Derivation of the Stress States for {111} ⟨112⟩ Multiple Slip and Twinning

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The yield equation of a slip or twinning system is represented by a surface in five dimensional stress coordinates. Because of crystallographic symmetry, the yield surfaces of the twelve {111}⟨112⟩ twin systems can be arranged as faces of three separate tetrahedra in three-dimensional shear stress coordinates σ_{23}, σ_{31}, σ_{12}. The other two stress components then determine the size of the tetrahedra. For {111}⟨112⟩ (or {112}⟨111⟩) slip, three additional tetrahedra are needed for slip in the reverse direction. A general shape change requirement of five or more active twin (or slip) systems may be found as intersections of five or more faces among the tetrahedra with the requirement that the stresses at the intersections do not exceed the yield value for the nonactive systems. These intersections have been obtained systematically. It is verified that the lists of stress states previously reported by Hosford and Chin for {111}⟨112⟩ multiple slip and twinning are complete.

PREVIOUSLY Hosford and Chin[1] have obtained a list of stress states for the simultaneous activation of five or more {111}⟨112⟩ slip or twin systems. More recently, Chin and Wonsiewicz[2] have obtained a similar list for {123}⟨111⟩ slip. These stress states were obtained by solving five or more simultaneous equations expressing the critical resolved shear stress for slip (or twin) in terms of the applied stresses, with the requirement that the critical stress is not exceeded in the remaining systems. The procedure was facilitated by the list of active systems for deformation in axisymmetric flow, obtained from our computer solutions of Taylor's minimum work analysis.[3,4] A hypothesis had been discovered empirically by one of us[5] that in the case of {111}⟨110⟩ slip, the list of active systems for enforcing *axisymmetric* flow is identical to the list previously obtained by Bishop and Hill[6-8] for enforcing an *arbitrary* shape change. Since the general applicability of this hypothesis is uncertain, it is important that a systematic search be made for all permissible stress states, such as that done by Bishop and Hill, before the lists for axisymmetric flow can be applied for arbitrary deformation. The present paper deals with such a search in the cases of {111}⟨112⟩ slip and twinning.

ANALYSIS OF {111}⟨112⟩ TWINNING

Using the previously adapted notations[1] for the twelve {111}⟨112⟩ twin systems, see Table I, and slightly altering the resolved shear stress equations, expressions for yielding on these systems are:

System	Yield Expression	Constraints
$a1$	$2F - G - H = S_1$	
$b1$	$-2F + G - H = S_1$	$-\frac{1}{2}S_1 \le F \le \frac{1}{2}S_1$
$c1$	$2F + G + H = S_1$	$-S_1 \le G \le S_1$
$d1$	$-2F - G + H = S_1$	$-S_1 \le H \le S_1$
$a2$	$-F + 2G - H = S_2$	
$b2$	$F - 2G - H = S_2$	$-S_2 \le F \le S_2$
$c2$	$-F - 2G + H = S_2$	$-\frac{1}{2}S_2 \le G \le \frac{1}{2}S_1$
$d2$	$F + 2G + H = S_2$	$-S_2 \le H \le S_2$
$a3$	$-F - G + 2H = S_3$	
$b3$	$F + G + 2H = S_3$	$-S_3 \le F \le S_3$
$c3$	$-F + G - 2H = S_3$	$-S_3 \le G \le S_3$
$d3$	$F - G - 2H = S_3$	$-\frac{1}{2}S_3 \le H \le \frac{1}{2}S_3$

[1]

where

$$S_1 = 1 - B + C \qquad S_2 = 1 - C + A \qquad S_3 = 1 - A + B \qquad [2]$$

and

$$A = \frac{\sigma_{22} - \sigma_{33}}{3\sqrt{2}\tau}, \quad B = \frac{\sigma_{33} - \sigma_{11}}{3\sqrt{2}\tau}, \quad C = \frac{\sigma_{11} - \sigma_{22}}{3\sqrt{2}\tau}$$

$$F = \frac{\sigma_{23}}{3\sqrt{2}\tau}, \quad G = \frac{\sigma_{31}}{3\sqrt{2}\tau}, \quad H = \frac{\sigma_{12}}{3\sqrt{2}\tau}$$

are the ratios of the external stress components σ_{ij} to the critical resolved shear stress for twinning τ. Although the yield expressions [1] represent surfaces in five-dimensional stress coordinates, the above arrangement permits their separation into three tetrahedra T_1, T_2, T_3 (in F, G, H coordinates) of varying size $S \ge 0$ depending on the values A, B, C. Note that since $A + B + C = 0$, the sizes of the three tetrahedra are not independent of one another; in particular $S_1 + S_2 + S_3 = 3$.

With the tetrahedra representation of the yield surface, twinning on a single system is represented by a point on one of the twelve faces of Eqs. [1]. Duplex twinning is represented by an edge, and simultaneous twinning on three systems is represented by a vertex. When the size of a tetrahedron shrinks to a point ($S = 0$), four systems become active simultaneously. Fig. 1 shows a perspective drawing of one such tetra-

Table I. The Twelve {111} <112> Twin (or Slip) Systems

Plane	111			$\bar{1}\bar{1}1$			$\bar{1}11$			$1\bar{1}1$		
Direction	$\bar{2}11$	$1\bar{2}1$	$11\bar{2}$	$2\bar{1}1$	$\bar{1}21$	$\bar{1}\bar{1}\bar{2}$	211	$\bar{1}\bar{2}1$	$\bar{1}1\bar{2}$	$\bar{2}\bar{1}1$	121	$1\bar{1}\bar{2}$
System	$a1$*	$a2$	$a3$	$b1$	$b2$	$b3$	$c1$	$c2$	$c3$	$d1$	$d2$	$d3$

*For slip, there are twelve additional systems $-a1$, $-a2$, and so forth, representing glide in the reverse direction.

