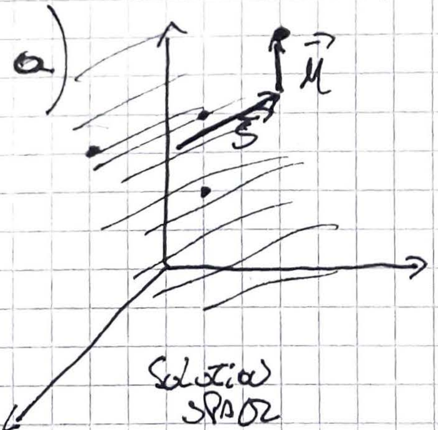
EXERCISE SHEET 03

Ex. 1

a)



OUTSIDE THE SPAN OF THE DATA $\left(\vec{\mu} \perp \text{SPAN(DATA)}\right)$

$$w_x = s_x + \mu_x \quad , \langle \mu_x , X \rangle = 0$$
$$w_y = s_y + \mu_y \quad , \langle \mu_y , Y \rangle = 0$$

$$w_x^T C_{xy} w_y = w_x^T X Y^T w_y =$$
$$= (s_x + \mu_x)^T X Y^T (s_y + \mu_y) =$$
$$= \Big(s_x^T X + \underbrace{\mu_x^T X}_{=0}\Big)\Big(Y^T s_y + \underbrace{Y^T \mu_y}_{=0}\Big) =$$
$$= s_x^T X Y^T s_y$$

WHERE $\vec{s_x}$ AND $\vec{s_y}$ LIE IN THE SPAN OF X AND Y, RESPECTIVELY.

b)

$$\max \; w_x^T C_{xy} w_y \quad \text{S.T.} \quad w_x^T C_{xx} w_x = 1$$
$$w_y^T C_{yy} w_y = 1$$

$$\mathcal{L}(w_x, w_y, \alpha, \beta) = w_x^T C_{xy} w_y - \alpha \tfrac{1}{2}\left(w_x^T C_{xx} w_x - 1\right) - \beta \tfrac{1}{2}\left(w_y^T C_{yy} w_y - 1\right)$$

$$\frac{\delta \mathcal{L}}{\delta w_x^T} = C_{xy} w_y - \alpha C_{xx} w_x = 0$$

$$\Rightarrow C_{xy} w_y = \alpha C_{xx} w_x$$

$$\frac{\delta \mathcal{L}}{\delta w_y^T} = C_{yx} w_x - \beta C_{yy} w_y = 0$$

$$\Rightarrow C_{yx} w_x = \beta C_{yy} w_y$$

AS A RESULT OF SETTING THE PARTIAL DERIVATIVES TO ZERO AND THE OPTIMIZATION PROBLEM'S CONSTRAINTS:

$$\alpha = \beta$$

HENCE:

$$\begin{bmatrix} 0 & C_{XY} \\ C_{YX} & 0 \end{bmatrix} \begin{bmatrix} \omega_X \\ \omega_Y \end{bmatrix} = \alpha \begin{bmatrix} C_{XX} & 0 \\ 0 & C_{YY} \end{bmatrix} \begin{bmatrix} \omega_X \\ \omega_Y \end{bmatrix}$$

FOR: $\begin{cases} \omega_X = X \alpha_X \\ \omega_Y = Y \alpha_Y \end{cases}$ AND $\begin{cases} A = X^T X \\ B = Y^T Y \end{cases}$

$$\Rightarrow \begin{cases} X Y^T Y \alpha_Y = \rho X X^T X \alpha_X \\ Y X^T X \alpha_X = \rho Y Y^T Y \alpha_Y \end{cases} = \begin{cases} X^T X Y^T Y \alpha_Y = \rho X^T X X^T X \alpha_X \\ Y^T Y X^T X \alpha_X = \rho Y^T Y Y^T Y \alpha_Y \end{cases}$$

$$= \begin{cases} A.B \, \alpha_Y = \rho A^2 \alpha_X \\ B.A \, \alpha_X = \rho B^2 \alpha_Y \end{cases}$$

HENCE:

$$\begin{bmatrix} 0 & A.B \\ B.A & 0 \end{bmatrix} \begin{bmatrix} \alpha_X \\ \alpha_Y \end{bmatrix} = \rho \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix} \begin{bmatrix} \alpha_X \\ \alpha_Y \end{bmatrix}$$

c) $\omega_X = X \alpha_X$ AND $\omega_Y = Y \alpha_Y$

$\boxed{Ex. 2}$

a) $K_x(X, X) = \phi(X)^T \phi(X)$
$K_Y(Y, Y) = \phi(Y)^T \phi(Y)$

From the solution of 1.b):

$$\begin{bmatrix} 0 & K_x K_Y \\ K_Y K_x & 0 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_Y \end{bmatrix} = \rho \begin{bmatrix} K_x^2 & 0 \\ 0 & K_Y^2 \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_Y \end{bmatrix}$$

b) $\max \ w_x^T C_{xy} w_y \quad$ s.t. $(\dots)$

$= \max \ (\phi(X)\alpha_x)^T \phi(X) \phi(Y)^T (\phi(Y) . \alpha_Y)$

$= \max \ \alpha_x^T \phi(X)^T \phi(X) \phi(Y)^T \phi(Y) . \alpha_Y$

$= \max \ \alpha_x^T K_x K_Y \alpha_Y$

$\boxed{Ex. 3}$