# EXERCISE 8

## Ex 1

MERCER'S CONDITION: $\sum_{i=1}^{m}\sum_{j=1}^{m} c_i c_j k(x_i, x_j) \geq 0$

a) i. $k(x,x') = a$, $a \in \mathbb{R}$:

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_i c_j a = a\left(\sum_{i=1}^{m} c_i\right)^2 \geq 0$$

ii. $k(x,x') = \langle x, x' \rangle$

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_i c_j \langle x, x' \rangle = \sum_{i=1}^{m}\sum_{j=1}^{m} c_i c_j \sum_{k=1}^{d} x_{ik} x_{jk}$$

$$= \sum_{k=1}^{d}\left(\sum_{i=1}^{m} c_i x_{ik}\right)^2 \geq 0$$

iii. $k(x,x') = f(x) \cdot f(x')$

WHERE $f: \mathbb{R}^d \to \mathbb{R}$ IS AN ARBITRARY CONTINUOUS FUNCTION

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_i c_j f(x) \cdot f(x') = \left(\sum_{i=1}^{m} c_i f(x_i)\right)^2 \geq 0$$

b) i. $k(x,x') = k_1(x,x') + k_2(x,x')$

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_i c_j \left(k_1(x_i,x_j) + k_2(x_i,x_j)\right) =$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{m} c_i c_j k_1(x_i,x_j) + \sum_{i=1}^{m}\sum_{j=1}^{m} c_i c_j k_2(x_i,x_j) \geq 0$$

ii. $k(x,x') = k_1(x,x') \cdot k_2(x,x')$

NOTE:
$k_1(x_i,x_j) = \sum_{k}^{d} \varphi_{ik} \varphi_{jk}$
$k_2(x_i,x_j) = \sum_{l}^{d} \psi_{il} \psi_{jl}$

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_i c_j \sum_{k=1}^{d} \varphi_{ik} \varphi_{jk} \sum_{l=1}^{d} \psi_{il} \psi_{jl} =$$

$$= \sum_{k=1}^{d}\sum_{l=1}^{d}\left(\sum_{i=1}^{m} c_i \varphi_{ik} \psi_{il}\right)^2 \geq 0$$

c) $k(x,x') = (\langle x, x' \rangle + \vartheta)^d$ and $\vartheta \in \mathbb{R}^+$

As shown in a) i. and ii:

$k(x,x') = \vartheta$, $\vartheta \in \mathbb{R}^+$

and

$k(x,x') = \langle x, x' \rangle$

are Mercer kernels.

As shown in b) i.:

$$k(x,x') = k_1(x,x') + k_2(x,x') =$$
$$= \langle x, x' \rangle + \vartheta$$

is too a Mercer kernel.

As shown in b) ii.:

$$k(x,x') = k_1(x,x') \cdot k_2(x,x')$$

is a Mercer kernel, which implies that a power of a Mercer kernel is also a Mercer kernel, hence:

$$k(x,x') = (\langle x, x' \rangle + \vartheta)^d$$

is a Mercer kernel.

d) $k(x,x') = \exp\left(-\frac{\|x-x'\|^2}{2\sigma^2}\right) =$

| Note: |
|---|
| $\|x-x'\|^2 =$ |
| $= \|x\|^2 + \|x'\|^2 - 2\langle x, x' \rangle$ |

$$= \exp\left(-\frac{\|x\|^2}{2\sigma^2} - \frac{\|x'\|^2}{2\sigma^2} + \frac{\langle x, x' \rangle}{\sigma^2}\right) =$$

$$= \underbrace{\exp\left(-\frac{\|x\|^2}{2\sigma^2}\right)}_{= f(x)} \underbrace{\exp\left(-\frac{\|x'\|^2}{2\sigma^2}\right)}_{= f(x')} \underbrace{\exp\left(\frac{\langle x, x' \rangle}{\sigma^2}\right)}_{\text{PSD}}$$

Hence, it follows from a) i. ii. and b) ii. that

$$k(x,x') = \exp\left(-\frac{\|x-x'\|^2}{2\sigma^2}\right)$$

is a Mercer kernel.