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hedron. Projections on the F, G plane are given in Fig. 2 for several values of H. In Fig. 2 the faces project as sides of a parallelogram, the edges as the four corners of the parallelogram plus the two diagonal lines at $H = \pm S$ for T_1 and T_2 and $H = \pm\frac{1}{2}S$ for T_3, and the vertices as corners of the solid-lined rectangle (or square) whose sides are the loci of the edges as H is varied. Values of F, G, H for all edges and vertices are listed in Tables II and III, respectively, together with the associated twin systems. Since the simultaneous activation of five or more systems corresponds to an intersection of five or more faces among the tetrahedra, a complete list of stress states can be obtained by systematically searching for all such intersections. This procedure is similar to that employed by Piehler[9] in a reexamination of the $\{111\}\langle 110\rangle$ problem worked out by Bishop.[8]

Let us denote p (point) for simultaneous activation of four systems, v (vertex) for three, e (edge) for two, f (face) for one, and o (outside) for zero systems. (The condition "o" means that the tetrahedron lies outside the other two, *i.e.*, faces from the other two tetrahedra form the yield surface.) All possible intersections of five systems can then be divided into the following three categories: 1) a point intersecting p, v, e, f, or o from the other two tetrahedra; 2) a vertex intersecting another v, e, or o; and 3) an edge intersecting another e or f:

1.a) $p_1 + p_2 + o_3$, $p_1 + p_2 + f_3$, $p_1 + p_2 + e_3$, $p_1 + p_2 + v_3$, $p_1 + p_2 + p_3$

b) $p_1 + v_2 + o_3$, $p_1 + v_2 + f_3$, $p_1 + v_2 + e_3$, $p_1 + v_2 + v_3$

c) $p_1 + e_2 + o_3$, $p_1 + e_2 + f_3$, $p_1 + e_2 + e_3$

d) $p_1 + f_2 + o_3$, $p_1 + f_2 + f_3$

2.a) $v_1 + v_2 + o_3$, $v_1 + v_2 + f_3$, $v_1 + v_2 + e_3$,

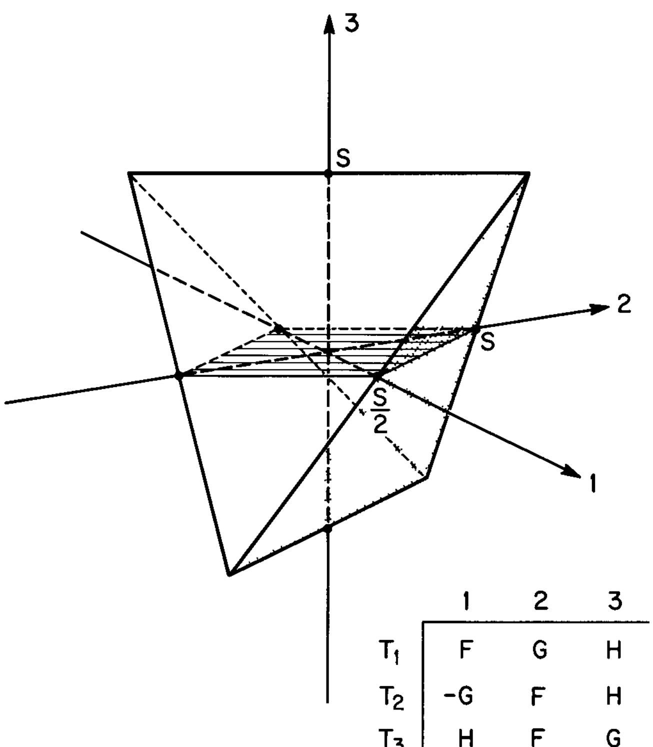

Fig. 1—Perspective view of yield tetrahedron in F, G, H stress coordinates for $\{111\}\langle 112\rangle$ twinning. Table indicates the axes for three separate tetrahedra for a total of twelve faces. A point on a face corresponds to yielding in one of the twelve twin systems. The other two stress components A and B (or C) determine the size of tetrahedron.

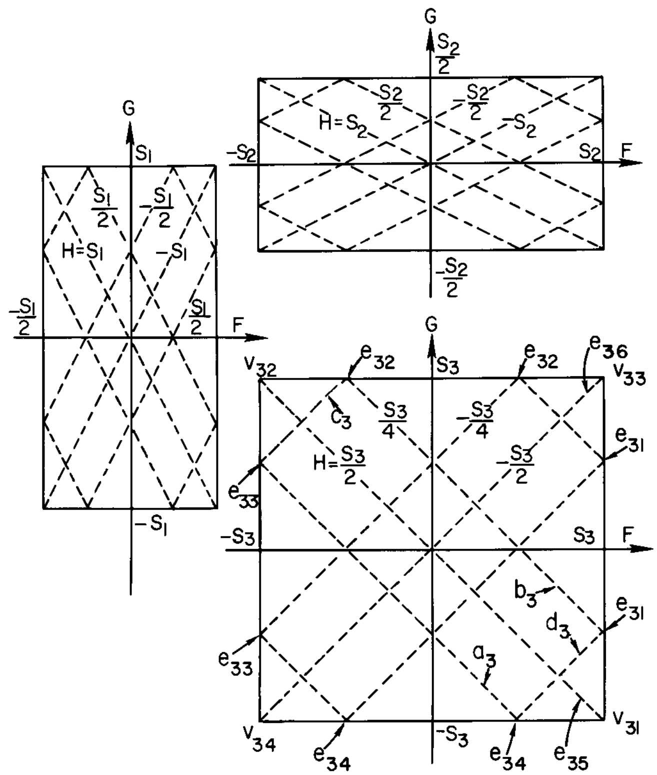

Fig. 2—Projections of the three yield tetrahedra of Fig. 1 onto the F, G plane, for several values of H. Solid-lined rectangle (or square) is the locus of the edges of tetrahedron as H is varied. Examples of faces, edges, and vertices are shown for the third tetrahedron of size S_3.

