

DOMAIN DECOMPOSITION FOR POROELASTICITY AND ELASTICITY WITH DG JUMPS AND MORTARS

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We couple a time-dependent poroelastic model in a region with an elastic model in adjacent regions. We discretize each model independently on non-matching grids and we realize a domain decomposition on the interface between the regions by introducing DG jumps and mortars. The unknowns are condensed on the interface, so that at each time step, the computation in each subdomain can be performed in parallel. In addition, by extrapolating the displacement, we present an algorithm where the computations of the pressure and displacement are decoupled. We show that the matrix of the interface problem is positive definite and establish error estimates for this scheme.

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1. Introduction

Land subsidence due to the exploitation of subsurface resources, its damage to surface infrastructures, and its impact on the environment are triggering extensive studies in the modeling of fluid flow and geomechanics. The consolidation of surface layers and fluid from groundwater pumping and withdrawals from oil and gas

reservoirs have resulted in significant destruction to infrastructure, building and private homes in the Houston and New Orleans areas over the last century. In addition, unexpectedly large (several meters) subsidence in the North Sea oil fields has resulted in multi-billion dollar adjustments and repairs of the infrastructure. In these examples, it has been well recognized that subsidence is a result of complex interactions between fluid flow, heat transfer, and perhaps chemical reactions and transport coupled with solid phases that include inelastic deformation of rock, possibly plasticity, creep, damage accumulation, strain localization and other mechanisms leading to instability.

Another important related class of problems involves CO₂ sequestration, which is proposed as a key strategy for mitigating climate change driven by high levels of anthropogenic CO₂ being added to the atmosphere. In geological CO₂ sequestration, fluid is injected into a deep subsurface reservoir so that inflation of the reservoir leads to uplift displacement of the overlying surface. As long as the sequestration site is removed from faults, this uplift is several centimeters, while its wavelength is in tens of kilometers, so that the uplift poses little danger to buildings and infrastructure. Nevertheless, the uplift displacements are of great interest for non-intrusive monitoring of CO₂ sequestration. Indeed, uplift can be measured using Interferometric Synthetic Aperture Radar (InSAR) technology^{9,34} as demonstrated by BP at In Salah Algeria,²¹ a commercial scale CO₂ sequestration project.

Clearly, the above-mentioned problems involving consolidation, fluid extraction and injection, require sophisticated and accurate multiscale chemo-thermo-mechanical multi-component models over large temporal (days to hundreds to thousand years) and spatial (pore scale to kilometers). Unfortunately, existing chemo-thermo-mechanical multi-component coupled models are in their infancy in terms of their formulation, computer implementation, calibration, and certainly validation.

There is an extensive bibliography of fluid-solid couplings in a deformable porous medium. We briefly mention the early formulations attributed to the work of Terzaghi³⁵ and Biot^{5,6} and subsequent extensions to nonlinear problems and thermal effects by Refs. 33, 13 and 23. An incomplete list of coupled geomechanics and reservoir flow models includes Refs. 31, 14, 11, 15, 10 and 30. Mathematical analyses of coupling finite element (both continuous and discontinuous Galerkin) with mixed finite element methods for the poroelastic problem have been treated in Refs. 25 and 29. In all of these papers the poroelastic problem is treated over one domain. In Ref. 19 however, Göransson considered the coupling of acoustic-elastic, wave propagation for a flexible porous material in a three-dimensional continuum and developed a model for treating interfaces between different media, i.e. an elastic solid (bonded and unbonded frame conditions) and a free exterior fluid (open boundary).

With petascale and future exascale computing power, parallel domain decomposition offers an opportunity for decoupling realistic subsurface field studies such as carbon sequestration. To reduce the complexity of the multiphysics problem, we apply the coupling in domains where it is only needed. For simplicity, we discuss this approach by first restricting our attention to the coupling of an elastic and

poroelastic models in adjacent regions. This approach is motivated by the observation that pore pressure variations and fluid content within the cap rock and higher layers are oftentimes unaffected by the injection or extraction of fluids within the reservoir. We discretize each model independently on non-matching grids and we realize a domain decomposition on the interface between the regions by introducing discontinuous Galerkin jumps and mortars. The unknowns are condensed on the interface, so that at each time step, the computation in each subdomain can be performed in parallel. In addition, by extrapolating the displacement, we present an algorithm where the computations of the pressure and displacement are decoupled. We show that the matrix of the interface problem is positive definite and establish error estimates for this scheme. This approach generalizes to multiple subdomains. For simplicity, we only analyze conforming Galerkin methods, but a similar analysis holds when mixed finite element methods are used to discretize the flow part of the problem.

The paper is organized as follows. In the subsection below we establish notation. In Sec. 2, a continuous time model involving the decoupling of the model into elastic and poroelastic domains with interface conditions is formulated. This model is extended to a discrete time model in Sec. 3. The interface problem for the discrete time formulation is discussed in Sec. 4. Computational results that confirm error estimates derived in Secs. 3 and 4 are presented in Sec. 5. A brief summary is provided in Sec. 6.

1.1. Notation

In the sequel, we shall use the following functional notation; for the sake of simplicity, we define the spaces in three dimensions. In a region Ω , the scalar product of $L^2(\Omega)$ is denoted by $(\cdot, \cdot)_\Omega$

$$\forall f, g \in L^2(\Omega), \quad (f, g)_\Omega = \int_\Omega f(\mathbf{x})g(\mathbf{x})d\mathbf{x},$$

and if the domain of integration is clear from the context, we suppress the index Ω . The space $\mathcal{D}(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω . Let (k_1, k_2, k_3) denote a triple of non-negative integers, set $|k| = k_1 + k_2 + k_3$ and define the partial derivative ∂^k by

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}}.$$

Then, for any non-negative integer m , recall the classical Sobolev space (cf. Ref. 2 or 27)

$$H^m(\Omega) = \{v \in L^2(\Omega); \partial^k v \in L^2(\Omega) \quad \forall |k| \leq m\},$$

equipped with the following seminorm and norm (for which it is a Hilbert space)

$$|v|_{H^m(\Omega)} = \left[\sum_{|k|=m} \int_\Omega |\partial^k v|^2 d\mathbf{x} \right]^{1/2}, \quad \|v\|_{H^m(\Omega)} = \left[\sum_{0 \leq |k| \leq m} |v|_{H^k(\Omega)}^2 \right]^{1/2}.$$

This definition is extended to any real number $s = m + s'$ for an integer $m \geq 0$ and $0 < s' < 1$ by defining in dimension d the fractional seminorm and norm:

$$|v|_{H^s(\Omega)} = \left(\sum_{|k|=m} \int_{\Omega} \int_{\Omega} \frac{|\partial^k v(\mathbf{x}) - \partial^k v(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{d+2s'}} d\mathbf{x} d\mathbf{y} \right)^{1/2},$$

$$\|v\|_{H^s(\Omega)} = \left(\|v\|_{H^m(\Omega)}^2 + |v|_{H^s(\Omega)}^2 \right)^{1/2}.$$

The reader can refer to Refs. 24 and 20 for properties of these spaces. In particular, we have the following trace property in a domain Ω with Lipschitz continuous boundary $\partial\Omega$: If v belongs to $H^s(\Omega)$ for some $s \in]1/2, 1]$, then its trace on $\partial\Omega$ belongs to $H^{s-1/2}(\partial\Omega)$ and there exists a constant C such that

$$\forall v \in H^s(\Omega), \quad \|v\|_{H^{s-1/2}(\partial\Omega)} \leq C \|v\|_{H^s(\Omega)}.$$

Finally, if Γ is a subset of $\partial\Omega$ with positive measure, $|\Gamma| > 0$, we say that a function g in $H^{1/2}(\partial\Omega)$ belongs to $H_{00}^{1/2}(\Gamma)$ if its extension by zero to $\partial\Omega$ belongs to $H^{1/2}(\partial\Omega)$.

Remark 1.1. When Γ is a polyhedral surface and S is any plane face of Γ , then for any real value of $s > 1/2$, $H^s(S)$ can be defined and the above trace inequality holds on S (see Ref. 4, Corollary 4.3). For s large, only $H^s(S)$ should be used; however, to simplify the notation, we shall write subsequently $H^s(\Gamma)$ as a shortcut, meaning $H^s(S)$ on each S .

All the above definitions are directly extended to vector functions, with the same notation. The space $H(\text{div}, \Omega)$ is the Hilbert space

$$H(\text{div}, \Omega) = \{\mathbf{v} \in L^2(\Omega)^d; \text{div } \mathbf{v} \in L^2(\Omega)\}, \tag{1.1}$$

equipped with the graph norm. The normal trace $\mathbf{v} \cdot \mathbf{n}$ of a function \mathbf{v} of $H(\text{div } \Omega)$ on $\partial\Omega$ belongs to $H^{-1/2}(\partial\Omega)$, the dual space of $H^{1/2}(\partial\Omega)$, see for instance Ref. 17. The same result holds when Γ is a part of $\partial\Omega$ and is a closed surface. But if Γ is not a closed surface, then $\mathbf{v} \cdot \mathbf{n}$ belongs to the dual of $H_{00}^{1/2}(\Gamma)$. We also recall Korn's inequality, valid for all functions \mathbf{v} in $H^1(\Omega)^d$ that vanish on Γ :

$$|\mathbf{v}|_{H^1(\Omega)} \leq C_1(\Omega, \Gamma) \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{L^2(\Omega)}, \tag{1.2}$$

for a constant $C_1(\Omega, \Gamma)$ depending only on Ω and Γ . Here $\boldsymbol{\varepsilon}(\mathbf{v})$ denotes the strain tensor. When combined with Poincaré's inequality, also valid for all functions v in $H^1(\Omega)$ that vanish on Γ , with another constant $C_2(\Omega, \Gamma)$ depending only on Ω and Γ :

$$\|v\|_{L^2(\Omega)} \leq C_2(\Omega, \Gamma) |v|_{H^1(\Omega)}, \tag{1.3}$$

we recover the full H^1 norm on the left-hand side of (1.2):

$$\|\mathbf{v}\|_{H^1(\Omega)} \leq C_1(\Omega, \Gamma) (1 + C_2^2(\Omega, \Gamma))^{1/2} \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{L^2(\Omega)}. \tag{1.4}$$

When \mathbf{v} does not vanish on Γ , a similar inequality is valid, by adding a surface norm in the right-hand side:

$$\forall \mathbf{v} \in H^1(\Omega)^d, \quad \|\mathbf{v}\|_{H^1(\Omega)} \leq C_3(\Omega, \Gamma)(\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{L^2(\Omega)} + \|\mathbf{v}\|_{L^2(\Gamma)}). \quad (1.5)$$

As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a, b[$ with values in a functional space, say X (cf. Ref. 24). More precisely, let $\|\cdot\|_X$ denote the norm of X ; then for any number r , $1 \leq r \leq \infty$, we define

$$L^r(a, b; X) = \left\{ f \text{ measurable in }]a, b[; \int_a^b \|f(t)\|_X^r dt < \infty \right\},$$

equipped with the norm

$$\|f\|_{L^r(a, b; X)} = \left(\int_a^b \|f(t)\|_X^r dt \right)^{1/r},$$

with the usual modification if $r = \infty$. It is a Banach space if X is a Banach space, and for $r = 2$, it is a Hilbert space if X is a Hilbert space. We denote derivatives with respect to time with a prime and we define for instance

$$H^1(a, b; X) = \{f \in L^2(a, b; X); f' \in L^2(a, b; X)\}.$$

2. Coupling Poroelasticity with Elasticity

Let Ω be a bounded, connected, Lipschitz domain in \mathbb{R}^d , $d = 2, 3$, decomposed into two non-overlapping, connected, Lipschitz subdomains, Ω_1 and Ω_2 , such that Ω_1 is a proper subset of Ω , as in Fig. 1(a):

$$\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}, \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad \overline{\Omega_1} \subset \Omega.$$

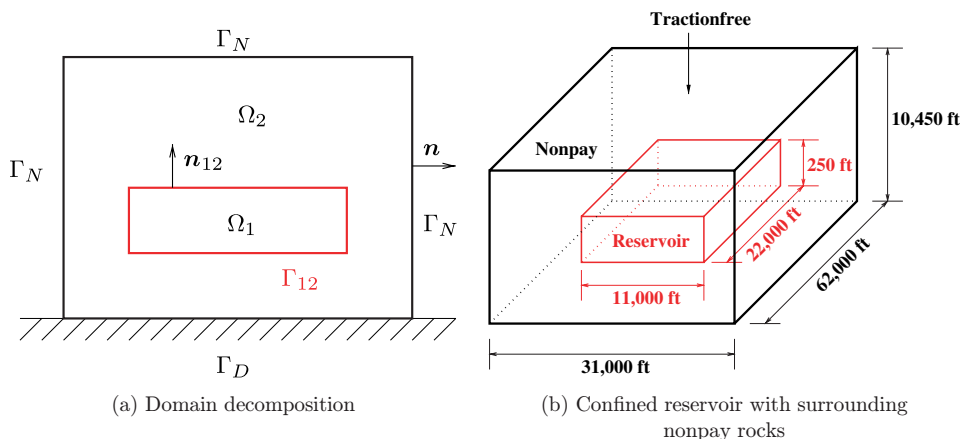


Fig. 1.

This work extends readily to more general configurations, but for simplicity, we focus on this situation. Let Γ_{12} denote the interface between Ω_1 and Ω_2 :

$$\Gamma_{12} = \overline{\Omega_1} \cap \overline{\Omega_2}$$

and let \mathbf{n}_{12} be the unit normal on Γ_{12} pointing into Ω_2 , i.e. exterior to Ω_1 . Let the boundary of Ω , $\partial\Omega$ be partitioned into two disjoint open regions, not necessarily connected,

$$\overline{\partial\Omega} = \overline{\Gamma_D} \cup \overline{\Gamma_N};$$

when $d = 3$, we assume that the boundaries of Γ_D and Γ_N are both Lipschitz-continuous. We denote by \mathbf{n}_Ω the unit outward normal vector to $\partial\Omega$. To simplify, we assume that the measure of Γ_D is positive: $|\Gamma_D| > 0$. We are interested in the situation where a poroelastic model holds in Ω_1 (the *pay-zone*) while an elastic model holds in Ω_2 (the *nonpay-zone*), see Fig. 1(b). To simplify the exposition, we present the problem in the three-dimensional case.

2.1. A model of poroelasticity

In the pay-zone Ω_1 , we use Biot's consolidation model for a linear elastic, homogeneous, isotropic, porous solid saturated with a slightly compressible fluid. The constitutive equation for the Cauchy stress tensor $\tilde{\boldsymbol{\sigma}}$ in terms of the displacement \mathbf{u} and fluid pressure p is

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\mathbf{u}) - \alpha p \mathbf{I}, \tag{2.1}$$

where \mathbf{I} is the identity tensor, $\boldsymbol{\sigma}(\mathbf{u})$ and $\boldsymbol{\varepsilon}(\mathbf{u})$ are the linear elasticity and strain tensor respectively

$$\boldsymbol{\sigma}(\mathbf{u}) = \lambda(\operatorname{div} \mathbf{u})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}), \quad \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T), \tag{2.2}$$

$\lambda > 0$ and $\mu > 0$ are the Lamé coefficients, and $\alpha > 0$ is the Biot–Willis constant, which is usually around one. The flux of the fluid \mathbf{v}_f is governed by Darcy's law in porous media

$$\mathbf{v}_f = -\frac{1}{\mu_f}\mathbf{K}(\nabla p - \rho_f\mathbf{g}), \tag{2.3}$$

where $\mu_f > 0$ and $\rho_f > 0$ are respectively the fluid viscosity and fluid density, both assumed to be constant, \mathbf{g} is the gravitational force, and \mathbf{K} is the permeability tensor, assumed to be symmetric, uniformly bounded, and uniformly positive definite, i.e. each eigenvalue λ_i of \mathbf{K} is real and there exist two constants $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ such that

$$\text{a.e. } \mathbf{x} \in \Omega_1, \quad \lambda_{\min} \leq \lambda_i(\mathbf{x}) \leq \lambda_{\max}. \tag{2.4}$$

The equation of mass conservation is

$$\frac{\partial\eta}{\partial t} = -\operatorname{div} \mathbf{v}_f + q,$$

where q is a volumetric fluid source term and η is the fluid content of the medium; η is related to the fluid pressure p and material volume $\operatorname{div} \mathbf{u}$ by

$$\eta = c_0 p + \alpha \operatorname{div} \mathbf{u}, \quad (2.5)$$

where $c_0 \geq 0$ is the constrained specific storage coefficient, that is assumed to be constant. As explained by Phillips and Wheeler in Ref. 29, $c_0 = 0$ may lead to locking, whatever the value of the Lamé coefficient λ . Although in practical situations, c_0 can vanish, we do not consider this possibility here and therefore we suppose that $c_0 > 0$. With (2.5) and (2.3), the equation of mass conservation reads

$$\frac{\partial}{\partial t}(c_0 p + \alpha \operatorname{div} \mathbf{u}) - \frac{1}{\mu_f} \operatorname{div} \mathbf{K}(\nabla p - \rho_f \mathbf{g}) = q. \quad (2.6)$$

Finally, the balance of linear momentum is derived by making a *quasi-static* assumption, namely by assuming that the material deformation is much slower than the flow rate, and hence the second-time derivative of the displacement (i.e. the acceleration) is zero. Denoting by \mathbf{f}_1 the body force in Ω_1 , this yields

$$-\operatorname{div} \tilde{\boldsymbol{\sigma}} = \mathbf{f}_1. \quad (2.7)$$

Thus, substituting the constitutive relation (2.1) into (2.7) and expanding $\tilde{\boldsymbol{\sigma}}$, we obtain

$$-(\lambda + \mu) \nabla(\operatorname{div} \mathbf{u}) - \mu \Delta \mathbf{u} + \alpha \nabla p = \mathbf{f}_1. \quad (2.8)$$

Collecting the above equations, we have the following system of equations a.e. in $\Omega_1 \times]0, T[$

$$\frac{\partial}{\partial t}(c_0 p + \alpha \operatorname{div} \mathbf{u}) - \frac{1}{\mu_f} \operatorname{div} \mathbf{K}(\nabla p - \rho_f \mathbf{g}) = q, \quad (2.9a)$$

$$-(\lambda + \mu) \nabla(\operatorname{div} \mathbf{u}) - \mu \Delta \mathbf{u} + \alpha \nabla p = \mathbf{f}_1. \quad (2.9b)$$

This system must be complemented by an initial condition

$$p(0) = p_0 \quad \text{in } \Omega_1. \quad (2.10)$$

In practice, the pressure is either measured or computed through a hydrostatic assumption and a *compatible* initial displacement is given (or otherwise it is computed satisfying (2.19)). As the boundary of Ω_1 is reduced to Γ_{12} , that is part of the boundary of Ω_2 , then the only boundary conditions on Γ_{12} are in fact transmission conditions between the two regions. They are discussed in the next subsection.

2.2. Coupling with elasticity

As stated at the beginning of this section, in the nonpay-zone Ω_2 , the governing equations are those of linear elasticity. The equations themselves are steady, but the unknowns depend on time through the transmission conditions that we shall propose

below. Therefore we have a.e. in $\Omega_2 \times]0, T[$

$$-(\lambda + \mu)\nabla(\operatorname{div} \mathbf{u}) - \mu\Delta \mathbf{u} = \mathbf{f}_2, \quad (2.11)$$

where \mathbf{f}_2 is the body force in Ω_2 . On the boundary $\partial\Omega$, we prescribe mixed boundary conditions:

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}_\Omega = \mathbf{t}_N \quad \text{on } \Gamma_N. \quad (2.12)$$

Now, we turn to the transmission conditions on the interface. First, recall the definition of jump through Γ_{12} in the direction of the normal, for any smooth enough function v

$$[v] = (v|_{\Omega_1} - v|_{\Omega_2})|_{\Gamma_{12}}. \quad (2.13)$$

Then, we prescribe the following transmission conditions:

$$\text{the medium is continuous } [\mathbf{u}] = \mathbf{0}, \quad (2.14)$$

$$\text{the normal stresses are continuous } [\boldsymbol{\sigma}(\mathbf{u})]\mathbf{n}_{12} = \alpha p \mathbf{n}_{12}, \quad (2.15)$$

$$\text{there is no flow at the interface } -\frac{1}{\mu_f} \mathbf{K}(\nabla p - \rho_f \mathbf{g}) \cdot \mathbf{n}_{12} = 0. \quad (2.16)$$

To simplify the notation, we denote by \mathbf{f} the function defined in Ω , with restriction \mathbf{f}_i to Ω_i , $i = 1, 2$. Summarizing, we propose to solve (2.9) in Ω_1 and (2.11) in Ω_2

$$\begin{aligned} \frac{\partial}{\partial t} (c_0 p + \alpha \operatorname{div} \mathbf{u}) - \frac{1}{\mu_f} \operatorname{div} \mathbf{K}(\nabla p - \rho_f \mathbf{g}) &= q \quad \text{in } \Omega_1, \\ -(\lambda + \mu)\nabla(\operatorname{div} \mathbf{u}) - \mu\Delta \mathbf{u} + \alpha \nabla p &= \mathbf{f} \quad \text{in } \Omega_1, \\ -(\lambda + \mu)\nabla(\operatorname{div} \mathbf{u}) - \mu\Delta \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega_2, \end{aligned}$$

with the initial condition (2.10)

$$p(0) = p_0 \quad \text{in } \Omega_1,$$

with the boundary conditions (2.12)

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}_\Omega = \mathbf{t}_N \quad \text{on } \Gamma_N,$$

and with the transmission conditions (2.14)–(2.16) on Γ_{12}

$$[\mathbf{u}] = \mathbf{0}, \quad [\boldsymbol{\sigma}(\mathbf{u})]\mathbf{n}_{12} = \alpha p \mathbf{n}_{12}, \quad \frac{1}{\mu_f} \mathbf{K}(\nabla p - \rho_f \mathbf{g}) \cdot \mathbf{n}_{12} = 0.$$

Concerning data regularity, we assume that \mathbf{f} belongs to $H^1(0, T; L^2(\Omega)^d)$, q belongs to $L^2(\Omega_1 \times]0, T[)$, \mathbf{t}_N is in $H^1(0, T; L^2(\Gamma_N)^d)$, and p_0 belongs to $H^1(\Omega)$. These are not the most general data assumptions, but they are a convenient simplification for the applications we have in mind. Of course, the gravitational force \mathbf{g} is smooth and is independent of time. Note that both \mathbf{f} and \mathbf{t}_N are continuous in time and therefore $\mathbf{f}(0)$ and $\mathbf{t}_N(0)$ are well-defined.

