_p(s)\|_{1/2}^2)\,ds \\ &\quad + \frac{1}{2}\left[\|A^{1/2}(\sigma_h(0)+\alpha p_h I(0))\|_Q^2 + c_0\|p_h(0)\|^2 + \|u_h(0)\|^2 + \|f(0)\|^2\right]. \end{aligned} \tag{3.10}$$

Using the inf-sup condition as in [3, 4] and (2.63), we obtain

$$\begin{aligned}\|u_h\| + \|\gamma_h\| &\le C\sup_{0\neq\tau\in\mathbb{X}_h}\frac{(u_h,\,\mathrm{div}\,\tau) + (\gamma_h,\,\mathrm{as}\;\tau)_Q}{\|\tau\|_{\mathrm{div}}} \\ &= C\sup_{0\neq\tau\in\mathbb{X}_h}\frac{-\left(A^{1/2}(\sigma_h+\alpha p_h I),\,A^{1/2}\tau\right)_Q + \langle\mathcal{P}_0 g_u,\,\tau\,n\rangle}{\|\tau\|_{\mathrm{div}}} \\ &\le C\|A^{1/2}(\sigma_h+\alpha p_h I)\| + \|\mathcal{P}_0 g_u\|_{\frac{1}{2}}, \end{aligned} \tag{3.11}$$

where in the last step we used equivalence of norms as stated in Corollary 2.1.
Similarly, using the inf-sup condition [11] and (2.66), we have

$$\begin{aligned}\|p_h\| &\le C \sup_{0\neq q\in Z_h} \frac{(p_h,\, \operatorname{div} q)}{\|q\|_{\mathrm{div}}} = C \sup_{0\neq q\in Z_h} \frac{\left(K^{-1}z_h,\, q\right)_Q + \langle \mathcal{P}_0 g_p,\, q\cdot n\rangle}{\|q\|_{\mathrm{div}}} \\ &\le C\|K^{-1/2}z_h\| + \|\mathcal{P}_0 g_p\|_{\frac12}. \end{aligned} \tag{3.12}$$

Combining (3.10)-(3.12), from equivalence of norms we have

$$\begin{aligned}&\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|^2 + \|u_h(t)\|^2 + \|\gamma_h(t)\|^2 + c_0\|p_h(t)\|^2 + \int_0^t (\|K^{-1/2}z_h(s)\|^2 + \|p_h(s)\|^2)\,ds \\ &\quad\le C\Big[\epsilon\left(\|u_h(t)\|^2 + \int_0^t (\|p_h(s)\|^2 + \|u_h(s)\|^2)\,ds\right) + \tilde\epsilon\int_0^t (\|\sigma_h(s)\,n\|_{-1/2} + \|z_h(s)\cdot n\|_{-1/2})\,ds \\ &\qquad + \frac{C}{\epsilon}\left(\|f(t)\|^2 + \int_0^t (\|g(s)\|^2 + \|\partial_t f(s)\|^2)\,ds\right) + \frac{C}{\tilde\epsilon}\int_0^t (\|\partial_t \mathcal{P}_0 g_u(s)\|_{1/2}^2 + \|\mathcal{P}_0 g_p(s)\|_{1/2}^2)\,ds \\ &\qquad + C\left[\|A^{1/2}(\sigma_h(0)+\alpha p_h I(0))\|_Q^2 + c_0\|p_h(0)\|^2 + \|u_h(0)\|^2 + \|f(0)\|^2\right] + \|\mathcal{P}_0 g_u(t)\|_{1/2}^2\Big].\end{aligned}$$

Finally, choosing ϵ small enough, we obtain the following inequality

$$\begin{aligned}&\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|^2 + \|u_h(t)\|^2 + \|\gamma_h(t)\|^2 + c_0\|p_h(t)\|^2 + \int_0^t (\|K^{-1/2}z_h(s)\|^2 + \|p_h(s)\|^2)\,ds \\ &\quad\le C\Big[\epsilon\int_0^t \|u_h(s)\|^2\,ds + \tilde\epsilon\int_0^t (\|\sigma_h(s)\,n\|_{-1/2}^2 + \|z_h(s)\cdot n\|_{-1/2}^2)\,ds \\ &\qquad + \frac{C}{\tilde\epsilon}\int_0^t (\|\partial_t \mathcal{P}_0 g_u(s)\|_{1/2}^2 + \|\mathcal{P}_0 g_p(s)\|_{1/2}^2)\,ds + \left(\|f(t)\|^2 + \int_0^t (\|g(s)\|^2 + \|\partial_t f(s)\|^2)\,ds\right) \\ &\qquad + \|\mathcal{P}_0 g_u(t)\|_{1/2}^2 + \|A^{1/2}(\sigma_h(0)+\alpha p_h I(0))\|_Q^2 + c_0\|p_h(0)\|^2 + \|u_h(0)\|^2 + \|f(0)\|^2\Big].\end{aligned} \tag{3.13}$$

Let us denote the right hand side of (3.13) by H_1. We proceed with deriving estimates for $\operatorname{div}\sigma_h$ and $\operatorname{div} z_h$.

Step 2: $H(\mathrm{div})$ in space estimate for the stress:
Testing (2.64) with $v = \operatorname{div}\sigma_h$, we immediately obtain a bound on divergence of stress:

$$\|\operatorname{div}\sigma_h\| \le \|f\|. \tag{3.14}$$

On the other hand setting $\tau = s_h$, $v = u_h$, $\xi = g_h$ in (2.63)-(2.65) and using equivalence of norms, we obtain

$$\|\sigma_h\|^2 \le C(\|p\|^2 + \|\mathcal{P}_0 g_u\|_{1/2}^2 + \|f\|^2) + \epsilon(\|\sigma_h\, n\|_{-1/2}^2 + \|u\|^2) \tag{3.15}$$

We combine (3.14)-(3.15) and integrate in time:

$$\begin{aligned}&\int_0^t (\|\sigma_h(s)\|^2 + \|\operatorname{div}\sigma_h(s)\|^2)\,ds \\ &\qquad\le C\int_0^t \Big((\|p(s)\|^2 + \|\mathcal{P}_0 g_u(s)\|_{1/2}^2 + \|f(s)\|^2) + \epsilon(\|\sigma_h(s)\,n\|_{-1/2}^2 + \|u(s)\|^2)\Big)\,ds\end{aligned}$$

Using (3.11), we obtain

$$\begin{aligned}\int_0^t (\|\sigma_h(s)\|_{\mathrm{div}}^2 + \|u_h(s)\|^2 + \|\gamma_h(s)\|^2)\,ds &\le C\int_0^t (\|p(s)\|^2 + \|\mathcal{P}_0 g_u(s)\|_{1/2}^2 + \|f(s)\|^2)\,ds \\ &\le H_1 + \int_0^t (\|\mathcal{P}_0 g_u(s)\|_{1/2}^2 + \|f(s)\|^2)\,ds \end{aligned} \tag{3.16}$$

Step 3: $H(\mathrm{div})$ **in space estimate for the velocity:**
It follows from equation (2.67) and Corollary 2.1 that

$$\| \operatorname{div} z_h\| \le C \left(c_0 \|\partial_t p_h\| + \|A^{1/2}\partial_t(\sigma_h + \alpha p_h I)\| + \|g\|\right). \tag{3.17}$$

To control the first two terms on the right hand side of (3.17), we differentiate equations (2.63)-(2.66) and combine (2.63)-(2.67) as it was done in (3.1)-(3.10), with the choice $(\tau, v, \xi, q, w) = (\partial_t\sigma_h, \partial_t u_h, \partial_t\gamma_h, z_h, \partial_t p_h)$:

$$\begin{aligned}
&\int_0^t \left(\|A^{1/2}\partial_t(\sigma_h(s) + \alpha p_h I(s))\|_Q^2 + c_0\|\partial_t p_h(s)\|^2 \right) ds + \frac{1}{2}\|K^{-1/2} z_h(t)\|_Q^2 \\
&\le \int_0^t \left(\|p_h(s)\|\|\partial_t g(s)\| + \|\partial_t u_h(s)\|\|\partial_t f(s)\| + \|\sigma_h\, n\|_{-1/2}\|\partial_t \mathcal{P}_0 g_u\|_{1/2} + \|z_h\cdot n\|_{-1/2}\|\partial_t\mathcal{P}_0 g_p\|_{1/2}\right) ds \\
&\quad + \|p_h(t)\|\|g(t)\| + \frac{1}{2}\|K^{-1/2} z_h(0)\|_Q^2 - \|p_h(0)\|\|g(0)\|.
\end{aligned} \tag{3.18}$$

Using the inf-sup condition [3, 4] and (2.63), differentiated in time, we get

$$\|\partial_t u_h\| + \|\partial_t\gamma_h\| \le C\|A^{1/2}\partial_t(\sigma_h + \alpha p_h I)\| + \|\partial_t\mathcal{P}_0 g_u\|_{\frac{1}{2}}. \tag{3.19}$$

Combining (3.12), (3.19) and (3.18), we get:

$$\begin{aligned}
&\int_0^t \left(\|A^{1/2}\partial_t(\sigma_h(s) + \alpha p_h I(s))\|^2 + \|\partial_t u_h(s)\|^2 + \|\partial_t\gamma_h(s)\|^2 + c_0\|\partial_t p_h(s)\|^2 \right) ds \\
&\quad + \|K^{-1/2} z_h(t)\|^2 + \|p_h(t)\|^2 \\
&\le \epsilon \left(\int_0^t (\|p_h(s)\|^2 + \|\partial_t u_h(s)\|^2)\, ds + \|p_h(t)\|^2 \right) + \tilde\epsilon \int_0^t (\|\sigma_h(s)\, n\|_{-1/2}^2 + \|z_h(s)\cdot n\|_{-1/2}^2)\, ds \\
&\quad + \frac{C}{\epsilon}\left(\int_0^t (\|\partial_t g(s)\|^2 + \|\partial_t f(s)\|^2)\, ds + \|g(t)\|^2 \right) + \frac{C}{\tilde\epsilon}\int_0^t (\|\partial_t\mathcal{P}_0 g_u(s)\|_{1/2}^2 + \|\partial_t\mathcal{P}_0 g_p(s)\|_{1/2}^2)\, ds \\
&\quad + C(\|z_h(0)\|^2 + \|p_h(0)\|^2 + \|g(0)\|^2).
\end{aligned}$$

Choosing ϵ small enough, we obtain

$$\begin{aligned}
&\int_0^t \left(\|A^{1/2}\partial_t(\sigma_h(s) + \alpha p_h I(s))\|^2 + \|\partial_t u_h(s)\|^2 + \|\partial_t\gamma_h(s)\|^2 + c_0\|\partial_t p_h(s)\|^2 \right) ds \\
&\quad + \|K^{-1/2} z_h(t)\|^2 + \|p_h(t)\|^2 \\
&\le \tilde\epsilon \int_0^t (\|\sigma_h(s)\, n\|_{-1/2}^2 + \|z_h(s)\cdot n\|_{-1/2}^2)\, ds + \frac{C}{\tilde\epsilon}\int_0^t (\|\partial_t\mathcal{P}_0 g_u(s)\|_{1/2}^2 + \|\partial_t\mathcal{P}_0 g_p(s)\|_{1/2}^2)\, ds \\
&\quad + C\left(\int_0^t (\|\partial_t g(s)\|^2 + \|\partial_t f(s)\|^2)\, ds + \|g(t)\|^2 + \|z_h(0)\|^2 + \|p_h(0)\|^2 + \|g(0)\|^2 + H_1 \right).
\end{aligned} \tag{3.20}$$

Integrating (3.17) in time and using (3.20), results in

$$\begin{aligned}
&\int_0^t \|\operatorname{div} z_h(s)\|^2\, ds + \|K^{-1/2} z_h(t)\|^2 + \|p_h(t)\|^2 \\
&\le \tilde\epsilon \int_0^t (\|\sigma_h(s)\, n\|_{-1/2}^2 + \|z_h(s)\cdot n\|_{-1/2}^2)\, ds + \frac{C}{\tilde\epsilon}\int_0^t (\|\partial_t\mathcal{P}_0 g_u(s)\|_{1/2}^2 + \|\partial_t\mathcal{P}_0 g_p(s)\|_{1/2}^2)\, ds \\
&+ C\left(\int_0^t (\|g(s)\|^2 + \|\partial_t g(s)\|^2 + \|\partial_t f(s)\|^2)\, ds + \|g(t)\|^2 + \|z_h(0)\|^2 + \|p_h(0)\|^2 + \|g(0)\|^2 + H_1 \right).
\end{aligned} \tag{3.21}$$

We note that initial condition for Darcy velocity can be computed as a suitable projection of $-K\nabla p(0)$, provided the initial condition is regular enough.