Table II. Values of F, G, H for All Edges of the Three Tetrahedra with Faces Given in Eq. [1]

Tetrahedron	Edge	Twin Systems	F	G	H
	e_{11}	$a1, c1$	$\frac{1}{2}S_1$	$-H$	H
	e_{12}	$b1, c1$	$-\frac{1}{2}H$	S_1	H
T_1	e_{13}	$b1, d1$	$-\frac{1}{2}S_1$	H	H
	e_{14}	$a1, d1$	$\frac{1}{2}H$	$-S_1$	H
	e_{15}	$c1, d1$	$-\frac{1}{2}G$	G	S_1
	e_{16}	$a1, b1$	$\frac{1}{2}G$	G	$-S_1$
	e_{21}	$b2, d2$	S_2	$-\frac{1}{2}H$	H
	e_{22}	$a2, d2$	$-H$	$\frac{1}{2}S_2$	H
T_2	e_{23}	$a2, c2$	$-S_2$	$\frac{1}{2}H$	H
	e_{24}	$b2, c2$	H	$-\frac{1}{2}S_2$	H
	e_{25}	$c2, d2$	F	$-\frac{1}{2}F$	S_2
	e_{26}	$a2, b2$	F	$\frac{1}{2}F$	$-S_2$
	e_{31}	$b3, d3$	S_3	$-2H$	H
	e_{32}	$b3, c3$	$-2H$	S_3	H
T_3	e_{33}	$a3, c3$	$-S_3$	$2H$	H
	e_{34}	$a3, d3$	$2H$	$-S_3$	H
	e_{35}	$a3, b3$	$-G$	G	$\frac{1}{2}S_3$
	e_{36}	$c3, d3$	G	G	$-\frac{1}{2}S_3$

$v_1 + v_2 + v_3$

b) $v_1 + e_2 + o_3$, $v_1 + e_2 + f_3$, $v_1 + e_2 + e_3$

c) $v_1 + f_2 + f_3$

3.a) $e_1 + e_2 + f_3$, $e_1 + e_2 + e_3$

1) Points

Two tetrahedra of zero size, *i.e.*, two points, intersect at the origin ($F = G = H = 0$), resulting in the activation of eight systems. Since $S_1 + S_2 + S_3 = 3$, we can have only two intersecting points at any one time. This corresponds to the case of $p_1 + p_2 + o_3$ above. There are three ways of obtaining two intersecting points:

	A	B	C	F	G	H
$S_1 = S_2 = 0$	-1	1	0	0	0	0
$S_2 = S_3 = 0$	0	-1	1	0	0	0
$S_3 = S_1 = 0$	1	0	-1	0	0	0

It may be noted that with values of S known, values of A, B, C can be calculated from Eqs. [2] together with the relation $A + B + C = 0$. The above three stress states correspond to states 1 to 3 in Table II of Ref. 1.

Since the only way for tetrahedron 1 (T_1) of zero size ($S_1 = 0$) to intersect a second tetrahedron T_2 is to have $S_2 = 0$ also (both at the origin), the above three stress states are the only ones involving a point. Hence all other combinations in category 1 degenerate to the case of $p_1 + p_2 + o_3$.

2) Vertices

As shown in Fig. 3, vertices of the three tetrahedra project along diagonal lines on the F, G plane depending on the size S. It may be seen that *two vertices* can intersect only at the origin. This corresponds to the case of $S_1 = S_2$ (or S_3) $= 0$ which has already been examined under category 1. Hence cases $v_1 + v_2 + o_3$, $v_1 + v_2 + f_3$, $v_1 + v_2 + e_3$, and $v_1 + v_2 + v_3$ need not be examined further.

The possible intersection of a *vertex* with an *edge* may be examined with the aid of Tables II and III by comparing the F, G, H values of a vertex with those of edges. For example, the values for v_{11} and e_{21} are $(\frac{1}{2}S_1, -S_1, S_1)$ and $(S_2, -\frac{1}{2}H, H)$, respectively. Equating the two values leads to the inconsistent result of $S_1 = 2S_2 = \frac{1}{2}H = H$ and hence an intersection is impossible (except for the previously examined case of $S_1 = S_2 = 0$). Similar comparisons for all other possible vertex-edge intersections lead to the same conclusion. Thus, cases $v_1 + e_2 + o_3$, $v_1 + e_2 + f_3$, and $v_1 + e_2 + e_3$ need not be examined further.

Table III. Values of *F*, *G*, *H* for All Vertices of the Three Tetrahedra with Faces Given in Eq. [1]

Tetrahedron	Vertex	Twin Systems	F	G	H
	v_{11}	$a1, c1, d1$	$\frac{1}{2}S_1$	$-S_1$	S_1
T_1	v_{12}	$b1, c1, d1$	$-\frac{1}{2}S_1$	S_1	S_1
	v_{13}	$a1, b1, c1$	$\frac{1}{2}S_1$	S_1	$-S_1$
	v_{14}	$a1, b1, d1$	$-\frac{1}{2}S_1$	$-S_1$	$-S_1$
	v_{21}	$b2, c2, d2$	S_2	$-\frac{1}{2}S_2$	S_2
T_2	v_{22}	$a2, c2, d2$	$-S_2$	$\frac{1}{2}S_2$	S_2
	v_{23}	$a2, b2, d2$	S_2	$\frac{1}{2}S_2$	$-S_2$
	v_{24}	$a2, b2, c2$	$-S_2$	$-\frac{1}{2}S_2$	$-S_2$
	v_{31}	$a3, b3, d3$	S_3	$-S_3$	$\frac{1}{2}S_3$
T_3	v_{32}	$a3, b3, c3$	$-S_3$	S_3	$\frac{1}{2}S_3$
	v_{33}	$b3, c3, d3$	S_3	S_3	$-\frac{1}{2}S_3$
	v_{34}	$a3, c3, d3$	$-S_3$	$-S_3$	$-\frac{1}{2}S_3$