2.3. Variational formulation

Let us introduce the space

$$H_{0D}^1(\Omega) = \{v \in H^1(\Omega); v|_{\Gamma_D} = 0\}. \quad (2.17)$$

The next lemma (whose proof uses standard tools) shows that problem (2.9)–(2.12), (2.14)–(2.16) has the equivalent variational formulation: Find \mathbf{u} in $L^\infty(0, T; H_{0D}^1(\Omega)^d)$ and p in $L^\infty(0, T; L^2(\Omega_1)) \cap L^2(0, T; H^1(\Omega_1))$ solving

$$\begin{aligned} \forall \mathbf{v} \in H_{0D}^1(\Omega)^d, \quad & \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) d\mathbf{x} - \alpha \int_{\Omega_1} p \operatorname{div} \mathbf{v} d\mathbf{x} \\ & = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N(s) \cdot \mathbf{v}(s) ds, \end{aligned} \quad (2.18a)$$

$$\begin{aligned} \forall \theta \in H^1(\Omega_1), \quad & \int_{\Omega_1} \frac{\partial}{\partial t} (c_0 p + \alpha \operatorname{div} \mathbf{u}) \theta d\mathbf{x} + \frac{1}{\mu_f} \int_{\Omega_1} \mathbf{K} \nabla p \cdot \nabla \theta d\mathbf{x} \\ & = \int_{\Omega_1} q \theta d\mathbf{x} + \frac{\rho_f}{\mu_f} \int_{\Omega_1} \mathbf{K} \mathbf{g} \cdot \nabla \theta d\mathbf{x}, \end{aligned} \quad (2.18b)$$

$$p(0) = p_0 \quad \text{in } \Omega_1. \quad (2.18c)$$

Lemma 2.1. *For \mathbf{f} in $H^1(0, T; L^2(\Omega)^d)$, q in $L^2(\Omega_1 \times]0, T[)$, \mathbf{t}_N in $H^1(0, T; L^2(\Gamma_N)^d)$, and p_0 in $H^1(\Omega)$, problem (2.9)–(2.12), (2.14)–(2.16) is equivalent to (2.18).*

Note that (2.18a) permits one to write the displacement \mathbf{u} as a function of p . Indeed, consider the problem: For a given p in $L^\infty(0, T; L^2(\Omega_1))$, find $\mathbf{u}(p)$ in $L^\infty(0, T; H_{0D}^1(\Omega)^d)$ solving for a.e. t in $]0, T[$

$$\forall \mathbf{v} \in H_{0D}^1(\Omega)^d, \quad (\boldsymbol{\sigma}(\mathbf{u}(p)), \boldsymbol{\varepsilon}(\mathbf{v}))_{\Omega} = \alpha (p, \operatorname{div} \mathbf{v})_{\Omega_1} + (\mathbf{f}, \mathbf{v})_{\Omega} + (\mathbf{t}_N, \mathbf{v})_{\Gamma_N}. \quad (2.19)$$

Owing to Korn's inequality (1.4), by Lax–Milgram's lemma, this problem has a unique solution $\mathbf{u}(p)$ in $L^\infty(0, T; H_{0D}^1(\Omega)^d)$, and the mapping $p \mapsto \mathbf{u}(p)$ is a continuous affine mapping from $L^\infty(0, T; L^2(\Omega_1))$ into $L^\infty(0, T; H_{0D}^1(\Omega)^d)$: There exists a constant C such that

$$\forall p_1, p_2 \in L^2(\Omega_1), \quad \|\mathbf{u}(p_1) - \mathbf{u}(p_2)\|_{H^1(\Omega)} \leq C \|p_1 - p_2\|_{L^2(\Omega_1)}. \quad (2.20)$$

Therefore, (2.18) has the equivalent implicit formulation: Find p in $L^\infty(0, T; L^2(\Omega_1)) \cap L^2(0, T; H^1(\Omega_1))$ satisfying (2.18c) and for a.e. t in $]0, T[$

$$\begin{aligned} \forall \theta \in H^1(\Omega_1), \quad & \int_{\Omega_1} \frac{\partial}{\partial t} (c_0 p + \alpha \operatorname{div} \mathbf{u}(p)) \theta d\mathbf{x} + \frac{1}{\mu_f} \int_{\Omega_1} \mathbf{K} \nabla p \cdot \nabla \theta d\mathbf{x} \\ & = \int_{\Omega_1} q \theta d\mathbf{x} + \frac{\rho_f}{\mu_f} \int_{\Omega_1} \mathbf{K} \mathbf{g} \cdot \nabla \theta d\mathbf{x}, \end{aligned} \quad (2.21)$$

with $\mathbf{u}(p)$ defined by (2.19). Finally, it is convenient to split $\mathbf{u}(p)$ as follows:

$$\mathbf{u}(p) = \bar{\mathbf{u}} + \tilde{\mathbf{u}}(p), \quad (2.22)$$

where $\bar{\mathbf{u}} \in L^\infty(0, T; H_{0D}^1(\Omega)^d)$ is the unique solution of

$$\text{a.e. } t \in]0, T[, \quad \forall \mathbf{v} \in H_{0D}^1(\Omega)^d, \quad (\boldsymbol{\sigma}(\bar{\mathbf{u}}), \boldsymbol{\varepsilon}(\mathbf{v}))_\Omega = (\mathbf{f}, \mathbf{v})_\Omega + (\mathbf{t}_N, \mathbf{v})_{\Gamma_N}, \quad (2.23)$$

and $\tilde{\mathbf{u}}(p) \in L^\infty(0, T; H_{0D}^1(\Omega)^d)$ is the unique solution of

$$\text{a.e. } t \in]0, T[, \quad \forall \mathbf{v} \in H_{0D}^1(\Omega)^d, \quad (\boldsymbol{\sigma}(\tilde{\mathbf{u}}(p)), \boldsymbol{\varepsilon}(\mathbf{v}))_\Omega = \alpha(p, \operatorname{div} \mathbf{v})_{\Omega_1}. \quad (2.24)$$

2.4. Construction of a solution

The purely poroelastic problem (2.9)–(2.10) has been analyzed by several authors, see the very thorough work of Showalter in Ref. 32. Here we propose to construct a solution of (2.18) by a semi-discrete Galerkin method. Let $(\theta_n)_{n \geq 1}$ be a smooth basis of $H^1(\Omega)$ and let Q_k be the space spanned by $(\theta_i)_{i=1}^k$. Then, our semi-discrete problem reads: Find

$$p_k(t) = \sum_{i=1}^k \pi_i(t) \theta_i \in H^1(0, T; Q_k),$$

such that

$$\begin{aligned} & \left(\frac{\partial}{\partial t} (c_0 p_k + \alpha \operatorname{div} \tilde{\mathbf{u}}_k), \theta_i \right)_{\Omega_1} + \frac{1}{\mu_f} (\mathbf{K} \nabla p_k, \nabla \theta_i)_{\Omega_1} \\ & = -\alpha \left(\frac{\partial}{\partial t} \operatorname{div} \bar{\mathbf{u}}, \theta_i \right)_{\Omega_1} + (q, \theta_i)_{\Omega_1} + \frac{\rho_f}{\mu_f} (\mathbf{K} \mathbf{g}, \nabla \theta_i)_{\Omega_1}, \quad 1 \leq i \leq k, \end{aligned} \quad (2.25a)$$

$$p_k(0) = p_{0k}, \quad (2.25b)$$

where $\bar{\mathbf{u}}$ is defined by (2.23), $\tilde{\mathbf{u}}_k = \tilde{\mathbf{u}}(p_k)$, i.e. $\mathbf{u}(p_k) = \bar{\mathbf{u}} + \tilde{\mathbf{u}}_k$, and $p_{0k} \in Q_k$ satisfies

$$\lim_{k \rightarrow \infty} \|p_{0k} - p_0\|_{H^1(\Omega)} = 0,$$

for example p_{0k} can be the projection of $p(0)$ on Q_k for the H^1 norm.

Proposition 2.1. *Let $p_0 \in H^1(\Omega)$, $\mathbf{f} \in H^1(0, T; L^2(\Omega)^d)$, $q \in L^2(\Omega_1 \times]0, T[)$ and $\mathbf{t}_N \in H^1(0, T; L^2(\Gamma_N)^d)$. The semi-discrete problem (2.25) has exactly one solution on $[0, T]$.*

Proof. Let us write (2.25) in matrix form; we define the following vectors and matrices:

$$\begin{aligned} (\mathbf{P})_i &= \pi_i, \quad 1 \leq i \leq k, \\ (\mathbf{C})_{i,j} &= c_0(\theta_j, \theta_i)_{\Omega_1}, \\ (\mathbf{A})_{i,j} &= \alpha(\operatorname{div} \tilde{\mathbf{u}}(\theta_j), \theta_i)_{\Omega_1}, \quad 1 \leq i, \quad j \leq k, \\ (\mathbf{D})_{i,j} &= \frac{1}{\mu_f} (\mathbf{K} \nabla \theta_j, \nabla \theta_i)_{\Omega_1}, \quad 1 \leq i, \quad j \leq k. \end{aligned}$$

With this notation, writing the time derivative with a prime, (2.25) is a square system of k linear ODEs of order one: Find $\mathbf{P} \in H^1(0, T)^k$ satisfying

$$\forall t \in [0, T], (\mathbf{C} + \mathbf{A})\mathbf{P}' + \mathbf{D}\mathbf{P} = \mathbf{H}, \quad \mathbf{P}(0) = \mathbf{P}_0, \quad (2.26)$$

where \mathbf{H} is the vector of the right-hand side of (2.25a). Note that \mathbf{C} is a square, symmetric, positive-definite matrix and \mathbf{D} is square, symmetric, and semi-positive definite. Regarding \mathbf{A} , by choosing $p = \theta_i$ and $\mathbf{v} = \tilde{\mathbf{u}}(\theta_j)$ in (2.24), we observe that

$$(\mathbf{A})_{i,j} = (\boldsymbol{\sigma}(\tilde{\mathbf{u}}(\theta_i)), \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}(\theta_j)))_{\Omega} = 2\mu(\boldsymbol{\varepsilon}(\tilde{\mathbf{u}}(\theta_i)), \boldsymbol{\varepsilon}(\tilde{\mathbf{u}}(\theta_j)))_{\Omega} + \lambda(\operatorname{div} \mathbf{u}(\theta_i), \operatorname{div} \mathbf{u}(\theta_j))_{\Omega}.$$

In view of Korn's inequality (1.4), \mathbf{A} is square, symmetric, positive definite. Hence, considering that the matrix multiplying \mathbf{P}' is square, symmetric, positive-definite, (2.26) has exactly one solution on $[0, T]$. \square

The next lemma derives *a priori* estimates for the solutions p_k and $\mathbf{u}(p_k)$.

Lemma 2.2. *Let $p_0 \in H^1(\Omega)$, $\mathbf{f} \in H^1(0, T; L^2(\Omega)^d)$, $q \in L^2(\Omega_1 \times]0, T[)$, and $\mathbf{t}_N \in H^1(0, T; L^2(\Gamma_N)^d)$. The solution p_k and $\mathbf{u}(p_k)$ of the semi-discrete problem (2.25) satisfy the following uniform bounds*

$$\begin{aligned} & \lambda \|\operatorname{div} \mathbf{u}(p_k)\|_{L^\infty(0, T; L^2(\Omega))}^2 + \mu \|\boldsymbol{\varepsilon}(\mathbf{u}(p_k))\|_{L^\infty(0, T; L^2(\Omega)^{d \times d})}^2 \\ & + c_0 \|p_k\|_{L^\infty(0, T; L^2(\Omega_1))}^2 + \frac{1}{\mu_f} \|\mathbf{K}^{1/2} \nabla p_k\|_{L^2(0, T; L^2(\Omega_1)^d)}^2 \leq C, \\ & \lambda \|\operatorname{div} \mathbf{u}(p'_k)\|_{L^2(\Omega \times]0, T])}^2 + \mu \|\boldsymbol{\varepsilon}(\mathbf{u}(p'_k))\|_{L^2(0, T; L^2(\Omega)^{d \times d})}^2 \\ & + c_0 \|p'_k\|_{L^2(\Omega_1 \times]0, T])}^2 + \frac{1}{\mu_f} \|\mathbf{K}^{1/2} \nabla p_k\|_{L^\infty(0, T; L^2(\Omega_1)^d)}^2 \leq C, \end{aligned} \quad (2.27)$$

where C depends on T , $\|\mathbf{f}\|_{H^1(0, T; L^2(\Omega)^d)}$, $\|q\|_{L^2(\Omega_1 \times]0, T])}$, $\|\mathbf{t}_N\|_{H^1(0, T; L^2(\Gamma_N)^d)}$, $\|p_0\|_{H^1(\Omega_1)}$, but not k .

Proof. Let us sketch the proof. Following Ref. 28, we test (2.19) with $\mathbf{u}'(p_k) = \mathbf{u}(p'_k)$, (2.25a) with p_k , add the two equations, and integrate with respect to time on $(0, t)$ for arbitrary $t \in [0, T]$. Note that, as a result of this choice, the scalar products of p with $\operatorname{div} \mathbf{u}'$ cancel. This yields

$$\begin{aligned} & \frac{\lambda}{2} \|\operatorname{div} \mathbf{u}(p_k(t))\|_{L^2(\Omega)}^2 + \mu \|\boldsymbol{\varepsilon}(\mathbf{u}(p_k(t)))\|_{L^2(\Omega)}^2 \\ & + \frac{c_0}{2} \|p_k(t)\|_{L^2(\Omega_1)}^2 + \frac{1}{\mu_f} \|\mathbf{K}^{1/2} \nabla p_k\|_{L^2(0, t; L^2(\Omega_1)^d)}^2 \\ & = \frac{\lambda}{2} \|\operatorname{div} \mathbf{u}(p_{0k})\|_{L^2(\Omega)}^2 + \mu \|\boldsymbol{\varepsilon}(\mathbf{u}(p_{0k}))\|_{L^2(\Omega)}^2 \\ & + \frac{c_0}{2} \|p_{0k}\|_{L^2(\Omega_1)}^2 + \int_0^t (\mathbf{f}, \mathbf{u}'(p_k))_{\Omega} dt + \int_0^t (q, p_k)_{\Omega_1} dt \\ & + \frac{\rho_f}{\mu_f} \int_0^t (\mathbf{K} \mathbf{g}, \nabla p_k)_{\Omega_1} dt + \int_0^t (\mathbf{t}_N, \mathbf{u}'(p_k))_{\Gamma_N} dt. \end{aligned}$$

As none of the terms on the left-hand side can absorb $\mathbf{u}'(p)$, we eliminate it by integrating by parts the two integrals in the above right-hand side involving $\mathbf{u}'(p)$, and we obtain

$$\begin{aligned}
& \frac{\lambda}{2} \|\operatorname{div} \mathbf{u}(p_k(t))\|_{L^2(\Omega)}^2 + \mu \|\varepsilon(\mathbf{u}(p_k(t)))\|_{L^2(\Omega)}^2 \\
& + \frac{c_0}{2} \|p_k(t)\|_{L^2(\Omega_1)}^2 + \frac{1}{\mu_f} \|\mathbf{K}^{1/2} \nabla p_k\|_{L^2(0,t;L^2(\Omega_1)^d)}^2 \\
& = \frac{\lambda}{2} \|\operatorname{div} \mathbf{u}(p_{0k})\|_{L^2(\Omega)}^2 + \mu \|\varepsilon(\mathbf{u}(p_{0k}))\|_{L^2(\Omega)}^2 \\
& + \frac{c_0}{2} \|p_{0k}\|_{L^2(\Omega_1)}^2 + (\mathbf{f}(t), \mathbf{u}(p_k(t)))_{\Omega} - (\mathbf{f}(0), \mathbf{u}(p_{0k}))_{\Omega} \\
& - \int_0^t (\mathbf{f}', \mathbf{u}(p_k))_{\Omega} dt + (\mathbf{t}_N(t), \mathbf{u}(p_k(t)))_{\Gamma_N} - (\mathbf{t}_N(0), \mathbf{u}(p_{0k}))_{\Gamma_N} \\
& - \int_0^t (\mathbf{t}'_N, \mathbf{u}(p_k))_{\Gamma_N} dt + \int_0^t (q, p_k)_{\Omega_1} dt + \frac{\rho_f}{\mu_f} \int_0^t (\mathbf{K} \mathbf{g}, \nabla p_k)_{\Omega_1} dt. \quad (2.28)
\end{aligned}$$

The first part of (2.27) follows readily from (2.28) by applying Korn's inequality, Young's inequality, the regularity assumptions on \mathbf{f} , q , \mathbf{t}_N , the convergence hypothesis on p_{0k} , and Gronwall's lemma. The second part is derived similarly by testing (2.25) with p'_k , differentiating in time (2.19) written for $p = p_k$, testing the resulting equation with $\mathbf{u}'(p_k)$, adding the two equations and integrating with respect to time. \square

Owing to the uniform bounds in (2.27), there exists a subsequence of k (still denoted by k), a function \bar{p} in $H^1(0, T; L^2(\Omega_1)) \cap L^\infty(0, T; H^1(\Omega_1))$, and a function \mathbf{w} in $H^1(0, T; H_{0D}^1(\Omega)^d)$ such that

$$\begin{aligned}
\lim_{k \rightarrow \infty} p_k &= \bar{p} \quad \text{weakly* in } L^\infty(0, T; H^1(\Omega_1)), \\
\lim_{k \rightarrow \infty} p_k &= \bar{p} \quad \text{weakly in } H^1(0, T; L^2(\Omega_1)), \\
\lim_{k \rightarrow \infty} \mathbf{u}(p_k) &= \mathbf{w} \quad \text{weakly in } H^1(0, T; H_{0D}^1(\Omega)^d).
\end{aligned}$$

By passing to the limit in (2.19) with $p = p_k$, we readily derive that $\mathbf{w} = \mathbf{u}(\bar{p})$. Then passing to the limit in (2.25), we also easily derive that \bar{p} solves (2.21) and the continuity in time of \bar{p} yields the initial condition (2.18c). This proves the main result of this subsection.

Theorem 2.1. *Let $p_0 \in H^1(\Omega)$, $\mathbf{f} \in H^1(0, T; L^2(\Omega)^d)$, $q \in L^2(\Omega_1 \times]0, T[)$ and $\mathbf{t}_N \in H^1(0, T; L^2(\Gamma_N)^d)$. Then problem (2.18) is well-posed and its solution p belongs to $L^\infty(0, T; H^1(\Omega_1)) \cap H^1(0, T; L^2(\Omega_1))$.*

2.5. Variational formulation with DG jumps

In view of decomposing Ω into Ω_1 and Ω_2 , let us relax the transmission conditions (2.14) and (2.15) by introducing discontinuous Galerkin jumps on the interface Γ_{12} .

This technique has been used for domain decomposition by several authors; see for instance Ref. 18 or 16.