Step 4: obtaining the final result:
We combine (3.13), (3.16) and (3.21):

$$
\begin{aligned}
&\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|^2+\|u_h(t)\|^2+\|\gamma_h(t)\|^2+\|z_h(t)\|^2+\|p_h(t)\|^2\\
&\quad+\int_0^t (\|\sigma_h(s)\|^2_{\mathrm{div}}+\|u_h(s)\|^2+\|\gamma_h(s)\|^2+\|z_h(s)\|^2_{\mathrm{div}}+\|p_h(s)\|^2)\,ds\\
&\le C\Big[\int_0^t \Big(\|\mathcal{P}_0 g_u(s)\|_{1/2}+\|\partial_t\mathcal{P}_0 g_u(s)\|_{1/2}+\|\mathcal{P}_0 g_p(s)\|_{1/2}+\|\partial_t\mathcal{P}_0 g_p(s)\|_{1/2}+\|g(s)\|^2\\
&\qquad+\|\partial_t g(s)\|^2+\|f(s)\|^2+\|\partial_t f(s)\|^2\Big)\,ds+\epsilon\int_0^t\|u_h(s)\|^2\,ds+\|f(t)\|^2+\|g(t)\|^2+\|\mathcal{P}_0 g_u(t)\|_{1/2}\\
&\quad+\|f(0)\|^2+\|g(0)\|^2+\|A^{1/2}(\sigma_h(0)+\alpha p_h I(0))\|^2_Q+\|p_h(0)\|^2+\|u_h(0)\|^2+\|z_h(0)\|^2\Big]. \qquad (3.22)
\end{aligned}
$$

Note that we can also obtain an estimate on $\|\sigma_h(t)\|$ as follows:

$$
\begin{aligned}
\|\sigma_h(t)\|\le C\|A^{1/2}\sigma_h(t)\| &\le C\left(\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|+\|A^{1/2}\alpha p_h I(t)\|\right)\\
&\le C\left(\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|+\|p_h(t)\|\right) \qquad (3.23)
\end{aligned}
$$

Then, (3.23) together with (3.14) yield

$$
\|\sigma_h(t)\|_{\mathrm{div}}\le C\left(\|A^{1/2}(\sigma_h(t)+\alpha p_h I(t))\|+\|p_h(t)\|+\|f(t)\|\right) \qquad (3.24)
$$

Finally, (3.22)-(3.24) yield the following result.

Theorem 3.1. *Let $(\sigma_h,u_h,\gamma_h,z_h,p_h)\in\mathbb{X}_h\times V_h\times\Theta_h\times Z_h\times W_h$ be the solution of* (2.63)-(2.67). *Then the following stability estimate holds:*

$$
\begin{aligned}
&\|\sigma_h\|_{L^\infty(0,T;H(\mathrm{div},\Omega))}+\|u_h\|_{L^\infty(0,T;L^2(\Omega))}+\|\gamma_h\|_{L^\infty(0,T;L^2(\Omega))}+\|z_h\|_{L^\infty(0,T;L^2(\Omega))}+\|p_h\|_{L^\infty(0,T;L^2(\Omega))}\\
&+\|\sigma_h\|_{L^2(0,T;H(\mathrm{div},\Omega))}+\|u_h\|_{L^2(0,T;L^2(\Omega))}+\|\gamma_h\|_{L^2(0,T;L^2(\Omega))}+\|z_h\|_{L^2(0,T;H(\mathrm{div},\Omega))}+\|p_h\|_{L^2(0,T;L^2(\Omega))}\\
&\le C\Big[\|p_h(0)\|+\|\sigma_h(0)\|+\|u_h(0)\|+\|z_h(0)\|+\|f\|_{L^\infty(0,T;L^2(\Omega))}+\|f\|_{H^1(0,T;L^2(\Omega))}\\
&+\|g_p\|_{H^1(0,T;H^{1/2}(\partial\Omega))}+\|g\|_{L^\infty(0,T;L^2(\Omega))}+\|g\|_{H^1(0,T;L^2(\Omega))}\\
&+\|g_u\|_{L^\infty(0,T;H^{1/2}(\partial\Omega))}+\|g_u\|_{H^1(0,T;H^{1/2}(\partial\Omega))}\Big]. \qquad (3.25)
\end{aligned}
$$

4 Reduction to a cell-centered displacement-pressure system

The choice of trapezoidal quadrature rule implies that on each element, the stress and velocity degrees of freedom associated with a vertex become decoupled from the rest of the degrees of freedom. As a result, the assembled velocity mass matrix in (2.66) has a block-diagonal structure with one block per grid vertex. The dimensions of each velocity block equals the number of velocity DOFs associated with the vertex. For example, this dimension is 4 for logically rectangular quadrilateral grids. Inverting each local block in mass matrix in (2.66) allows for expressing the velocity DOF associated with a vertex in terms of the pressures at the centers of the elements that share the vertex.

Similarly, inverting each local block in mass matrix in (2.63) allows for expressing the stress DOF associated with a vertex in terms of the corresponding displacements, rotations and pressures. By substituting these expressions into equations (2.64)-(2.65) one gets the intermediate step, where the elasticity system was reduced to a cell-centered displacement-rotation system. Due to the choice of the quadrature rule, the rotation basis functions corresponding to each vertex of the grid become decoupled from the rest of the variables other than the stress DOF at this same vertex, leading to matrix $A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T$ being diagonal (see [3, 4]). With this, one obtains the expression for the rotation DOF in terms of the displacements and pressures, which can be further substituted into (2.64) leading to a final displacement-pressure system.

More precisely, in matrix form we have

$$\begin{pmatrix} A_{\sigma\sigma} & A_{\sigma u}^T & A_{\sigma\gamma}^T & 0 & A_{\sigma p}^T \\ -A_{\sigma u} & 0 & 0 & 0 & 0 \\ -A_{\sigma\gamma} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{zz} & A_{zp}^T \\ A_{\sigma p} & 0 & 0 & -A_{zp} & A_{pp} \end{pmatrix} \begin{pmatrix} \sigma \\ u \\ \gamma \\ z \\ p \end{pmatrix}$$

$$\xrightarrow{\sigma=-A_{\sigma\sigma}^{-1}A_{\sigma u}^T u - A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T\gamma - A_{\sigma\sigma}^{-1}A_{\sigma p}^T p} \begin{pmatrix} A_{\sigma u}A_{\sigma\sigma}^{-1}A_{\sigma u}^T & A_{\sigma u}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T & 0 & A_{\sigma u}A_{\sigma\sigma}^{-1}A_{\sigma p}^T \\ A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma u}^T & A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T & 0 & A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma p}^T \\ 0 & 0 & A_{zz} & A_{zp}^T \\ -A_{\sigma p}A_{\sigma\sigma}^{-1}A_{\sigma u}^T & -A_{\sigma p}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T & -A_{zp} & A_{pp} - A_{\sigma p}A_{\sigma\sigma}^{-1}A_{\sigma p}^T \end{pmatrix} \begin{pmatrix} u \\ \gamma \\ z \\ p \end{pmatrix}$$

$$\xrightarrow{z=-A_{zz}^{-1}A_{zp}^T p} \begin{pmatrix} A_{u\sigma u} & A_{u\sigma\gamma} & A_{u\sigma p} \\ A_{u\sigma\gamma}^T & A_{\gamma\sigma\gamma} & A_{\gamma\sigma p} \\ -A_{u\sigma p}^T & -A_{\gamma\sigma p}^T & A_{p\sigma zp} \end{pmatrix} \begin{pmatrix} u \\ \gamma \\ p \end{pmatrix}$$

$$\xrightarrow{\gamma=-A_{\gamma\sigma\gamma}^{-1}A_{\gamma\sigma p}p - A_{\gamma\sigma\gamma}^{-1}A_{u\sigma\gamma}^T u} \begin{pmatrix} A_{u\sigma u} - A_{u\sigma\gamma}A_{\gamma\sigma\gamma}^{-1}A_{u\sigma\gamma}^T & A_{u\sigma p} - A_{u\sigma\gamma}A_{\gamma\sigma\gamma}^{-1}A_{\gamma\sigma p} \\ -A_{u\sigma p}^T + A_{u\sigma p}^T A_{\gamma\sigma\gamma}^{-1}A_{u\sigma\gamma}^T & A_{p\sigma zp} + A_{\gamma\sigma p}^T A_{\gamma\sigma\gamma}^{-1}A_{\gamma\sigma p} \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix}.$$

And finally, the displacement-pressure system for the Biot poroelasticity model reads as follows

$$\begin{pmatrix} A_{u\sigma u} - A_{u\sigma\gamma}A_{\gamma\sigma\gamma}^{-1}A_{u\sigma\gamma}^T & A_{u\sigma p} - A_{u\sigma\gamma}A_{\gamma\sigma\gamma}^{-1}A_{\gamma\sigma p} \\ -A_{u\sigma p}^T + A_{u\sigma p}^T A_{\gamma\sigma\gamma}^{-1}A_{u\sigma\gamma}^T & A_{p\sigma zp} + A_{\gamma\sigma p}^T A_{\gamma\sigma\gamma}^{-1}A_{\gamma\sigma p} \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F_u \\ F_p, \end{pmatrix} \tag{4.1}$$

where

$$\begin{aligned} A_{u\sigma u} &:= A_{\sigma u}A_{\sigma\sigma}^{-1}A_{\sigma u}^T, & A_{u\sigma\gamma} &:= A_{\sigma u}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T, \\ A_{\gamma\sigma\gamma} &:= A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T, & A_{u\sigma p} &:= A_{\sigma u}A_{\sigma\sigma}^{-1}A_{\sigma p}^T, \\ A_{\gamma\sigma p} &:= A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma p}^T, & A_{p\sigma zp} &:= A_{pp} - A_{\sigma p}A_{\sigma\sigma}^{-1}A_{\sigma p}^T + A_{zp}A_{\sigma\sigma}^{-1}A_{zp}^T, \end{aligned}$$

and F_u, F_p are the right-hand side functions transformed accordingly to the procedure above.

