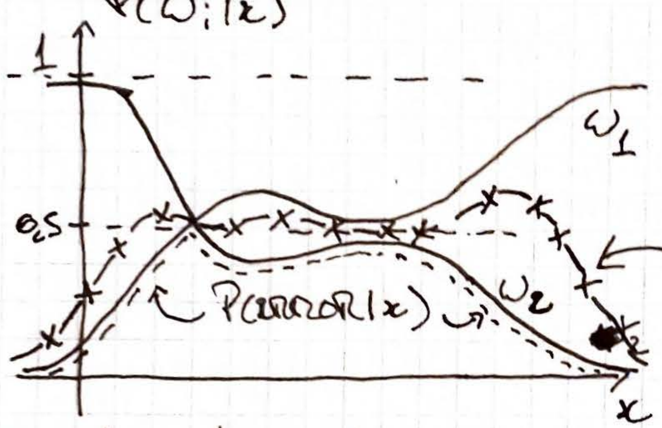
• EXERCISE SHEET (1)

Ex. 1

a) $P(\text{ERROR}) = \int_{-\infty}^{+\infty} P(\text{ERROR}, x)\, dx =$

$= \int_{-\infty}^{+\infty} P(\text{ERROR} \mid x)\, P(x)\, dx$



BAYES' DECISION RULE:

$P(\text{ERROR} \mid x) = \min\left(P(\omega_1 \mid x), P(\omega_2 \mid x)\right)$

HARMONIC MEAN OF CLASS POSTERIORS:

$$\frac{2 \cdot P(\omega_1 \mid x) \cdot P(\omega_2 \mid x)}{P(\omega_1 \mid x) + P(\omega_2 \mid x)}$$

GRAPH SHOWS HOW THE HARMONIC MEAN OF CLASS POSTERIORS IS AN UPPER-BOUND TO THE PROBABILITY OF MISS-CLASSIFICATION, FOR A GIVEN OBSERVATION 'x_0'.

$$P(\text{ERROR} \mid x) = \min\left(P(\omega_1 \mid x), P(\omega_2 \mid x)\right) \leq \frac{2 \cdot P(\omega_1 \mid x) \cdot P(\omega_2 . x)}{P(\omega_1 \mid x) + P(\omega_2 \mid x)} = \frac{2}{\frac{P(\omega_1 \mid x) + P(\omega_2 \mid x)}{P(\omega_1 \mid x) \cdot P(\omega_2 \mid x)}} =$$

$$= \frac{2}{\frac{1}{P(\omega_1 \mid x)} + \frac{1}{P(\omega_2 \mid x)}}$$

HENCE: $P(\text{ERROR}) \leq \int \frac{2}{\frac{1}{P(\omega_1 \mid x)} + \frac{1}{P(\omega_2 \mid x)}} P(x) \cdot dx$

b) BAYES' FORMULA: $P(\omega_i|x) = \dfrac{P(x|\omega_i) \times P(\omega_i)}{P(x)}$

- IN ENGLISH:

$$\text{POSTERIOR} = \frac{\text{LIKELIHOOD} \times \text{PRIOR}}{\text{EVIDENCE}}$$

- UNIVARIATE PROBABILITY DISTRIBUTIONS:

$$P(x|\omega_1) = \frac{\pi^{-1}}{1+(x-\mu)^2}$$

$$P(x|\omega_2) = \frac{\pi^{-1}}{1+(x+\mu)^2}$$

$$P(\text{ERROR}) \leq \int \frac{2}{\frac{1}{P(\omega_1|x)} + \frac{1}{P(\omega_2|x)}} \cdot P(x)\, dx = {}^{*}$$

$$= \int \frac{2}{\frac{\cancel{P(x)}}{P(x|\omega_1)P(\omega_1)} + \frac{\cancel{P(x)}}{P(x|\omega_2)P(\omega_2)}} \cancel{P(x)}\, dx =$$

$$= \int \frac{2}{\frac{1+(x-\mu)^2}{\pi^{-1} \cdot P(\omega_1)} + \frac{1+(x+\mu)^2}{\pi^{-1} \cdot P(\omega_2)}}\, dx =$$

$$= \int \frac{2 \cdot \pi^{-1} \cdot P(\omega_1) \cdot P(\omega_2)}{(1+(x-\mu)^2) \cdot P(\omega_2) + (1+(x+\mu)^2) \cdot P(\omega_1)}\, dx =$$

$$= \int \frac{2 \cdot P(\omega_1) \cdot P(\omega_2) \cdot \pi^{-1}}{1+x^2+\mu^2 + 2x\mu(P(\omega_1) - P(\omega_2))} \cdot dx$$