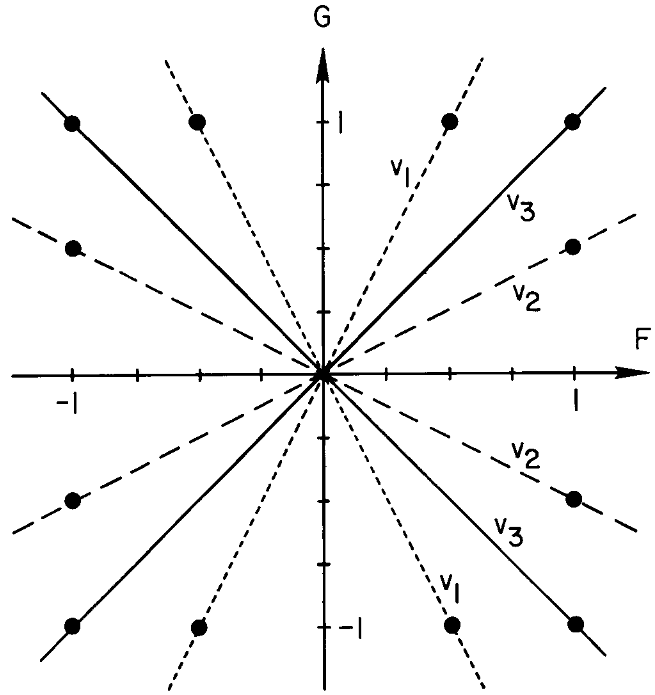

Fig. 3—Loci of vertices for the three tetrahedron of Figs. 1 and 2 for various values of S (size). Solid circles indicate positions for $S_1 = S_2 = S_3 = 1$. Note that two vertices can intersect only at the origin, for which $S = 0$.

The possible intersection of a *vertex* with a *face* may be examined by inserting the F, G, H values of a vertex, Table III, into Eqs. [1] expressing the faces. As an example, consider v_{11} $(\frac{1}{2}S_1, -S_1, S_1)$ vs the four faces of T_2.

Face $a2$: $-F + 2G - H = S_2$

Hence

$$-\tfrac{1}{2}S_1 + 2(-S_1) - S_1 = S_2$$

or

$$-\tfrac{7}{2}S_1 = S_2$$

The intersection is not permissible since it violates the condition $S_1, S_2 \geq 0$.

Face b_2: $F - 2G - H = S_2$

Hence

$$\tfrac{1}{2}S_1 - 2(-S_1) - S_1 = S_2$$

or

$$\tfrac{3}{2}S_1 = S_2$$

which leads to values of $F = S_2/3$, $G = -\frac{2}{3}S_2$, $H = \frac{2}{3}S_2$ for T_2. The value $G = -\frac{2}{3}S_2$ lies outside the limit $-\frac{1}{2}S_2 \leq G \leq \frac{1}{2}S_2$ for T_2, see Eqs. [1], and hence the intersection is not permitted.

Face $c2$: $-F - 2G + H = S_2$

Hence

$-\frac{1}{2}S_1 - 2(-S_1) + S_1 = S_2$

or

$\frac{5}{2}S_1 = S_2$

which leads to $F = S_2/5$, $G = -2S_2/5$, $H = 2S_2/5$ for T_2. These values lie within the limits for T_2 and hence the intersection is permissible.

Face $d2$: $F + 2G + H = S_2$

Hence

$\frac{1}{2}S_1 + 2(-S_1) + S_1 = S_2$

or

$-\frac{1}{2}S_1 = S_2$

which violates the condition S_1, $S_2 \geq 0$ and, hence, impermissible.

The conclusion from the foregoing examination is that a vertex from T_1 can meet only one face from T_2. Further examination shows that the same vertex also intersects one other face from T_3. In the above example of v_{11} and $c2$ intersecting at $\frac{5}{2}S_1 = S_2$, it is found that v_{11} also intersects $a3$ at $\frac{5}{2}S_1 = S_3$. Thus, the simultaneous activation of five twin systems is possible with the intersection of a vertex and two faces. In the above example, it may be calculated that $A = 0$, $B = \frac{1}{4}$, $C = -\frac{1}{4}$, $F = \frac{1}{4}$, $G = -\frac{1}{2}$, $H = \frac{1}{2}$ which corresponds to stress state 19 in Table II of Ref. 1. The active systems $c2$, $a3$, $a1$, $c1$, $d1$, (last three from v_{11}, see Table III) also are in agreement with previous work. Fig. 4 illustrates the above example of $v_{11} + c_2 + a_3$.

Since each vertex intersects two other faces and there are a total of twelve vertices for the three tetrahedra, the twelve stress states nos. 14 through 25 of Ref. 1 are accounted for.

3) Edges

As may be noted in Fig. 2 and deduced from Table II, the four sides of the solid outline of the rectangle (or square) formed on the F, G plane represent the loci of projections of four of the six edges of a tetrahedron. The other two edges project as diagonals; they are the top and bottom extremities along the H axis. The edges e_{11}, e_{13}, e_{21}, e_{23}, e_{31}, and e_{33} project along the vertical sides of the rectangle; e_{12}, e_{14}, e_{22}, e_{24}, e_{32}, and e_{34} are the "horizontal" edges; and e_{15}, e_{16}, e_{25}, e_{26}, e_{35}, and e_{36} are the "diagonal" edges.

a) VERTICAL EDGES

Consider the tetrahedra T_1 and T_3. If $S_3 < \frac{1}{2}S_1$, Fig. 5(a), there is no intersection of the vertical edges. If $S_3 = \frac{1}{2}S_1$, Fig. 5(b), the vertical edges can meet at $H = 0$ only. A third (vertical) edge from T_2 can also meet here if $S_2 = S_3 = \frac{1}{2}S_1$, yielding a total of six twin systems. It may be deduced that these intersections are:

$e_{11} + e_{21} + e_{31}$; $A = 0$, $B = -\frac{1}{4}$, $C = \frac{1}{4}$, $F = \frac{3}{4}$, $G = 0$, $H = 0$

$e_{13} + e_{23} + e_{33}$; $A = 0$, $B = -\frac{1}{4}$, $C = \frac{1}{4}$, $F = -\frac{3}{4}$, $G = 0$, $H = 0$

These correspond to stress states 6 and 7 of Table II, Ref. 1.