Lemma 4.1. *The cell-centered finite difference system for the displacement and pressure obtained from* (2.63)-(2.67) *using the procedure described above is symmetric and positive definite.*

Proof. The proof follows from the inf-sup conditions for the MSMFE and MFMFE methods, Corollary 2.1 and the combined stress-pressure coercivity estimate, see [42, 3, 4] for details. □

5 Error analysis

5.1 Preliminaries

Due to the reduced approximation properties of the MFE spaces on general quadrilaterals [5], we restrict the quadrilateral elements to be $O(h^2)$-perturbations of parallelograms:

$$\|\mathbf{r}_{34} - \mathbf{r}_{21}\| \le Ch^2.$$

In this case it is easy to verify (see [42] for details) that

$$|DF_E|_{1,\infty,\hat{E}} \le Ch^2 \quad \text{and} \quad \left|\frac{1}{J_E}DF_E\right|_{j,\infty,\hat{E}} \le Ch^{j-1},\ j = 1, 2. \tag{5.1}$$

We introduce the L^2-projection operators $Q^0 : L^2(\Omega) \to W_h$ and $Q^1 : L^2(\Omega) \to \mathbb{W}_h$ satisfying

$$(\phi - Q^0\phi, \psi_h) = 0, \qquad \forall \psi_h \in W_h, \tag{5.2}$$

$$(\phi - Q^1\phi, \psi_h) = 0, \qquad \forall \psi_h \in \mathbb{W}_h. \tag{5.3}$$

We will use projection operator Q^1 for approximation of the rotation variable, and Q^0 operator for approximation of the pressure. Notice also, that the same operator Q^0 applied component-wise can be used for approximation of the displacement variable.

In the error analysis of we will utilize the elliptic projection $\tilde{\Pi} : H^1(\Omega, \mathbb{M}) \to \mathbb{X}_h$ introduced in [8]. Given $\sigma \in \mathbb{X}$ there exists a unique triple $(\sigma_h, u_h, \gamma_h) \in \mathbb{X}_h \times V_h \times \mathbb{W}_h$ such that

$$(\sigma_h,\, \tau)_Q + (u_h,\, \operatorname{div} \tau) + (\gamma_h,\, \tau)_Q = (\sigma,\, \tau) \qquad \forall \tau \in \mathbb{X}_h \tag{5.4}$$

$$(\operatorname{div} \sigma_h,\, v) = (\operatorname{div} \sigma,\, v)\,, \qquad \forall v \in V_h \tag{5.5}$$

$$(\sigma_h,\, \xi)_Q = (\sigma,\, \xi) \qquad \forall \xi \in \mathbb{W}_h. \tag{5.6}$$

Namely, $(\sigma_h, u_h, \gamma_h)$ is a multipoint stress mixed finite element (see MSMFE-1 in [3, 4]) method approximation of $(\sigma, 0, 0)$. We then define $\tilde{\Pi}\sigma = \sigma_h$. If $\sigma \in \mathbb{X}_h$ we have $\sigma_h = \sigma$, $u_h = 0$ and $\gamma_h = 0$ so $\tilde{\Pi}$ is indeed a projection. It follows from (5.5)-(5.6) and the inf-sup condition of the MSMFE-1 method that

$$\left(\operatorname{div} \tilde{\Pi}\sigma,\, v\right) = (\operatorname{div} \sigma,\, v)\,, \qquad v \in V_h, \tag{5.7}$$

$$\left(\tilde{\Pi}\sigma,\, \xi\right) = (\sigma,\, \xi)\,, \qquad \xi \in \mathbb{W}_h. \tag{5.8}$$

Moreover, the error estimate for the MSMFE method, allows us to show that there exists a positive constant C such that

$$\|\tilde{\Pi}\sigma\|_{\operatorname{div}} \le \|\sigma\|_{\operatorname{div}}, \quad \|\sigma - \tilde{\Pi}\sigma\| \le C\|\sigma - \Pi\sigma\|, \quad \sigma \in H^1(\Omega, \mathbb{R}^{d\times d}). \tag{5.9}$$

The following lemma summarizes well-known continuity and approximation properties of the projection operators.

Lemma 5.1. *There exists a constant $C > 0$ such that on simplices and h^2-parallelograms*

$$\|\phi - Q^0\phi\| \le C\|\phi\|_r h^r, \qquad \forall \phi \in H^r(\Omega), \qquad 0 \le r \le 1, \tag{5.10}$$

$$\|\phi - Q^1\phi\| \le C\|\phi\|_r h^r, \qquad \forall \phi \in H^r(\Omega), \qquad 0 \le r \le 1, \tag{5.11}$$

$$\|\psi - \Pi\psi\| \le C\|\psi\|_r h^r, \qquad \forall \psi \in H^r(\Omega), \qquad 1 \le r \le 2, \tag{5.12}$$

$$\|\psi - \Pi^0\psi\| \le C\|\psi\|_1 h, \qquad \forall \psi \in H^1(\Omega), \tag{5.13}$$

$$\|\operatorname{div}(\psi - \Pi\psi)\| + \|\operatorname{div}(\psi - \Pi^0\psi)\| \le C\|\operatorname{div}\psi\|_r h^r, \qquad \forall \psi \in H^{r+1}(\Omega), \qquad 0 \le r \le 1. \tag{5.14}$$

Moreover, for all elements $E \in \mathcal{T}_h$, there exists a constant $c > 0$, such that

$$\|\Pi\psi\|_{j,E} \le C\|\psi\|_j, \qquad \forall \psi \in H^j(\Omega), \quad j = 1, 2, \tag{5.15}$$

$$\|\Pi^0\psi\|_{1,E} \le C\|\psi\|_1, \qquad \forall \psi \in H^1(\Omega). \tag{5.16}$$

Proof. Proof of bounds for the L^2-projections (5.10)-(5.11) can be found in [13]; and bounds (5.12)-(5.14) can be found in [11, 35] for affine elements and [38, 5] for h^2-parallelograms. Finally, the proof of (5.15)-(5.16) was presented in [42]. □

The next result summarizes the error bounds for the terms arising from the use of quadrature rule.

Lemma 5.2. *If $K^{-1} \in W^{1,\infty}_{\mathcal{T}_h}$ and $A \in W^{1,\infty}_{\mathcal{T}_h}$, then there is a constant $C > 0$ such that*

$$\left|\theta\left(K^{-1}q,\, v\right)\right| \le C \sum_{E \in \mathcal{T}_h} h\|K^{-1}\|_{1,\infty,E}\|q\|_{1,E}\|v\|_E, \qquad \forall q \in V_h,\, v \in V_h^0, \tag{5.17}$$

$$|\theta\,(A\tau,\, \chi + wI)| \le C \sum_{E \in \mathcal{T}_h} h\|A\|_{1,\infty,E}\|\tau\|_{1,E}\|\chi + wI\|_E, \qquad \forall \tau \in \mathbb{X}_h,\, \chi \in \mathbb{X}_h^0, w \in W_h, \tag{5.18}$$

$$|\theta\,(AwI,\, r)| \le C \sum_{E \in \mathcal{T}_h} h\|A\|_{1,\infty,E}\|w\|_E\|r\|_E, \qquad \forall w, r \in W_h, \tag{5.19}$$

$$|\theta\,(\operatorname{as}\chi,\, \xi)| \le C \sum_{E \in \mathcal{T}_h} h\|\chi\|_{1,E}\|\xi\|_E, \qquad \forall \chi \in \mathbb{X}_h^0, \xi \in \Theta_h. \tag{5.20}$$

Moreover, on h^2-parallelograms, if $K^{-1} \in W^{1,\infty}_{\mathcal{T}_h}$ and $A \in W^{1,\infty}_{\mathcal{T}_h}$, there is a constant $c > 0$ such that

$$\left|\left(K^{-1}\Pi u,\, v - \Pi^0 v\right)_Q\right| \le ch\|q\|_1\|v\|, \qquad v \in V_h, \tag{5.21}$$

$$\left|\left(A(\tilde{\Pi}\sigma + Q^0 p),\, \chi - \Pi^0\chi\right)_Q\right| \le ch(\|\sigma\|_1 + \|p\|)\|\chi\|, \qquad \forall \chi \in \mathbb{X}_h, \tag{5.22}$$

$$\left|\left(\mathrm{as}\,(\chi - \Pi^0\chi),\, Q^1\gamma\right)_Q\right| \le ch\|\gamma\|_1\|\chi\|, \qquad \forall \chi \in \mathbb{X}_h. \tag{5.23}$$

Proof. The estimates (5.17) and (5.21) can be found in [42], while (5.18), (5.20), (5.22) and (5.23) were proven in [3, 4] for $p = w = 0$.

Next we prove (5.18) for the case $w \neq 0$. We note that (5.19) can be obtained in the say way. We compute for any $E \in \mathcal{T}_h$

$$|\theta\,(A\tau,\, wI)_E| = \left|\theta\left(\hat{A}DF_E\hat{\tau},\, \hat{w}I\right)_{\hat{E}}\right| \le \left|\theta\left((\hat{A}DF_E - \overline{\hat{A}DF_E})\hat{\tau},\, \hat{w}I\right)_{\hat{E}}\right| + \left|\theta\left(\overline{\hat{A}DF_E}\hat{\tau},\, \hat{w}I\right)_{\hat{E}}\right|,$$

where the overline notation stands for the mean value. For the first term on the right hand side, we use Taylor expansion, (2.21) and (5.1):

$$\begin{aligned}
\left|\theta\left((\hat{A}\,DF_E - \overline{\hat{A}\,DF_E})\hat{\tau},\, \hat{w}I\right)_{\hat{E}}\right| &\le C|\hat{A}\,DF_E|_{1,\infty,\hat{E}}\|\hat{\tau}\|_{\hat{E}}\|\hat{w}\|_{\hat{E}} \\
&\le C(|\hat{A}\,|_{1,\infty,\hat{E}}\|DF_E\|_{0,\infty,\hat{E}} + |DF_E|_{1,\infty,\hat{E}}\|\hat{A}\|_{0,\infty,\hat{E}})\|\hat{\tau}\|_{\hat{E}}\|\hat{w}\|_{\hat{E}} \\
&\le Ch\|A\|_{1,\infty,E}\|\tau\|_E\|w\|_E.
\end{aligned} \tag{5.24}$$

For the second term we note that since the quadrature rule is exact for linears, $\theta\left(\overline{\hat{A}DF_E}\hat{\Pi}^0\hat{\tau},\, \hat{w}I\right)_{\hat{E}} = 0$. Therefore, using (2.21) and (5.13) we obtain

$$\begin{aligned}
\left|\theta\left(\overline{\hat{A}DF_E}\hat{\tau},\, \hat{w}I\right)_{\hat{E}}\right| = \left|\theta\left(\overline{\hat{A}DF_E}(\hat{\tau} - \hat{\Pi}^0\hat{\tau}),\, \hat{w}I\right)_{\hat{E}}\right| &\le C\|\hat{A}\,DF_E\|_{0,\infty,\hat{E}}\|\hat{\tau} - \hat{\Pi}^0\hat{\tau}\|_{\hat{E}}\|\hat{w}\|_{\hat{E}} \\
&\le Ch\|A\|_{0,\infty,E}\|\tau\|_{1,E}\|w\|_E.
\end{aligned} \tag{5.25}$$

Combining (5.24)-(5.25) and summing over all $E \in \mathcal{T}_h$, we get

$$|\theta\,(A\tau,\, wI)| \le C\sum_{E\in\mathcal{T}_h} h\|A\|_{1,\infty,E}\|\tau\|_{1,E}\|w\|_E,$$

as desired. We use similar arguments to prove (5.22) with nonzero p. First, we write:

$$\begin{aligned}
\left|(A\,Q^0p,\, \chi - \Pi^0\chi)_{Q,E}\right| &= \left|\left(DF_E^T\,\hat{A}\,\widehat{Q^0p},\, \hat{\chi} - \hat{\Pi}^0\hat{\chi}\right)_{\hat{\mathcal{Q}},\hat{E}}\right| \\
&\le \left|\left(\overline{DF_E^T\,\hat{A}}\,\widehat{Q^0p},\, \hat{\chi} - \hat{\Pi}^0\hat{\chi}\right)_{\hat{\mathcal{Q}},\hat{E}}\right| + \left|\left((DF_E^T\,\hat{A} - \overline{DF_E^T\,\hat{A}})\,\widehat{Q^0p},\, \hat{\chi} - \hat{\Pi}^0\hat{\chi}\right)_{\hat{\mathcal{Q}},\hat{E}}\right|.
\end{aligned}$$