Beforehand, we observe that (2.8) and (2.11) have the same form when α is extended by zero in Ω_2 . Thus we set

$$\tilde{\alpha} = \alpha \quad \text{in } \Omega_1, \quad \tilde{\alpha} = 0 \quad \text{in } \Omega_2,$$

and the transmission condition (2.15) becomes

$$[\boldsymbol{\sigma}(\mathbf{u}) - \tilde{\alpha} p \mathbf{I}] \mathbf{n}_{12} = \mathbf{0}. \quad (2.29)$$

Now, to illustrate the idea, in addition to the jump of a function v through Γ_{12} defined in (2.13), recall the standard notation for the average of v on Γ_{12} :

$$\{v\} = \frac{1}{2}(v|_{\Omega_1} + v|_{\Omega_2})|_{\Gamma_{12}}. \quad (2.30)$$

Let \mathbf{v} be a smooth function in Ω_1 and Ω_2 , but possibly not globally smooth in Ω , and let \mathbf{v}_i denote its restriction to Ω_i . The smoothness of \mathbf{v}_i guarantees that $\boldsymbol{\sigma}(\mathbf{v}_i) \cdot \mathbf{n}_{12}$ has a trace on Γ_{12} ; for instance $\mathbf{v}_i \in H^s(\Omega_i)^d$ with $s > 3/2$. Let (\mathbf{u}, p) be the solution of (2.18), set $\mathbf{u}_i := \mathbf{u}|_{\Omega_i}$, take the scalar product of (2.8) with \mathbf{v}_1 in Ω_1 , the scalar product of (2.11) with \mathbf{v}_2 in Ω_2 , apply Green's formula in each domain, and add the two equations. Assuming that all traces involved are sufficiently smooth, this gives the following term on the interface Γ_{12}

$$- \int_{\Gamma_{12}} [(\boldsymbol{\sigma}(\mathbf{u}) - \tilde{\alpha} p \mathbf{I}) \mathbf{n}_{12} \cdot \mathbf{v}] ds.$$

Then using (2.29) and the standard identity $[ab] = [a]\{b\} + \{a\}[b]$, the interface term becomes

$$- \int_{\Gamma_{12}} \{\boldsymbol{\sigma}(\mathbf{u}) - \tilde{\alpha} p \mathbf{I}\} \mathbf{n}_{12} \cdot [\mathbf{v}] ds$$

and leads to the following system

$$\begin{aligned} & \sum_{i=1}^2 \int_{\Omega_i} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}_i) d\mathbf{x} - \alpha \int_{\Omega_1} p \operatorname{div} \mathbf{v}_1 d\mathbf{x} \\ & - \int_{\Gamma_{12}} \{\boldsymbol{\sigma}(\mathbf{u}) - \tilde{\alpha} p \mathbf{I}\} \mathbf{n}_{12} \cdot [\mathbf{v}] ds - \int_{\Gamma_D} \boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}_\Omega \cdot \mathbf{v}_2 ds \\ & = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N \cdot \mathbf{v}_2 ds. \end{aligned} \quad (2.31)$$

It is easy to see that the solution (\mathbf{u}, p) of (2.9)–(2.12), (2.14)–(2.16) solves (2.31), (2.18b), (2.18c) but not conversely because (2.31) does not imply Dirichlet boundary and transmission conditions. The Dirichlet transmission condition can be imposed

weakly, see for example Ref. 3, by inserting appropriate consistent terms involving jumps. Since by (2.14), the jump of \mathbf{u} through Γ_{12} vanishes, we can add the term

$$\int_{\Gamma_{12}} \{\boldsymbol{\sigma}(\mathbf{v})\} \mathbf{n}_{12} \cdot [\mathbf{u}] ds.$$

Note that this brings antisymmetry to (2.31), but when condensing the unknowns on the interface, a symmetric term (with a minus sign) on Γ_{12} leads to a less suitable matrix. It is also useful to add stabilizing jumps, but for this we need to triangulate Γ_{12} . To guarantee unconditional stability of our scheme (see Proposition 3.1), we choose to define two triangulations of Γ_{12} , one for each subdomain. For $i = 1, 2$ let $\Gamma_{h,i}$ be a conforming triangulation of Γ_{12} (i.e. with no hanging nodes) consisting of triangles γ_i with diameter $h_{\gamma_i} \leq h_i$, and let us add the consistent stabilizing jumps (that for convenience are multiplied by $\lambda + 2\mu$)

$$(\lambda + 2\mu) \sum_{\gamma_i \in \Gamma_{h_i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} [\mathbf{u}] \cdot [\mathbf{v}] ds.$$

As observed in Ref. 29, these are not sufficient because \mathbf{u}' is used as a test function for establishing stability, see for instance the proof of Lemma 2.2. Therefore similar consistent stabilizing jumps are added for the time derivative of the displacement:

$$(\lambda + 2\mu) \sum_{\gamma_i \in \Gamma_{h_i}} \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} [\mathbf{u}'] \cdot [\mathbf{v}] ds.$$

Here, $\sigma_{\gamma_i} > 0$ and $\bar{\sigma}_{\gamma_i} > 0$ are well-chosen parameters bounded above and bounded away from zero: There exist constants m_i and M_i , independent of h_i , such that

$$\forall \gamma_i \in \Gamma_{h,i}, \quad 0 < m_i \leq \sigma_{\gamma_i}, \quad \bar{\sigma}_{\gamma_i} \leq M_i. \quad (2.32)$$

The essential Dirichlet boundary condition on Γ_D is analogously relaxed. We construct a conforming triangulation $\Gamma_{h,D}$ of Γ_D made of triangles γ with diameter $h_\gamma \leq h_2$ and add the consistent trace terms

$$-\int_{\Gamma_D} \boldsymbol{\sigma}(\mathbf{v}_2) \mathbf{n}_\Omega \cdot \mathbf{u}_2 ds, \quad (\lambda + 2\mu) \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma \mathbf{u}_2 \cdot \mathbf{v}_2 ds,$$

where the parameters $\sigma_\gamma > 0$ are also suitably chosen, and are bounded above and away from zero: There exist constants m_D and M_D , independent of h_2 , such that

$$\forall \gamma \in \Gamma_{h,D}, \quad 0 < m_D \leq \sigma_\gamma \leq M_D. \quad (2.33)$$

This leads to the following formulation with jumps, in addition to (2.18b) and (2.18c) that hold in Ω_1 : Find \mathbf{u} , with $\mathbf{u}_i := \mathbf{u}|_{\Omega_i}$, and p such that for all \mathbf{v} with

$\mathbf{v}_i \in H^s(\Omega_i)^d$ for some $s > 3/2$,

$$\begin{aligned}
 & \sum_{i=1}^2 \left(\int_{\Omega_i} \boldsymbol{\sigma}(\mathbf{u}_i) : \boldsymbol{\varepsilon}(\mathbf{v}_i) d\mathbf{x} + (\lambda + 2\mu) \right. \\
 & \quad \cdot \sum_{\gamma_i \in \Gamma_{h,i}} \left(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} [\mathbf{u}] \cdot [\mathbf{v}] ds + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} [\mathbf{u}'] \cdot [\mathbf{v}] ds \right) \Bigg) - \alpha \int_{\Omega_1} p \operatorname{div} \mathbf{v}_1 d\mathbf{x} \\
 & \quad - \int_{\Gamma_{12}} (\{\boldsymbol{\sigma}(\mathbf{u}) - \tilde{\alpha} p \mathbf{I}\} \mathbf{n}_{12} \cdot [\mathbf{v}] - \{\boldsymbol{\sigma}(\mathbf{v})\} \mathbf{n}_{12} \cdot [\mathbf{u}]) ds \\
 & \quad - \int_{\Gamma_D} (\boldsymbol{\sigma}(\mathbf{u}_2) \mathbf{n}_\Omega \cdot \mathbf{v}_2 + \boldsymbol{\sigma}(\mathbf{v}_2) \mathbf{n}_\Omega \cdot \mathbf{u}_2) ds + (\lambda + 2\mu) \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma \mathbf{u}_2 \cdot \mathbf{v}_2 ds \\
 & = \int_\Omega \mathbf{f} \cdot \mathbf{v} d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N \cdot \mathbf{v}_2 ds. \tag{2.34}
 \end{aligned}$$

Note that, on the one hand, the trace term on Γ_D is symmetric, on the other hand, the additional trace term involving \mathbf{u}'_2 is missing. Owing to the symmetry, this additional term is not necessary. To simplify, all surface terms in (2.34) are written as integrals, but some of them are dualities. As Γ_{12} is a closed surface, all terms on Γ_{12} are meaningful when \mathbf{u} is the solution of (2.18). On the other hand, Γ_D is not necessarily closed and in this case the first term on Γ_D is not defined without assuming that the solution \mathbf{u}_2 is a little smoother. Then, assuming that the solution \mathbf{u} of (2.18) is sufficiently smooth in a neighborhood of Γ_D , we easily prove that (\mathbf{u}, p) satisfies (2.34).

Lemma 2.3. *Let \mathbf{f} , q , \mathbf{t}_N and p_0 be given in $H^1(0, T; L^2(\Omega)^d)$, $L^2(\Omega_1 \times]0, T[)$, $H^1(0, T; L^2(\Gamma_N)^d)$ and $H^1(\Omega_1)$ respectively. If the solution \mathbf{u} of problem (2.18) is sufficiently smooth in a neighborhood of Γ_D , then the pair (\mathbf{u}, p) also solves (2.34), (2.18b) and (2.18c).*

2.6. Variational formulation with DG jumps and mortars

Problem (2.34) relaxes the continuity of \mathbf{u} through the interface Γ_{12} , and if we discretize it, we can use different triangulations and different discretizations in each subdomain. But it does not entirely decouple the computation in each subdomain, because the interface terms involve values coming from both sides. There are several decoupling algorithms; here we propose a mortar technique introduced in Ref. 18 and applied for linear elasticity in Ref. 16. Loosely speaking, it consists in treating the interface as a boundary:

- (1) In Ω_1 , the unknown values of \mathbf{u}_2 on Γ_{12} coming from Ω_2 are treated as a Dirichlet data and replaced by a multiplier;
- (2) In Ω_1 , the average of the total stress on Γ_{12} is replaced by its trace;

- (3) By interchanging Ω_1 and Ω_2 , the same strategy is used in Ω_2 , with the same multiplier;
- (4) The system is closed by prescribing an appropriate equation relating the multiplier and the total stress jump on Γ_{12} .

In the formulation with DG jumps and mortars, (2.34) is replaced by the system of subdomain equations: Find $\mathbf{u}_1, \mathbf{u}_2, p$, and $\boldsymbol{\lambda}$ such that for all \mathbf{v} with $\mathbf{v}_i \in H^s(\Omega_i)^d$ for some $s > 3/2$,

$$\begin{aligned}
 & \sum_{i=1}^2 \left[\int_{\Omega_i} \boldsymbol{\sigma}(\mathbf{u}_i) : \boldsymbol{\varepsilon}(\mathbf{v}_i) d\mathbf{x} - \int_{\Gamma_{12}} (\boldsymbol{\sigma}(\mathbf{u}_i) \mathbf{n}_i \cdot \mathbf{v}_i - \boldsymbol{\sigma}(\mathbf{v}_i) \mathbf{n}_i \cdot \mathbf{u}_i) ds \right. \\
 & \quad \left. + (\lambda + 2\mu) \sum_{\gamma_i \in \Gamma_{h,i}} \left(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} \mathbf{u}_i \cdot \mathbf{v}_i ds + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} \mathbf{u}'_i \cdot \mathbf{v}_i ds \right) \right] \\
 & \quad - \alpha \int_{\Omega_1} p \operatorname{div} \mathbf{v}_1 d\mathbf{x} + \alpha \int_{\Gamma_{12}} p \mathbf{n}_{12} \cdot \mathbf{v}_1 ds \\
 & \quad - \int_{\Gamma_D} (\boldsymbol{\sigma}(\mathbf{u}_2) \mathbf{n}_\Omega \cdot \mathbf{v}_2 + \boldsymbol{\sigma}(\mathbf{v}_2) \mathbf{n}_\Omega \cdot \mathbf{u}_2) ds + (\lambda + 2\mu) \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma \mathbf{u}_2 \cdot \mathbf{v}_2 ds \\
 & = \int_\Omega \mathbf{f} \cdot \mathbf{v} d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N \cdot \mathbf{v}_2 ds + \sum_{i=1}^2 \left[\int_{\Gamma_{12}} \boldsymbol{\sigma}(\mathbf{v}_i) \mathbf{n}_i \cdot \boldsymbol{\lambda} ds + (\lambda + 2\mu) \right. \\
 & \quad \left. \times \sum_{\gamma_i \in \Gamma_{h,i}} \left(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} \boldsymbol{\lambda} \cdot \mathbf{v}_i ds + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} \boldsymbol{\lambda}' \cdot \mathbf{v}_i ds \right) \right], \tag{2.35}
 \end{aligned}$$

and the interface condition for all $\boldsymbol{\mu} \in H^{1/2}(\Gamma_{12})^d$,

$$\begin{aligned}
 & \int_{\Gamma_{12}} ([\boldsymbol{\sigma}(\mathbf{u})] - \alpha p \mathbf{I}) \mathbf{n}_{12} \cdot \boldsymbol{\mu} ds - (\lambda + 2\mu) \\
 & \quad \cdot \sum_{i=1}^2 \sum_{\gamma_i \in \Gamma_{h,i}} \left(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} (\mathbf{u}_i - \boldsymbol{\lambda}) \cdot \boldsymbol{\mu} ds + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} (\mathbf{u}'_i - \boldsymbol{\lambda}') \cdot \boldsymbol{\mu} ds \right) = 0. \tag{2.36}
 \end{aligned}$$

Here again, if the solution \mathbf{u} of (2.18) is sufficiently smooth near Γ_D , then $\mathbf{u}_i = \mathbf{u}|_{\Omega_i}$, $i = 1, 2, p$, and $\boldsymbol{\lambda} = \mathbf{u}|_{\Gamma_{12}}$ satisfy (2.35) and (2.36).

Lemma 2.4. *If the assumptions of Lemma 2.3 hold, then $\mathbf{u}_i = \mathbf{u}|_{\Omega_i}$, $i = 1, 2, p$, and $\boldsymbol{\lambda} = \mathbf{u}|_{\Gamma_{12}}$ solve (2.35), (2.36), (2.18b) and (2.18c).*

2.7. Time-stepping algorithm

Even with domain decomposition, computing simultaneously \mathbf{u}_i, p and $\boldsymbol{\lambda}$ is not easy to implement and can be computationally expensive. However, since the problem is time-dependent and a time-stepping scheme must be used, we can at least dissociate the computation of \mathbf{u}_i from that of p by time-lagging either \mathbf{u}_i or p . Considering that

the displacement varies slowly, we choose to time-lag \mathbf{u}_i . More precisely, we divide the interval $]0, T[$ into N equal time steps and set

$$k = \frac{T}{N}, \quad t^n = nk, \quad \mathbf{f}^n = \mathbf{f}(t^n), \quad \mathbf{t}_N^n = \mathbf{t}_N(t^n), \quad q^n = q(t^n), \quad 0 \leq n \leq N.$$

The starting procedure is

- (1) Set $p^0 = p_0$, $\mathbf{u}^0 = \mathbf{u}_0$, and $\boldsymbol{\lambda}^0 = \mathbf{u}_0|_{\Gamma_{12}}$; here we assume that $\mathbf{u}_0 = \mathbf{u}(p_0)$, hence it satisfies (2.19).
- (2) With p^0 , \mathbf{u}_i^0 , and $\boldsymbol{\lambda}^0$, compute \mathbf{u}_i^1 , and $\boldsymbol{\lambda}^1$ by solving (2.35) and (2.36).
- (3) With p^0 , \mathbf{u}_i^0 , $\boldsymbol{\lambda}^0$, \mathbf{u}_i^1 , and $\boldsymbol{\lambda}^1$, compute p^1 by solving

$$\begin{aligned} & \frac{c_0}{k}(p^1 - p^0, \theta)_{\Omega_1} + \frac{1}{\mu_f}(\mathbf{K}\nabla p^1, \nabla \theta)_{\Omega_1} \\ &= -\frac{\alpha}{k}(\operatorname{div}(\mathbf{u}_1^1 - \mathbf{u}_1^0), \theta)_{\Omega_1} + \frac{\rho_f}{\mu_f}(\mathbf{K}\mathbf{g}, \nabla \theta)_{\Omega_1} + (q^1, \theta)_{\Omega_1}. \end{aligned} \quad (2.37)$$

Then for $n \geq 1$, knowing \mathbf{u}_i^{n-1} , \mathbf{u}_i^n , p^n and $\boldsymbol{\lambda}^n$, solve the reaction-diffusion equation for p^{n+1}

$$\begin{aligned} & \frac{c_0}{k}(p^{n+1} - p^n, \theta)_{\Omega_1} + \frac{1}{\mu_f}(\mathbf{K}\nabla p^{n+1}, \nabla \theta)_{\Omega_1} \\ &= -\frac{\alpha}{k}(\operatorname{div}(\mathbf{u}_1^n - \mathbf{u}_1^{n-1}), \theta)_{\Omega_1} + \frac{\rho_f}{\mu_f}(\mathbf{K}\mathbf{g}, \nabla \theta)_{\Omega_1} + (q^{n+1}, \theta)_{\Omega_1}. \end{aligned}$$

Once p^{n+1} is known, \mathbf{u}_i^{n+1} and $\boldsymbol{\lambda}^{n+1}$ are computed by solving

$$\begin{aligned} & \sum_{i=1}^2 \left[\int_{\Omega_i} \boldsymbol{\sigma}(\mathbf{u}_i^{n+1}) : \boldsymbol{\varepsilon}(\mathbf{v}_i) d\mathbf{x} - \int_{\Gamma_{12}} (\boldsymbol{\sigma}(\mathbf{u}_i^{n+1}) \mathbf{n}_i \cdot \mathbf{v}_i - \boldsymbol{\sigma}(\mathbf{v}_i) \mathbf{n}_i \cdot \mathbf{u}_i^{n+1}) ds \right. \\ & \quad \left. + (\lambda + 2\mu) \sum_{\gamma_i \in \Gamma_{h,i}} \int_{\gamma_i} \left(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \frac{1}{k} \right) \mathbf{u}_i^{n+1} \cdot \mathbf{v}_i ds \right] \\ & \quad - \int_{\Gamma_D} (\boldsymbol{\sigma}(\mathbf{u}_2^{n+1}) \mathbf{n}_\Omega \cdot \mathbf{v}_2 + \boldsymbol{\sigma}(\mathbf{v}_2) \mathbf{n}_\Omega \cdot \mathbf{u}_2^{n+1}) ds \\ & \quad + (\lambda + 2\mu) \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma \mathbf{u}_2^{n+1} \cdot \mathbf{v}_2 ds \\ &= \int_\Omega \mathbf{f}^{n+1} \cdot \mathbf{v} d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N^{n+1} \cdot \mathbf{v}_2 ds + \alpha \int_{\Omega_1} p^{n+1} \operatorname{div} \mathbf{v}_1 d\mathbf{x} \\ & \quad - \alpha \int_{\Gamma_{12}} p^{n+1} \mathbf{n}_{12} \cdot \mathbf{v}_1 ds + \sum_{i=1}^2 \left[\int_{\Gamma_{12}} \boldsymbol{\sigma}(\mathbf{v}_i) \mathbf{n}_i \cdot \boldsymbol{\lambda}^{n+1} ds \right. \\ & \quad \left. + (\lambda + 2\mu) \sum_{\gamma_i \in \Gamma_{h,i}} \left(\int_{\gamma_i} \left(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \frac{1}{k} \right) \boldsymbol{\lambda}^{n+1} \cdot \mathbf{v}_i ds + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} \frac{1}{k} (\mathbf{u}_i^n - \boldsymbol{\lambda}^n) \cdot \mathbf{v}_i ds \right) \right], \end{aligned}$$

$$\begin{aligned}
& \int_{\Gamma_{12}} ([\boldsymbol{\sigma}(\mathbf{u}^{n+1})] - \alpha p^{n+1} \mathbf{I}) \mathbf{n}_{12} \cdot \boldsymbol{\mu} \, ds \\
& - (\lambda + 2\mu) \sum_{i=1}^2 \sum_{\gamma_i \in \Gamma_{h,i}} \left[\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} (\mathbf{u}_i^{n+1} - \boldsymbol{\lambda}^{n+1}) \cdot \boldsymbol{\mu} \, ds \right. \\
& \left. + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} \frac{1}{k} (\mathbf{u}_i^{n+1} - \mathbf{u}_i^n - \boldsymbol{\lambda}^{n+1} + \boldsymbol{\lambda}^n) \cdot \boldsymbol{\mu} \, ds \right] = 0.
\end{aligned}$$

These equations can be expressed more compactly by introducing the following bilinear and linear forms, for $i = 1, 2$:

$$\begin{aligned}
B_i(\mathbf{u}, \mathbf{v}) &= \int_{\Omega_i} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\mathbf{x} - \int_{\Gamma_{12}} (\boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}_i \cdot \mathbf{v} - \boldsymbol{\sigma}(\mathbf{v}) \mathbf{n}_i \cdot \mathbf{u}) \, ds, \\
L_i(\boldsymbol{\lambda}, \mathbf{v}) &= \int_{\Gamma_{12}} \boldsymbol{\sigma}(\mathbf{v}) \mathbf{n}_i \cdot \boldsymbol{\lambda} \, ds, \\
J_i(\mathbf{w}, \mathbf{v}) &= (\lambda + 2\mu) \sum_{\gamma_i \in \Gamma_{h,i}} \int_{\gamma_i} \left(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \frac{1}{k} \right) \mathbf{w} \cdot \mathbf{v} \, ds, \\
\bar{J}_i(\mathbf{w}, \mathbf{v}) &= (\lambda + 2\mu) \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \int_{\gamma_i} \frac{1}{k} \mathbf{w} \cdot \mathbf{v} \, ds, \\
b_D(\mathbf{u}, \mathbf{v}) &= - \int_{\Gamma_D} (\boldsymbol{\sigma}(\mathbf{u}) \mathbf{n}_\Omega \cdot \mathbf{v} + \boldsymbol{\sigma}(\mathbf{v}) \mathbf{n}_\Omega \cdot \mathbf{u}) \, ds \\
&\quad + (\lambda + 2\mu) \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_\gamma}{h_\gamma} \int_\gamma \mathbf{u} \cdot \mathbf{v} \, ds, \\
l(p, \mathbf{v}) &= \alpha \int_{\Omega_1} p \operatorname{div} \mathbf{v} \, d\mathbf{x} - \alpha \int_{\Gamma_{12}} p \mathbf{n}_{12} \cdot \mathbf{v} \, ds, \\
l_{f,N}^n(\mathbf{v}) &= \int_\Omega \mathbf{f}^n \cdot \mathbf{v} \, d\mathbf{x} + \int_{\Gamma_N} \mathbf{t}_N^n \cdot \mathbf{v}_2 \, ds, \\
S^n(\theta) &= \frac{\rho_f}{\mu_f} (\mathbf{K} \mathbf{g}, \nabla \theta)_{\Omega_1} + (q^n, \theta)_{\Omega_1}, \\
K^n(p, \theta) &= \frac{c_0}{k} (p^n - p^{n-1}, \theta)_{\Omega_1} + \frac{1}{\mu_f} (\mathbf{K} \nabla p^n, \nabla \theta)_{\Omega_1}, \\
D^n(\mathbf{v}, \theta) &= \frac{\alpha}{k} (\operatorname{div}(\mathbf{v}^n - \mathbf{v}^{n-1}), \theta)_{\Omega_1}.
\end{aligned}$$