If $S_3 > \frac{1}{2}S_1$, Fig. 5(c), there is no possibility of intersection of the vertical edges.

b) HORIZONTAL EDGES

Again consider T_1 and T_3 first. If $S_3 < S_1$, Figs. 5(a) to (c), horizontal edges from the two tetrahedra do not meet. At $S_3 = S_1$, Fig. 5(d), they can meet at $H = 0$ only. A third (horizontal) edge from T_2 can also meet here at $H = 0$ if $S_2 = 2S_1 = 2S_3$, yielding a total of six systems. The intersections are:

$e_{12} + e_{22} + e_{32}$; $A = \frac{1}{4}$, $B = 0$, $C = -\frac{1}{4}$, $F = 0$, $G = \frac{3}{4}$, $H = 0$

$e_{14} + e_{24} + e_{34}$; $A = \frac{1}{4}$, $B = 0$, $C = -\frac{1}{4}$, $F = 0$, $G = -\frac{3}{4}$, $H = 0$

These correspond to stress states 8 and 9 of Table II, Ref. 1.

If $S_3 > S_1$, Fig. 5(e), there is no intersection of the horizontal edges.

c) DIAGONAL EDGES

The diagonal edges lie at fixed values of H. These are, from Table II.

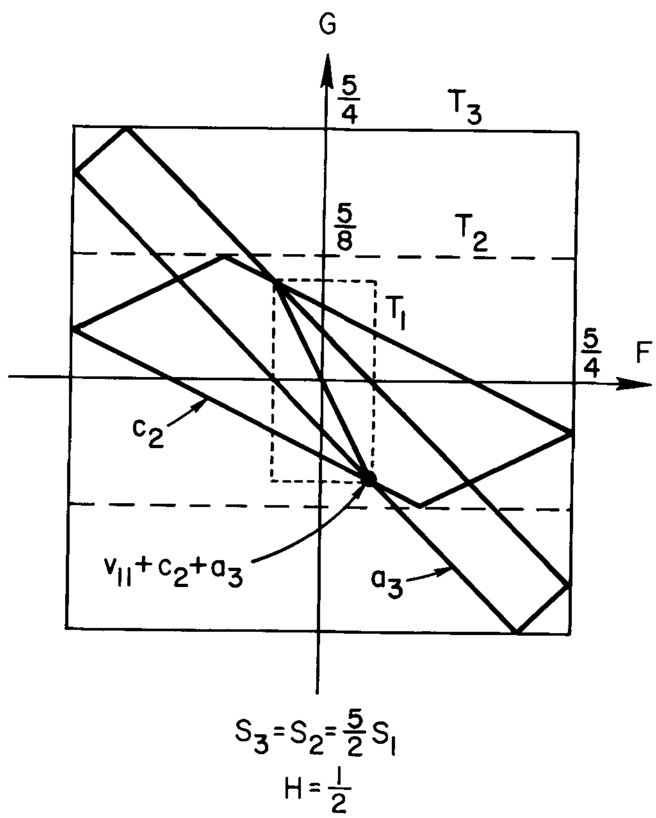

Fig. 4—Example of activation of five twin systems as a result of intersection of a vertex with two faces. Heavy weight parallelograms indicate projections of tetrahedra for $H = \frac{1}{2}$ where intersection occurs. Lighter outlines are loci of tetrahedron edges. Intersection corresponds to stress state 19 of Table II of Ref. 1.

Edge: e_{15} e_{16} e_{25} e_{26} e_{35} e_{36}

H: S_1 $-S_1$ S_2 $-S_2$ $\frac{1}{2}S_3$ $-\frac{1}{2}S_3$

Hence there are two intersections of three diagonal edges at values of $S_1 = S_2 = \frac{1}{2}S_3$. They are:

$$e_{15} + e_{25} + e_{35};\quad A = -\tfrac{1}{4},\ B = \tfrac{1}{4},\ C = 0,\ F = 0,\ G = 0,\ H = \tfrac{3}{4}$$

$$e_{16} + e_{26} + e_{36};\quad A = -\tfrac{1}{4},\ B = \tfrac{1}{4},\ C = 0,\ F = 0,\ G = 0,\ H = -\tfrac{3}{4}$$

These correspond to stress states 4 and 5 of Table II, Ref. 1.

d) HORIZONTAL PLUS VERTICAL (AND DIAGONAL) EDGES

Consider tetrahedra T_1 and T_2. It may be deduced from Table II that the vertical edges of T_1 (e_{11}, e_{13}) can meet the horizontal edges of T_2 (e_{22}, e_{24}) only if $S_1 = S_2$. There are four such intersections as shown in Fig. 5(f). At these intersections, the diagonal edges e_{35} and e_{36} of T_3 may also pass through if $S_3 = S_1 = S_2$, Fig. 5(f), since e_{35} and e_{36} require values of $H = \pm\frac{1}{2}S_3$ and the intersections at T_1 and T_2 require $H = \pm\frac{1}{2}S_1 = \pm\frac{1}{2}S_2$.