The first term on the right is equal to zero due to Lemma 2.2. For the second term we use Taylor expansion, equivalence of norms, (2.21) and (2.43):

$$\left|\left((DF_E^T\,\hat{A} - \overline{DF_E^T\,\hat{A}})\,\widehat{Q^0p},\, \hat{\chi} - \hat{\Pi}^0\hat{\chi}\right)_{\hat{\mathcal{Q}},\hat{E}}\right| \le C|DF_E^T\,\hat{A}|_{1,\infty,\hat{E}}\|\widehat{Q^0p}\|_{\hat{E}}\|\hat{\chi} - \hat{\Pi}^0\hat{\chi}\|_{\hat{E}} \le Ch\|p\|_E\|\chi\|_E.$$

□

5.2 Optimal convergence

We form the error system by subtracting the discrete problem (2.63)-(2.67) from the continuous one (2.44)-(2.48)

$$
\begin{aligned}
&(A\sigma,\,\tau)-(A\sigma_h,\,\tau)_Q+(A\alpha pI,\,\tau)-(A\alpha p_hI,\,\tau)_Q+(u-u_h,\,\operatorname{div}\tau)\\
&\qquad+(\gamma,\,\text{as }\tau)-(\gamma_h,\,\text{as }\tau)_Q=\langle g_u-\mathcal{P}_0g_u,\,\tau\,n\rangle, && \forall\tau\in\mathbb{X}_h, && (5.26)\\
&(\operatorname{div}\sigma-\operatorname{div}\sigma_h,\,v)=0, && \forall v\in V_h, && (5.27)\\
&(\text{as }\sigma,\,\xi)-(\text{as }\sigma_h,\,\xi)_Q=0, && \forall\xi\in\Theta_h, && (5.28)\\
&\left(K^{-1}z,\,q\right)-\left(K^{-1}z_h,\,q\right)_Q-(p-p_h,\,\operatorname{div}q)=\langle g_p-\mathcal{P}_0g_p,\,q\cdot n\rangle, && \forall q\in Z_h, && (5.29)\\
&c_0\,(\partial_tp-\partial_tp_h,\,w)+\alpha\,(\partial_t\operatorname{tr}(A\sigma),\,w)-\alpha\,(\partial_t\operatorname{tr}(A\sigma_h),\,w)_Q+\alpha\,(\partial_t\operatorname{tr}(A\alpha pI),\,w)\\
&\qquad-\alpha\,(\partial_t\operatorname{tr}(A\alpha p_hI),\,w)_Q+(\operatorname{div}z-\operatorname{div}z_h,\,w)=0, && \forall w\in W_h. && (5.30)
\end{aligned}
$$

We split the errors into approximation and truncation errors as follows:

$$
\begin{aligned}
e_s&=\sigma-\sigma_h=(\sigma-\tilde{\Pi}\sigma)+(\tilde{\Pi}\sigma-\sigma_h):=\psi_s+\phi_s,\\
e_u&=u-u_h=(u-Q^0u)+(Q^0u-u_h):=\psi_u+\phi_u,\\
e_\gamma&=\gamma-\gamma_h=(\gamma-Q^1\gamma)+(Q^1\gamma-\gamma_h):=\psi_\gamma+\phi_\gamma,\\
e_z&=z-z_h=(z-\Pi z)+(\Pi z-z_h):=\psi_z+\phi_z,\\
e_p&=p-p_h=(p-Q^0p)+(Q^0p-p_h):=\psi_p+\phi_p.
\end{aligned}
$$

Step 1: L^2 in space estimates:
With these notations we can rewrite the first equation (5.26) in the error system in the following way:

$$
\begin{aligned}
&(A\phi_s,\,\tau)_Q+\alpha\,(A\phi_pI,\,\tau)_Q+(\phi_u,\,\operatorname{div}\tau)+(\phi_\gamma,\,\text{as }\tau)_Q\\
&\quad=\left(A\tilde{\Pi}\sigma,\,\tau\right)_Q-(A\sigma,\,\tau)+\alpha\left(AQ^0pI,\,\tau\right)_Q-\alpha\,(ApI,\,\tau)+(\psi_u,\,\operatorname{div}\tau)\\
&\qquad+\left(Q^1\gamma,\,\text{as }\tau\right)_Q-(\gamma,\,\text{as }\tau)+\langle g_u-\mathcal{P}_0g_u,\,\tau\,n\rangle.
\end{aligned}
$$

It follows from the definition of Q^0 operator (5.2) that $(\psi_u,\,\operatorname{div}\tau)=0$. Combining the rest of the terms, we write

$$
\begin{aligned}
&(A\phi_s,\,\tau)_Q+\alpha\,(A\phi_pI,\,\tau)_Q+(\phi_u,\,\operatorname{div}\tau)+(\phi_\gamma,\,\text{as }\tau)_Q\\
&\quad=-\left(A(\sigma+\alpha pI),\,\tau-\Pi^0\tau\right)-\left(A(\psi_s+\alpha\psi_pI),\,\Pi^0\tau\right)-\left(A(\tilde{\Pi}\sigma+\alpha Q^0pI),\,\Pi^0\tau\right)\\
&\qquad+\left(A(\tilde{\Pi}\sigma+\alpha Q^0pI),\,\Pi^0\tau\right)_Q+\left(A(\tilde{\Pi}\sigma+\alpha Q^0pI),\,\tau-\Pi^0\tau\right)_Q-\left(\gamma,\,\text{as}\,(\tau-\Pi^0\tau)\right)-\left(\psi_\gamma,\,\text{as }\Pi^0\tau\right)\\
&\qquad-\left(Q^1\gamma,\,\text{as }\Pi^0\tau\right)+\left(Q^1\gamma,\,\text{as }\Pi^0\tau\right)_Q+\left(Q^1\gamma,\,\text{as}\,(\tau-\Pi^0\tau)\right)_Q+\langle g_u,\,(\tau-\Pi^0\tau)\,n\rangle, \qquad (5.31)
\end{aligned}
$$

where we also used (2.61). Taking $\tau-\Pi^0\tau$ as a test function in (2.44), we obtain

$$
\left(A(\sigma+\alpha pI),\,\tau-\Pi^0\tau\right)+\left(u,\,\operatorname{div}(\tau-\Pi^0\tau)\right)+\left(\gamma,\,\text{as}\,(\tau-\Pi^0\tau)\right)=\langle g_u,\,(\tau-\Pi^0\tau)\,n\rangle.
$$

Hence, due to (5.7) and (2.43),

$$
-\left(A(\sigma+\alpha pI),\,\tau-\Pi^0\tau\right)-\left(\gamma,\,\text{as}\,(\tau-\Pi^0\tau)\right)+\langle g_u,\,(\tau-\Pi^0\tau)\,n\rangle=0. \qquad (5.32)
$$

Combining (5.31)-(5.32) and rewriting terms, coming from the use of quadrature rule, we get

$$
\begin{aligned}
&(A\phi_s,\,\tau)_Q+\alpha\,(A\phi_pI,\,\tau)_Q+(\phi_u,\,\operatorname{div}\tau)+(\phi_\gamma,\,\text{as }\tau)_Q\\
&\quad=-\left(A(\psi_s+\alpha\psi_pI),\,\Pi^0\tau\right)-\left(\psi_\gamma,\,\text{as }\Pi^0\tau\right)-\theta\left(A\tilde{\Pi}\sigma,\,\Pi^0\tau\right)-\theta\left(A\alpha Q^0pI,\,\Pi^0\tau\right)\\
&\qquad-\theta\left(Q^1\gamma,\,\text{as }\Pi^0\tau\right)+\left(A(\Pi\sigma+\alpha Q^0pI),\,\tau-\Pi^0\tau\right)_Q+\left(Q^1\gamma,\,\text{as}\,(\tau-\Pi^0\tau)\right)_Q. \qquad (5.33)
\end{aligned}
$$

From (2.33) and (5.27) we have

$$\operatorname{div}\phi_s = 0. \tag{5.34}$$

It also follows from (2.46) (2.65) that

$$(\text{as}\ \phi_s,\, \xi)_Q = \left(\text{as}\ \tilde{\Pi}\sigma,\, \xi\right)_Q - (\text{as}\ \sigma_h,\, \xi)_Q = 0, \tag{5.35}$$

where we used the property (5.8). We rewrite (5.29) similarly to how it was done in (5.31)-(5.33):

$$\begin{aligned}\left(K^{-1}\phi_z,\, q\right)_q - (\phi_p,\, \operatorname{div} q) &= (\psi_p,\, \operatorname{div} q) - \left(K^{-1}z,\, q - \Pi^0 q\right) - \left(K^{-1}(z - \Pi z),\, \Pi^0 q\right) - \left(K^{-1}\Pi z,\, \Pi^0 q\right)\\ &\quad + \left(K^{-1}\Pi z,\, \Pi_0 q\right)_Q + \left(K^{-1}\Pi z,\, q - \Pi^0 q\right)_Q - \langle g_p,\, (q - \Pi^0 q)\cdot n\rangle.\end{aligned}$$

Using (5.2), we conclude that $(\psi_p,\, \operatorname{div} q) = 0$. Moreover, testing (2.44) with $q - \Pi^0 q$, we also obtain

$$-\left(K^{-1}z,\, q - \Pi^0 q\right) - \langle g_p,\, (q - \Pi^0 q)\cdot n\rangle = 0.$$

Hence, we have

$$\left(K^{-1}\phi_z,\, q\right)_Q - (\phi_p,\, \operatorname{div} q) = -\left(K^{-1}\psi_z,\, \Pi^0 q\right) - \theta\left(K^{-1}\Pi z,\, \Pi^0 q\right) + \left(K^{-1}\Pi z,\, q - \Pi^0 q\right)_Q. \tag{5.36}$$

Finally, using (5.2) and (5.7), we rewrite the last equation, (5.30), in the error system as follows

$$\begin{aligned} c_0\left(\partial_t\phi_p,\, w\right) &+ \alpha\left(\partial_t \operatorname{tr}(A\phi_s),\, w\right)_Q + \alpha^2\left(\partial_t \operatorname{tr}(A\phi_p),\, w\right)_Q + (\operatorname{div}\phi_z,\, w) - \alpha\left(\partial_t \operatorname{tr}(A\psi_s),\, w\right)\\ &= -\alpha\theta\left(\partial_t \operatorname{tr}(A\tilde{\Pi}\sigma),\, w\right) - \alpha^2\left(\partial_t \operatorname{tr}(A\psi_p I),\, w\right) - \alpha^2\theta\left(\partial_t \operatorname{tr}(AQ^0 pI),\, w\right).\end{aligned} \tag{5.37}$$

Next we differentiate (5.33), set $\tau = \phi_s$, $\xi = \partial_t\phi_\gamma$, $q = \phi_z$, $w = \phi_p$ and combine (5.33)-(5.36):

$$\begin{aligned}&\frac{1}{2}\partial_t\left[\|A^{1/2}(\phi_s + \alpha\phi_p I)\|_Q^2 + c_0\|\phi_p\|^2\right] + \left(K^{-1}\phi_z,\, \phi_z\right)_Q\\ &\quad = -\left(A\partial_t(\psi_s + \alpha\psi_p I),\, \Pi^0\phi_s\right) - \left(\partial_t\psi_\gamma,\, \text{as}\ \Pi^0\phi_s\right) - \theta\left(A\partial_t\tilde{\Pi}\sigma,\, \Pi^0\phi_s + \alpha\phi_p I\right)\\ &\qquad - \theta\left(\partial_t Q^1\gamma,\, \text{as}\ \Pi^0\phi_s\right) + \left(A\partial_t(\tilde{\Pi}\sigma + \alpha Q^0 pI),\, \phi_s - \Pi^0\phi_s\right)_Q + \left(\partial_t Q^1\gamma,\, \text{as}\,(\phi_s - \Pi^0\phi_s)\right)_Q\\ &\qquad - \left(K^{-1}\psi_z,\, \Pi^0\phi_z\right) - \theta\left(K^{-1}\Pi z,\, \Pi^0\phi_z\right) + \left(K^{-1}\Pi z,\, \phi_z - \Pi^0\phi_z\right)_Q - \alpha\left(\partial_t \operatorname{tr}(A\psi_s),\, \phi_p\right)\\ &\qquad - \alpha^2\left(\partial_t \operatorname{tr}(A\psi_p I),\, \phi_p\right) - \alpha\theta\left(\partial_t AQ^0 pI,\, \Pi^0\phi_s + \alpha\phi_p\right).\end{aligned} \tag{5.38}$$

