# EXERCISE SHEET 04

## Ex.1

a) $P(x) = e^{S(x)}$ AND $\mathbb{E}[x] = 0$

$$\Lambda(S(x), l_1, l_2, l_3) = -\int e^{S(x)} \log e^{S(x)} dx + l_1\left(\int e^{S(x)} dx - 1\right) + l_2 \int x e^{S(x)} dx + l_3\left(\int x^2 e^{S(x)} dx - \sigma^2\right)$$

b) $$\frac{\delta \Lambda}{\delta S(x)} = -e^{S(x)}(1 + S(x)) + l_1 e^{S(x)} + l_2 x . e^{S(x)} + l_3 x^2 e^{S(x)} = 0$$

$$\Rightarrow e^{S(x)} . \left(-1 - S(x) + l_1 + l_2 x + l_3 x^2\right) = 0$$

$$S(x) = -1 + l_1 + l_2 x + l_3 x^2$$

c) $$\frac{\delta \Lambda}{\delta l_1} = \int e^{-1 + l_1 + l_2 x + l_3 x^2} dx - 1 = 0$$

1 $\Rightarrow \dfrac{\sqrt{\pi} . e^{l_1 - \frac{l_2^2}{4 l_3} - 1}}{\sqrt{-l_3}} = 1$ , WHERE WE ASSUME $l_3 < 0$

$$\frac{\delta \Lambda}{\delta l_2} = \int x . e^{-1 + l_1 + l_2 x + l_3 x^2} dx = 0$$

2 $\Rightarrow \dfrac{\sqrt{\pi}\, l_2\, e^{l_1 - \frac{l_2^2}{4 l_3} - 1}}{2 . (-l_3)^{3/2}} = 0$

$$\frac{\partial \Lambda}{\partial l_3} = \int x^2 . e^{-1 + l_1 + l_2 x + l_3 x^2} dx - \sigma^2 = 0$$

$$\Rightarrow \frac{\sqrt{\pi} (l_2 - 2 l_3) . e^{l_1 - \frac{l_2^2}{4 l_3} - 1}}{4 . (-l_3)^{5/2}} = \sigma^2 \quad (3)$$

FROM (1) WE OBTAIN: $e^{l_1 - \frac{l_2^2}{4 l_3} - 1} = \frac{\sqrt{-l_3}}{\sqrt{\pi}}$

FROM (2) WE OBTAIN: $l_2 = 0$

FROM (1) AND (3), WE HAVE:

$$\sqrt{\pi} . (-2 l_3) . \frac{\sqrt{-l_3}}{\sqrt{\pi}} = \sigma^2 . 4 . (-l_3)^{5/2}$$

$$-2 . l_3 . \sqrt{-l_3} = \sigma^2 . 4 . (-l_3)^{5/2}$$

$$2 . (-l_3)^{3/2} = \sigma^2 . 4 (-l_3)^{5/2}$$

$$l_3 = -\frac{1}{2\sigma^2}$$

INSERTING INTO (1):

$$e^{l_1 - 1} = \frac{1}{\sqrt{2\pi} . \sigma}$$

$$l_1 - 1 = -\log(\sqrt{2\pi} \sigma)$$

$$l_1 = 1 - \log(\sqrt{2\pi} \sigma)$$

WE THEN OBTAIN:

$$P(x) = e^{-1 + 1 - \log(\sqrt{2\pi} . \sigma) - \frac{1}{2\sigma^2} x^2} =$$

$$= \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2\sigma^2} x^2}$$

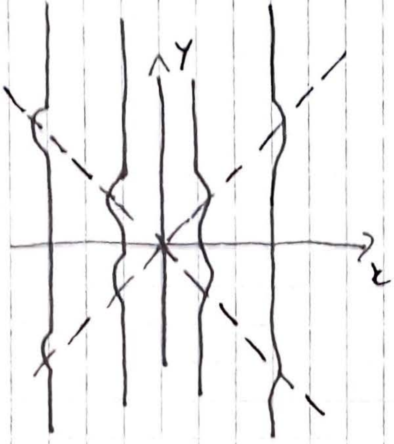
d) $J(x) = H(x^*) - H(x)$

$H(x^*) > H(x)$ Where $P_x \neq P_{x^*}$

And $H(x^*) = H(x)$ Where $P_x = P_{x^*}$

Ex. 2

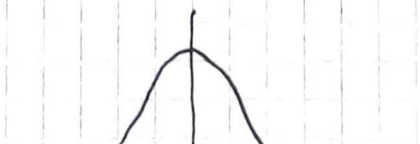
$P(x) \sim N(0,1)$, $P(y|x) \sim \frac{1}{2}\delta(y-x) + \frac{1}{2}\delta(y+x)$





