EXERCISE SHEET (06)

1.a)

$$X = \{ \vec{x}_1, \dots, \vec{x}_m \} , \quad \hat{X} = \vec{\omega}_x X$$
$$Y = \{ \vec{Y}_1, \dots, \vec{Y}_m \} , \quad \hat{Y} = \vec{\omega}_Y Y$$

$$\mathrm{COV}(\hat{X}, \hat{Y}) = \mathbb{E}[(\vec{\omega}_x X - \underbrace{\mathbb{E}(\vec{\omega}_x X)}_{=0})(\vec{\omega}_Y Y - \underbrace{\mathbb{E}(\vec{\omega}_Y Y)}_{=0})^T] =$$

$$= \mathbb{E}[(\vec{\omega}_x X)(\vec{\omega}_Y Y)^T]$$

$$\mathrm{VAR}(\hat{X}) = \mathbb{E}[(\vec{\omega}_x X)^2]$$
$$\mathrm{VAR}(\hat{Y}) = \mathbb{E}[(\vec{\omega}_Y Y)^2]$$

$$\mathrm{CORR}(\hat{X}, \hat{Y}) = \frac{\mathbb{E}[(\vec{\omega}_x X)(\vec{\omega}_Y Y)^T]}{\sqrt{\mathbb{E}[(\vec{\omega}_x X)^2] \cdot \mathbb{E}[(\vec{\omega}_Y Y)^2]}} =$$

$$= \frac{\frac{1}{N} \sum_i^N (\vec{\omega}_x x_i)(\vec{\omega}_Y Y_i)}{\sqrt{\frac{1}{N} \sum_i^N (\vec{\omega}_x x_i)^2} \sqrt{\frac{1}{N} \sum_i^N (\vec{\omega}_Y Y_i)}} =$$

NOTE:

$$\boxed{\Sigma = \begin{bmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{XY} & \Sigma_{YY} \end{bmatrix}}$$

$$= \frac{\vec{\omega}_x^T \Sigma_{XY} \vec{\omega}_Y}{\sqrt{\vec{\omega}_x^T \Sigma_{XX} \vec{\omega}_x} \sqrt{\vec{\omega}_Y^T \Sigma_{YY} \vec{\omega}_Y}}$$

$$\Longrightarrow \max_{\vec{\omega}_x, \vec{\omega}_Y} : \vec{\omega}_x \Sigma_{XY} \vec{\omega}_Y$$

$$\text{S.T.}: \quad \vec{\omega}_x \Sigma_{XX} \vec{\omega}_x = \vec{\omega}_Y \Sigma_{YY} \vec{\omega}_Y = 1$$

1.b)

$$f(\vec{\omega}_x, \vec{\omega}_Y) = \vec{\omega}_x^T \Sigma_{XY} \vec{\omega}_Y$$
$$g_x(\vec{\omega}_x) = \vec{\omega}_x^T \Sigma_{XX} \vec{\omega}_x - 1$$
$$g_Y(\vec{\omega}_Y) = \vec{\omega}_Y^T \Sigma_{YY} \vec{\omega}_Y - 1$$

$$\mathcal{L}(\vec{\omega}_x, \vec{\omega}_Y, \alpha, \beta) = f(\vec{\omega}_x, \vec{\omega}_Y) - \alpha g_x(\vec{\omega}_x) - \beta g_Y(\vec{\omega}_Y) =$$
$$= \vec{\omega}_x^T \Sigma_{XY} \vec{\omega}_Y - \alpha(\vec{\omega}_x^T \Sigma_{XX} \vec{\omega}_x) - \beta(\vec{\omega}_Y^T \Sigma_{YY} \vec{\omega}_Y)$$

$$\frac{\partial \mathcal{L}}{\partial \vec{\omega}_X} = \Sigma_{XY} \vec{\omega}_Y - 2\alpha \Sigma_{XX} \vec{\omega}_X = 0$$
$$\Rightarrow \Sigma_{XX}^{-1} \Sigma_{XY} \vec{\omega}_Y = 2 . \alpha . \vec{\omega}_x$$

$$\frac{\partial \mathcal{L}}{\partial \vec{\omega}_Y} = \Sigma_{XY}^T \vec{\omega}_x - 2\beta \Sigma_{YY} \vec{\omega}_Y = 0$$
$$\Rightarrow \Sigma_{YY}^{-1} \Sigma_{XY}^T \vec{\omega}_x = 2 . \beta . \vec{\omega}_Y$$