Using (5.10)-(5.12) and (5.16), we have

$$\begin{aligned}&\Big|\left(A\partial_t(\psi_s + \alpha\psi_p I),\, \Pi^0\phi_s\right) + \left(\partial_t\psi_\gamma,\, \text{as}\ \Pi^0\phi_s\right) + \left(K^{-1}\psi_z,\, \Pi^0\phi_z\right)\\ &\qquad\qquad + \alpha\left(\partial_t \operatorname{tr}(A\psi_s),\, \phi_p\right) - \alpha^2\left(\partial_t \operatorname{tr}(A\psi_p I),\, \phi_p\right)_Q\Big|\\ &\quad \le Ch^2(\|\partial_t\sigma\|_1^2 + \|\partial_t p\|_1^2 + \|\partial_t\gamma\|_1^2 + \|z\|_1^2) + \epsilon(\|\phi_s\|^2 + \|\phi_p\|^2 + \|\phi_z\|^2).\end{aligned} \tag{5.39}$$

Applying (5.17)-(5.20) and continuity of projection operators

$$\begin{aligned}&\left|\theta\left(A\partial_t\tilde{\Pi}\sigma,\, \Pi^0\phi_s + \alpha\phi_p I\right) + \theta\left(K^{-1}\Pi z,\, \Pi^0\phi_z\right) - \alpha\theta\left(\partial_t AQ^0 pI,\, \Pi^0\phi_s + \alpha\phi_p\right) - \theta\left(\partial_t Q^1\gamma,\, \text{as}\ \Pi^0\phi_s\right)\right|\\ &\quad \le Ch^2(\|\partial_t\sigma\|_1^2 + \|z\|_1^2 + \|\partial_t p\|_0^2 + \|\partial_t\gamma\|_0^2) + \epsilon(\|\phi_s\|^2 + \|\phi_p\|^2 + \|\phi_z\|^2).\end{aligned} \tag{5.40}$$

Due to (5.21) -(5.23) ,

$$\begin{aligned}&\left|\left(A\partial_t(\tilde{\Pi}\sigma + \alpha Q^0 pI),\, \phi_s - \Pi^0\phi_s\right)_Q + \left(\partial_t Q\gamma,\, \text{as}\,(\phi_s - \Pi^0\phi_s)\right)_Q + \left(K^{-1}\Pi z,\, \phi_z - \Pi^0\phi_z\right)_Q\right|\\ &\quad \le Ch^2(\|\partial_t\sigma\|_1^2 + \|\partial_t p\|_1^2 + \|\partial_t\gamma\|_1^2 + \|z\|_1^2) + \epsilon(\|\phi_s\|^2 + \|\phi_z\|^2).\end{aligned} \tag{5.41}$$

Next, we combine (5.38)-(5.41) and integrate the result in time from 0 to arbitrary $t \in (0, T]$:

$$\begin{aligned}
\|A^{1/2}(\phi_s(t)+\alpha\phi_p I(t))\|_Q^2 + c_0\|\phi_p(t)\|^2 + \int_0^t \|K^{-1/2}\phi_z(s)\|_Q^2\, ds \\
\le \epsilon \int_0^t (\|\phi_s(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_z(s)\|^2)\, ds \\
+ Ch^2 \int_0^t (\|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2 + \|z(s)\|_1^2)\, ds \\
+ \|A^{1/2}(\phi_s(0) + \alpha\phi_p I(0))\|_Q^2 + c_0\|\phi_p(0)\|^2.
\end{aligned} \tag{5.42}$$

Choosing $\sigma_h(0) = \Pi\sigma(0)$ and $p_h(0) = Q^0 p(0)$, we obtain

$$\|A^{1/2}(\phi_s(0) + \alpha\phi_p I(0))\|_Q^2 + c_0\|\phi_p(0)\|^2 = 0. \tag{5.43}$$

Hence, we can write (5.42) as

$$\begin{aligned}
\|A^{1/2}(\phi_s(t) + \alpha\phi_p I(t))\|_Q^2 + c_0\|\phi_p(t)\|^2 + \int_0^t \|K^{-1/2}\phi_z(s)\|_Q^2\, ds \\
\le \epsilon \int_0^t (\|\phi_s(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_z(s)\|^2)\, ds \\
+ Ch^2 \int_0^t (\|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2 + \|z(s)\|_1^2)\, ds.
\end{aligned} \tag{5.44}$$

Using the inf-sup condition [3, 4] and (5.26), we get

$$\begin{aligned}
\|\phi_u\| + \|\phi_\gamma\| \le& C \sup_{0\neq\tau\in\mathbb{X}_h} \frac{(\phi_u, \operatorname{div}\tau) + (\phi_\gamma, \operatorname{as}\tau)_Q}{\|\tau\|_{\operatorname{div}}} \\
&= C \sup_{0\neq\tau\in\mathbb{X}_h} \Bigg(\frac{(A(\sigma_h + \alpha p_h I), \tau)_Q - (A(\sigma + \alpha p I), \tau)}{\|\tau\|_{\operatorname{div}}} \\
&\qquad + \frac{\big(Q^1\gamma, \operatorname{as}\tau\big) - (\gamma, \operatorname{as}\tau)_Q + \langle g_u - Q^0 g_u, \tau\, n\rangle}{\|\tau\|_{\operatorname{div}}} \Bigg).
\end{aligned} \tag{5.45}$$

Using the calculations as in (5.31)-(5.33), (2.61) and (2.42), we have

$$\begin{aligned}
(A(\sigma_h + \alpha p_h I), \tau)_Q - (A(\sigma + \alpha p I), \tau) + \big(Q^1\gamma, \operatorname{as}\tau\big) - (\gamma, \operatorname{as}\tau)_Q + \langle g_u - \mathcal{P}_0 g_u, \tau\, n\rangle \\
= -(A(\phi_s + \alpha\phi_p I), \tau)_Q - \big(A(\psi_s + \alpha\psi_p I), \Pi^0\tau\big) - \big(\psi_\gamma, \operatorname{as}\Pi^0\tau\big) - \theta\left(A\tilde{\Pi}\sigma, \Pi^0\tau\right) \\
+ \left(A(\tilde{\Pi}\sigma + \alpha Q^0 p I), \tau - \Pi^0\tau\right)_Q + \big(Q^1\gamma, \operatorname{as}(\tau - \Pi^0\tau)\big)_Q \\
\le Ch(\|\sigma\|_1 + \|p\|_1 + \|\gamma\|_1)\|\tau\| + C\|A^{1/2}(\phi_s + \alpha\phi_p I)\|\|\tau\|
\end{aligned} \tag{5.46}$$

Combining (5.45) and (5.46) and using orthogonality of projections, we get

$$\|\phi_u\| + \|\phi_\gamma\| \le Ch(\|\sigma\|_1 + \|p\|_1 + \|\gamma\|_1) + C\|A^{1/2}(\phi_s + \alpha\phi_p I)\|.$$

Thus, (5.44) becomes

$$\begin{aligned}
\|A^{1/2}(\phi_s(t) + \alpha\phi_p I(t))\|^2 + \|\phi_u(t)\|^2 + \|\phi_\gamma(t)\|^2 + c_0\|\phi_p(t)\|^2 + \int_0^t \|\phi_z(s)\|^2\, ds \\
\le \epsilon \int_0^t (\|\phi_s(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_z(s)\|^2)\, ds + Ch^2(\|\sigma(t)\|_1^2 + \|p(t)\|_1^2 + \|\gamma(t)\|_1^2), \\
+Ch^2 \int_0^t (\|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2 + \|z(s)\|_1^2)\, ds,
\end{aligned} \tag{5.47}$$

where we also used the equivalence of norms, see Corollary 2.1.

Using the fact that $Z_h^0 \times W_h$ is a stable Darcy pair, (5.29), (2.62), (5.12) and (5.17) we also obtain

$$\begin{aligned} \|\phi_p\| &\le C \sup_{0\neq q\in Z_h^0} \frac{(\operatorname{div} q,\, \phi_p)}{\|q\|_{\text{div}}} = C \sup_{0\neq q\in Z_h^0} \frac{\left(K^{-1}z,\, q\right) - \left(K^{-1}z_h,\, q\right)_Q}{\|q\|_{\text{div}}} \\ &= C \sup_{0\neq q\in Z_h^0} \frac{\left(K^{-1}\phi_z,\, q\right)_Q - \left(K^{-1}\psi_z,\, q\right) + \theta\left(K^{-1}\Pi z,\, q\right)}{\|q\|_{\text{div}}} \le Ch\|z\|_1 + \|\phi_z\|. \end{aligned} \tag{5.48}$$

Therefore, we have

$$\begin{aligned} &\|A^{1/2}(\phi_s(t) + \alpha\phi_p I(t))\|^2 + \|\phi_u(t)\|^2 + \|\phi_\gamma(t)\|^2 + c_0\|\phi_p(t)\|^2 + \int_0^t (\|\phi_z(s)\|^2 + \|\phi_p(s)\|^2)\, ds \\ &\qquad \le \epsilon \int_0^t (\|\phi_s(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_z(s)\|^2)\, ds + Ch^2(\|\sigma(t)\|_1^2 + \|p(t)\|_1^2 + \|\gamma(t)\|_1^2), \\ &\qquad\qquad + Ch^2 \int_0^t (\|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2 + \|z(s)\|_1^2). \end{aligned} \tag{5.49}$$

Next, we choose $\tau = \phi_s$ in (5.33) and use (5.34)- (5.35) and (5.39)-(5.41):

$$\begin{aligned} C\|\phi_s\|^2 &\le -\alpha\left(A\phi_p I,\, \phi_s\right)_Q - \left(A(\psi_s + \alpha\psi_p I),\, \Pi^0\phi_s\right) - \left(\psi_\gamma,\, \text{as}\, \Pi^0\phi_s\right) - \theta\left(A\Pi\sigma,\, \Pi^0\phi_s\right) \\ &\quad - \theta\left(A\alpha Q^0 pI,\, \Pi^0\phi_s\right) - \theta\left(Q^1\gamma,\, \text{as}\, \Pi^0\phi_s\right) + \left(A(\Pi\sigma + \alpha Q^0 pI),\, \phi_s - \Pi^0\phi_s\right)_Q \\ &\quad + \left(Q^1\gamma,\, \text{as}\,(\phi_s - \Pi^0\phi_s)\right)_Q \le Ch^2(\|\sigma\|_1^2 + \|p\|_1^2 + \|\gamma\|_1^2) + C\|\phi_p\|^2 + \epsilon\|\phi_s\|^2, \end{aligned}$$

where in the last step we used (5.10)-(5.12) and Lemma 5.2. Thus, we have

$$\int_0^t \|\phi_s(s)\|^2\, ds \le C\int_0^t h^2(\|\sigma(s)\|_1^2 + \|p(s)\|_1^2 + \|\gamma(s)\|_1^2)\, ds + C\int_0^t \|\phi_p(s)\|^2\, ds. \tag{5.50}$$