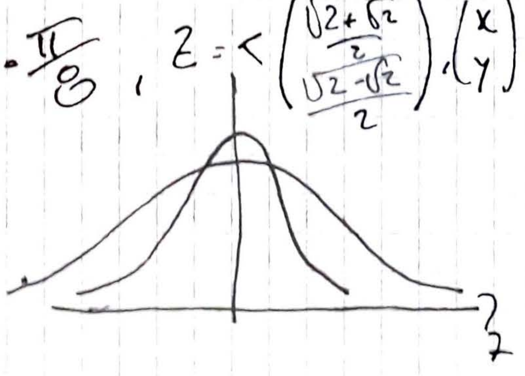
a) $z = \left\langle \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$

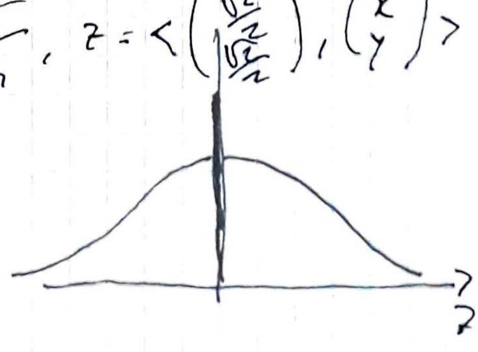
$\theta = 0$, $z = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = x \Rightarrow P(z) = P(x)$




$\theta = \frac{\pi}{8}$, $z = \left\langle \begin{pmatrix} \frac{\sqrt{2+\sqrt{2}}}{2} \\ \frac{\sqrt{2-\sqrt{2}}}{2} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$

$\theta = \frac{\pi}{4}$, $z = \left\langle \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$



b) $\mathrm{VAR}(Z) = \mathbb{E}[(Z - \mathbb{E}[Z])^2]$

$$\mathbb{E}[Z] = \int_{x,y} z\, P(x,y)\, dx\, dy =$$

$$= \int_{x,y} (x.\cos\theta + y\sin\theta) \cdot \left[\tfrac{1}{2}\delta(x-y) + \tfrac{1}{2}\delta(x+y)\right] P(x)\, dy\, dx$$

$$= \int_x \left[\tfrac{1}{2}(x.\cos\theta + x\sin\theta) + \tfrac{1}{2}(x.\cos\theta + x\sin\theta)\right] P(x)\, dx$$

$$= \int_x x.\cos\theta . P(x)\, dx = \cos\theta \underbrace{\int x P(x)\, dx}_{\mathbb{E}[X]=0} = 0$$

$$\mathrm{VAR}[Z] = \mathbb{E}[Z^2] = \int_{x,y} z^2 P(x,y)\, dx\, dy$$

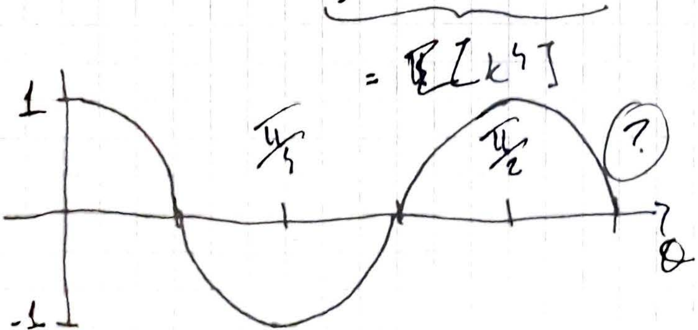
$$= \int_x x^2 [\cos^2\theta + \sin^2\theta]\, P(x)\, dx = \int x^2 P(x)\, dx = 1$$

c) $\mathrm{kurt}[Z(\theta)] = \mathbb{E}[Z(\theta)^4]$

$$\int x^4 [\cos^4\theta + \sin^4\theta + 6\cos^2\theta\sin^2\theta]\, P(x)\, dx$$

$$(\underbrace{\cos^2\theta + \sin^2\theta}_{=1})^2 + \underbrace{4\cos^2\theta\sin^2\theta}_{=\sin(2\theta)}$$

$$= \sin^2 2\theta \underbrace{\int x^4 P(x)\, dx}_{=\mathbb{E}[x^4]} = \frac{1-\cos 4\theta}{2}\, \mathbb{E}[x]$$




$$\theta \in \left\{\frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$$