The intersections are:

$$e_{13} + e_{22} + e_{35};\quad A = B = C = 0,\ F = -\tfrac{1}{2},\ G = H = \tfrac{1}{2}$$

$$e_{11} + e_{24} + e_{35};\quad A = B = C = 0,\ F = \tfrac{1}{2},\ G = -\tfrac{1}{2},\ H = \tfrac{1}{2}$$

$$e_{11} + e_{22} + e_{36};\quad A = B = C = 0,\ F = G = \tfrac{1}{2},\ H = -\tfrac{1}{2}$$

$$e_{13} + e_{24} + e_{35};\quad A = B = C = 0,\ F = G = H = -\tfrac{1}{2}$$

These correspond to stress states 10 to 13 of Table II, Ref. 1.

There are no additional intersections involving five or more twin systems. Since the twenty-five stress states derived in this paper correspond to the twenty-five states obtained in Table II of Ref. 1, the previous list was complete, accounting for all possible stress

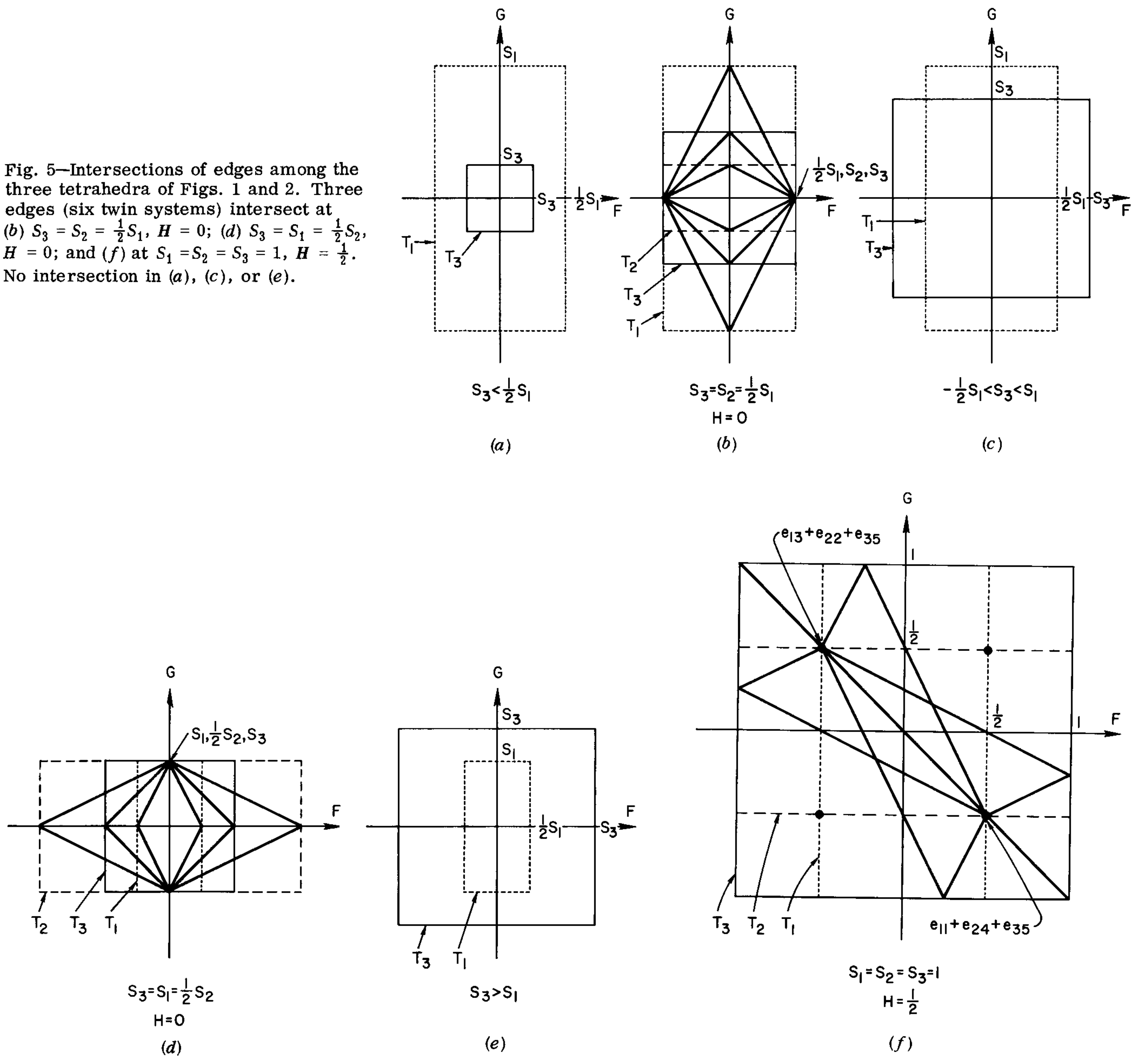

Fig. 5—Intersections of edges among the three tetrahedra of Figs. 1 and 2. Three edges (six twin systems) intersect at (b) $S_3 = S_2 = \frac{1}{2}S_1$, $H = 0$; (d) $S_3 = S_1 = \frac{1}{2}S_2$, $H = 0$; and (f) at $S_1 = S_2 = S_3 = 1$, $H = \frac{1}{2}$. No intersection in (a), (c), or (e).

states. Briefly summarized, the twenty-five stress states are represented by the following intersections among the three tetrahedra.