With this notation, the elasticity, interface, and fluid systems read

$$\begin{aligned}
& \sum_{i=1}^2 (B_i(\mathbf{u}_i^{n+1}, \mathbf{v}_i) - L_i(\boldsymbol{\lambda}^{n+1}, \mathbf{v}_i) + J_i(\mathbf{u}_i^{n+1} - \boldsymbol{\lambda}^{n+1}, \mathbf{v}_i)) + b_D(\mathbf{u}_2^{n+1}, \mathbf{v}_2) \\
& = l(p^{n+1}, \mathbf{v}_1) + l_{f,N}^{n+1}(\mathbf{v}) + \sum_{i=1}^2 \bar{J}_i(\mathbf{u}_i^n - \boldsymbol{\lambda}^n, \mathbf{v}_i),
\end{aligned}$$

$$\sum_{i=1}^2 (L_i(\boldsymbol{\mu}, \mathbf{u}^{n+1}) - J_i(\mathbf{u}_i^{n+1} - \boldsymbol{\lambda}^{n+1}, \boldsymbol{\mu}) + \bar{J}_i(\mathbf{u}_i^n - \boldsymbol{\lambda}^n, \boldsymbol{\mu})) - \alpha \int_{\Gamma_{12}} p^{n+1} \mathbf{n}_{12} \cdot \boldsymbol{\mu} \, ds = 0,$$

$$K^{n+1}(p, \theta) = -D^n(\mathbf{u}, \theta) + S^{n+1}(\theta).$$

3. Fully Discrete Problem

To simplify the presentation, we describe here the case $d = 3$, but everything carries over to $d = 2$. From now on, we assume that Ω is a Lipschitz polyhedron and that the boundaries of Γ_D and Γ_N are polygonal. Let $\mathcal{T}_{h,i}$ be a regular family of conforming triangulations of $\bar{\Omega}_i$ consisting of tetrahedra, E , of maximum diameter h_i , and such that no element adjacent to $\partial\Omega$ intersects both Γ_D and Γ_N . The restriction to simplicial elements is only a matter of convenience; the analysis below extends readily to other shapes such as hexahedra or prisms. As usual, we assume that these triangulations are regular in the sense of Ciarlet¹²: There exists a constant $\varphi_i > 0$, independent of h_i , such that

$$\forall E \in \mathcal{T}_{h,i}, \quad \frac{h_E}{\rho_E} := \varphi_E \leq \varphi_i, \quad (3.1)$$

where $h_E \leq h_i$ denotes the diameter of E and ρ_E denotes the diameter of the ball inscribed in E . For $i = 1, 2$, by taking the trace of $\mathcal{T}_{h,i}$ on Γ_{12} , we automatically generate a triangulation $\Gamma_{h,i}$ of Γ_{12} , and (3.1) implies that the family $\Gamma_{h,i}$ is also regular

$$\forall \gamma_i \in \Gamma_{h,i}, \quad \frac{h_{\gamma_i}}{\rho_{\gamma_i}} := \varphi_{\gamma_i} \leq \varphi_i, \quad (3.2)$$

where ρ_{γ_i} is the diameter of the disc inscribed in γ_i . We also consider a regular family of conforming triangulations Γ_H of Γ_{12} made of triangles τ with diameter H_τ bounded by H :

$$\forall \tau \in \Gamma_H, \quad \frac{H_\tau}{\rho_\tau} \leq \varphi, \quad (3.3)$$

where ρ_τ is the radius of the circle inscribed in τ . The mesh Γ_H can be chosen independently of $\mathcal{T}_{h,i}$, the case of interest being $H \geq h_i, i = 1, 2$. On the other hand, we take for $\Gamma_{h,D}$ the set of all boundary faces of $\mathcal{T}_{h,2}$ on Γ_D . Then $\Gamma_{h,D}$ is regular by virtue of (3.1):

$$\forall \gamma \in \Gamma_{h,D}, \quad \frac{h_\gamma}{\rho_\gamma} \leq \varphi_2. \quad (3.4)$$

Since we focus on the domain-decomposition aspect of the problem, for the sake of simplicity, we choose an H^1 conforming Galerkin method for the discretization of each subproblem in Ω_i . Thus, we introduce the following finite element spaces for

given integers $k \geq 1$, $\ell \geq 1$ and $m \geq 1$:

$$\begin{aligned}\Lambda_H &= \{\boldsymbol{\lambda}_H \in L^2(\Gamma_{12})^3 : \forall \tau \in \Gamma_H, \boldsymbol{\lambda}_H|_\tau \in \mathbb{P}_\ell(\tau)^3\}, \\ X_{h_i} &= \{\mathbf{v}_{h_i} \in H^1(\Omega_i)^3 : \forall E \in \mathcal{T}_{h,i}, \mathbf{v}_{h_i}|_E \in \mathbb{P}_k(E)^3\}, \\ X_h &= \{\mathbf{v}_h \in L^2(\Omega)^3 : \mathbf{v}_{h_i} := \mathbf{v}_h|_{\Omega_i} \in X_{h_i}, i = 1, 2\}, \\ M_{h_1} &= \{\theta_{h_1} \in H^1(\Omega_1) : \forall E \in \mathcal{T}_{h,1}, \theta_{h_1}|_E \in \mathbb{P}_m(E)\}.\end{aligned}$$

As the exact solution is assumed to be sufficiently smooth (in particular it is continuous), we shall approximate it with standard nodal Lagrange interpolation operators $I_h \in \mathcal{L}(\mathcal{C}^0(\overline{\Omega_i})^d, X_{h,i})$, $i = 1, 2$, $r_h \in \mathcal{L}(\mathcal{C}^0(\overline{\Omega}), M_{h_1})$, and $\rho_H \in \mathcal{L}(\mathcal{C}^0(\Gamma_{12})^d, \Lambda_H)$. Considering the degree of the polynomial functions in $X_{h,i}$ and Λ_H , these interpolants have the following local approximation errors:

$$\begin{aligned}\forall E \in \mathcal{T}_{h,i}, \quad \forall \mathbf{v} \in H^s(E)^d, \\ |\mathbf{v} - I_h(\mathbf{v})|_{H^j(E)} \leq Ch_E^{s-j} |\mathbf{v}|_{H^s(E)}, \quad \frac{3}{2} < s \leq k+1, \quad 0 \leq j \leq s, \quad i = 1, 2, \quad (3.5)\end{aligned}$$

$$\begin{aligned}\forall E \in \mathcal{T}_{h,1}, \quad \forall q \in H^s(E), \\ |q - r_h(q)|_{H^j(E)} \leq Ch_E^{s-j} |q|_{H^s(E)}, \quad \frac{3}{2} < s \leq m+1, \quad 0 \leq j \leq s, \quad (3.6)\end{aligned}$$

$$\begin{aligned}\forall \tau \in \Gamma_H, \quad \forall \boldsymbol{\mu} \in H^s(\tau)^d, \\ |\boldsymbol{\mu} - \rho_H(\boldsymbol{\mu})|_{H^j(\tau)} \leq CH_\tau^{s-j} |\boldsymbol{\mu}|_{H^s(\tau)}, \quad 1 < s \leq \ell+1, \quad 0 \leq j \leq s, \quad (3.7)\end{aligned}$$

with constants C independent of E , τ , H and h_i , $i = 1, 2$. In these spaces, the fully discrete problem is:

- (1) Set $p_{h_1}^0 = r_{h_1}(p_0)$, $\mathbf{u}_{h_i}^0 = I_{h_i}(\mathbf{u}_0)$, and $\boldsymbol{\lambda}_H^0 = \rho_H(\boldsymbol{\lambda}_0)$.
- (2) Compute $\mathbf{u}_{h_i}^1$, and $\boldsymbol{\lambda}_H^1$ by solving

$$\begin{aligned}\forall \mathbf{v}_h \in X_h, \quad \sum_{i=1}^2 (B_i(\mathbf{u}_{h_i}^1, \mathbf{v}_{h_i}) - L_i(\boldsymbol{\lambda}_H^1, \mathbf{v}_{h_i}) + J_i(\mathbf{u}_{h_i}^1 - \boldsymbol{\lambda}_H^1, \mathbf{v}_{h_i})) + b_D(\mathbf{u}_{h_2}^1, \mathbf{v}_{h_2}) \\ = l(p_{h_1}^0, \mathbf{v}_{h_1}) + l_{f,N}^1(\mathbf{v}_h) + \sum_{i=1}^2 \bar{J}_i(\mathbf{u}_{h_i}^0 - \boldsymbol{\lambda}_H^0, \mathbf{v}_{h_i}), \quad (3.8)\end{aligned}$$

$$\begin{aligned}\forall \boldsymbol{\mu}_H \in \Lambda_H, \quad \sum_{i=1}^2 (L_i(\boldsymbol{\mu}_H, \mathbf{u}_{h_i}^1) - J_i(\mathbf{u}_{h_i}^1 - \boldsymbol{\lambda}_H^1, \boldsymbol{\mu}_H) + \bar{J}_i(\mathbf{u}_{h_i}^0 - \boldsymbol{\lambda}_H^0, \boldsymbol{\mu}_H)) \\ - \alpha \int_{\Gamma_{12}} p_{h_1}^0 \mathbf{n}_{12} \cdot \boldsymbol{\mu}_H ds = 0. \quad (3.9)\end{aligned}$$

- (3) Compute $p_{h_1}^1$ by solving

$$\forall \theta_{h_1} \in M_{h_1}, \quad K^1(p_{h_1}, \theta_{h_1}) = -D^1(\mathbf{u}_{h_1}, \theta_{h_1}) + S^1(\theta_{h_1}). \quad (3.10)$$

Then for $n \geq 1$, knowing $\mathbf{u}_{h_i}^{n-1}$, $\mathbf{u}_{h_i}^n$, $p_{h_i}^n$ and $\boldsymbol{\lambda}_H^n$, compute $p_{h_i}^{n+1}$ by solving

$$\forall \theta_{h_1} \in M_{h_1}, \quad K_{n+1}(p_{h_1}, \theta_{h_1}) = -D^n(\mathbf{u}_{h_1}, \theta_{h_1}) + S_{n+1}(\theta_{h_1}). \quad (3.11)$$

Once $p_{h_i}^{n+1}$ is known, $\mathbf{u}_{h_i}^{n+1}$ and $\boldsymbol{\lambda}_H^{n+1}$ are computed by solving

$$\begin{aligned} \forall \mathbf{v}_h \in X_h, \quad & \sum_{i=1}^2 (B_i(\mathbf{u}_{h_i}^{n+1}, \mathbf{v}_{h_i}) - L_i(\boldsymbol{\lambda}_H^{n+1}, \mathbf{v}_{h_i}) + J_i(\mathbf{u}_{h_i}^{n+1} - \boldsymbol{\lambda}_H^{n+1}, \mathbf{v}_{h_i})) \\ & + b_D(\mathbf{u}_{h_2}^{n+1}, \mathbf{v}_{h_2}) \\ & = l(p_{h_1}^{n+1}, \mathbf{v}_{h_1}) + l_{f,N}^{n+1}(\mathbf{v}_h) + \sum_{i=1}^2 \bar{J}_i(\mathbf{u}_{h_i}^n - \boldsymbol{\lambda}_H^n, \mathbf{v}_{h_i}), \end{aligned} \quad (3.12)$$

$$\begin{aligned} & \sum_{i=1}^2 (L_i(\boldsymbol{\mu}_H, \mathbf{u}_{h_i}^{n+1}) - J_i(\mathbf{u}_{h_i}^{n+1} - \boldsymbol{\lambda}_H^{n+1}, \boldsymbol{\mu}_H) + \bar{J}_i(\mathbf{u}_{h_i}^n - \boldsymbol{\lambda}_H^n, \boldsymbol{\mu}_H)) \\ & - \alpha \int_{\Gamma_{12}} p_{h_1}^{n+1} \mathbf{n}_{12} \cdot \boldsymbol{\mu}_H \, ds = 0. \end{aligned} \quad (3.13)$$

Note that at each time step, this algorithm solves:

- (1) An elasticity and interface system with the same matrix;
- (2) A standard reaction diffusion system for the pressure.

3.1. Norms and properties of the discrete spaces

The space X_h is equipped with the norms and seminorms:

$$\forall \mathbf{v}_h \in X_h, \quad |\mathbf{v}_h|_{X_h} = \left(\sum_{i=1}^2 |\mathbf{v}_{h,i}|_{h,i}^2 \right)^{1/2}, \quad \|\mathbf{v}_h\|_{X_h} = \left(\sum_{i=1}^2 \|\mathbf{v}_{h,i}\|_{h,i}^2 \right)^{1/2},$$

where for $i = 1, 2$,

$$|\mathbf{v}_{h,i}|_{h,i}^2 = \lambda \|\operatorname{div} \mathbf{v}_{h,i}\|_{L^2(\Omega_i)}^2 + 2\mu \|\boldsymbol{\varepsilon}(\mathbf{v}_{h,i})\|_{L^2(\Omega_i)}^2, \quad (3.14)$$

$$\begin{aligned} \|\mathbf{v}_{h,1}\|_{h,1}^2 &= |\mathbf{v}_{h,1}|_{h,1}^2 + (\lambda + 2\mu) \sum_{\gamma_i \in \Gamma_{h,1}} \left(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \frac{1}{k} \right) \|\mathbf{v}_{h,1}\|_{L^2(\gamma_i)}^2, \\ \|\mathbf{v}_{h,2}\|_{h,2}^2 &= |\mathbf{v}_{h,2}|_{h,2}^2 + (\lambda + 2\mu) \left(\sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_{\gamma}}{h_{\gamma}} \|\mathbf{v}_{h,2}\|_{L^2(\gamma)}^2 \right. \\ & \quad \left. + \sum_{\gamma_2 \in \Gamma_{h,2}} \left(\frac{\sigma_{\gamma_2}}{h_{\gamma_2}} + \frac{\bar{\sigma}_{\gamma_2}}{h_{\gamma_2}} \frac{1}{k} \right) \|\mathbf{v}_{h,2}\|_{L^2(\gamma_2)}^2 \right). \end{aligned} \quad (3.15)$$

As all parameters $\lambda, \mu, \sigma_{\gamma_i}$ and $\bar{\sigma}_{\gamma_i}$ are strictly positive, it is clear that (3.15) defines norms on these spaces. Furthermore, the following Korn's inequality follows easily from (1.19) in Ref. 8, (2.33), (2.32), and the fact that necessarily $h_{\gamma_i} < 1$: There exists

a constant C , depending only on Ω_i , k and ℓ , such that

$$\forall \mathbf{v}_{h,i} \in X_{h,i}, \quad \|\nabla \mathbf{v}_{h,i}\|_{L^2(\Omega_i)} \leq C \left(\|\boldsymbol{\varepsilon}(\mathbf{v}_{h,i})\|_{L^2(\Omega_i)}^2 + \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{v}_{h,i}\|_{L^2(\gamma_i)}^2 \right)^{1/2}. \quad (3.16)$$

Similarly, the following Poincaré's inequality follows from (1.10) in Ref. 7: There exists a constant C , depending only on Ω_i , k and ℓ , such that

$$\forall \mathbf{v}_{h,i} \in X_{h,i}, \quad \|\mathbf{v}_{h,i}\|_{L^2(\Omega_i)} \leq C \left(\|\nabla \mathbf{v}_{h,i}\|_{L^2(\Omega_i)}^2 + \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{v}_{h,i}\|_{L^2(\gamma_i)}^2 \right)^{1/2}.$$

Then (3.16) implies: There exists a constant C , depending only on Ω_i , k and ℓ , such that

$$\forall \mathbf{v}_{h,i} \in X_{h,i}, \quad \|\mathbf{v}_{h,i}\|_{L^2(\Omega_i)} \leq C \left(\|\boldsymbol{\varepsilon}(\mathbf{v}_{h,i})\|_{L^2(\Omega_i)}^2 + \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{v}_{h,i}\|_{L^2(\gamma_i)}^2 \right)^{1/2}. \quad (3.17)$$

Note that (3.16) and (3.17) do not involve the boundary Γ_D . Consequently, they apply to any subdomain of Ω and in particular to any floating subdomain.

Finally, if we replace the trace terms on Γ_{12} by jumps and insert the trace terms on Γ_D , according to Refs. 8 and 7, we obtain a global Poincaré–Korn's inequality on the entire domain Ω for all $\mathbf{v}_h \in X_h$:

$$\begin{aligned} & \sum_{i=1}^2 \|\nabla \mathbf{v}_{h,i}\|_{L^2(\Omega_i)}^2 + \|\mathbf{v}_h\|_{L^2(\Omega)}^2 \\ & \leq C \left(\sum_{i=1}^2 (\|\boldsymbol{\varepsilon}(\mathbf{v}_{h,i})\|_{L^2(\Omega_i)}^2 + \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\llbracket \mathbf{v}_h \rrbracket\|_{L^2(\gamma_i)}^2) + \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_{\gamma}}{h_{\gamma}} \|\mathbf{v}_{h,2}\|_{L^2(\gamma)}^2 \right). \end{aligned} \quad (3.18)$$

The following lemma established in Ref. 16, valid for $d = 2$ or 3 , controls the integrals

$$\int_{\gamma_i} \boldsymbol{\sigma}(\mathbf{u}_{h,i}) \mathbf{n}_i \cdot \mathbf{v}_{h,i} \, ds, \quad \int_{\gamma} \boldsymbol{\sigma}(\mathbf{u}_{h,2}) \mathbf{n}_{\Omega} \cdot \mathbf{v}_{h,2} \, ds.$$

Lemma 3.1. *Let $\mathcal{T}_{h,i}$ be regular. Then, for any $\eta > 0$, we can associate with each γ_i in $\Gamma_{h,i}$ a positive number σ_{γ_i} , bounded above and below independently of h_i , λ and μ , such that the following inequality holds for all $\mathbf{v}_{h,i} \in X_{h,i}$ and all $\mathbf{q} \in L^2(\gamma_i)^d$:*

$$\begin{aligned} \left| \int_{\gamma_i} \boldsymbol{\sigma}(\mathbf{v}_{h,i}) \mathbf{n}_i \cdot \mathbf{q} \, ds \right| & \leq \frac{\eta}{2} (\lambda + 2\mu) \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{q}\|_{L^2(\gamma_i)}^2 \\ & + \frac{\eta}{2} (\lambda \|\operatorname{div} \mathbf{v}_{h,i}\|_{L^2(E)}^2 + 2\mu \|\boldsymbol{\varepsilon}(\mathbf{v}_{h,i})\|_{L^2(E)}^2), \end{aligned} \quad (3.19)$$

where E denotes the element of $\mathcal{T}_{h,i}$ that is adjacent to γ_i . The number σ_{γ_i} has the form

$$\sigma_{\gamma_i} = \frac{1}{\eta^2} \hat{c}^2 \frac{|\gamma_i| h_{\gamma_i}}{|E|}, \quad (3.20)$$

with a constant \hat{c} depending only on the reference element. The same statement is true for

$$\int_{\gamma} \boldsymbol{\sigma}(\mathbf{v}_{h,2}) \mathbf{n}_{\Omega} \cdot \mathbf{q} \, ds.$$

As an immediate application, we fix $\eta > 0$ and choose

$$\sigma_{\gamma} \geq \frac{1}{\eta^2} \hat{c}^2 \text{Max}_{\gamma \in \Gamma_{h,D}} \frac{|\gamma| h_{\gamma}}{|E|}. \quad (3.21)$$

Then

$$\begin{aligned} & 2 \left| \int_{\Gamma_D} \boldsymbol{\sigma}(\mathbf{v}_{h,2}) \mathbf{n}_{\Omega} \cdot \mathbf{v}_{h,2} \, ds \right| \\ & \leq \eta \left(\lambda \|\text{div } \mathbf{v}_{h,2}\|_{L^2(\Delta_D)}^2 + 2\mu \|\boldsymbol{\varepsilon}(\mathbf{v}_{h,2})\|_{L^2(\Delta_D)}^2 + (\lambda + 2\mu) \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_{\gamma}}{h_{\gamma}} \|\mathbf{v}_{h,2}\|_{L^2(\gamma)}^2 \right), \end{aligned} \quad (3.22)$$

where Δ_D is the layer of elements of $\mathcal{T}_{h,2}$ that is adjacent to Γ_D .

