EXERCISE 9

Ex.1

$$\min_{\omega,\theta} \|w\|^2$$

$$\text{S.T. } y_i(\omega^T x_i + \theta) \geq 1 \quad \text{, FOR } 1 \leq i \leq m$$

a) $\Lambda(\omega, \theta, \alpha) = \|w\|^2 + \sum_i^m \alpha_i (1 - y_i(\omega^T x_i + \theta))$

b) $\min_{\omega,\theta} \left(\|w\|^2 + \max_{\alpha_i \geq 0} \sum_i^m \alpha_i (1 - y_i(\omega^T x_i + \theta)) \right)$

- SLATER'S CONDITION: $\exists x_0 \in D : f_i(x_0) < 0$
(STRICT FEASIBILITY) $\qquad h_i(x_0) = 0$

i.e. THERE'S A DATA POINT THAT SATISFIES ALL CONSTRAINTS (LINEARLY SEPARABLE)

WHICH GARANTEES A ZERO DUALITY GAP.

$$\mathcal{L}(\omega, \theta, \alpha) = \frac{1}{2}\|w\|^2 + \sum_i \alpha_i (1 - y_i(\omega^T x_i + \theta))$$

$$\frac{\partial \mathcal{L}}{\partial \omega} = \omega - \sum_i^m \alpha_i y_i x_i = 0 \Rightarrow \omega = \sum_i^m \alpha_i y_i x_i$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -\sum_i^m \alpha_i y_i = 0 \Rightarrow \sum_i^m \alpha_i y_i = 0$$

NOTE: $\frac{1}{2}\|w\|^2 = \frac{1}{2}\omega^T\omega = \frac{1}{2}\left(\sum_i \alpha_i y_i x_i\right)^T\left(\sum_j^m \alpha_j y_j x_j\right)$

$$= \frac{1}{2}\sum_{i,j}^m \alpha_i \alpha_j y_i y_j x_i^T x_j$$

$$\max_{\alpha_1, \dots, \alpha_m} \min \left[\frac{1}{2}\|w\|^2 + \sum_i \alpha_i (1 - y_i(\omega^T x_i - \theta))\right]$$

$$\min_{\alpha_1 \ldots \alpha_m} \frac{1}{2}\sum_{ij} \alpha_i \alpha_j y_i y_j x_i^T x_j + \sum_i \alpha_i - \sum_i \alpha_i y_i \left(\sum_j \alpha_j y_j x_j\right)^T x_i - \underbrace{\theta \sum \alpha_i y_i}_{=0}$$

$$= \sum_i \alpha_i - \frac{1}{2}\sum_{ij} \alpha_i \alpha_j y_i y_j x_i^T x_j$$

S.T. $\sum_i^m \alpha_i y_i = 0$

AND $\alpha_i \geq 0, \; \forall i$

When:

$d \gg m \rightarrow$ DUAL PROBLEM (m-DIMENSIONAL PROBLEM)

$m \gg d \rightarrow$ PRIMAL PROBLEM (d-DIMENSIONAL PROBLEM)

NOTE: $w = \sum_i \alpha_i y_i x_i$

$\theta = ? \rightarrow w^T x^{(i)} \leq \ldots \leq w^T x^{(i)}$

$\hookrightarrow \theta = w^T x^{(i)} - 1$

c) $x \rightarrow \phi(x)$

$x^T x' \rightarrow \langle \phi(x), \phi(x') \rangle = k(x, x')$

$\min \frac{1}{2}\|w\|^2$ S.T. $y_i(w^T . \phi(x) + \theta) \geq 1$

$$\min_{\alpha} \frac{1}{2}\sum_{ij} \alpha_i \alpha_j y_i y_j \, k(x_i, x_j)$$

S.T. $y_i(w^T . \phi(x) + \theta) \geq 1$

AND $\alpha_i \geq 0, \; \forall i$

Ex.2

a) Let:

$x = [\alpha_1, \dots, \alpha_m]^T$

$P \in \mathbb{R}^{m \times m}$, where $P_{ij} = y_i y_j k(x_i, x_j)$

$q \in \mathbb{R}^m$, where every element equals -1.

$G = -I_m = \begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix}_{m \times m}$

$h \in \mathbb{R}^m$, where every element equals 0

$A = y^T = [y_1, \dots, y_m]$

$b = 0$

then the dual SVM can be written as:

$$\min_x \frac{1}{2} x^T P x + q^T x$$

$$\text{s.t. } Gx \preceq h$$

$$\text{and } Ax = b$$