On the other hand, it follows from (5.45)-(5.46) and (5.50) that

$$\int_0^t (\|\phi_u(s)\| + \|\phi_\gamma(s)\|)\, ds \le C\int_0^t (h(\|\sigma(s)\|_1 + \|p(s)\|_1 + \|\gamma(s)\|_1) + \|\phi_s(s)\| + \|\phi_p(s)\|)\, ds. \tag{5.51}$$

Combining (5.49)-(5.51), we obtain

$$\begin{aligned} &\|A^{1/2}(\phi_s(t) + \alpha\phi_p I(t))\|^2 + \|\phi_u(t)\|^2 + \|\phi_\gamma(t)\|^2 + c_0\|\phi_p(t)\|^2 \\ &\quad + \int_0^t (\|\phi_z(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_s(s)\|^2 + \|\phi_u(s)\|^2 + \|\phi_\gamma(s)\|^2)\, ds \\ &\le \epsilon \int_0^t (\|\phi_s(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_z(s)\|^2)\, ds + Ch^2(\|\sigma(t)\|_1^2 + \|p(t)\|_1^2 + \|\gamma(t)\|_1^2), \\ &\quad + Ch^2 \int_0^t (\|\sigma(s)\|_1^2 + \|\partial_t\sigma(s)\|_1^2 + \|p(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\gamma(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2 + \|z(s)\|_1^2). \end{aligned} \tag{5.52}$$

Choosing ϵ small enough, we get

$$\begin{aligned} &\|A^{1/2}(\phi_s(t) + \alpha\phi_p I(t))\|^2 + \|\phi_u(t)\|^2 + \|\phi_\gamma(t)\|^2 + c_0\|\phi_p(t)\|^2 \\ &\quad + \int_0^t (\|\phi_z(s)\|^2 + \|\phi_p(s)\|^2 + \|\phi_s(s)\|^2 + \|\phi_u(s)\|^2 + \|\phi_\gamma(s)\|^2)\, ds \\ &\le Ch^2(\|\sigma(t)\|_1^2 + \|p(t)\|_1^2 + \|\gamma(t)\|_1^2), \\ &\quad + Ch^2 \int_0^t (\|\sigma(s)\|_1^2 + \|\partial_t\sigma(s)\|_1^2 + \|p(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\gamma(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2 + \|z(s)\|_1^2). \end{aligned} \tag{5.53}$$

Step 2: $H(div)$ **in space estimate for stress and velocity:**
Estimate for stress error follows immediately due to (5.34).
It follows from (5.37) that

$$\|\operatorname{div}\phi_z\| \leq c_0\|\partial_t\phi_p\| + \|\partial_t A^{1/2}(\phi_s + \alpha\phi_p I)\| + Ch(\|\sigma\|_1 + \|\partial_t\sigma\|_1). \tag{5.54}$$

Next we differentiate (5.33)-(5.36) , set $\tau = \partial_t\phi_s$, $\xi = \partial_t\phi_\gamma$, $q = \phi_z$, $w = \partial_t\phi_p$ and combine (5.33)-(5.37):

$$\begin{aligned}
&\frac{1}{2}\partial_t\|K^{-1/2}\phi_z\|_Q^2 + \|A^{1/2}\partial_t(\phi_s + \alpha\phi_p I)\|_Q^2 + c_0\|\partial_t\phi_p\|^2 \\
&= -\left(A\partial_t(\psi_s + \alpha\psi_p I),\, \Pi^0\partial_t\phi_s\right) - \left(\partial_t\psi_\gamma,\, \operatorname{as}\Pi^0\partial_t\phi_s\right) - \theta\left(A\partial_t\tilde{\Pi}\sigma,\, \Pi^0\partial_t\phi_s + \alpha\partial_t\phi_p I\right) \\
&\quad + \left(A\partial_t(\tilde{\Pi}\sigma + \alpha Q^0 pI),\, \partial_t\phi_s - \Pi^0\partial_t\phi_s\right)_Q - \theta\left(\partial_t Q^1\gamma,\, \operatorname{as}\partial_t\Pi^0\phi_s\right) + \left(\partial_t Q^1\gamma,\, \operatorname{as}(\partial_t\phi_s - \Pi^0\partial_t\phi_s)\right)_Q \\
&\quad - \left(K^{-1}\psi_z,\, \Pi^0\partial_t\phi_z\right) - \theta\left(K^{-1}\Pi z,\, \partial_t\Pi^0\phi_z\right) + \left(K^{-1}\Pi z,\, \partial_t\phi_z - \partial_t\Pi^0\phi_z\right)_Q - \alpha\left(\partial_t\operatorname{tr}(A\psi_s),\, \partial_t\phi_p\right) \\
&\quad - \alpha^2\left(\partial_t\operatorname{tr}(A\psi_p I),\, \partial_t\phi_p\right) - \alpha\theta\left(\partial_t AQ^0 p,\, \partial_t\Pi^0\phi_s + \alpha\partial_t\phi_p\right).
\end{aligned} \tag{5.55}$$

For all terms not corresponding to error in Darcy velocity, we repeat the arguments from (5.38)-(5.42), combining stress and pressure errors into one.

$$\begin{aligned}
&\left| -\theta\left(A\partial_t\tilde{\Pi}\sigma,\, \Pi^0\partial_t\phi_s + \alpha\partial_t\phi_p I\right) - \theta\left(\partial_t Q^1\gamma,\, \operatorname{as}\partial_t\Pi^0\phi_s\right) - \alpha\theta\left(\partial_t AQ^0 p,\, \partial_t\Pi^0\phi_s + \alpha\partial_t\phi_p\right)\right| \\
&\quad = \left|\sum_{E\in\mathcal{T}_h}\left(\theta\left(A\partial_t\tilde{\Pi}\sigma,\, \Pi^0\partial_t(\phi_s + \alpha\phi_p I)\right)_E + \theta\left(\partial_t Q^1\gamma,\, \operatorname{as}\Pi^0\partial_t(\phi_s + \alpha\phi_p I)\right)_E\right.\right. \qquad (5.56)\\
&\qquad\qquad \left.\left. + \alpha\theta\left(\partial_t AQ^0 p,\, \Pi^0\partial_t(\phi_s + \alpha\phi_p I)\right)_E\right)\right| \\
&\quad \leq Ch^2(\|\partial_t\sigma\|_1^2 + \|\partial_t p\|_1^2 + \|\partial_t\gamma\|_1^2) + \epsilon\|\Pi^0\partial_t\phi_s + \alpha\partial_t\phi_p I\|^2,
\end{aligned} \tag{5.57}$$

where we used the fact that on every $E \in \mathcal{T}_h$, $\phi_p I|_E \in \mathbb{X}_h^0(E)$ and also that $\operatorname{as}(\phi_p I) = 0$. Similarly,

$$\begin{aligned}
&\left| -\left(A\partial_t(\psi_s + \alpha\psi_p I),\, \Pi^0\partial_t\phi_s\right) - \left(\partial_t\psi_\gamma,\, \operatorname{as}\Pi^0\partial_t\phi_s\right) - \alpha\left(\partial_t\operatorname{tr}(A\psi_s),\, \partial_t\phi_p\right) - \alpha^2\left(\partial_t\operatorname{tr}(A\psi_p I),\, \partial_t\phi_p\right)\right| \\
&\quad = \left| -\left(A\partial_t(\psi_s + \alpha\psi_p I),\, \partial_t(\Pi^0\phi_s + \alpha\phi_p)\right) - \left(\partial_t\psi_\gamma,\, \operatorname{as}\partial_t(\Pi^0\phi_s + \phi_p)\right)\right| \\
&\quad = \left|\sum_{E\in\mathcal{T}_h}\left(\left(A\partial_t(\psi_s + \alpha\psi_p I),\, \partial_t\Pi^0(\phi_s + \alpha\phi_p)\right)_E + \left(\partial_t\psi_\gamma,\, \operatorname{as}\partial_t\Pi^0(\phi_s + \phi_p)\right)_E\right)\right| \\
&\quad \leq Ch^2(\|\partial_t\sigma\|_1^2 + \|\partial_t p\|_1^2 + \|\partial_t\gamma\|_1^2) + \epsilon\|\partial_t\phi_s + \alpha\partial_t\phi_p I\|^2,
\end{aligned} \tag{5.58}$$

and

$$\begin{aligned}
&\left|\left(A\partial_t(\tilde{\Pi}\sigma + \alpha Q^0 pI),\, \partial_t\phi_s - \Pi^0\partial_t\phi_s\right)_Q + \left(\partial_t Q^1\gamma,\, \operatorname{as}(\partial_t\phi_s - \Pi^0\partial_t\phi_s)\right)_Q\right| \\
&\quad = \left|\sum_{E\in\mathcal{T}_h}\left(\left(A\partial_t(\tilde{\Pi}\sigma + \alpha Q^0 pI),\, \partial_t(\phi_s + \phi_p I) - \Pi^0\partial_t(\phi_s + \phi_p I)\right)_{Q,E}\right.\right. \\
&\qquad\qquad \left.\left. + \left(\partial_t Q^1\gamma,\, \operatorname{as}(\partial_t(\phi_s + \phi_p I) - \Pi^0\partial_t(\phi_s + \phi_p I))\right)_{E,Q}\right)\right| \\
&\quad \leq Ch^2(\|\partial_t\sigma\|_1^2 + \|\partial_t p\|_1^2 + \|\partial_t\gamma\|_1^2) + \epsilon\|\partial_t\phi_s + \alpha\partial_t\phi_p I\|^2.
\end{aligned} \tag{5.59}$$

Combining (5.55)-(5.59), we obtain

$$
\begin{aligned}
&\|K^{-1/2}\phi_z(t)\|_Q^2 + \int_0^t \Big(\|A^{1/2}\partial_t(\phi_s(s) + \alpha\phi_p I(s))\|_Q^2 + c_0\|\partial_t\phi_p(s)\|^2\Big)\, ds \\
&\le C\Big(\|K^{-1/2}\phi_z(0)\|_Q^2 + \epsilon\int_0^t \|\partial_t\phi_s(s) + \alpha\partial_t\phi_p(s)I\|^2\, ds \\
&\quad + Ch^2\int_0^t (\|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2)\, ds \\
&\quad + \int_0^t \big(- \big(K^{-1}\psi_z(s),\, \Pi^0\partial_t\phi_z(s)\big) - \theta\,\big(K^{-1}\Pi z(s),\, \partial_t\Pi^0\phi_z(s)\big) \\
&\qquad + \big(K^{-1}\Pi z(s),\, \partial_t\phi_z(s) - \partial_t\Pi^0\phi_z(s)\big)_Q \,\big)\, ds\Big).
\end{aligned}
\tag{5.60}
$$

We integrate by parts the terms involving error in Darcy velocity

$$
\begin{aligned}
&\int_0^t \Big(- \big(K^{-1}\psi_z(s),\, \Pi^0\partial_t\phi_z(s)\big) - \theta\,\big(K^{-1}\Pi z(s),\, \partial_t\Pi^0\phi_z(s)\big) + \big(K^{-1}\Pi z(s),\, \partial_t\phi_z(s) - \partial_t\Pi^0\phi_z(s)\big)_Q\Big)\, ds \\
&\quad = - \big(K^{-1}\psi_z(t),\, \Pi^0\phi_z(t)\big) - \theta\,\big(K^{-1}\Pi z(t),\, \Pi^0\phi_z(t)\big) + \big(K^{-1}\Pi z(t),\, \phi_z(t) - \Pi^0\phi_z(t)\big)_Q \\
&\qquad + \big(K^{-1}\psi_z(0),\, \Pi^0\phi_z(0)\big) + \theta\,\big(K^{-1}\Pi z(0),\, \Pi^0\phi_z(0)\big) + \big(K^{-1}\Pi z(0),\, \phi_z(0) - \Pi^0\phi_z(0)\big)_Q \\
&\qquad - \int_0^t \Big(- \big(K^{-1}\partial_t\psi_z(s),\, \Pi^0\phi_z(s)\big) - \theta\,\big(K^{-1}\partial_t\Pi z(s),\, \Pi^0\phi_z(s)\big) \\
&\qquad\quad + \big(K^{-1}\partial_t\Pi z(s),\, \phi_z(s) - \Pi^0\phi_z(s)\big)_Q\Big)\, ds
\end{aligned}
$$