3.2. Consistency in time

When the solution is sufficiently smooth, it is easy to see that each step of (3.8)–(3.13) is either consistent or asymptotically consistent in time as k tends to zero, provided we set $\boldsymbol{\lambda}(t^n) = \mathbf{u}(t^n)$ on Γ_{12} for all n , $0 \leq n \leq N$. We use the following formulas valid for any sufficiently smooth function w :

$$\begin{aligned} w(t^n) &= w(t^{n+1}) - \int_{t^n}^{t^{n+1}} w'(t) dt, \quad 0 \leq n \leq N-1, \\ w(t^{n+1}) - w(t^n) &= kw'(t^{n+1}) - \int_{t^n}^{t^{n+1}} (t - t^n) w''(t) dt, \quad 0 \leq n \leq N-1, \\ w(t^n) - w(t^{n-1}) &= kw'(t^{n+1}) - k \int_{t^n}^{t^{n+1}} w''(t) dt \\ &\quad - \int_{t^{n-1}}^{t^n} (t - t^{n-1}) w''(t) dt, \quad 1 \leq n \leq N-1. \end{aligned}$$

At time $t = t^1 = k$, time-lagging of p produces a consistency error: The triple $(\mathbf{u}(k), p(0), \boldsymbol{\lambda}(k))$ satisfies

$$\begin{aligned} \forall \mathbf{v}_h \in X_h, \quad & \sum_{i=1}^2 [B_i(\mathbf{u}_i(k), \mathbf{v}_{h_i}) - L_i(\boldsymbol{\lambda}(k), \mathbf{v}_{h_i}) + J_i(\mathbf{u}_i(k) - \boldsymbol{\lambda}(k), \mathbf{v}_{h_i})] \\ & + b_D(\mathbf{u}_2(k), \mathbf{v}_{h_2}) \\ & = l(p(0), \mathbf{v}_{h_1}) + l_{f,N}^1(\mathbf{v}_h) + \sum_{i=1}^2 \bar{J}_i(\mathbf{u}_i(0) - \boldsymbol{\lambda}(0), \mathbf{v}_{h_i}) \\ & + l\left(\int_0^k p'(t) dt, \mathbf{v}_{h_1}\right), \end{aligned} \quad (3.23)$$

$$\begin{aligned} \forall \boldsymbol{\mu}_H \in \Lambda_H, \quad & \sum_{i=1}^2 [L_i(\boldsymbol{\mu}_H, \mathbf{u}_{h_i}(k)) - J_i(\mathbf{u}_i(k) - \boldsymbol{\lambda}(k), \boldsymbol{\mu}_H) + \bar{J}_i(\mathbf{u}_i(0) - \boldsymbol{\lambda}(0), \boldsymbol{\mu}_H)] \\ & - \alpha \int_{\Gamma_{12}} p(0) \mathbf{n}_{12} \cdot \boldsymbol{\mu}_H ds \\ & = \alpha \int_{\Gamma_{12}} \left(\int_0^k p'(t) dt \right) \mathbf{n}_{12} \cdot \boldsymbol{\mu}_H ds. \end{aligned} \quad (3.24)$$

Next p is computed by approximating derivatives; that introduces a consistency error: $p(k)$ satisfies

$$\begin{aligned} \forall \theta_{h_1} \in M_{h_1}, \quad & K^1(p, \theta_{h_1}) = -D^1(\mathbf{u}, \theta_{h_1}) + S^1(\theta_{h_1}) - \frac{c_0}{k} \left(\int_0^k t p''(t) dt, \theta_{h_1} \right)_{\Omega_1} \\ & - \frac{\alpha}{k} \left(\int_0^k t \operatorname{div} \mathbf{u}''(t) dt, \theta_{h_1} \right)_{\Omega_1}. \end{aligned} \quad (3.25)$$

For the general step, p is computed both by time lagging and by approximating derivatives, and that introduces a double consistency error: $p(t^{n+1})$ satisfies

$$\begin{aligned} \forall \theta_{h_1} \in M_{h_1}, \quad & K_{n+1}(p, \theta_{h_1}) = -D^n(\mathbf{u}, \theta_{h_1}) + S_{n+1}(\theta_{h_1}) - \frac{c_0}{k} \left(\int_{t^n}^{t^{n+1}} (t - t^n) p''(t) dt, \theta_{h_1} \right)_{\Omega_1} \\ & - \frac{\alpha}{k} \left(\int_{t^n}^{t^{n+1}} k \operatorname{div} \mathbf{u}''(t) dt + \int_{t^{n-1}}^{t^n} (t - t^{n-1}) \operatorname{div} \mathbf{u}''(t) dt, \theta_{h_1} \right)_{\Omega_1}. \end{aligned} \quad (3.26)$$

Finally, there is no consistency error in the elasticity equations: The pair $(\mathbf{u}(t^{n+1}), \boldsymbol{\lambda}(t^{n+1}))$ satisfies:

$$\begin{aligned} & \sum_{i=1}^2 (B_i(\mathbf{u}_i(t^{n+1}), \mathbf{v}_{h_i}) - L_i(\boldsymbol{\lambda}(t^{n+1}), \mathbf{v}_{h_i}) + J_i(\mathbf{u}_i(t^{n+1}) - \boldsymbol{\lambda}(t^{n+1}), \mathbf{v}_{h_i})) \\ & + b_D(\mathbf{u}_2(t^{n+1}), \mathbf{v}_{h_2}) \\ & = l(p(t^{n+1}), \mathbf{v}_{h_1}) + l_{f,N}^{n+1}(\mathbf{v}_h) + \sum_{i=1}^2 \bar{J}_i(\mathbf{u}_i(t^n) - \boldsymbol{\lambda}(t^n), \mathbf{v}_{h_i}), \end{aligned} \quad (3.27)$$

$$\sum_{i=1}^2 (L_i(\boldsymbol{\mu}_H, \mathbf{u}_i(t^{n+1})) - J_i(\mathbf{u}_i(t^{n+1}) - \boldsymbol{\lambda}(t^{n+1}), \boldsymbol{\mu}_H) + \bar{J}_i(\mathbf{u}_i(t^n) - \boldsymbol{\lambda}(t^n), \boldsymbol{\mu}_H)) - \alpha \int_{\Gamma_{12}} p(t^{n+1}) \mathbf{n}_{12} \cdot \boldsymbol{\mu}_H ds = 0, \quad (3.28)$$

for all $\mathbf{v}_h \in X_h$ and $\boldsymbol{\mu}_H \in \Lambda_H$. If both p and \mathbf{u} are sufficiently smooth, the consistency errors tend to zero with k .

3.3. Existence and uniqueness of the discrete solution

Since at each time step, (3.8)–(3.13) solves separately an elasticity-interface problem and a reaction-diffusion problem, it suffices to check that each problem has a unique solution. For the first problem, this is achieved by appropriately choosing σ_γ .

The elasticity–interface problem has the form: Find $\mathbf{u}_h \in X_h$ and $\boldsymbol{\lambda}_H \in \Lambda_H$ such that

$$\begin{aligned} \forall \mathbf{v}_h \in X_h, \quad & \sum_{i=1}^2 (B_i(\mathbf{u}_{h_i}, \mathbf{v}_{h_i}) - L_i(\boldsymbol{\lambda}_H, \mathbf{v}_{h_i}) + J_i(\mathbf{u}_{h_i} - \boldsymbol{\lambda}_H, \mathbf{v}_{h_i})) \\ & + b_D(\mathbf{u}_{h_2}, \mathbf{v}_{h_2}) = (\mathbf{F}, \mathbf{v}_h), \\ \forall \boldsymbol{\mu}_H \in \Lambda_H, \quad & \sum_{i=1}^2 (L_i(\boldsymbol{\mu}_H, \mathbf{u}_{h_i}) - J_i(\mathbf{u}_{h_i} - \boldsymbol{\lambda}_H, \boldsymbol{\mu}_H)) = (\mathbf{G}, \boldsymbol{\mu}_H), \end{aligned}$$

where \mathbf{F} and \mathbf{G} are known discrete quantities. As this is a square system in finite dimension, uniqueness implies existence. Thus, let $\mathbf{F} = \mathbf{0}$, $\mathbf{G} = \mathbf{0}$, choose $\mathbf{v}_h = \mathbf{u}_h$, $\boldsymbol{\mu}_H = \boldsymbol{\lambda}_H$, and add the resulting equations:

$$\sum_{i=1}^2 (|\mathbf{u}_{h_i}|_{h,i}^2 + J_i(\mathbf{u}_{h_i} - \boldsymbol{\lambda}_H, \mathbf{u}_{h_i} - \boldsymbol{\lambda}_H)) + b_D(\mathbf{u}_{h,2}, \mathbf{u}_{h,2}) = 0. \quad (3.29)$$

Now fix some number $\eta \in]0, 1[$, choose σ_γ according to (3.21), apply (3.22) and substitute into (3.29):

$$\begin{aligned} & |\mathbf{u}_{h_1}|_{h,1}^2 + (1 - \eta) \left(|\mathbf{u}_{h_2}|_{h,2}^2 + (\lambda + 2\mu) \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_\gamma}{h_\gamma} \|\mathbf{u}_{h,2}\|_{L^2(\gamma)}^2 \right) \\ & + \sum_{i=1}^2 J_i(\mathbf{u}_{h_i} - \boldsymbol{\lambda}_H, \mathbf{u}_{h_i} - \boldsymbol{\lambda}_H) \leq 0, \end{aligned} \quad (3.30)$$

whence $\mathbf{u}_h = \mathbf{0}$ and $\boldsymbol{\lambda}_H = \mathbf{0}$. The second problem has the form: Find $p_{h_1} \in M_{h_1}$ such that

$$\forall \theta_{h_1} \in M_{h_1}, \quad \frac{c_0}{k} (p_{h_1}, \theta_{h_1})_{\Omega_1} + \frac{1}{\mu_f} (\mathbf{K} \nabla p_{h_1}, \nabla \theta_{h_1})_{\Omega_1} = (\mathbf{F}_1, \theta_{h_1})_{\Omega_1},$$

where \mathbf{F}_1 is a given discrete function. The assumptions on \mathbf{K} and c_0 guarantee that this problem has a unique solution. Hence all steps of (3.8)–(3.13) have a unique solution.

3.4. Error equations

For all $n, 0 \leq n \leq N$, we introduce the following notation for the scheme's and interpolations errors,

$$\begin{aligned} \mathbf{e}_{\mathbf{u}_i}^n &= \mathbf{u}_{h_i}^n - I_h(\mathbf{u}_i(t^n)), & \mathbf{e}_{\boldsymbol{\lambda}}^n &= \boldsymbol{\lambda}_H^n - \rho_H(\boldsymbol{\lambda}(t^n)), & e_p^n &= p_{h_1}^n - r_{h_1}(p(t^n)), \\ \mathbf{a}_{\mathbf{u}_i}^n &= \mathbf{u}_i(t^n) - I_h(\mathbf{u}_i(t^n)), & \mathbf{a}_{\boldsymbol{\lambda}}^n &= \boldsymbol{\lambda}(t^n) - \rho_H(\boldsymbol{\lambda}(t^n)), & a_p^n &= p(t^n) - r_{h_1}(p(t^n)). \end{aligned}$$

The error equations are obtained by taking the differences between (3.8) and (3.23), (3.9) and (3.24), (3.10) and (3.25), (3.11) and (3.26), (3.12) and (3.27), (3.13) and (3.28), and inserting suitable interpolation terms. This gives for all $\mathbf{v}_h \in X_h$, for all $\boldsymbol{\mu}_H \in \Lambda_H$, and for all $\theta_{h_1} \in M_{h_1}$,

$$\begin{aligned} & \sum_{i=1}^2 [B_i(\mathbf{e}_{\mathbf{u}_i}^1 - \mathbf{a}_{\mathbf{u}_i}^1, \mathbf{v}_{h_i}) - L_i(\mathbf{e}_{\boldsymbol{\lambda}}^1 - \mathbf{a}_{\boldsymbol{\lambda}}^1, \mathbf{v}_{h_i}) + J_i(\mathbf{e}_{\mathbf{u}_i}^1 - \mathbf{a}_{\mathbf{u}_i}^1 - \mathbf{e}_{\boldsymbol{\lambda}}^1 + \mathbf{a}_{\boldsymbol{\lambda}}^1, \mathbf{v}_{h_i}) \\ & \quad - \bar{J}_i(\mathbf{e}_{\mathbf{u}_i}^0 - \mathbf{a}_{\mathbf{u}_i}^0 - \mathbf{e}_{\boldsymbol{\lambda}}^0 + \mathbf{a}_{\boldsymbol{\lambda}}^0, \mathbf{v}_{h_i})] + b_D(\mathbf{e}_{\mathbf{u}_2}^1 - \mathbf{a}_{\mathbf{u}_2}^1, \mathbf{v}_{h_2}) \\ & = -l(a_p^0, \mathbf{v}_{h_1}) - l\left(\int_0^k p'(t)dt, \mathbf{v}_{h_1}\right), \end{aligned} \quad (3.31)$$

$$\begin{aligned} & \sum_{i=1}^2 (L_i(\boldsymbol{\mu}_H, \mathbf{e}_{\mathbf{u}_i}^1 - \mathbf{a}_{\mathbf{u}_i}^1) - J_i(\mathbf{e}_{\mathbf{u}_i}^1 - \mathbf{a}_{\mathbf{u}_i}^1 - \mathbf{e}_{\boldsymbol{\lambda}}^1 + \mathbf{a}_{\boldsymbol{\lambda}}^1, \boldsymbol{\mu}_H) \\ & \quad + \bar{J}_i(\mathbf{e}_{\mathbf{u}_i}^0 - \mathbf{a}_{\mathbf{u}_i}^0 - \mathbf{e}_{\boldsymbol{\lambda}}^0 + \mathbf{a}_{\boldsymbol{\lambda}}^0, \boldsymbol{\mu}_H)) \\ & = -\alpha \int_{\Gamma_{12}} a_p^0 \mathbf{n}_{12} \cdot \boldsymbol{\mu}_H ds - \alpha \int_{\Gamma_{12}} \left(\int_0^k p'(t)dt\right) \mathbf{n}_{12} \cdot \boldsymbol{\mu}_H ds, \end{aligned} \quad (3.32)$$

$$\begin{aligned} & K^1(e_p - a_p, \theta_{h_1}) + D^1(\mathbf{e}_{\mathbf{u}_1} - \mathbf{a}_{\mathbf{u}_1}, \theta_{h_1}) \\ & = \frac{1}{k} \left(\int_0^k t(c_0 p''(t) + \alpha \operatorname{div} \mathbf{u}''(t)) dt, \theta_{h_1} \right)_{\Omega_1}, \end{aligned} \quad (3.33)$$

$$\begin{aligned} & K^{n+1}(e_p - a_p, \theta_{h_1}) + D^n(\mathbf{e}_{\mathbf{u}_1} - \mathbf{a}_{\mathbf{u}_1}, \theta_{h_1}) = \frac{c_0}{k} \left(\int_{t^n}^{t^{n+1}} (t - t^n) p''(t) dt, \theta_{h_1} \right)_{\Omega_1} \\ & \quad + \frac{\alpha}{k} \left(\int_{t^n}^{t^{n+1}} k \operatorname{div} \mathbf{u}''(t) dt + \int_{t^{n-1}}^{t^n} (t - t^{n-1}) \operatorname{div} \mathbf{u}''(t) dt, \theta_{h_1} \right)_{\Omega_1}, \end{aligned} \quad (3.34)$$

$$\begin{aligned} n \geq 1, \quad & \sum_{i=1}^2 [B_i(\mathbf{e}_{\mathbf{u}_i}^{n+1} - \mathbf{a}_{\mathbf{u}_i}^{n+1}, \mathbf{v}_{h_i}) - L_i(\mathbf{e}_{\boldsymbol{\lambda}}^{n+1} - \mathbf{a}_{\boldsymbol{\lambda}}^{n+1}, \mathbf{v}_{h_i}) \\ & \quad + J_i(\mathbf{e}_{\mathbf{u}_i}^{n+1} - \mathbf{a}_{\mathbf{u}_i}^{n+1} - \mathbf{e}_{\boldsymbol{\lambda}}^{n+1} + \mathbf{a}_{\boldsymbol{\lambda}}^{n+1}, \mathbf{v}_{h_i}) - \bar{J}_i(\mathbf{e}_{\mathbf{u}_i}^n - \mathbf{a}_{\mathbf{u}_i}^n - \mathbf{e}_{\boldsymbol{\lambda}}^n + \mathbf{a}_{\boldsymbol{\lambda}}^n, \mathbf{v}_{h_i})] \\ & \quad + b_D(\mathbf{e}_{\mathbf{u}_2}^{n+1} - \mathbf{a}_{\mathbf{u}_2}^{n+1}, \mathbf{v}_{h_2}) = l(e_p^{n+1} - a_p^{n+1}, \mathbf{v}_{h_1}), \end{aligned} \quad (3.35)$$

$$\begin{aligned}
 & \sum_{i=1}^2 [L_i(\boldsymbol{\mu}_H, \mathbf{e}_{\mathbf{u}_i}^{n+1} - \mathbf{a}_{\mathbf{u}_i}^{n+1}) - J_i(\mathbf{e}_{\mathbf{u}_i}^{n+1} - \mathbf{a}_{\mathbf{u}_i}^{n+1} - \mathbf{e}_{\boldsymbol{\lambda}}^{n+1} + \mathbf{a}_{\boldsymbol{\lambda}}^{n+1}, \boldsymbol{\mu}_H) \\
 & \quad + \bar{J}_i(\mathbf{e}_{\mathbf{u}_i}^n - \mathbf{a}_{\mathbf{u}_i}^n - \mathbf{e}_{\boldsymbol{\lambda}}^n + \mathbf{a}_{\boldsymbol{\lambda}}^n, \boldsymbol{\mu}_H)] \\
 & = \alpha \int_{\Gamma_{12}} (e_p^{n+1} - a_p^{n+1}) \mathbf{n}_{12} \cdot \boldsymbol{\mu}_H \, ds.
 \end{aligned} \tag{3.36}$$

3.5. Error inequality

To simplify, we denote by δ the discrete difference in time operator, e.g. $\delta \mathbf{e}_{\mathbf{u}_i}^m = \mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\mathbf{u}_i}^{m-1}$. Following Ref. 29, at step n , we test the equations relative to p multiplied by k with e_p^n , the equations relative to \mathbf{u} with $\delta \mathbf{e}_{\mathbf{u}_i}^n$, the interface equation with $\delta \mathbf{e}_{\boldsymbol{\lambda}}^n$, and we add all resulting equations. Considering that $\mathbf{a}_{\mathbf{u}_2}^m = \mathbf{0}$ on Γ_D and $e_p^0 = 0$, this gives an equation with the following right-hand side and left-hand side for each $n \geq 0$, with the convention that all terms involving superscripts larger than n vanish:

$$\begin{aligned}
 \text{R.H.S.} = & \sum_{i=1}^2 \sum_{m=1}^n [(\mathbf{a}_{\mathbf{u}_i}^m, \delta \mathbf{e}_{\mathbf{u}_i}^m)_{h,i} + (\boldsymbol{\sigma}(\delta \mathbf{e}_{\mathbf{u}_i}^m) \mathbf{n}_i, \mathbf{a}_{\mathbf{u}_i}^m - \mathbf{a}_{\boldsymbol{\lambda}}^m)_{\Gamma_{12}} \\
 & - (\boldsymbol{\sigma}(\mathbf{a}_{\mathbf{u}_i}^m) \mathbf{n}_i, \delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\boldsymbol{\lambda}}^m))_{\Gamma_{12}}] \\
 & - \alpha \left[(a_p^0, \operatorname{div} \mathbf{e}_{\mathbf{u}_1}^1)_{\Omega_1} + \sum_{m=2}^n (a_p^m, \operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^m))_{\Omega_1} \right] - \sum_{m=1}^n (\boldsymbol{\sigma}(\mathbf{a}_{\mathbf{u}_2}^m) \mathbf{n}_{\Omega}, \delta \mathbf{e}_{\mathbf{u}_2}^m)_{\Gamma_D} \\
 & + (\lambda + 2\mu) \sum_{i=1}^2 \sum_{m=1}^n \sum_{\gamma_i \in \Gamma_{h,i}} \left[\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} (\mathbf{a}_{\mathbf{u}_i}^m - \mathbf{a}_{\boldsymbol{\lambda}}^m, \delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\boldsymbol{\lambda}}^m))_{\gamma_i} \right. \\
 & \left. + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \frac{1}{k} (\delta(\mathbf{a}_{\mathbf{u}_i}^m - \mathbf{a}_{\boldsymbol{\lambda}}^m), \delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\boldsymbol{\lambda}}^m))_{\gamma_i} \right] \\
 & + \alpha \left[\left(e_p^1, \operatorname{div}(\delta \mathbf{a}_{\mathbf{u}_1}^1) \right)_{\Omega_1} - (a_p^0 \mathbf{n}_{12}, \delta(\mathbf{e}_{\mathbf{u}_1}^1 - \mathbf{e}_{\boldsymbol{\lambda}}^1))_{\Gamma_{12}} \right. \\
 & \left. + \sum_{m=2}^n ((e_p^m, \operatorname{div}(\delta \mathbf{a}_{\mathbf{u}_1}^{m-1}))_{\Omega_1} - (a_p^m \mathbf{n}_{12}, \delta(\mathbf{e}_{\mathbf{u}_1}^m - \mathbf{e}_{\boldsymbol{\lambda}}^m))_{\Gamma_{12}}) \right] \\
 & + \sum_{m=1}^n (c_0(\delta a_p^m, e_p^m)_{\Omega_1} + \frac{k}{\mu_f} (\mathbf{K} \nabla a_p^m, \nabla e_p^m)_{\Omega_1}) \\
 & + \sum_{m=2}^n \left(\int_{t^{m-1}}^{t^m} (c_0(t - t^{m-1}) p''(t) + \alpha k \operatorname{div} \mathbf{u}''(t)) dt \right. \\
 & \left. + \int_{t^{m-2}}^{t^{m-1}} \alpha(t - t^{m-2}) \operatorname{div} \mathbf{u}''(t) dt, e_p^m \right)_{\Omega_1} \\
 & + \int_0^k [(t(c_0 p''(t) + \alpha \operatorname{div} \mathbf{u}''(t)), e_p^1)_{\Omega_1} - \alpha(p'(t), \operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^1))_{\Omega_1} \\
 & + \alpha(p'(t) \mathbf{n}_{12}, \delta(\mathbf{e}_{\mathbf{u}_1}^1 - \mathbf{e}_{\boldsymbol{\lambda}}^1))_{\Gamma_{12}}] dt,
 \end{aligned} \tag{3.37}$$

$$\begin{aligned}
\text{L.H.S.} = & \frac{1}{2} \left(|\mathbf{e}_{\mathbf{u}}^n|_{X_h}^2 + \sum_{m=1}^n |\delta \mathbf{e}_{\mathbf{u}}^m|_{X_h}^2 \right) + \frac{c_0}{2} \left(\|e_p^n\|_{L^2(\Omega_1)}^2 + \sum_{m=1}^n \|\delta e_p^m\|_{L^2(\Omega_1)}^2 \right) \\
& + \frac{k}{\mu_f} \sum_{m=1}^n \|\mathbf{K}^{1/2} \nabla e_p^m\|_{L^2(\Omega_1)}^2 + \frac{1}{2} (\lambda + 2\mu) \\
& \times \sum_{i=1}^2 \sum_{\gamma_i \in \Gamma_{h,i}} \left[\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{e}_{\mathbf{u}_i}^n - \mathbf{e}_{\lambda}^n\|_{L^2(\gamma_i)}^2 + \sum_{m=1}^n \left(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \frac{1}{k} \right) \|\delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\lambda}^m)\|_{L^2(\gamma_i)}^2 \right] \\
& + \frac{1}{2} (\lambda + 2\mu) \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_{\gamma}}{h_{\gamma}} \left(\|\mathbf{e}_{\mathbf{u}_2}^n\|_{L^2(\gamma)}^2 + \sum_{m=1}^n \|\delta \mathbf{e}_{\mathbf{u}_2}^m\|_{L^2(\gamma)}^2 \right) \\
& - \sum_{m=1}^n ((\sigma(\delta \mathbf{e}_{\mathbf{u}_2}^m) \mathbf{n}_{\Omega}, \mathbf{e}_{\mathbf{u}_2}^m)_{\Gamma_D} + (\sigma(\mathbf{e}_{\mathbf{u}_2}^m) \mathbf{n}_{\Omega}, \delta \mathbf{e}_{\mathbf{u}_2}^m)_{\Gamma_D}) \\
& + \sum_{i=1}^2 \sum_{m=1}^n ((\sigma(\delta \mathbf{e}_{\mathbf{u}_i}^m) \mathbf{n}_i, \mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\lambda}^m)_{\Gamma_{12}} - (\sigma(\mathbf{e}_{\mathbf{u}_i}^m) \mathbf{n}_i, \delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\lambda}^m))_{\Gamma_{12}}) \\
& + \alpha(e_p^1, \operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^1))_{\Omega_1} - \alpha \sum_{m=2}^n ((e_p^m, \operatorname{div}(\delta(\mathbf{e}_{\mathbf{u}_1}^m)))_{\Omega_1} \\
& - (e_p^m \mathbf{n}_{12}, \delta(\mathbf{e}_{\mathbf{u}_1}^m - \mathbf{e}_{\lambda}^m))_{\Gamma_{12}}). \tag{3.38}
\end{aligned}$$