Stress States Table II, Ref. 1	No. Systems	Intersections	Size of Tetrahedra
1-3	8	two points	*e.g.*, $S_1 = S_2 = 0$, $S_3 = 3$
4-9	6	three edges	*e.g.*, $S_1 = S_2 = \frac{3}{4}$, $S_3 = \frac{3}{2}$
10-13	6	three edges	$S_1 = S_2 = S_3 = 1$
14-25	5	vertex plus two faces	*e.g.*, $S_1 = S_2 = \frac{5}{4}$, $S_3 = \frac{1}{2}$

ANALYSIS OF {111}⟨112⟩ (OR {112}⟨111⟩) SLIP

The case of {111}⟨112⟩ slip is more complicated than twinning since slip can be activated in both forward and reverse directions, effectively doubling the number of systems involved. The yield expressions for slip in the reverse directions of those listed in Eqs. [1] are:

System	Yield Expression	Constraints
$-a1$	$2F - G - H = -S_1'$	
$-b1$	$-2F + G - H = -S_1'$	$-\frac{1}{2}S_1' \le F \le \frac{1}{2}S_1'$
$-c1$	$2F + G + H = -S_1'$	$-S_1' \le G \le S_1'$
$-d1$	$-2F - G + H = -S_1'$	$-S_1' \le H \le S_1'$
$-a2$	$-F + 2G - H = -S_2'$	
$-b2$	$F - 2G - H = -S_2'$	$-S_2' \le F \le S_2'$
$-c2$	$-F - 2G + H = -S_2'$	$-\frac{1}{2}S_2' \le G \le \frac{1}{2}S_2'$
$-d2$	$F + 2G + H = -S_2'$	$-S_2' \le H \le S_2'$
$-a3$	$-F - G + 2H = -S_3'$	
$-b3$	$F + G + 2H = -S_3'$	$-S_3' \le F \le S_3'$
$-c3$	$-F + G - 2H = -S_3'$	$-S_3' \le G \le S_3'$
$-d3$	$F - G - 2H = -S_3'$	$-\frac{1}{2}S_3' \le H \le \frac{1}{2}S_3'$

[3]

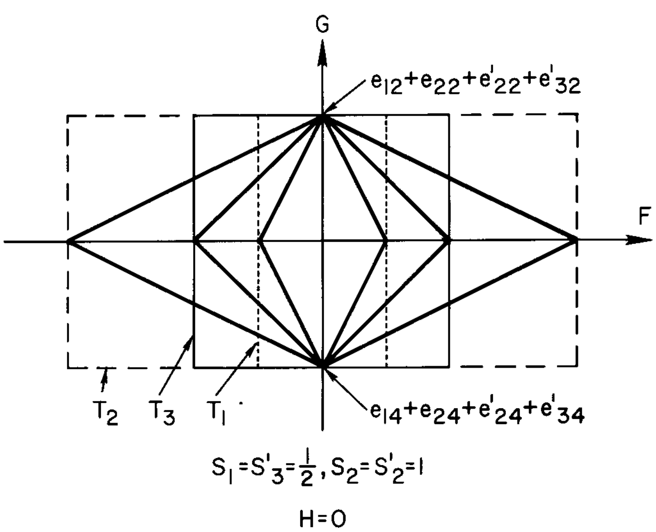

Fig. 6—Examples of activation of eight {111} ⟨112⟩ (or {112} ⟨111⟩) slip systems as a result of intersection of four edges. Conditions for intersection indicated below figure. For slip analysis, three tetrahedra, T_1', T_2', T_3', similar to those for Fig. 1, have been added to describe slip in the reverse direction. The two intersections shown correspond to stress states 6 and 7 of Table III of Ref. 1.

with:

$$S_1' = 1 + B - C$$
$$S_2' = 1 + C - A \qquad [4]$$
$$S_3' = 1 + A - B$$

Thus, in the case of slip, Eqs. [1] and [3] together represent six tetrahedra, and the simultaneous activation of five or more slip systems required consideration of all possible intersections among them. Fortunately, there are additional constraints governing the forward and reverse systems. In particular, a slip system cannot be activated simultaneously with its reverse. Thus, in considering possible intersections of T_1 and T_1', for example, a point from T_1, *i.e.*, $S_1 = 0$ cannot intersect T_1' at all, a vertex can intersect only one face from T_1'; an edge can intersect only one edge from T_1'; and so forth. Since the procedure for obtaining all permissible intersections is the same as that for twinning, it is not repeated here. It may be pointed out that the forty-five stress states listed for {111}⟨112⟩ slip in Table III of Ref. 1 are complete. The types of intersections are listed below:

Stress States, Table III, Ref. 1	No. Systems	Type of Intersection
1-3	8	two points
4-9	8	four edges
10-21	5	five faces
22-33	5	edge + three faces
34-45	5	vertex + two faces

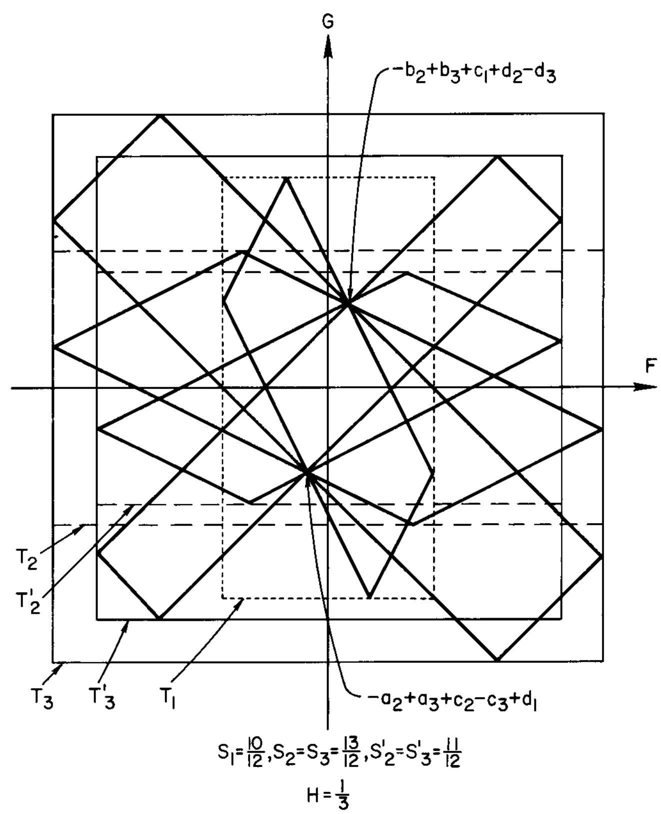

Fig. 7—Examples of activation of five {111} ⟨112⟩ slip systems as a result of intersection of five faces. Conditions for intersection indicated below figure. Intersections correspond to stress states −11 and −12 of Table III of Ref. 1.