Choosing $z_h(0) = \Pi z(0)$, we obtain

$$
\big(K^{-1}\psi_z(0),\, \Pi^0\phi_z(0)\big) + \theta\,\big(K^{-1}\Pi z(0),\, \Pi^0\phi_z(0)\big) + \big(K^{-1}\Pi z(0),\, \phi_z(0) - \Pi^0\phi_z(0)\big)_Q = 0,
\tag{5.61}
$$

and for the rest of the terms we use (5.12), (5.17) and (5.21):

$$
\begin{aligned}
&- \big(K^{-1}\psi_z(t),\, \Pi^0\phi_z(t)\big) - \theta\,\big(K^{-1}\Pi z(t),\, \Pi^0\phi_z(t)\big) + \big(K^{-1}\Pi z(t),\, \phi_z(t) - \Pi^0\phi_z(t)\big)_Q \\
&\quad - \int_0^t \Big(- \big(K^{-1}\partial_t\psi_z(s),\, \Pi^0\phi_z(s)\big) - \theta\,\big(K^{-1}\partial_t\Pi z(s),\, \Pi^0\phi_z(s)\big) \\
&\qquad + \big(K^{-1}\partial_t\Pi z(s),\, \phi_z(s) - \Pi^0\phi_z(s)\big)_Q\Big)\, ds \\
&\le C(h^2\|z(t)\|_1^2 + \epsilon\|\phi_z(t)\|^2) + \int_0^t (h^2\|\partial_t z(s)\|_1^2 + \epsilon\|\phi_z(s)\|^2)\, ds.
\end{aligned}
\tag{5.62}
$$

From (5.60)-(5.62) we obtain:

$$
\begin{aligned}
&\|K^{-1/2}\phi_z(t)\|_Q^2 + \int_0^t \Big(\|A^{1/2}\partial_t(\phi_s(s) + \alpha\phi_p I(s))\|_Q^2 + c_0\|\partial_t\phi_p(s)\|^2\Big)\, ds \\
&\le C(h^2\|z(t)\|_1^2 + \epsilon\|\phi_z(t)\|^2) + C\int_0^t (h^2(\|\partial_t z(s)\|_1^2 + h^2\|\partial_t\sigma(s)\|_1^2 + \epsilon\|\phi_z(s)\|^2)\, ds..
\end{aligned}
\tag{5.63}
$$

Combining (5.63), (5.60),(5.48) and using the equivalence of norms, we get

$$
\begin{aligned}
&\|\phi_z(t)\|^2 + \|\phi_p(t)\|^2 + \int_0^t \big(\|\partial_t(\phi_s(s) + \alpha\phi_p I(s))\|^2 + c_0\|\partial_t\phi_p(s)\|^2\big)\, ds \\
&\quad \le C\int_0^t h^2(\|\partial_t z(s)\|_1^2 + \|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2)\, ds \\
&\qquad + \epsilon\int_0^t (\|\phi_z(s)\|^2 + \|\partial_t(\phi_s(s) + \alpha\phi_p I(s))\|^2)\, ds + C(h^2\|z(t)\|_1^2 + \epsilon\|\phi_z(t)\|^2).
\end{aligned}
\tag{5.64}
$$

Hence, (5.54) and (5.64) yield

$$\begin{aligned}
&\|\phi_z(t)\|^2 + \|\phi_p(t)\|^2 + \int_0^t \|\operatorname{div}\phi_z\|^2\, ds \\
&\quad \le \epsilon \int_0^t \|\phi_z(s)\|^2\, ds \\
&\qquad + C\left(\int_0^t h^2(\|\partial_t z(s)\|_1^2 + \|\sigma(s)\|_1^2 + \|\partial_t\sigma(s)\|_1^2 + \|\partial_t p(s)\|_1^2 + \|\partial_t\gamma(s)\|_1^2)\, ds + \|z(t)\|_1^2\right). \quad (5.65)
\end{aligned}$$

Step 3: obtaining the final result:
We note that

$$\begin{aligned}
\|\phi_s\| \le C\|A^{1/2}\phi_s\| &\le C\left(\|A^{1/2}(\phi_s + \alpha\phi_p I)\| + \|A^{1/2}\alpha\phi_p I\|\right) \\
&\le C\left(\|A^{1/2}(\phi_s + \alpha\phi_p I)\| + \|\phi_p\|\right). \quad (5.66)
\end{aligned}$$

Therefore, combining (5.53), (5.65) and (5.66), we obtain the following result.

Theorem 5.1. *Let* $(\sigma_h, u_h, \gamma_h, z_h, p_h) \in \mathbb{X}_h \times V_h \times \Theta_h \times Z_h \times W_h$ *be the solution of* (2.63)-(2.67) *and* $(\sigma, u, \gamma, z, p) \in \mathbb{X} \times V \times \mathbb{W} \times Z \times W \cap\ H^1(0,T;(H^1(\Omega))^{d\times d}) \times H^1(0,T;(H^1(\Omega))^d)$ $\times H^1(0,T;H^1(\Omega)^{d\times d,skew}) \times H^1(0,T;(H^1(\Omega))^d) \times H^1(0,T;H^1(\Omega))$ *be the solution of* (2.8)-(2.12)*. Then the following error estimate holds:*

$$\begin{aligned}
&\|\sigma - \sigma_h\|_{L^\infty(0,T;H(\mathrm{div},\Omega))} + \|u - u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\gamma - \gamma_h\|_{L^\infty(0,T;L^2(\Omega))} + \|z - z_h\|_{L^\infty(0,T;L^2(\Omega))} \\
&\quad + \|p - p_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\sigma - \sigma_h\|_{L^2(0,T;H(\mathrm{div},\Omega))} + \|u - u_h\|_{L^2(0,T;L^2(\Omega))} + \|\gamma - \gamma_h\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad + \|z - z_h\|_{L^2(0,T;H(\mathrm{div},\Omega))} + \|p - p_h\|_{L^2(0,T;L^2(\Omega))} \\
&\le Ch\Big(\|s\|_{H^1(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^1(\Omega))} + \|\gamma\|_{H^1(0,T;H^1(\Omega))} + \|z\|_{H^1(0,T;H^1(\Omega))} \\
&\quad + \|p\|_{H^1(0,T;H^1(\Omega))} + \|\sigma\|_{L^\infty(0,T;H^1(\Omega))} + \|u\|_{L^\infty(0,T;L^2(\Omega))} \\
&\qquad + \|\gamma\|_{L^\infty(0,T;H^1(\Omega))} + \|z\|_{L^\infty(0,T;H^1(\Omega))} + \|p\|_{L^\infty(0,T;H^1(\Omega))}\Big). \quad (5.67)
\end{aligned}$$

6 Numerical results

In this section we provide several numerical tests verifying the theoretically predicted convergence rates and illustrating the behavior of the proposed method on simplicial and quadrilateral grids. We also briefly address the issue of locking when dealing with small storativity coefficients.

6.1 Example 1.

We first verify the method's convergence on simplicial grids in 3 dimensions. For this, we use a unit cube as a computational domain, and choose the analytical solution for pressure and displacement as follows:

$$p = \cos(t)(x + y + z + 1.5), \quad u = \sin(t)\begin{pmatrix} -0.1(e^x - 1)\sin(\pi x)\sin(\pi y) \\ -(e^x - 1)(y - \cos(\frac{\pi}{12})(y - 0.5) + \sin(\frac{\pi}{12})(z - 0.5) - 0.5) \\ -(e^x - 1)(z - \sin(\frac{\pi}{12})(y - 0.5) - \cos(\frac{\pi}{12})(z - 0.5) - 0.5) \end{pmatrix}.$$

The permeability tensor is of the form

$$K = \begin{pmatrix} x^2 + y^2 + 1 & 0 & 0 \\ 0 & z^2 + 1 & \sin(xy) \\ 0 & \sin(xy) & x^2y^2 + 1 \end{pmatrix},$$

and the rest of the parameters are presented in Table 1.

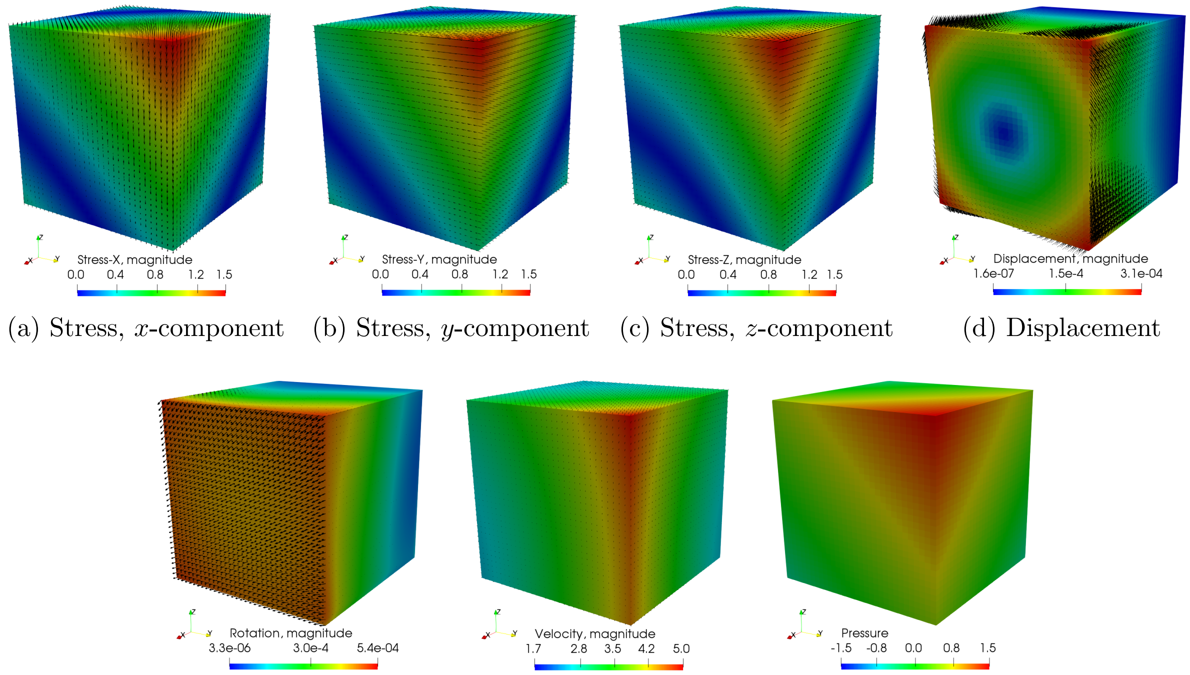
Parameter	Symbol	Values
Lame coefficient	μ	100.0
Lame coefficient	λ	100.0
Mass storativity	c_0	1.0
Biot-Willis constant	α	1.0
Total time	T	10^{-3}
Time step	Δt	10^{-4}

Table 1: Physical parameters, Examples 1 and 2

Using the analytical solution provided above and equations (2.1)-(2.2) we recover the rest of variables and right-hand side functions. Dirichlet boundary conditions for the pressure and the displacement are specified on the entire boundary of the domain.