3.5.1. Lower bound for the left-hand side

We must estimate all terms in (3.38) that are not norms or seminorms. Let us call T_1 the terms involving σ on Γ_{12} , T_2 those involving σ on Γ_D , T_3 those involving e_p in Ω_1 and T_4 those involving e_p on Γ_{12} . For T_1 , we use the identity, valid for any numbers a_1, a_2, b_1, b_2 :

$$a_1(b_1 - b_2) - (a_1 - a_2)b_1 = (a_1b_1 - a_2b_2) - 2(a_1 - a_2)b_1 + (a_1 - a_2)(b_1 - b_2),$$

with $a_1 = \mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\lambda}^m$, $a_2 = \mathbf{e}_{\mathbf{u}_i}^{m-1} - \mathbf{e}_{\lambda}^{m-1}$, $b_1 = \sigma(\mathbf{e}_{\mathbf{u}_i}^m) \mathbf{n}_i$, $b_2 = \sigma(\mathbf{e}_{\mathbf{u}_i}^{m-1}) \mathbf{n}_i$. We have,

$$\begin{aligned}
T_1 := T_{11} - 2T_{12} + T_{13} = & \sum_{i=1}^2 \left[(\sigma(\mathbf{e}_{\mathbf{u}_i}^n) \mathbf{n}_i, \mathbf{e}_{\mathbf{u}_i}^n - \mathbf{e}_{\lambda}^n)_{\Gamma_{12}} \right. \\
& \left. + \sum_{m=1}^n (-2(\sigma(\mathbf{e}_{\mathbf{u}_i}^m) \mathbf{n}_i, \delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\lambda}^m))_{\Gamma_{12}} + (\sigma(\delta \mathbf{e}_{\mathbf{u}_i}^m) \mathbf{n}_i, \delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\lambda}^m))_{\Gamma_{12}}) \right].
\end{aligned}$$

Applying Lemma 3.1, we obtain, for some $\eta_1 > 0$ to be specified later on,

$$|T_{12}| \leq \sum_{i=1}^2 \sum_{m=1}^n \left(\frac{\eta_1}{2} k |\mathbf{e}_{\mathbf{u}_i}^m|_{h,i,\Delta_{12}^i}^2 + (\lambda + 2\mu) \frac{\eta_1}{2} \frac{1}{k} \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \|\delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\lambda}^m)\|_{L^2(\gamma_i)}^2 \right), \tag{3.39}$$

where Δ_{12}^i stands for the union of elements of $\mathcal{T}_{h,i}$ adjacent to Γ_{12} , and provided $\bar{\sigma}_{\gamma_i}$ is compatible with (3.20), i.e.

$$\bar{\sigma}_{\gamma_i} \geq \frac{1}{\eta_1^2} \hat{c}^2 \frac{|\gamma_i| h_{\gamma_i}}{|E_i|}, \quad (3.40)$$

with E_i the element of Δ_{12}^i adjacent to γ_i . Similarly,

$$|T_{13}| \leq \sum_{i=1}^2 \sum_{m=1}^n \left(\frac{\eta_2}{2} |\delta \mathbf{e}_{\mathbf{u}_i}^m|^2_{h,i,\Delta_{12}^i} + (\lambda + 2\mu) \frac{\eta_2}{2} \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\lambda}^m)\|_{L^2(\gamma_i)}^2 \right), \quad (3.41)$$

$$|T_{11}| \leq \sum_{i=1}^2 \left(\frac{\eta_2}{2} |\mathbf{e}_{\mathbf{u}_i}^n|^2_{h,i,\Delta_{12}^i} + (\lambda + 2\mu) \frac{\eta_2}{2} \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{e}_{\mathbf{u}_i}^n - \mathbf{e}_{\lambda}^n\|_{L^2(\gamma_i)}^2 \right), \quad (3.42)$$

provided

$$\sigma_{\gamma_i} \geq \frac{1}{\eta_2^2} \hat{c}^2 \frac{|\gamma_i| h_{\gamma_i}}{|E_i|}. \quad (3.43)$$

The identity, valid for any numbers a_1, a_2, b_1, b_2 :

$$(a_1 - a_2)b_1 + a_1(b_1 - b_2) = (a_1b_1 - a_2b_2) + (a_1 - a_2)(b_1 - b_2),$$

gives for T_2 ,

$$T_2 = -(\boldsymbol{\sigma}(\mathbf{e}_{\mathbf{u}_2}^n) \mathbf{n}_{\Omega}, \mathbf{e}_{\mathbf{u}_2}^n)_{\Gamma_D} - \sum_{m=1}^n (\boldsymbol{\sigma}(\delta \mathbf{e}_{\mathbf{u}_2}^m) \mathbf{n}_{\Omega}, \delta \mathbf{e}_{\mathbf{u}_2}^m)_{\Gamma_D}.$$

Therefore

$$\begin{aligned} |T_2| \leq & \frac{\eta_2}{2} \left(|\mathbf{e}_{\mathbf{u}_2}^n|^2_{h,2,\Delta_D} + \sum_{m=1}^n |\delta \mathbf{e}_{\mathbf{u}_2}^m|^2_{h,2,\Delta_D} \right) \\ & + (\lambda + 2\mu) \frac{\eta_2}{2} \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_{\gamma}}{h_{\gamma}} \left(\|\mathbf{e}_{\mathbf{u}_2}^n\|_{L^2(\gamma)}^2 + \sum_{m=1}^n \|\delta \mathbf{e}_{\mathbf{u}_2}^m\|_{L^2(\gamma)}^2 \right), \end{aligned} \quad (3.44)$$

where Δ_D denotes the union of elements of $\mathcal{T}_{h,2}$ adjacent to Γ_D , and provided σ_{γ} is chosen as in (3.43):

$$\sigma_{\gamma} \geq \frac{1}{\eta_2^2} \hat{c}^2 \frac{|\gamma| h_{\gamma}}{|E|}.$$

The term $T_4 = 0$ if $n = 1$, and if $n \geq 2$, it is treated as T_{12} :

$$|T_4| \leq \frac{\eta_4}{2} \sum_{m=2}^n \left(c_0 k \|e_p^m\|_{L^2(\Delta_{12}^1)}^2 + (\lambda + 2\mu) \frac{1}{k} \sum_{\gamma_1 \in \Gamma_{h,1}} \frac{\bar{\sigma}_{\gamma_1}}{h_{\gamma_1}} \|\delta(\mathbf{e}_{\mathbf{u}_1}^m - \mathbf{e}_{\lambda}^m)\|_{L^2(\gamma_1)}^2 \right), \quad (3.45)$$

provided $\bar{\sigma}_{\gamma_1}$ satisfies

$$\bar{\sigma}_{\gamma_1} \geq \frac{1}{\eta_4^2} \frac{1}{c_0} \frac{\hat{c}^2}{\lambda + 2\mu} \frac{|\gamma_1| h_{\gamma_1}}{|E_1|}. \quad (3.46)$$

The next lemma estimates the time-lagging effect T_3 on the pressure equation. It relies on assumption (3.47) that is often met in practical situations when c_0 is not too small because λ is usually large.

Lemma 3.2. *If the Lamé constant λ , the Biot–Willis constant α and the constrained specific storage coefficient c_0 are such that*

$$c_0 \lambda = \frac{8}{\beta} \alpha^2, \quad (3.47)$$

for some constant $\beta \in]0, 1[$, then when $n = 1$,

$$|T_3| \leq \frac{1}{2} \beta c_0 \|\delta e_p^1\|_{L^2(\Omega_1)}^2 + \frac{1}{8} \lambda \|\operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^1)\|_{L^2(\Omega_1)}^2.$$

When $n \geq 2$,

$$\begin{aligned} |T_3| \leq & \frac{1}{2} \beta c_0 \left(\|e_p^n\|_{L^2(\Omega_1)}^2 + \sum_{m=1}^n \|\delta e_p^m\|_{L^2(\Omega_1)}^2 \right) \\ & + \frac{1}{16} \lambda \sum_{m=1}^n \|\operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^m)\|_{L^2(\Omega_1)}^2 + \frac{1}{4} \lambda \|\operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^1)\|_{L^2(\Omega_1)}^2. \end{aligned} \quad (3.48)$$

Proof. Considering that $e_p^0 = 0$, a discrete summation by parts yields

$$T_3 = \alpha \sum_{m=2}^n (\delta e_p^m, \operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^{m-1}))_{\Omega_1} - \alpha (e_p^n, \operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^n))_{\Omega_1} + 2\alpha (\delta e_p^1, \operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^1))_{\Omega_1}.$$

We apply Cauchy–Schwarz inequality to all terms above and for any positive β , we split each product as follows:

$$\begin{aligned} & \alpha \|\delta e_p^m\|_{L^2(\Omega_1)} \|\operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^{m-1})\|_{L^2(\Omega_1)} \\ & \leq \frac{1}{2} \left(\beta c_0 \|\delta e_p^m\|_{L^2(\Omega_1)}^2 + \left(\frac{\alpha^2}{c_0 \beta \lambda} \right) \lambda \|\operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^{m-1})\|_{L^2(\Omega_1)}^2 \right) \end{aligned}$$

and (3.48) follows by choosing β according to (3.47). □

The choice $\eta_1 = \eta_2 = \eta_4 = \beta = \frac{1}{4}$ and $k \leq \frac{1}{2}$ yield the next lower bound for the left-hand side. The proof is straightforward.

Proposition 3.1. *If $k \leq \frac{1}{2}$ and*

$$\sigma_{\gamma_i} \geq A_i, \quad i = 1, 2, \quad \sigma_{\gamma} \geq A, \quad \bar{\sigma}_{\gamma_2} \geq A_2, \quad \bar{\sigma}_{\gamma_1} \geq A_1 \max\left(1, \frac{1}{c_0(\lambda + 2\mu)}\right), \quad (3.49)$$

where

$$A_i = 16\hat{c}^2 \frac{|\gamma_i| h_{\gamma_i}}{|E_i|}, \quad i = 1, 2, \quad A = 16\hat{c}^2 \frac{|\gamma| h_\gamma}{|E|},$$

then

$$\begin{aligned} \text{L.H.S.} \geq & \frac{1}{4} |\mathbf{e}_{\mathbf{u}_1}^n|_{h,1}^2 + \frac{1}{8} |\mathbf{e}_{\mathbf{u}_2}^n|_{h,2}^2 + \frac{3}{8} \sum_{m=1}^n |\delta \mathbf{e}_{\mathbf{u}_2}^m|_{h,2}^2 \\ & + \frac{1}{16} \lambda \|\operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^1)\|_{L^2(\Omega_1)}^2 - \frac{1}{4} \sum_{m=1}^{n-1} k |\mathbf{e}_{\mathbf{u}_1}^n|_{X_h}^2 \\ & + \frac{5}{16} \sum_{m=1}^n \lambda \|\operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^m)\|_{L^2(\Omega_1)}^2 + \frac{3}{8} \sum_{m=1}^n 2\mu \|\varepsilon(\delta \mathbf{e}_{\mathbf{u}_1}^m)\|_{L^2(\Omega_1)}^2 \\ & + \frac{3}{8} (\lambda + 2\mu) \sum_{i=1}^2 \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \left(\|\mathbf{e}_{\mathbf{u}_i}^n - \mathbf{e}_{\boldsymbol{\lambda}}^n\|_{L^2(\gamma_i)}^2 + \sum_{m=1}^n \|\delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\boldsymbol{\lambda}}^m)\|_{L^2(\gamma_i)}^2 \right) \\ & + \frac{1}{8} (\lambda + 2\mu) \sum_{i=1}^2 \sum_{m=1}^n \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \frac{1}{k} \|\delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\boldsymbol{\lambda}}^m)\|_{L^2(\gamma_i)}^2 \\ & + \frac{1}{4} (\lambda + 2\mu) \sum_{m=1}^n \sum_{\gamma_2 \in \Gamma_{h,2}} \frac{\bar{\sigma}_{\gamma_2}}{h_{\gamma_2}} \frac{1}{k} \|\delta(\mathbf{e}_{\mathbf{u}_2}^m - \mathbf{e}_{\boldsymbol{\lambda}}^m)\|_{L^2(\gamma_2)}^2 \\ & + \frac{3}{8} (\lambda + 2\mu) \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_\gamma}{h_\gamma} \left(\|\mathbf{e}_{\mathbf{u}_2}^n\|_{L^2(\gamma)}^2 + \sum_{m=1}^n \|\delta \mathbf{e}_{\mathbf{u}_2}^m\|_{L^2(\gamma)}^2 \right) \\ & + \frac{1}{\mu_f} \sum_{m=1}^n k \|\mathbf{K}^{1/2} \nabla e_p^m\|_{L^2(\Omega_1)}^2 + \frac{5}{16} c_0 \|e_p^n\|_{L^2(\Omega_1)}^2 \\ & - \frac{c_0}{8} \sum_{m=2}^{n-1} k \|e_p^m\|_{L^2(\Omega_1)}^2 + \frac{3c_0}{8} \sum_{m=1}^n \|\delta e_p^m\|_{L^2(\Omega_1)}^2. \end{aligned} \quad (3.50)$$

Owing to the regularity assumption (3.1) on \mathcal{T}_h , (3.49) can be stated independently of $h_i, i = 1, 2$.

3.5.2. Upper bound for the right-hand side

Let $T_i, 1 \leq i \leq 7$ (resp. $10 \leq i \leq 12$) denote the terms in the first three (resp. last three) lines of (3.37); in the fourth line it is convenient to denote by T_8 the terms involving e_p and by T_9 those involving a_p . For T_1 , we use a discrete summation by

parts:

$$\begin{aligned} |T_1| &= \left| \sum_{i=1}^2 \left((\mathbf{a}_{\mathbf{u}_i}^n, \mathbf{e}_{\mathbf{u}_i}^n)_{h,i} - \sum_{m=1}^n (\delta \mathbf{a}_{\mathbf{u}_i}^m, \mathbf{e}_{\mathbf{u}_i}^{m-1})_{h,i} \right) \right| \\ &\leq |\mathbf{a}_{\mathbf{u}}^n|_{X_h} |\mathbf{e}_{\mathbf{u}}^n|_{X_h} + \left(\int_0^{t^n} |\mathbf{u}'(t) - I_h(\mathbf{u}'(t))|_{X_h}^2 dt \right)^{1/2} \left(\sum_{m=1}^{n-1} k |\mathbf{e}_{\mathbf{u}}^m|_{X_h}^2 \right)^{1/2}. \end{aligned}$$

Similarly,

$$\begin{aligned} |T_4| &= \left| \alpha \left((a_p^n, \operatorname{div} \mathbf{e}_{\mathbf{u}_1}^n)_{\Omega_1} + (a_p^0, \operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^1))_{\Omega_1} + \sum_{m=1}^n (\delta \mathbf{a}_p^m, \operatorname{div} \mathbf{e}_{\mathbf{u}_1}^{m-1})_{\Omega_1} \right) \right| \\ &\leq \alpha \left(\|\mathbf{a}_p^n\|_{L^2(\Omega_1)} \|\operatorname{div} \mathbf{e}_{\mathbf{u}_1}^n\|_{L^2(\Omega_1)} + \|a_p^0\|_{L^2(\Omega_1)} \|\operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^1)\|_{L^2(\Omega_1)} \right. \\ &\quad \left. \cdot \|p' - r_{h_1}(p')\|_{L^2(\Omega_1 \times]0, t^n])} \left(\sum_{m=1}^{n-1} k \|\operatorname{div} \mathbf{e}_{\mathbf{u}_1}^m\|_{L^2(\Omega_1)}^2 \right)^{1/2} \right). \end{aligned}$$

In the same fashion,

$$\begin{aligned} |T_6| &= (\lambda + 2\mu) \left| \sum_{i=1}^2 \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \left((\mathbf{a}_{\mathbf{u}_i}^n - \mathbf{a}_{\lambda}^n, \mathbf{e}_{\mathbf{u}_i}^n - \mathbf{e}_{\lambda}^n)_{\gamma_i} \right. \right. \\ &\quad \left. \left. - \sum_{m=1}^n (\delta(\mathbf{a}_{\mathbf{u}_i}^m - \mathbf{a}_{\lambda}^m), \mathbf{e}_{\mathbf{u}_i}^{m-1} - \mathbf{e}_{\lambda}^{m-1})_{\gamma_i} \right) \right| \\ &\leq (\lambda + 2\mu) \sum_{i=1}^2 \left[\left(\sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{u}'_i - I_h(\mathbf{u}'_i) - (\boldsymbol{\lambda}' - \rho_H(\boldsymbol{\lambda}'))\|_{L^2(\gamma_i \times]0, t^n])}^2 \right)^{1/2} \right. \\ &\quad \cdot \left(\sum_{m=1}^{n-1} k \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\lambda}^m\|_{L^2(\gamma_i)}^2 \right)^{1/2} \\ &\quad \left. + \left(\sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{a}_i^n - \mathbf{a}_{\lambda}^n\|_{L^2(\gamma_i)}^2 \right)^{1/2} \left(\sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{e}_{\mathbf{u}_i}^n - \mathbf{e}_{\lambda}^n\|_{L^2(\gamma_i)}^2 \right)^{1/2} \right]. \end{aligned}$$

We cannot use Lemma 3.1 to estimate T_3 , because $\mathbf{a}_{\mathbf{u}_i}^m$ does not belong to a finite-dimensional space. Instead, we write

$$\begin{aligned} |T_4| &\leq \sum_{i=1}^2 \left[\left(\sum_{m=1}^n k \sum_{\gamma_i \in \Gamma_{h,i}} \frac{h_{\gamma_i}}{\bar{\sigma}_{\gamma_i}} |\mathbf{a}_{\mathbf{u}_i}^m|_{h,i,\gamma_i}^2 \right)^{1/2} \right. \\ &\quad \left. \times (\lambda + 2\mu)^{1/2} \left(\sum_{m=1}^n \frac{1}{k} \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \|\delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\lambda}^m)\|_{L^2(\gamma_i)}^2 \right)^{1/2} \right]. \end{aligned}$$

The bound for T_9 follows the same pattern

$$\begin{aligned}
 |T_9| &\leq \frac{\alpha}{(\lambda + 2\mu)^{1/2}} \left[\left(\sum_{\gamma_1 \in \Gamma_{h,1}} k \frac{h_{\gamma_1}}{\bar{\sigma}_{\gamma_1}} \|a_p^0\|_{L^2(\gamma_1)}^2 \right)^{1/2} \right. \\
 &\quad \cdot (\lambda + 2\mu)^{1/2} \left(\sum_{\gamma_1 \in \Gamma_{h,1}} \frac{1}{k} \frac{\bar{\sigma}_{\gamma_1}}{h_{\gamma_1}} \|\delta(\mathbf{e}_{\mathbf{u}_1}^1 - \mathbf{e}_{\boldsymbol{\lambda}}^1)\|_{L^2(\gamma_1)}^2 \right)^{1/2} \\
 &\quad + \left(\sum_{m=2}^n k \sum_{\gamma_1 \in \Gamma_{h,1}} \frac{h_{\gamma_1}}{\bar{\sigma}_{\gamma_1}} \|a_p^m\|_{L^2(\gamma_1)}^2 \right)^{1/2} \\
 &\quad \left. \cdot (\lambda + 2\mu)^{1/2} \left(\sum_{m=2}^n \frac{1}{k} \sum_{\gamma_1 \in \Gamma_{h,1}} \frac{\bar{\sigma}_{\gamma_1}}{h_{\gamma_1}} \|\delta(\mathbf{e}_{\mathbf{u}_1}^m - \mathbf{e}_{\boldsymbol{\lambda}}^m)\|_{L^2(\gamma_1)}^2 \right)^{1/2} \right].
 \end{aligned}$$