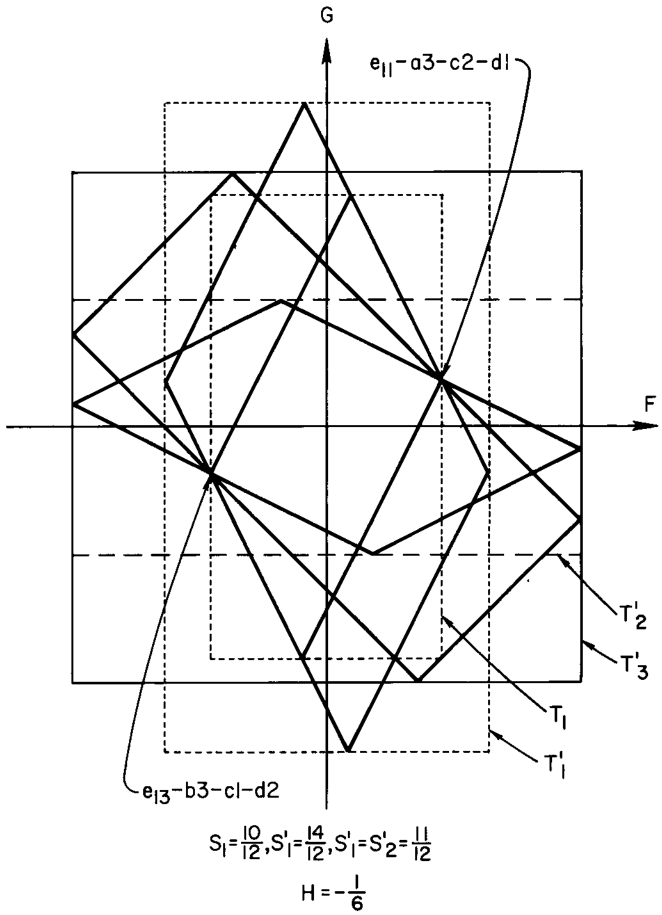

Fig. 8—Examples of activation of five {111} ⟨112⟩ slip systems as a result of intersection of an edge plus three faces. Intersections correspond to stress states –22 and –25 of Table III of Ref. 1.

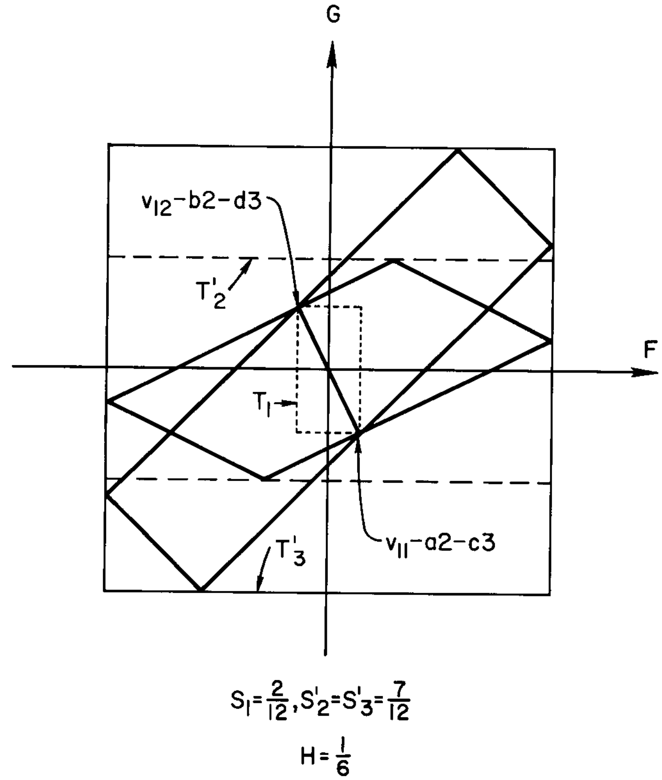

Fig. 9—Examples of activation of five {111} ⟨112⟩ slip systems as a result of intersection of a vertex and two faces. Intersections correspond to stress states –35 and –36 of Table III of Ref. 1.

Several typical intersections are illustrated in Figs. 6 to 9.

SUMMARY AND CONCLUSIONS

By arranging the yield surfaces of {111}⟨112⟩ slip and twin systems as faces of tetrahedra in three dimensional stress coordinates, with two remaining independent stress components determining their size, we have systematically derived all stress states which can activate five or more systems simultaneously. These stress states represent intersections of five or more faces of the tetrahedra. It has been verified that the lists of stress states previously reported by Hosford and Chin for {111}⟨112⟩ multiple slip and twinning are complete. Thus, the hypothesis that the stress states necessary and sufficient to enforce axisymmetric flow are necessary and sufficient to enforce any shape change, which was the basis of the previous work, has now been found valid for {111}⟨110⟩ and {112}⟨111⟩ slips and {111}⟨112⟩ twinning. Such a hypothesis requires that *each* stress state from the list derived for arbitrary deformation, be capable of enforcing axisymmetric flow. Although a given strain can always be expressed in terms of principal strain components (with suitable choice of axes) in the ratio $1{:}-x{:}-(1-x)$, $0 \leq x \leq 1$, axisymmetric flow requires that this ratio include the value $x = \frac{1}{2}$. Thus, it is clear that only under certain conditions will axisymmetric flow be included in the range of strains enforcible by a stress state. Unfortunately, we have not been able to deduce these conditions which are evidently fulfilled in the cases of {111}⟨110⟩ and {112}⟨111⟩ slips and {111}⟨112⟩ twinning.

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