Universal Portfolios

Jamal Bajwa

05/08/2020

Section 1

Modern Portfolio Theory

Mean-Variance Portfolio Optimization

- Sharpe and Markowitz (1960s), based on Capital Asset Pricing Model (CAPM)
- Maximize expected arithmetic return subject to a variance constraint (in a fixed window)
- Modern portfolio theory tries to estimate expected return mean and variance of various stocks – Implicit in this assumption is that the return distribution is *Gaussian*, the mean and variance of which are estimated from empirical history of a stock
- These estimates are inputs into an optimization model that allocates a portfolio subject to an *overall portfolio risk – A rolling variance of a given stock is estimated, and used as a measure of “risk”

Mathematical Formulation (1)

- Assumptions
- $Let\ b_i\ =\ the\ portfolio\ fraction\ invested\ in\ the\ i^{th}\ stock$
- $\sum_{i=1}^{N} b_i = 1$
- $Assume\ each\ stock\ is\ Gaussian\ and\ Independent:\ N(\mu_i, \sigma_i)$
- $\mu_i = mean\ expected\ return\ from\ i^{th}\ stock$
- $\sigma_i = variance\ of\ i^{th}\ stock$

Mathematical Formulation (2)

$$maximize \sum_{i=1}^{N} b_i \mu_i$$
$$s.t.\ \sigma_{portfolio} < \sigma_{max}$$

Efficient Frontier

- As you tweek desired portfolio variance, the expected return of the new portfolio will go up (high risk, high return)

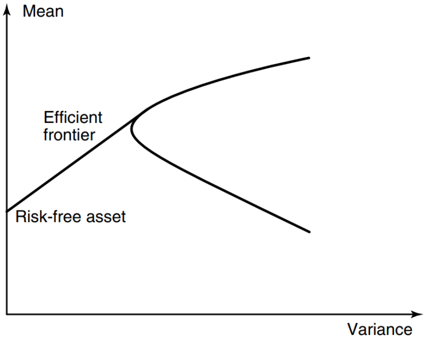

Figure 1: Sample Efficient Frontier

Shortcomings

- How can we make such specific distributional assumptions about the future?
- Even if we make distributional assumptions, how can we estimate the mean and variance using past data?
- Why are we only optimizing over one investment horizon? Don't returns compound?

Section 2

The Kelly Criterion

Derivation

- Kelly Criterion doesn't get rid of distributional assumptions but instead focuses on introducing compounding and multi-horizon decision into the optimization problem
- Also known as log-optimal portfolio theory, because of the role that logarithms play
- Let's derive the criterion for a bernoulli coin-toss, with probability p of heads, payoff o if heads and 0 if tails. We would like to invest a fraction f of our portfolio. Assume portfolio has 1 dollar.

Derivation

$$S_n = S_o \cdot (1 + f \cdot o)^S (1 - f)^F$$

$$(\frac{S_n}{S_o})^{1/n} = (1 + f \cdot o)^{S/n} (1 - f)^{F/n}$$

$$\frac{1}{n} log \frac{S_n}{S_o} = \frac{S}{n} log(1 + f \cdot o) + \frac{F}{n} log(1 - f)$$

$$\lim_{n \to \infty} \frac{1}{n} log \frac{S_n}{S_o} = p \cdot log(1 + f \cdot o) + (1 - p) \cdot log(1 - f)$$

$$G = p \cdot log(1 + f \cdot o) + (1 - p) \cdot log(1 - f)$$

- *G is the growth rate*
- *maximize G $\rightarrow$ obtain $\frac{\mathrm{d}G}{\mathrm{d}f}$ and set it to 0 $\rightarrow$ optimal f^* is optimal allocation*

Section 3

Shannon's Demon

Section 4

Cover's Universal Portfolio

What does it accomplish?

- Proposes a rebalancing strategy that beats the best optimally rebalanced portfolio (even in hindsight!)
- Simply put, this algorithm does as well as a lazy portfolio you might construct in hindsight that follows Shannon's demon
- The mathematical intuition required to see this was nothing short of genius
- Simple and elegant

Optimal, Constant Rebalanced Portfolio

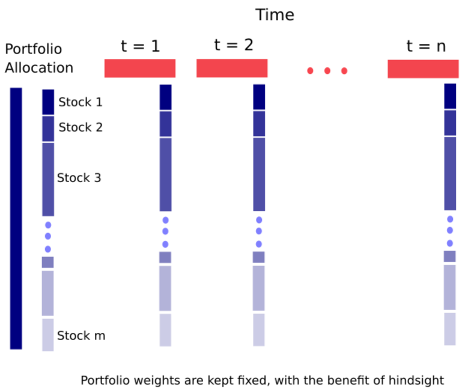

Figure 2: Constant Rebalanced Portfolio

Universal Portfolio

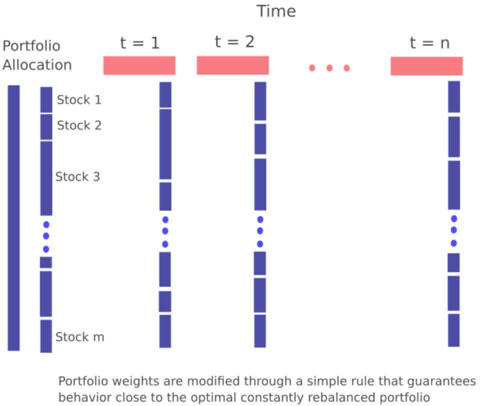

Figure 3: Universal Portfolio - asymptotically approaches optimal Constantly Rebalanced portfolio