	$\\|\sigma - \sigma_h\\|_{L^2(0,T;L^2(\Omega))}$		$\\|\operatorname{div}(\sigma - \sigma_h)\\|_{L^2(0,T;L^2(\Omega))}$		$\\|u - u_h\\|_{L^2(0,T;L^2(\Omega))}$	
h	error	rate	error	rate	error	rate
1/4	3.07E-02	-	2.29E-01	-	8.54E-01	-
1/8	9.92E-03	1.6	1.14E-01	1.0	2.32E-01	1.9
1/16	4.90E-03	1.0	5.68E-02	1.0	7.44E-02	1.6
1/32	2.50E-03	1.0	2.84E-02	1.0	2.97E-02	1.3
	$\\|\gamma - \gamma_h\\|_{L^2(0,T;L^2(\Omega))}$		$\\|z - z_h\\|_{L^2(0,T;L^2(\Omega))}$		$\\|\operatorname{div}(z - z_h)\\|_{L^2(0,T;L^2(\Omega))}$	
h	error	rate	error	rate	error	rate
1/4	7.65E-01	-	1.06E-02	-	5.85E-02	-
1/8	2.32E-01	1.7	2.66E-03	2.0	2.31E-02	1.3
1/16	7.00E-02	1.7	6.64E-04	2.0	7.70E-03	1.6
1/32	2.12E-02	1.7	1.66E-04	2.0	2.71E-03	1.5
	$\\|p - p_h\\|_{L^2(0,T;L^2(\Omega))}$		$\\|\sigma - \sigma_h\\|_{L^\infty(0,T;L^2(\Omega))}$		$\\|p - p_h\\|_{L^\infty(0,T;L^2(\Omega))}$	
h	error	rate	error	rate	error	rate
1/4	1.92E-04	-	2.29E-01	-	2.18E-04	-
1/8	5.56E-05	1.8	1.14E-01	1.0	6.39E-05	1.8
1/16	1.28E-05	2.1	5.70E-02	1.0	1.30E-05	2.3
1/32	2.55E-06	2.3	2.85E-02	1.0	2.78E-06	2.2

Table 2: Example 1, computed numerical errors and convergence rates.



(a) Stress, x-component (b) Stress, y-component (c) Stress, z-component (d) Displacement

(e) Rotation (f) Darcy velocity (g) Darcy pressure

Figure 1: Example 1, computed solution at the final time step.

In Table 2 we present computed relative errors and rates for this example. For the sake of space we report only the errors that would normally be of interest in studying the behavior of this problem. As one can observe, the results agree with theory of the previous section.

6.2 Example 2.

The second test case is to study the convergence of the method on an h^2-parallelogram grid. We consider the following analytical solution

$$p = \exp(t)(\sin(\pi x)\cos(\pi y) + 10), \quad u = \exp(t)\begin{pmatrix} x^3y^4 + x^2 + \sin((1-x)(1-y))\cos(1-y) \\ (1-x)^4(1-y)^3 + (1-y)^2 + \cos(xy)\sin(x) \end{pmatrix}.$$

and the permeability tensor of the form

$$\begin{pmatrix} (x+1)^2 + y^2 & \sin(xy) \\ \sin(xy) & (x+1)^2 \end{pmatrix}.$$

The Poisson ratio is set to be $\nu = 0.2$ and Young's modulus varies over the domain as $E = \sin(5\pi x)\sin(5\pi y) + 5$. The Lamé parameters are then computed using the well known relations

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

The time discretization parameters are the same as in Table 1.

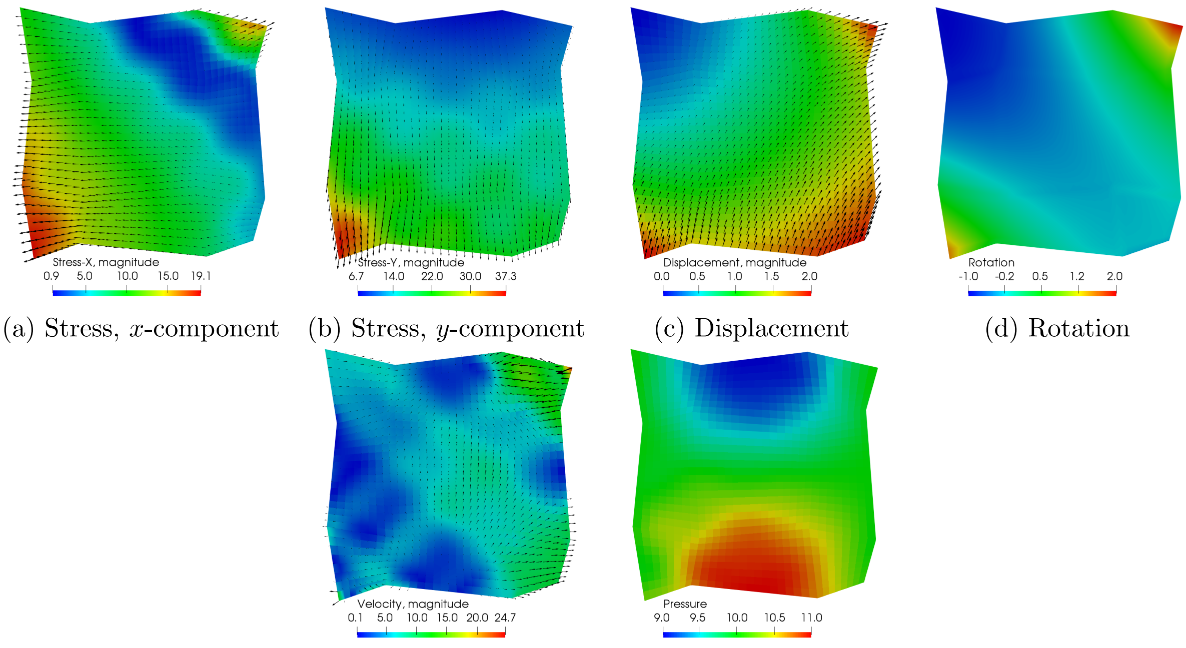
The computational grid for this case is obtained by taking a unit square with initial partitioning into a mesh with $h = \frac{1}{4}$, and further transforming it by the following map (see Figure 2):

$$x = \hat{x} + 0.03\cos(3\pi\hat{x})\cos(3\pi\hat{y}), \quad y = \hat{y} - 0.04\cos(3\pi\hat{x})\cos(3\pi\hat{y}).$$

As in the previous test case we observe optimal convergence rates for all variables in their respective norms.

h	$\|\sigma - \sigma_h\|_{L^2(0,T;L^2(\Omega))}$ error	rate	$\|\operatorname{div}(\sigma - \sigma_h)\|_{L^2(0,T;L^2(\Omega))}$ error	rate	$\|u - u_h\|_{L^2(0,T;L^2(\Omega))}$ error	rate
1/8	6.505e-02	-	4.305e-01	-	7.985e-02	-
1/16	3.130e-02	1.1	2.336e-01	0.9	3.959e-02	1.0
1/32	1.506e-02	1.1	1.172e-01	1.0	1.975e-02	1.0
1/64	7.435e-03	1.0	5.856e-02	1.0	9.869e-03	1.0
1/128	3.709e-03	1.0	2.927e-02	1.0	4.934e-03	1.0
h	$\|\gamma - \gamma_h\|_{L^2(0,T;L^2(\Omega))}$ error	rate	$\|z - z_h\|_{L^2(0,T;L^2(\Omega))}$ error	rate	$\|\operatorname{div}(z - z_h)\|_{L^2(0,T;L^2(\Omega))}$ error	rate
1/8	1.964e-01	-	5.321e-01	-	2.531e+00	-
1/16	7.444e-02	1.4	2.935e-01	0.9	1.599e+00	0.7
1/32	2.767e-02	1.4	9.757e-02	1.6	5.864e-01	1.5
1/64	1.016e-02	1.5	2.999e-02	1.7	1.767e-01	1.7
1/128	3.697e-03	1.5	1.080e-02	1.5	4.984e-02	1.8
h	$\|p - p_h\|_{L^2(0,T;L^2(\Omega))}$ error	rate	$\|\sigma - \sigma_h\|_{L^\infty(0,T;L^2(\Omega))}$ error	rate	$\|p - p_h\|_{L^\infty(0,T;L^2(\Omega))}$ error	rate
1/8	1.588e-02	-	6.595e-02	-	2.519e-02	-
1/16	6.755e-03	1.2	3.180e-02	1.1	1.170e-02	1.1
1/32	2.647e-03	1.4	1.516e-02	1.1	3.863e-03	1.6
1/64	1.178e-03	1.2	7.449e-03	1.0	1.387e-03	1.5
1/128	5.680e-04	1.1	3.710e-03	1.0	5.973e-04	1.2

Table 3: Example 2, computed numerical errors and convergence rates.



(a) Stress, x-component (b) Stress, y-component (c) Displacement (d) Rotation

(e) Darcy velocity (f) Darcy pressure

Figure 2: Example 2, computed solution at the final time step.

6.3 Example 3.

Our third example is to confirm that the coupled MFMFE-MSMFE method for the Biot system inherits is locking free, due to its mixed nature. It was shown in [33] that with continuous finite elements used for the elasticity part of the system, locking occurs when the storativity coefficient is very small. One of the typical model problems that illustrates such behavior is the cantilever bracket problem [26].

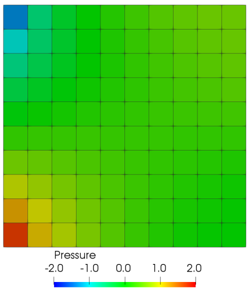
The computational domain is a unit square $[0,1]\times[0,1]$. We impose a no-flow boundary condition along all sides, the deformation is fixed along the left edge, and a downward traction is applied at the top of the unit square. The bottom and right sides are enforced to be traction-free. More precisely, with the sides of the domain being labeled as Γ_1 to Γ_4, going counterclockwise from the bottom side, we have

$$
\begin{aligned}
&z\cdot n = 0, && \text{on } \partial\Omega = \Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4,\\
&\sigma\, n = (0,-1)^T, && \text{on } \Gamma_3,\\
&\sigma\, n = (0,0)^T, && \text{on } \Gamma_1\cup\Gamma_2,\\
&u = (0,0)^T, && \text{on } \Gamma_4.
\end{aligned}
$$

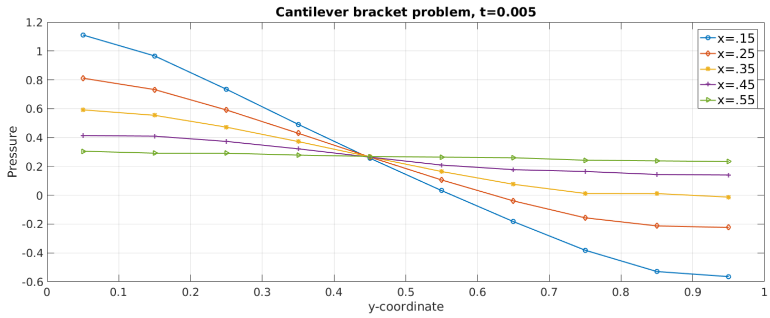
We use the same physical parameters as in [33], as they typically induce locking:

$$E = 10^5, \quad \nu = 0.4, \quad \alpha = 0.93, \quad c_0 = 0, \quad K = 10^{-7},$$

The time step is set to be $\Delta t = 0.001$ and the total simulation time is $T = 1$.



(a) Pressure field, $t = 0.001$.



(b) Pressure along different $x-$lines, $t = 0.005$.

Figure 3: Example 3, computed pressure solutions.

Figure 3a shows that the coupled MSMFE-MFMFE method yields a smooth pressure field, without a typically arising checkerboard pattern that one obtains with a CG-mixed method for the Biot system (see [33]) on early time steps. In addition, Figure 3b shows the pressure solution along different $x-$lines at time $t = 0.005$. The latter illustrates the lack of oscillations and that the solution of the coupled mixed method agrees with the one obtained by DG-mixed or stabilized CG-mixed [33, 26]

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