For T_7 , we write

$$\begin{aligned}
 T_7 &= (\lambda + 2\mu) \sum_{i=1}^2 \sum_{\gamma_i \in \Gamma_{h,i}} \frac{1}{k} \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \\
 &\quad \cdot \sum_{m=1}^n \left(\int_{t^{m-1}}^{t^m} (\mathbf{u}'_i(t) - I_h(\mathbf{u}'_i(t)) - (\boldsymbol{\lambda}'(t) - \rho_H(\boldsymbol{\lambda}'(t)))) dt, \delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\boldsymbol{\lambda}}^m) \right)_{\gamma_i}, \\
 |T_7| &\leq (\lambda + 2\mu) \sum_{i=1}^2 \left(\sum_{m=1}^n \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{u}'_i - I_h(\mathbf{u}'_i) - (\boldsymbol{\lambda}' - \rho_H(\boldsymbol{\lambda}'))\|_{L^2(\gamma_i \times]t^{m-1}, t^m])}^2 \right)^{1/2} \\
 &\quad \cdot \left(\sum_{m=1}^n \frac{1}{k} \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \|\delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\boldsymbol{\lambda}}^m)\|_{L^2(\gamma_i)}^2 \right)^{1/2}.
 \end{aligned}$$

The bound for T_2 starts with

$$\begin{aligned}
 T_2 &= \sum_{i=1}^2 ((\boldsymbol{\sigma}(\mathbf{e}_{\mathbf{u}_i}^n) \mathbf{n}_i, \mathbf{a}_{\mathbf{u}_i}^n - \mathbf{a}_{\boldsymbol{\lambda}}^n)_{\Gamma_{12}} \\
 &\quad - \sum_{m=1}^n \left(\boldsymbol{\sigma}(\mathbf{e}_{\mathbf{u}_i}^{m-1}) \mathbf{n}_i, \int_{t^{m-1}}^{t^m} (\mathbf{u}'_i(t) - I_h(\mathbf{u}'_i(t)) - (\boldsymbol{\lambda}'(t) - \rho_H(\boldsymbol{\lambda}'(t)))) dt \right)).
 \end{aligned}$$

Applying Lemma 3.1 here is not necessary because the \mathbf{a}^m are interpolation errors and their coefficients need not be sharply controlled. Instead, we write on each segment γ_i of $\Gamma_{h,i}$:

$$\begin{aligned}
 (\boldsymbol{\sigma}(\mathbf{e}_{\mathbf{u}_i}^n) \mathbf{n}_i, \mathbf{a}_{\mathbf{u}_i}^n - \mathbf{a}_{\boldsymbol{\lambda}}^n)_{\gamma_i} &\leq C \left(\frac{|\gamma_i|}{|E_i|} \right)^{1/2} \|\boldsymbol{\sigma}(\mathbf{e}_{\mathbf{u}_i}^n)\|_{L^2(E_i)} \|\mathbf{a}_{\mathbf{u}_i}^n - \mathbf{a}_{\boldsymbol{\lambda}}^n\|_{L^2(\gamma_i)} \\
 &\leq C \left(\frac{|\gamma_i|}{|E_i|} \right)^{1/2} (\lambda + 2\mu)^{1/2} |\mathbf{e}_{\mathbf{u}_i}^n|_{h,i,E_i} \|\mathbf{a}_{\mathbf{u}_i}^n - \mathbf{a}_{\boldsymbol{\lambda}}^n\|_{L^2(\gamma_i)},
 \end{aligned}$$

where E_i is the element of $\mathcal{T}_{h,i}$ adjacent to γ_i . By taking into account the regularity of \mathcal{T}_h , this becomes

$$\begin{aligned} & (\boldsymbol{\sigma}(\mathbf{e}_{\mathbf{u}_i}^n) \mathbf{n}_i, \mathbf{a}_{\mathbf{u}_i}^n - \mathbf{a}_{\boldsymbol{\lambda}}^n)_{\Gamma_{12}} \\ & \leq C \varphi_i (\lambda + 2\mu)^{1/2} |\mathbf{e}_{\mathbf{u}_i}^n|_{h,i,\Delta_{12}^i} \left(\sum_{\gamma_i \in \Gamma_{h,i}} \frac{1}{h_i} \|\mathbf{a}_{\mathbf{u}_i}^n - \mathbf{a}_{\boldsymbol{\lambda}}^n\|_{L^2(\gamma_i)}^2 \right)^{1/2}, \end{aligned}$$

where Δ_{12}^i is the layer of elements of $\mathcal{T}_{h,i}$ adjacent to Γ_{12} . Therefore

$$\begin{aligned} |T_2| & \leq C(\lambda + 2\mu)^{1/2} \sum_{i=1}^2 \varphi_i \left[|\mathbf{e}_{\mathbf{u}_i}^n|_{h,i,\Delta_{12}^i} \left(\sum_{\gamma_i \in \Gamma_{h,i}} \frac{1}{h_i} \|\mathbf{a}_{\mathbf{u}_i}^n - \mathbf{a}_{\boldsymbol{\lambda}}^n\|_{L^2(\gamma_i)}^2 \right)^{1/2} \right. \\ & \quad + \left(\sum_{m=1}^{n-1} k |\mathbf{e}_{\mathbf{u}_i}^m|_{h,i,\Delta_{12}^i}^2 \right)^{1/2} \left(\sum_{m=1}^n \sum_{\gamma_i \in \Gamma_{h,i}} \frac{1}{h_i} \|\mathbf{u}'_i - I_h(\mathbf{u}'_i) \right. \\ & \quad \left. \left. - (\boldsymbol{\lambda}' - \rho_H(\boldsymbol{\lambda}'))\|_{L^2(\gamma_i \times]t^{m-1}, t^m])}^2 \right)^{1/2} \right]. \end{aligned}$$

The term T_5 on Γ_D is treated like T_3 . First we write

$$T_5 = -(\boldsymbol{\sigma}(\mathbf{a}_{\mathbf{u}_2}^n \mathbf{n}_\Omega, \mathbf{e}_{\mathbf{u}_2}^n)_{\Gamma_D} + \sum_{m=1}^n \left(\int_{t^{m-1}}^{t^m} \boldsymbol{\sigma}(\mathbf{u}'_2(t) - I_h(\mathbf{u}'_2(t))) \mathbf{n}_\Omega dt, \mathbf{e}_{\mathbf{u}_2}^{m-1} \right)_{\Gamma_D}.$$

Then

$$\begin{aligned} T_5 & \leq (\lambda + 2\mu)^{1/2} \left[\left(\sum_{\gamma \in \Gamma_{h,D}} \frac{h_\gamma}{\sigma_\gamma} |\mathbf{a}_{\mathbf{u}_2}^n|_{h,\gamma}^2 \right)^{1/2} \left(\sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_\gamma}{h_\gamma} \|\mathbf{e}_{\mathbf{u}_2}^n\|_{L^2(\gamma)}^2 \right)^{1/2} \right. \\ & \quad \left. + \left(\sum_{m=1}^n \sum_{\gamma \in \Gamma_{h,D}} \frac{h_\gamma}{\sigma_\gamma} \int_{t^{m-1}}^{t^m} |\mathbf{u}'_2 - I_h(\mathbf{u}'_2)|_{h,\gamma}^2 dt \right)^{1/2} \left(\sum_{m=1}^{n-1} k \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_\gamma}{h_\gamma} \|\mathbf{e}_{\mathbf{u}_2}^m\|_{L^2(\gamma)}^2 \right)^{1/2} \right]. \end{aligned}$$

For T_8 , we write

$$\begin{aligned} T_8 & = \alpha \left(\left(\int_0^k \operatorname{div}(\mathbf{u}'_1(t) - I_h(\mathbf{u}'_1(t))) dt, e_p^1 \right)_{\Omega_1} \right. \\ & \quad \left. + \sum_{m=2}^n \left(\int_{t^{m-2}}^{t^{m-1}} \operatorname{div}(\mathbf{u}'_1(t) - I_h(\mathbf{u}'_1(t))) dt, e_p^m \right)_{\Omega_1} \right). \end{aligned}$$

Therefore

$$|T_8| \leq \alpha \left(\frac{c_0}{\lambda} \right)^{1/2} \left(\sum_{m=1}^n k \|e_p^m\|_{L^2(\Omega_1)}^2 \right)^{1/2} \left(\frac{\lambda}{c_0} \right)^{1/2} \|\operatorname{div}(\mathbf{u}'_1 - I_h(\mathbf{u}'_1))\|_{L^2(\Omega_1 \times]0, t_n])}.$$

The bounds for T_{10} and T_{11} are straightforward

$$\begin{aligned}
 |T_{10}| &= c_0 \left| \sum_{m=1}^n \left(\int_{t^{m-1}}^{t^m} (p'(t) - r_{h_1}(p'(t))) dt, e_p^m \right)_{\Omega_1} + \frac{1}{\mu_f} \sum_{m=1}^n k (\mathbf{K} \nabla a_p^m, \nabla e_p^m)_{\Omega_1} \right| \\
 &\leq c_0 \left(\sum_{m=1}^n k \|e_p^m\|_{L^2(\Omega_1)}^2 \right)^{1/2} \|p' - r_{h_1}(p')\|_{L^2(\Omega_1 \times]0, t^n])} \\
 &\quad + \frac{1}{\mu_f} \left(\sum_{m=1}^n k \|\mathbf{K}^{1/2} \nabla e_p^m\|_{L^2(\Omega_1)}^2 \right)^{1/2} \left(\sum_{m=1}^n k \|\mathbf{K}^{1/2} \nabla a_p^m\|_{L^2(\Omega_1)}^2 \right)^{1/2}, \\
 |T_{11}| &\leq \left(\frac{c_0}{3} \right)^{1/2} k \left(\sum_{m=2}^n k \|e_p^m\|_{L^2(\Omega_1)}^2 \right)^{1/2} \\
 &\quad \cdot \left(c_0^{1/2} \|p''\|_{L^2(\Omega_1 \times]k, t^n])} + 2 \frac{\alpha}{c_0^{1/2}} \|\operatorname{div} \mathbf{u}''\|_{L^2(\Omega_1 \times]0, t^n])} \right).
 \end{aligned}$$

Finally, for T_{12} , we have

$$\begin{aligned}
 |T_{12}| &\leq k \left[\left(\frac{k}{3} \right)^{1/2} \left(c_0^{1/2} \|p''\|_{L^2(\Omega_1 \times]0, k])} + \frac{\alpha}{c_0^{1/2}} \|\operatorname{div} \mathbf{u}''\|_{L^2(\Omega_1 \times]0, k])} \right) c_0^{1/2} \|e_p^1\|_{L^2(\Omega_1)} \right. \\
 &\quad + \frac{\alpha}{\lambda^{1/2}} \|p'\|_{L^\infty(0, k; L^2(\Omega_1))} \lambda^{1/2} \|\operatorname{div}(\delta \mathbf{e}_{\mathbf{u}_1}^1)\|_{L^2(\Omega_1)} \\
 &\quad + \frac{\alpha}{(\lambda + 2\mu)^{1/2}} (\lambda + 2\mu)^{1/2} \left(\sum_{\gamma_1 \in \Gamma_{h,1}} \frac{\bar{\sigma}_{\gamma_1}}{h_{\gamma_1}} \frac{1}{k} \|\delta(\mathbf{e}_{\mathbf{u}_1}^1 - \mathbf{e}_{\lambda}^1)\|_{L^2(\gamma_1)}^2 \right)^{1/2} \\
 &\quad \cdot \left. \left(\sum_{\gamma_1 \in \Gamma_{h,1}} \frac{h_{\gamma_1}}{\bar{\sigma}_{\gamma_1}} \|p'\|_{L^2(\gamma_1 \times]0, k])}^2 \right)^{1/2} \right].
 \end{aligned}$$

3.5.3. Error estimates

By combining (3.50), (3.37), and the upper bounds of the preceding section, and by suitable applications of Young's inequality and Gronwall's lemma, we obtain the following proposition. To simplify, we do not specify the constants.

Proposition 3.2. *Let the triangulations satisfy (3.1). Under the assumptions of Proposition 3.1, there exist constants C_1 and C_2 independent of $h_1, h_2, k, \lambda, \mu, c_0, \alpha, \mu_f$ and n such that for all $n \geq 2$,*

$$\begin{aligned}
 |\mathbf{e}_{\mathbf{u}}^n|_{X_h}^2 &+ \sum_{m=1}^n |\delta \mathbf{e}_{\mathbf{u}}^m|_{X_h}^2 + \frac{1}{\mu_f} \sum_{m=1}^n k \|\mathbf{K}^{1/2} \nabla e_p^m\|_{L^2(\Omega_1)}^2 \\
 &+ c_0 \left(\|e_p^n\|_{L^2(\Omega_1)}^2 + \sum_{m=1}^n \|\delta e_p^m\|_{L^2(\Omega_1)}^2 \right)
 \end{aligned}$$

$$\begin{aligned}
& + (\lambda + 2\mu) \sum_{i=1}^2 \sum_{\gamma_i \in \Gamma_{h,i}} \left[\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{e}_{\mathbf{u}_i}^n - \mathbf{e}_{\boldsymbol{\lambda}}^n\|_{L^2(\gamma_i)}^2 + \sum_{m=1}^n \left(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \frac{1}{k} \right) \|\delta(\mathbf{e}_{\mathbf{u}_i}^m - \mathbf{e}_{\boldsymbol{\lambda}}^m)\|_{L^2(\gamma_i)}^2 \right] \\
& + (\lambda + 2\mu) \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_{\gamma}}{h_{\gamma}} \left(\|\mathbf{e}_{\mathbf{u}_2}^n\|_{L^2(\gamma)}^2 + \sum_{m=1}^n \|\delta \mathbf{e}_{\mathbf{u}_2}^m\|_{L^2(\gamma)}^2 \right) \\
& \leq C_1 \exp(C_2 t^n) \left[\|\mathbf{u} - I_h(\mathbf{u})\|_{H^1(0,t^n;X_h)}^2 + \|p - r_{h_1}(p)\|_{H^1(0,t^n;L^2(\Omega_1))}^2 \right. \\
& \quad + \sum_{\gamma \in \Gamma_{h,D}} \frac{h_{\gamma}}{\sigma_{\gamma}} \|\mathbf{u}_2 - I_h(\mathbf{u}_2)\|_{H^1(0,t^n;X_h(\gamma))}^2 \\
& \quad + (\lambda + 2\mu) \sum_{i=1}^2 \sum_{\gamma_i \in \Gamma_{h,i}} \frac{1}{h_{\gamma_i}} (\sigma_{\gamma_i} + \bar{\sigma}_{\gamma_i} + \varphi_i^2) \|\mathbf{u}_i - I_h(\mathbf{u}_i) - (\boldsymbol{\lambda} - \rho_H(\boldsymbol{\lambda}))\|_{H^1(0,t^n;L^2(\gamma_i)^3)}^2 \\
& \quad + \sum_{i=1}^2 \sum_{\gamma_i \in \Gamma_{h,i}} \left(\frac{h_{\gamma_i}}{\sigma_{\gamma_i}} \|\mathbf{u}_i(0) - I_h(\mathbf{u}_i(0))\|_{X_h(\gamma_i)}^2 + t^n \frac{h_{\gamma_i}}{\bar{\sigma}_{\gamma_i}} \|\mathbf{u}_i - I_h(\mathbf{u}_i)\|_{L^\infty(0,t^n;X_h(\gamma_i))}^2 \right) \\
& \quad + \frac{\alpha^2}{\lambda + 2\mu} \sum_{\gamma_1 \in \Gamma_{h,1}} \left(\frac{h_{\gamma_1}}{\sigma_{\gamma_1}} \|p(0) - r_{h_1}(p(0))\|_{L^2(\gamma_1)}^2 + t^n \frac{h_{\gamma_1}}{\bar{\sigma}_{\gamma_1}} \|p - r_{h_1}(p)\|_{L^\infty(0,t^n;L^2(\gamma_1))}^2 \right) \\
& \quad + \alpha^2 \frac{\lambda}{c_0} \|\operatorname{div}(\mathbf{u}'_1 - I_h(\mathbf{u}'_1))\|_{L^2(\Omega_1 \times]0,t^n])}^2 \\
& \quad + c_0 \|p' - r_{h_1}(p')\|_{L^2(\Omega_1 \times]0,t^n])}^2 + \frac{t^n}{\mu_f} \|\mathbf{K}^{1/2} \nabla(p - r_{h_1}(p))\|_{L^\infty(0,t^n;L^2(\Omega_1))}^2 \\
& \quad + k^2 \left(c_0 \|p''\|_{L^2(\Omega_1 \times]0,t^n])}^2 + \frac{\alpha^2}{c_0} \|\operatorname{div} \mathbf{u}_1''\|_{L^2(\Omega_1 \times]0,t^n])}^2 \right. \\
& \quad \left. + \frac{1}{\lambda} \|p'\|_{L^\infty(0,k;L^2(\Omega_1))}^2 + \frac{\alpha^2}{\lambda + 2\mu} \sum_{\gamma_1 \in \Gamma_{h,1}} \frac{h_{\gamma_1}}{\sigma_{\gamma_1}} \|p'\|_{L^2(\gamma_1 \times]0,k])}^2 \right) \Bigg]. \tag{3.51}
\end{aligned}$$

This proposition yields the main result of this section.

Theorem 3.1. *Let the triangulations satisfy (3.1) and (3.3) and suppose that Γ_H and $\Gamma_{h,i}$ are related as follows: There exists a constant C independent of h_1, h_2 and H such that for any $\tau \in \Gamma_H$ and any $\gamma_i \in \Gamma_{h,i}$ that intersects τ ,*

$$\frac{H_{\tau}}{h_{\gamma_i}} \leq C \frac{H}{h_i}. \tag{3.52}$$

Let $k \leq 1/2$ and let the parameters be chosen according to (3.49). We assume that the exact solution has the following regularity:

$$\begin{aligned} \mathbf{u}_i &\in H^1(0, T; H^{s_u}(\Omega_i)^3), \quad i = 1, 2, \quad \operatorname{div} \mathbf{u}_1'' \in L^2(\Omega_1 \times]0, T[), \\ p &\in H^1(0, T; H^{s_p}(\Omega_1)), \quad p'' \in L^2(\Omega_1 \times]0, T[), \end{aligned} \quad (3.53)$$

with $s_u > 3/2$ and $s_p > 1/2$. Then there exists constants C_1 and C_2 , independent of h_1, h_2, H, k and n such that for all $n \geq 2$,

$$\begin{aligned} & \|\mathbf{u}(t^n) - \mathbf{u}_h^n\|_{X_h}^2 + \sum_{m=1}^n \|\delta(\mathbf{u}(t^m) - \mathbf{u}_h^m)\|_{X_h}^2 + \frac{1}{\mu_f} \sum_{m=1}^n k \|\mathbf{K}^{1/2} \nabla(p(t^m) - p_{h_1}^m)\|_{L^2(\Omega_1)}^2 \\ & + c_0 \left(\|p(t^n) - p_{h_1}^n\|_{L^2(\Omega_1)}^2 + \sum_{m=1}^n \|\delta(p(t^m) - p_{h_1}^m)\|_{L^2(\Omega_1)}^2 \right) \\ & + (\lambda + 2\mu) \sum_{i=1}^2 \sum_{\gamma_i \in \Gamma_{h,i}} \left[\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\mathbf{u}_i(t^n) - \mathbf{u}_{h_i}^n - (\boldsymbol{\lambda}(t^n) - \boldsymbol{\lambda}_H^n)\|_{L^2(\gamma_i)}^2 \right. \\ & \left. + \sum_{m=1}^n \left(\frac{\sigma_{\gamma_i}}{h_{\gamma_i}} + \frac{\bar{\sigma}_{\gamma_i}}{h_{\gamma_i}} \frac{1}{k} \right) \|\delta(\mathbf{u}_i(t^m) - \mathbf{u}_{h_i}^m - (\boldsymbol{\lambda}(t^m) - \boldsymbol{\lambda}_H^m))\|_{L^2(\gamma_i)}^2 \right] \\ & + (\lambda + 2\mu) \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_\gamma}{h_\gamma} \left(\|\mathbf{u}_2(t^n) - \mathbf{u}_{h_2}^n\|_{L^2(\gamma)}^2 + \sum_{m=1}^n \|\delta(\mathbf{u}_2(t^m) - \mathbf{u}_{h_2}^m)\|_{L^2(\gamma)}^2 \right) \\ & \leq C_1 \exp(C_2 t^n) \left(k^2 + h_1^{2(r_u-1)} + h_2^{2(r_u-1)} + h_1^{2(r_p-1)} + \left(\frac{H}{h_1} + \frac{H}{h_2} \right) H^{2r_\lambda-1} \right), \end{aligned} \quad (3.54)$$

where

$$r_u = \min(k+1, s_u), \quad r_p = \min(m+1, s_p), \quad r_\lambda = \min\left(\ell+1, s_u - \frac{1}{2}\right).$$

