EXERCISE SHEET (5)

EX. 1

- CONSIDERING:

$$x = \begin{cases} 1 & \text{if } X = \text{'heads'} \\ 0 & \text{if } X = \text{'tails'} \end{cases} \quad (1)$$

$$z = \begin{cases} 1 & \text{if } Z = \text{'heads'} \\ 0 & \text{if } Z = \text{'tails'} \end{cases} \quad (2)$$

$$P(Z = z^{(i)} \mid \theta) = l^{z^{(i)}} \cdot (1-l)^{1-z^{(i)}} \quad (3)$$

$$P(X = x_j^{(i)} \mid Z = z^{(i)}, \theta) = \left[P_1^{x_j^i} \cdot (1-P_1)^{1-x_j^i}\right]^{z_i} \cdot \left[P_2^{x_j^i} \cdot (1-P_2)^{1-x_j^i}\right]^{1-z_i} \quad (4)$$

$$q^{old}(z) := P(z_i = 1 \mid x_i, \theta^{old}) =$$

$$= \frac{P(x_i \mid z_i, \theta^{old})\, P(z_i = 1 \mid \theta^{old})}{P(x_i \mid \theta^{old})} = \quad (5)$$

$$= \frac{l^{old} \cdot [P_1^{old}]^{x_i} \cdot [(1-P^{old})]^{1-x_i}}{l^{old} \cdot [P_1^{old}]^{x_i} \cdot [(1-P_1^{old})]^{1-x_i} + (1-l^{old})[P_2^{old}]^{x_i} \cdot [(1-P_2^{old})]^{1-x_i}}$$

$$\log\left[P(X = x_i, Z = z_i \mid \theta)\right] =$$

$$= \log\left[\prod_{i=1}^{m} l^{z_i} \cdot (1-l)^{1-z_i} \cdot \prod_{j=1}^{m} [P_1^{x_j} \cdot (1-P_1)^{1-x_j}]^{z_i} \cdot [P_2^{x_j} \cdot (1-P_2)^{1-x_j}]^{1-z_j}\right] =$$

$$= \sum_{i=1}^{m}\Big[z_i \cdot \log(l) + (1-z_i) \cdot \log(1-l) + \quad (6)$$

$$\sum_{j=1}^{m}\Big[z_i \cdot \big(x_j \cdot \log(P_1) + (1-x_j) \cdot \log(1-P_1)\big) +$$

$$(1-z_i) \cdot \big(x_j \cdot \log(P_2) + (1-x_j) \cdot \log(1-P_2)\big)\Big]\Big]$$

$$Q(\Theta, \Theta^{old}) = \quad (7)$$

$$= \sum_{i=1}^{n} \Big[q^{old}_{(z)} \cdot \Big(\log(l) + \sum_{j=1}^{m} [x_j \cdot \log(P_1) + (1 - x_j) \cdot \log(1-P_1)] \Big) +$$

$$(1 - q^{old}_{(z)}) \cdot \Big(\log(1-l) + \sum_{j=1}^{m} [x_j \cdot \log(P_2) + (1-x_j) \cdot \log(1-P_2)] \Big) \Big]$$

$$\frac{dQ(\Theta, \Theta^{old})}{\partial l} = 0 \quad (8)$$

$$\Rightarrow \sum_{i=1}^{n} \frac{q^{old}_{(z)}}{l} - \frac{(1 - q^{old}_{(z)})}{1-l} = 0$$

$$\sum_{i=1}^{m} \frac{q^{old}_{(z)}}{l} = \sum_{i=1}^{m} \frac{(1 - q^{old}_{(z)})}{1-l}$$

$$\sum_{i=1}^{m} \frac{1-l}{l} = \sum_{i=1}^{m} \frac{1 - q^{old}_{(z)}}{q^{old}_{(z)}}$$

$$\sum_{i=1}^{m} l = \sum_{i=1}^{m} q^{old}_{(z)}$$

$$\boxed{\hat{l} = \frac{1}{m} \sum_{i=1}^{m} q^{old}_{(z)}}$$

$$\frac{d \cdot Q(\Theta, \Theta^{old})}{\partial P_1} = 0 \quad (9)$$

$$\Rightarrow \sum_{i=1}^{m} q^{old}_{(z)} \times \Big(\sum_{j=1}^{m} \Big[\frac{x_j^{(i)}}{P_1} - \frac{1 - x_j^{(i)}}{1 - P_1} \Big] \Big) = 0$$

$$\sum_{i=1}^{n} q^{old}_{(z)} \times \sum_{j=1}^{m} \frac{x_j^{(i)}}{P_1} = \sum_{i=1}^{m} q^{old}_{(z)} \times \sum_{j=1}^{m} \frac{1 - x_j^{(i)}}{1 - P_1}$$

$$\sum_{i=1}^{m} q^{old}_{(z)} \times \sum_{j=1}^{m} P_1 = \sum_{i=1}^{m} q^{old}_{(z)} \times \sum_{j=1}^{m} x_j^{(i)}$$

$$\boxed{\hat{P}_1 = \frac{1}{m} \cdot \frac{\sum_{i=1}^{m} q^{old}_{(z)} \times \sum_{j=1}^{m} x_j^{(i)}}{\sum_{i=1}^{m} q^{old}_{(z)}}}$$

$$\frac{dQ(\theta, \theta^{old})}{\delta p_2} = 0 \qquad (10)$$

$$\Rightarrow \sum_{i=1}^{n} (1 - q_{(z)}^{old}) \times \left(\sum_{j=1}^{m} \left[\frac{x_j^{(i)}}{p_2} - \frac{(1-x_j^{(i)})}{p_2} \right] \right) = 0$$

(...) SAME STEPS AS IN (9) (...)

$$\boxed{\hat{p}_2 = \frac{1}{m} \cdot \frac{\sum_{i=1}^{n} (1 - q_{(z)}^{old}) \times \sum_{j=1}^{m} x_j^{(i)}}{\sum_{i=1}^{n} (1 - q_{(z)}^{old})}}$$