HINT: (!) $\int \frac{1}{ax^2+bx+c}\,dx = \frac{2\pi}{\sqrt{4ac-b^2}}$ FOR $b^2 < 4ac$

$$\left.\begin{aligned} a &= 1 \\ b &= 2\mu\,(P(\omega_1)-P(\omega_2)) \\ c &= 1+\mu^2 \end{aligned}\right\} \begin{aligned} &\text{With: } b^2 < 4ac \\ &4\mu^2 (P(\omega_1)-P(\omega_2))^2 < 4\mu^2+4 \quad (!) \end{aligned}$$

$$\Rightarrow \int \frac{2.P(\omega_1).P(\omega_2).\pi^{-1}}{1+x^2+\mu^2+2x\mu(P(\omega_1)-P(\omega_2))}\,dx =$$

$$= \frac{\cancel{2}.2.P(\omega_1).P(\omega_2).\cancel{\pi^{-1}}.\cancel{\pi}}{\sqrt{\cancel{4}(1+\mu^2)-\cancel{4}.\mu^2.(P(\omega_1)-P(\omega_2))^2}} =$$

$$= \frac{2P(\omega_1).P(\omega_2)}{\sqrt{1+\underbrace{(P(\omega_1)+P(\omega_2))^2}_{=1}.\mu^2-\mu^2(P(\omega_1)-P(\omega_2))^2}} =$$

$$= \frac{2.P(\omega_1).P(\omega_2)}{\sqrt{1+4\mu^2.P(\omega_1).P(\omega_2)}}$$

HENCE: $P(\text{ERROR}) \leq \frac{2.P(\omega_1).P(\omega_2)}{\sqrt{1+4\mu^2 P(\omega_1).P(\omega_2)}}$

c) (1) LOW-DIMENSIONAL CASE: NUMERICALLY COMPUTE THE ERROR BOUND

(2) HIGH-DIMENSIONAL CASE: $x_1, \ldots, x_N \sim P(x)$

[redacted]

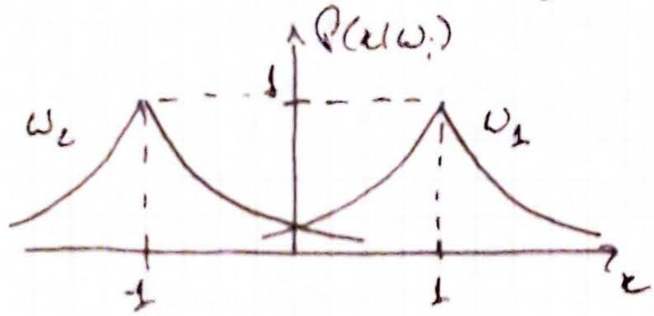
$P(\text{ERROR}) = \frac{1}{N}\sum_i^N P(\text{ERROR} \mid x_i)$

Ex. 2

a) $P(x|\omega_1) = \frac{1}{2\sigma} \exp\left(-\frac{|x-\mu|}{\sigma}\right)$

$P(x|\omega_2) = \frac{1}{2\sigma} \exp\left(-\frac{|x+\mu|}{\sigma}\right)$

- EXAMPLE: FOR $\mu = 1$ AND $\sigma = \frac{1}{2}$



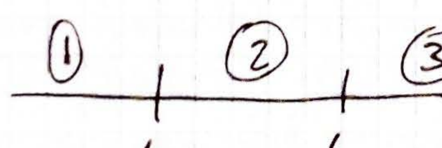
- OPTIMAL DECISION BOUNDARY:

$$P(\omega_1|x) = P(\omega_2|x)$$

Baye's Formula $\Rightarrow$ $\frac{P(x|\omega_1) \cdot P(\omega_1)}{\cancel{P(x)}} = \frac{P(x|\omega_2) \cdot P(\omega_2)}{\cancel{P(x)}}$

$$P(\omega_1)\left(\cancel{\frac{1}{2\sigma}} \exp\left(-\frac{|x-\mu|}{\sigma}\right)\right) = P(\omega_2)\left(\cancel{\frac{1}{2\sigma}} \exp\left(-\frac{|x+\mu|}{\sigma}\right)\right)$$