Proof. All interpolation errors in (3.51) are straightforward except the part involving $\boldsymbol{\lambda}$ in

$$(\lambda + 2\mu) \sum_{i=1}^2 \sum_{\gamma_i \in \Gamma_{h,i}} \frac{1}{h_{\gamma_i}} (\sigma_{\gamma_i} + \bar{\sigma}_{\gamma_i} + \varphi_i^2) \|\mathbf{u}_i - I_h(\mathbf{u}_i) - (\boldsymbol{\lambda} - \rho_H(\boldsymbol{\lambda}))\|_{H^1(0, t^n; L^2(\gamma_i)^3)}^2.$$

The difficulty comes from the interpolation of $\boldsymbol{\lambda}$ on the mesh \mathcal{T}_H and its division by h_{γ_i} , much smaller than H . To handle this term we observe that the regularity of \mathcal{T}_H implies that an arbitrary γ_i intersects at most L elements of \mathcal{T}_H , where L is a fixed number that is independent of γ_i and H . Thus, let τ_j , $1 \leq j \leq L$, be the elements of \mathcal{T}_H that intersect γ_i ; to simplify, we suppress the index i . In view of (3.52), we can

write

$$\begin{aligned}
 \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\boldsymbol{\lambda}(t^n) - \rho_H(\boldsymbol{\lambda}(t^n))\|_{L^2(\gamma_i)}^2 &= \sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \sum_{j=1}^L \|\boldsymbol{\lambda}(t^n) - \rho_H(\boldsymbol{\lambda}(t^n))\|_{L^2(\gamma_i \cap \tau_j)}^2 \\
 &= \sum_{\gamma_i \in \Gamma_{h,i}} \sum_{j=1}^L \frac{H_{\tau_j}}{h_{\gamma_i}} \frac{\sigma_{\gamma_i}}{H_{\tau_j}} \|\boldsymbol{\lambda}(t^n) - \rho_H(\boldsymbol{\lambda}(t^n))\|_{L^2(\gamma_i \cap \tau_j)}^2 \\
 &\leq C \sum_{\gamma_i \in \Gamma_{h,i}} \frac{H}{h_i} \sum_{j=1}^L \frac{\sigma_{\gamma_i}}{H_{\tau_j}} \|\boldsymbol{\lambda}(t^n) - \rho_H(\boldsymbol{\lambda}(t^n))\|_{L^2(\gamma_i \cap \tau_j)}^2 \\
 &\leq CM_i \frac{H}{h_i} \sum_{\tau \in \Gamma_H} \frac{1}{H_{\tau}} \|\boldsymbol{\lambda}(t^n) - \rho_H(\boldsymbol{\lambda}(t^n))\|_{L^2(\tau)}^2.
 \end{aligned}$$

Then a standard interpolation argument gives

$$\sum_{\gamma_i \in \Gamma_{h,i}} \frac{\sigma_{\gamma_i}}{h_{\gamma_i}} \|\boldsymbol{\lambda}(t^n) - \rho_H(\boldsymbol{\lambda}(t^n))\|_{L^2(\gamma_i)}^2 \leq C \left(\frac{H}{h_i} \right) H^{2r_{\lambda}-1} \|\boldsymbol{\lambda}\|_{L^\infty(0,t^n; H^{r_{\lambda}}(\Gamma_{12}^3))}^2,$$

and (3.54) is easily deduced from this. \square

Remark 3.1. When H is much larger than h , the estimate (3.54) is not optimal. The lack of optimality arises from the term in the second line of the right-hand side of (3.51), studied in the above proof. Unfortunately, as was observed in Ref. 18, there seems to be no way for improving this estimate. However, by choosing mortar space of sufficiently high degree, optimality can be achieved. For example, choosing linear approximation for displacements ($k = 1$), quadratic mortars ($\ell = 2$), and $H = \min(h_1, h_2)^{1/2}$, we get optimal error provided sufficient regularity holds.

4. The Interface Problem

Since our poroelastic problem is linear, we propose to reduce the elasticity equations in (3.8)–(3.13) to a system on the interface according to a superposition principle, such as the one studied in Refs. 16 and 18. It consists in splitting the solution into two parts: One part that only depends on the Lagrange multiplier, and another part that only depends on the data and previously computed quantities. Then these two parts are assembled when substituted into the interface equations at each time step, with only the Lagrange multiplier as unknown. The matrix of the system is generally not sparse, but its dimension is substantially smaller than that of the original system, and we shall see below that it is positive definite. More precisely, for each basis function $\boldsymbol{\lambda}_H = \boldsymbol{\lambda}_{H,j} \in \Lambda_H$, $1 \leq j \leq P_H$, where P_H is the dimension of Λ_H , we solve the

problem: Find $\mathbf{u}_h(\boldsymbol{\lambda}_{H,j}) \in X_h$ solution of

$$\begin{aligned} \forall \mathbf{v}_h \in X_h, \quad & \sum_{i=1}^2 (B_i(\mathbf{u}_{h_i}(\boldsymbol{\lambda}_{H,j}), \mathbf{v}_{h_i}) + J_i(\mathbf{u}_{h_i}(\boldsymbol{\lambda}_{H,j}), \mathbf{v}_{h_i})) + b_D(\mathbf{u}_{h_2}(\boldsymbol{\lambda}_{H,j}), \mathbf{v}_{h_2}) \\ & = \sum_{i=1}^2 (L_i(\boldsymbol{\lambda}_{H,j}, \mathbf{v}_{h_i}) + J_i(\boldsymbol{\lambda}_{H,j}, \mathbf{v}_{h_i})). \end{aligned} \quad (4.1)$$

This basis, which corresponds to the first part of the solution and is independent of time, will be used at all time steps. It can be computed and stored in parallel.

Next for $n \geq 2$, knowing \mathbf{u}_h^{n-1} , \mathbf{u}_h^{n-2} , $p_{h_1}^n$, and $\boldsymbol{\lambda}_H^{n-1}$, the part $\bar{\mathbf{u}}_h^n$ of the solution depending on the data and other known quantities is computed by solving: Find $\bar{\mathbf{u}}_h^n \in X_h$ such that

$$\begin{aligned} \forall \mathbf{v}_h \in X_h, \quad & \sum_{i=1}^2 (B_i(\bar{\mathbf{u}}_h^n, \mathbf{v}_{h_i}) + J_i(\bar{\mathbf{u}}_h^n, \mathbf{v}_{h_i})) + b_D(\bar{\mathbf{u}}_h^n, \mathbf{v}_{h_2}) \\ & = \sum_{i=1}^2 \bar{J}_i(\mathbf{u}_{h_i}^{n-1} - \boldsymbol{\lambda}_H^{n-1}, \mathbf{v}_{h_i}) + l(p_{h_1}^n, \mathbf{v}_{h_1}) + l_{f,N}^n(\mathbf{v}_h). \end{aligned} \quad (4.2)$$

The functions $\bar{\mathbf{u}}_h^n$ and $\mathbf{u}_h(\boldsymbol{\lambda}_H)$ are combined by solving the interface system for $\boldsymbol{\lambda}_H^n$: Find $\boldsymbol{\lambda}_H^n \in \Lambda_H$ such that

$$\begin{aligned} \forall \boldsymbol{\mu}_H \in \Lambda_H, \quad & \sum_{i=1}^2 (L_i(\boldsymbol{\mu}_H, \mathbf{u}_{h_i}(\boldsymbol{\lambda}_H^n)) - J_i(\mathbf{u}_{h_i}(\boldsymbol{\lambda}_H^n) - \boldsymbol{\lambda}_H^n, \boldsymbol{\mu}_H)) \\ & = \alpha \int_{\Gamma_{12}} p_{h_1}^n \mathbf{n}_{12} \cdot \boldsymbol{\mu}_H \, ds \\ & \quad - \sum_{i=1}^2 (L_i(\boldsymbol{\mu}_H, \bar{\mathbf{u}}_{h_i}) - J_i(\bar{\mathbf{u}}_{h_i}, \boldsymbol{\mu}_H) + \bar{J}_i(\mathbf{u}_{h_i}^{n-1} - \boldsymbol{\lambda}_H^{n-1}, \boldsymbol{\mu}_H)), \end{aligned} \quad (4.3)$$

and setting

$$\mathbf{u}_h^n = \mathbf{u}_h(\boldsymbol{\lambda}_H^n) + \bar{\mathbf{u}}_h^n. \quad (4.4)$$

Note that this requires storing the basis $\mathbf{u}_h(\boldsymbol{\lambda}_{H,j})$ for each $1 \leq j \leq P_H$. If the storage cost is prohibitive, the projection $\zeta_{H,j}$ of the relevant interface term onto Λ_H ,

$$\forall \boldsymbol{\mu}_H \in \Lambda_H, \quad (\zeta_{H,j}, \boldsymbol{\mu}_H)_{\Gamma_{12}} = \sum_{i=1}^2 (L_i(\boldsymbol{\mu}_H, \mathbf{u}_{h_i}(\boldsymbol{\lambda}_{H,j})) - J_i(\mathbf{u}_{h_i}(\boldsymbol{\lambda}_{H,j}) - \boldsymbol{\lambda}_{H,j}, \boldsymbol{\mu}_H)),$$

may be computed and stored instead. But since $\mathbf{u}_h(\boldsymbol{\lambda}_{H,j})$ is no longer stored, $\mathbf{u}_h(\boldsymbol{\lambda}_H^n)$ must be computed at each time step by solving the elasticity problem (3.12) with data $\boldsymbol{\lambda}_H^n$.

Next for $n \geq 2$, we compute $p_{h_1}^n$ by: Find $p_{h_1}^n \in M_{h_1}$ solution of (3.11):

$$\forall \theta_{h_1} \in M_{h_1}, \quad K^n(p_{h_1}, \theta_{h_1}) = -D^{n-1}(\mathbf{u}_{h_1}, \theta_{h_1}) + S^n(\theta_{h_1}).$$

When $n = 1$, (4.2) and (4.3) are solved with $p_{h_1}^0$ instead of $p_{h_1}^1$ which is not known yet, and $p_{h_1}^1$ is computed by solving (3.10), i.e. by replacing the undefined quantity D^0 by D^1 . Summarizing, the algorithm proceeds as follows:

- (1) Compute the basis $\{\mathbf{u}_h(\boldsymbol{\lambda}_{H,j}); 1 \leq j \leq P_H\}$.
- (2) Set $p_{h_1}^0 = r_{h_1}(p_0)$ and $\mathbf{u}_{h_1}^0 = I_{h_1}(\mathbf{u}_0)$.
- (3) Compute $\bar{\mathbf{u}}_h^1$ by solving (4.2) with $p_{h_1}^0$. Next compute $\boldsymbol{\lambda}_H^1$ by solving (4.3) also with $p_{h_1}^0$, and compute \mathbf{u}_h^1 by (4.4).
- (4) Compute $p_{h_1}^1$ by solving (3.10).
- (5) For $2 \leq n \leq N$,
 - (a) Compute $\bar{\mathbf{u}}_h^n$ by solving (4.2), next compute $\boldsymbol{\lambda}_H^n$ by solving (4.3), and compute \mathbf{u}_h^n by (4.4).
 - (b) Compute $p_{h_1}^n$ by solving (3.11).

Note that (4.1) and (4.2) have the same left-hand side; these are standard elasticity operators with additional trace terms on the interface. Owing to these trace terms, namely J_i , that eliminate the rigid body motions, (4.1) and (4.2) are *both uniquely solvable regardless of the boundary term b_D* . Of course, (3.10) and (3.11) are also uniquely solvable. Therefore this method applies to floating subdomains. As proved below, the matrix of (4.3) is positive definite, hence (4.3) is also uniquely solvable. Admitting this result for the moment, it can be readily checked that the functions \mathbf{u}_h^n , $p_{h_1}^n$ and $\boldsymbol{\lambda}_H^n$ constructed by this algorithm solve (3.8)–(3.13). It remains to prove that the matrix of (4.3) is positive definite.

Theorem 4.1. *If σ_γ is chosen according to (3.49), the matrix of (4.3) is positive definite.*

Proof. Let d_H denote the bilinear form of the left-hand side of (4.3):

$$d_H(\boldsymbol{\mu}_H, \boldsymbol{\mu}_H) = \sum_{i=1}^2 (L_i(\boldsymbol{\mu}_H, \mathbf{u}_{h_i}(\boldsymbol{\lambda}_H)) - J_i(\mathbf{u}_{h_i}(\boldsymbol{\lambda}_H) - \boldsymbol{\lambda}_H, \boldsymbol{\mu}_H)).$$

On the one hand,

$$d_H(\boldsymbol{\lambda}_H, \boldsymbol{\lambda}_H) = \sum_{i=1}^2 (L_i(\boldsymbol{\lambda}_H, \mathbf{u}_{h_i}(\boldsymbol{\lambda}_H)) - J_i(\mathbf{u}_{h_i}(\boldsymbol{\lambda}_H) - \boldsymbol{\lambda}_H, \boldsymbol{\lambda}_H)).$$

On the other hand, we infer from (4.1) with $\mathbf{v}_h = \mathbf{u}_h(\boldsymbol{\lambda}_H)$ that

$$\begin{aligned} d_H(\boldsymbol{\lambda}_H, \boldsymbol{\lambda}_H) &= \sum_{i=1}^2 (B_i(\mathbf{u}_{h_i}(\boldsymbol{\lambda}_H), \mathbf{u}_{h_i}(\boldsymbol{\lambda}_H)) + J_i(\mathbf{u}_{h_i}(\boldsymbol{\lambda}_H) - \boldsymbol{\lambda}_H, \mathbf{u}_{h_i}(\boldsymbol{\lambda}_H) - \boldsymbol{\lambda}_H)) \\ &\quad + b_D(\mathbf{u}_{h_2}(\boldsymbol{\lambda}_H), \mathbf{u}_{h_2}(\boldsymbol{\lambda}_H)). \end{aligned}$$

Thus $d_H(\boldsymbol{\lambda}_H, \boldsymbol{\lambda}_H)$ coincides with the expression on the left-hand side of (3.29) when $\mathbf{u}_h = \mathbf{u}_h(\boldsymbol{\lambda}_H)$. Then reverting to this situation, by choosing σ_γ satisfying (3.49),

$$\begin{aligned}
d_H(\boldsymbol{\lambda}_H, \boldsymbol{\lambda}_H) &\geq |\mathbf{u}_{h_1}(\boldsymbol{\lambda}_H)|_{h,1}^2 \\
&+ \frac{3}{4} \left(|\mathbf{u}_{h_2}(\boldsymbol{\lambda}_H)|_{h,2}^2 + (\lambda + 2\mu) \sum_{\gamma \in \Gamma_{h,D}} \frac{\sigma_\gamma}{h_\gamma} \|\mathbf{u}_{h,2}(\boldsymbol{\lambda}_H)\|_{L^2(\gamma)}^2 \right) \\
&+ \sum_{i=1}^2 J_i(\mathbf{u}_{h_i}(\boldsymbol{\lambda}_H) - \boldsymbol{\lambda}_H, \mathbf{u}_{h_i}(\boldsymbol{\lambda}_H) - \boldsymbol{\lambda}_H).
\end{aligned} \tag{4.5}$$
☐

The governing equations for our poroelastic/elastic system represent a complicated coupled system for which no analytical results have been derived. For the poroelasticity model alone, a small number of analytical solutions exist, see, e.g. Refs. 1, 26 and 22. The canonical analytical example in two dimensions is Mandel’s problem because its implementation is simple and its numerical solution exhibits the unexpected Mandel–Creyer effect which has been verified experimentally for poroelastic media. In this paper, we consider an extension of Mandel’s problem by constructing a manufactured solution in an adjacent elasticity domain such that the interface conditions are satisfied with Mandel’s solution in the poroelasticity domain. This solution allows us to verify the convergence rates proven in Sec. 3.5.

First, we describe Mandel’s problem for the poroelasticity domain. Following Ref. 28, consider a three-dimensional computational domain of length $2a$ in the x -direction, $2b$ in the y -direction, and infinite in the z -direction. The domain is bounded from above and below by an impermeable rigid plate. At time $t = 0$, a uniform load of magnitude $2F$ is applied to the top and the bottom plates. Since the problem is symmetric, we consider only the upper-right quadrant in the xy -plane reducing the equations to the 2D domain $(0, a) \times (0, b)$ as shown in Fig. 2(a). Let $\mathbf{u} = (u_x, u_y)^T$; the boundary conditions are $u_x = 0$ along the left boundary



$(x = 0)$, $u_y = 0$ along the bottom boundary ($y = 0$), $\mathbf{t}_N = (0, -F/a)^T$ along the top boundary ($y = b$), $p = 0$ along the right boundary ($x = a$), and $-\frac{1}{\mu_f} K \nabla p = 0$ along the left, bottom and top boundaries. The initial conditions are $p_0 = 0$ and $\mathbf{u}_0 = \mathbf{0} = \mathbf{u}(p_0)$.

Mandel's original paper²⁶ formulated an analytical solution for the pressure while an analytical expression for the displacement was later derived in Ref. 1. For completeness, we provide these solution components below:

$$p = \frac{2FB(1 + \nu_u)}{3a} \sum_{n=1}^{\infty} \frac{\sin \alpha_n}{\alpha_n - \sin \alpha_n \cos \alpha_n} \left(\cos \frac{\alpha_n x}{a} - \cos \alpha_n \right) \exp \left(-\frac{\alpha_n^2}{a^2} c_0 t \right),$$

$$u_x^{(1)} = F \left(\frac{\nu}{2\mu a} x - \sum_{n=1}^{\infty} \frac{\cos \alpha_n}{\alpha_n - \sin \alpha_n \cos \alpha_n} \exp \left(-\frac{\alpha_n^2}{a^2} c_0 t \right) \cdot \left(\frac{\nu_u}{\nu a} x - \frac{1}{\nu} \sin \left(\frac{\alpha_n x}{a} \right) \right) \right),$$

$$u_y^{(1)} = \left(\frac{-F(1 - \nu)}{2\mu a} + \frac{F(1 - \nu_u)}{\nu a} \sum_{n=1}^{\infty} \frac{\sin \alpha_n \cos \alpha_n}{\alpha_n - \sin \alpha_n \cos \alpha_n} \exp \left(-\frac{\alpha_n^2}{a^2} c_0 t \right) \right) y,$$

where

$$K = \lambda + \frac{2}{3}\mu, \quad E = \mu \frac{9K}{3K + \mu}, \quad \nu = \frac{3K - 2\mu}{6K + 2\mu}, \quad B = \frac{\alpha}{c_0 K + \alpha^2},$$

are the skeleton bulk modulus, Young's modulus, Poisson's coefficient, and Skempton's coefficient respectively. The undrained version of Poisson's coefficient, ν_u , is related to Skempton's coefficient via

$$\frac{\alpha B(1 - 2\nu)}{3} = \frac{\nu_u - \nu}{1 + \nu_u}.$$

Finally, the α_n are the smallest positive solutions to

$$\tan \alpha_n = \frac{1 - \nu}{\nu_u - \nu} \alpha_n.$$

We augment Mandel's problem by defining an elastic domain above the poroelastic domain with height b , see Fig. 2(b), and choose the manufactured solution

$$u_x^{(2)} = u_x^{(1)}, \quad u_y^{(2)} = u_y^{(1)} - \frac{\alpha}{\lambda + 2\mu} (y - b)p,$$

in the elasticity domain satisfying the interface conditions $[\mathbf{u}] = 0$ and $[\sigma(\mathbf{u})\mathbf{n}_{12}] = \alpha p \mathbf{n}_{12}$ on Γ_{12} . As shown in Fig. 2(b), we prescribe the following traction and

Table 1. Numerical errors for the extension of Mandel’s problem.

h	Degrees of freedom	Error	Conv. rate
1/20	1197	1.07E-2	—
1/40	4387	4.91E-3	1.09
1/80	16767	2.34E-3	1.05
1/160	65527	1.14E-3	1.03
1/320	259047	5.65E-4	1.01

displacement boundary conditions along the top boundary of the elasticity domain,

$$\mathbf{g}_1 = \left(-\frac{\alpha\mu}{\lambda+2\mu}(y-b)\frac{\partial}{\partial x}p, -\frac{F}{a} \right)^T, \quad \mathbf{h} = \left(u_1^{(1)}, u_2^{(2)} - \frac{\alpha}{\lambda+2\mu}(y-b)p \right)^T.$$

Differentiating the manufactured solution in the elasticity domain gives the loading function,

$$\mathbf{f}_2 = \left(-(\lambda+2\mu)\frac{\partial^2}{\partial x^2}u_x^{(1)} + (\lambda+\mu)\frac{\alpha}{\lambda+2\mu}\frac{\partial}{\partial x}p, \frac{\alpha\mu}{\lambda+2\mu}(y-b)\frac{\partial^2}{\partial x^2}p \right)^T.$$

To verify the error estimates in Sec. 3.5, we use the following model parameters $E = 1 \times 10^4$, $\nu = 0.2$, $a = 1$, $b = 0.5$, $F = 2000$, $K = 100I$, $\mu_f = 1$, $\alpha = 1$, $c_0 = 0.1$, and discretize each subdomain using piecewise linear finite elements. For simplicity, the grids are chosen to match along the interface and the mortar mesh is defined to be the trace of the subdomain meshes with piecewise linear mortars. We take $\Delta t = 1 \times 10^{-8}$ to isolate the spatial discretization errors and take 100 time steps. We use the analytical/manufactured solution to calculate the left-hand side of the error estimate (3.51) on a sequence of uniformly refined meshes and report the results in Table 1. We see that the order of convergence of the numerical solution is $O(h)$ which confirms the theoretical result.

6. Summary

A model coupling elasticity and poroelasticity has been formulated and analyzed. A simplified model based on Mandel’s problem has been employed for verifying computationally convergence of the scheme. More realistic three-dimensional studies have been considered by the authors but are not included here because of the current length of the paper. Future work will address generalizing the coupling between elasticity and multiphase flow problems and will include both an energy balance and reactive transport.

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