Universal Portfolio

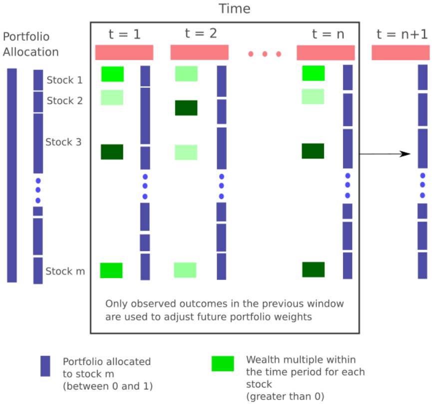

Figure 4: Universal Portfolio - only causal information is utilized

Significance

- Cover (1991) proved that this algorithm asymptotically approaches **optimal, constantly rebalanced portfolio in hindsight**
- Although Cover excludes Transaction costs, it is still unclear why this isn't more popular in practice

Outline of Proof (Definitions)

- Assume we have m stocks, so that **b** and $\mathbf{x_i}$ are m-dimensional vectors representing portfolio fractions and daily wealth multiples of the stocks
- *Wealth after n investment periods*

$$S_n(\mathbf{b},\ \mathbf{x}^n) \ = \ \prod_{i=1}^{n} \mathbf{b}^t \mathbf{x}_i$$

- *Wealth if using optimal constant rebalanced portfolio*

$$S_n^*(\mathbf{b}^*,\ \mathbf{x}^n) \ = \ \prod_{i=1}^{n} \mathbf{b}^{*t} \mathbf{x}_i$$

- *Wealth if using causally rebalanced portfolio*

$$\hat{S}_n(\widehat{\mathbf{b}},\ \mathbf{x}^n) \ = \ \prod_{i=1}^{n} \widehat{\mathbf{b}}^t \mathbf{x}_i$$

Outline of Proof (Definition)

The optimal causal portfolio is defined as :

$$\widehat{\mathbf{b}}_{\mathbf{i+1}}(\mathbf{x^i}) = \frac{\int_{b \in B} \mathbf{b} S_i(b, \mathbf{x}^i) d\mu(\mathbf{b})}{\int S_i(b, \mathbf{x}^i) d\mu(\mathbf{b})}$$

$$\textit{We wish to show } \frac{\hat{S}_n}{S_n^*} \textit{ is bounded}$$

Outline of Proof

- Assume $m = 2$ for simplicity (two stocks)
- Let's call the multiplication of the dot-product in the wealth equation **product of sums**
- We can alternatively write this **product of sums** as **sum of products**
- This converts the problem from allocating the portfolio over n investment periods, to investing in 2^m possible trajectores in **one** investment period!

Outline of Proof

$$\prod_{i=1}^{2} \mathbf{b_i^t} \mathbf{x}_i = ([b_{1,1}, b_{1,2}] \cdot [x_{1,1}, x_{1,2}])([b_{2,1}, b_{2,2}] \cdot [x_{2,1}, x_{2,2}])$$

$$\prod_{i=1}^{2} \mathbf{b_i^t} \mathbf{x}_i = (b_{1,1} \cdot x_{1,1} + b_{1,2} \cdot x_{1,2})(b_{2,1} \cdot x_{2,1} + b_{2,2} \cdot x_{2,2})$$

- Zooming into the RHS, re-arranging and using $b_{1,i} + b_{2,i} = 1$

$$\begin{aligned} b_{1,1} \cdot b_{2,1} \cdot x_{1,1} \cdot x_{2,1} + \\ b_{1,1} \cdot b_{2,2} \cdot x_{1,1} \cdot x_{2,2} + \\ b_{1,2} \cdot b_{2,1} \cdot x_{1,2} \cdot x_{2,1} + \\ b_{1,2} \cdot b_{2,2} \cdot x_{1,2} \cdot x_{2,2} \end{aligned}$$

- We are optimally allocating bets (in blue) over a sequence of outcomes (in green) in one investment period
- The proof rests on viewing investment in this way!

Some examples

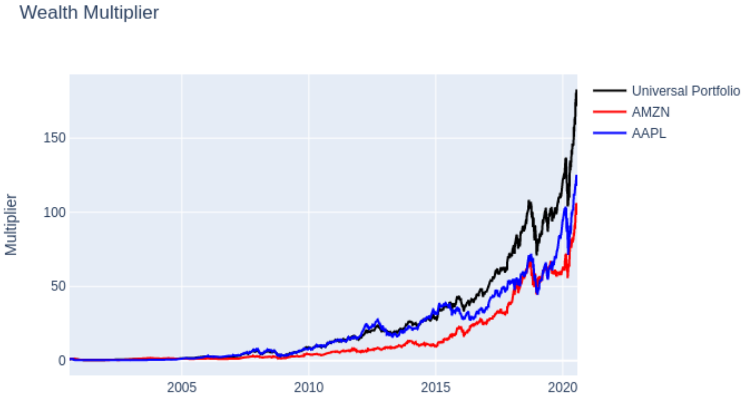

Figure 5: Universal Portfolio - Amazon and Apple