Log $\Rightarrow$ $-\frac{|x-\mu|}{\sigma} + \log P(\omega_1) = -\frac{|x+\mu|}{\sigma} + \log P(\omega_2)$



1. $\frac{\cancel{x}-\mu}{\sigma} + \log P(\omega_1) = \frac{\cancel{x}+\mu}{\sigma} + \log P(\omega_2)$ NOT DEPENDENT ON x !
2. $\frac{x+\mu}{\sigma} + \log P(\omega_1) = \frac{-x+\mu}{\sigma} + \log P(\omega_2)$ V
3. $\frac{-\cancel{x}+\mu}{\sigma} + \log P(\omega_1) = \frac{-\cancel{x}-\mu}{\sigma} + \log P(\omega_2)$ NOT DEPENDENT ON x !

FOR INTERVAL ②:

$$\frac{x+\mu}{\sigma} + \log P(\omega_1) = \frac{-x+\mu}{\sigma} + \log P(\omega_2)$$

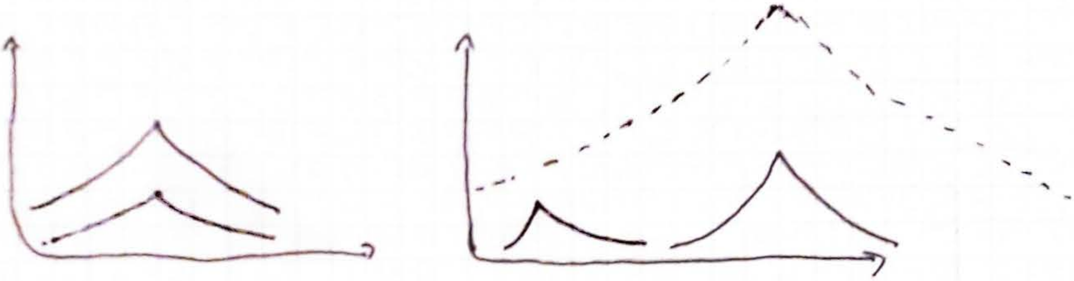
$$\frac{2x - \cancel{2\mu}}{\sigma} = \log \frac{P(\omega_2)}{P(\omega_2)}$$

$$x = \frac{\sigma}{2} \log\left(\frac{P(\omega_2)}{P(\omega_1)}\right)$$

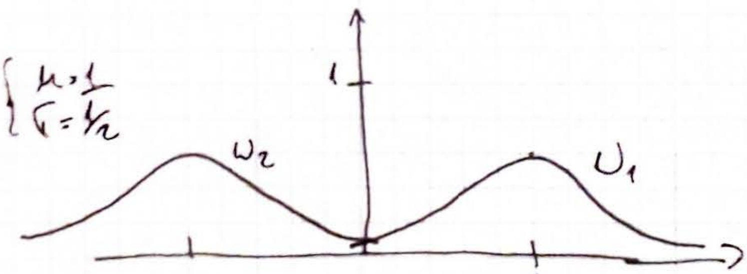
b) FOR INTERVAL ①: $2\mu \leq \sigma . \log \frac{P_{\omega_1}}{P_{\omega_2}}$

" ②: $2x \leq \sigma . \log \frac{P_{\omega_1}}{P_{\omega_2}}$

" ③: $-2\mu \leq \sigma . \log \frac{P_{\omega_1}}{P_{\omega_2}}$



c) EXAMPLE: FOR $\begin{cases} \mu = 1 \\ \sigma = 1/2 \end{cases}$



$$g_1(x) = \frac{(x-\mu)^2}{2\sigma^2} + \log P_{\omega_1}$$

$$g_2(x) = \frac{(x+\mu)^2}{2\sigma^2} + \log P_{\omega_2}$$

$$g_1(x) > g_2(x)$$

$$(\ldots) \Longrightarrow 2\mu x > \sigma^2 \log \frac{P_{\omega_1}}{P_{\omega_2}}$$

—— // ——