· SUBSTITUTING BOTH EQUATIONS YIELDS:

$$\Sigma_{XX}^{-1} \Sigma_{XY} \Sigma_{YY}^{-1} \Sigma_{XY}^T . \vec{\omega}_x = U_x \vec{\omega}_x = l \vec{\omega}_x$$
$$\Sigma_{YY}^{-1} \Sigma_{XY}^T \Sigma_{XX}^{-1} \Sigma_{XY} . \vec{\omega}_Y = U_X \vec{\omega}_X = l \omega_Y$$

With: $l = 4\alpha\beta$

1.c)

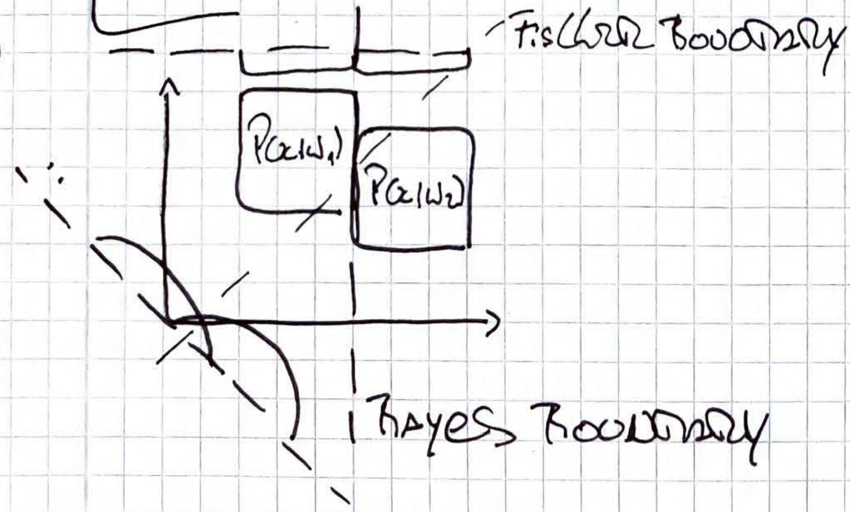
$$\underbrace{\vec{\omega}_x^T \Sigma_{XY} \vec{\omega}_Y + \vec{\omega}_Y^T \Sigma_{YX} \vec{\omega}_x}_{=} = 2\alpha \underbrace{\vec{\omega}_x^T \Sigma_{XX} \vec{\omega}_X}_{=1} + 2\beta \underbrace{\vec{\omega}_Y^T \Sigma_{YY} \vec{\omega}_Y}_{=1}$$

$$\Rightarrow \vec{\omega}_X^T \Sigma_{XY} \vec{\omega}_Y = \alpha + \beta$$

HENCE, THE CORRELATION COEFFICIENT AT AN OPTIMUM:

$$\text{CORR} = \alpha + \beta$$

2. a)



2. b)

$$P_1(x|\omega_1) = N(\mu_1, \Sigma_1)$$
$$P_2(x|\omega_2) = N(\mu_2, \Sigma_2)$$
$$g_1(x) = \log(P(x|\omega_1)) + \log(P(\omega_1))$$
$$g_2(x) = \log(P(x|\omega_2)) + \log(P(\omega_2))$$

$$\varphi(x) = g_1(x) - g_2(x) - \log(P(\omega_1)) + \log(P(\omega_2))$$
$$= \log P(x|\omega_1) - \log P(x|\omega_2)$$
$$= -\log(\sqrt{(2\pi)^d|\Sigma_1|}) - \frac{1}{2}(x-\mu_1)^T\Sigma(x-\mu_1) +$$
$$\log(\sqrt{(2\pi)^d|\Sigma_2|}) - \frac{1}{2}(x-\mu_2)^T\Sigma(x-\mu_2)$$