Ex. 2

$$k(x,y) = \langle x, y\rangle^2 = \left(\sum_{i=1}^{d} x_i y_i\right)^2$$

$d=2$

$$= x_1^2 y_1^2 + x_2^2 y_2^2 + 2 x_1 y_1 x_2 y_2$$

a)

$$\varphi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 \\ \sqrt{2}\, x_1 x_2 \\ x_2^2 \end{pmatrix}$$

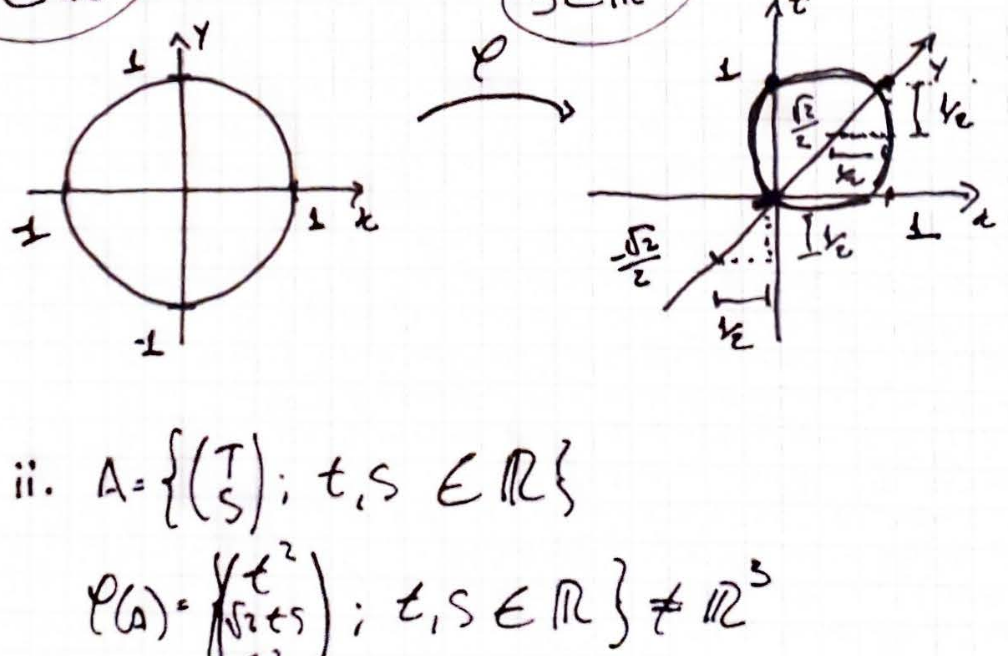
$$k(x,y) = \langle \varphi(x), \varphi(y)\rangle = \begin{pmatrix} x_1^2 \\ \sqrt{2}\, x_1 x_2 \\ x_2^2 \end{pmatrix}^T \begin{pmatrix} y_1^2 \\ \sqrt{2}\, y_1 y_2 \\ y_2^2 \end{pmatrix} =$$

$$= x_1^2 y_1^2 + 2 x_1 x_2 y_1 y_2 + x_2^2 y_2^2$$

b) i. $C = \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} ; \ 0 \le \theta < 2\pi \right\}$

$$\varphi(C) = \varphi\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta \\ \sqrt{2}\cos\theta\sin\theta \\ \sin^2\theta \end{pmatrix}, \ \forall \theta \in [0, 2\pi)$$

$C \in \mathbb{R}^2$ $\qquad\qquad$ $F \in \mathbb{R}^3$




ii. $A = \left\{ \begin{pmatrix} t \\ s \end{pmatrix} ; \ t, s \in \mathbb{R} \right\}$

$$\varphi(A) = \left\{ \begin{pmatrix} t^2 \\ \sqrt{2}\, t s \\ s^2 \end{pmatrix} ; \ t, s \in \mathbb{R} \right\} \neq \mathbb{R}^3$$

c) $\varphi(C) = \left\{ \begin{pmatrix} \cos^2\theta \\ \sqrt{2}\sin\theta\cos\theta \\ \sin^2\theta \end{pmatrix} ; 0 \leq \theta < 2\pi \right\}$

**Nota:** $\cos^2\theta = 1 - \sin^2\theta$

$\varphi(C) \subseteq \left\{ \begin{pmatrix} t \\ s \\ 1-t \end{pmatrix} ; t, s \in \mathbb{R} \right\}$

d) $P \in \mathbb{R}^3 \wedge P \notin F$

e.g.: $P = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \notin \varphi(A) \wedge \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$

**Nota:** $\varphi(A) = \left\{ \begin{pmatrix} t^2 \\ \sqrt{2}ts \\ s^2 \end{pmatrix} ; t, s \in \mathbb{R} \right\}$