Exercise Sheet 02

Ex.1

$$E(x, W) = \sum_i |x_i - \sum_j W_{ij} x_j|^2$$

$$\underset{W}{\text{ARGmin}}\; E(x, W) = \underset{W}{\text{ARGmin}}\; E(\phi(x), W)$$

i)
$$E(\alpha \cdot x, W) = \sum_i |\alpha \cdot x_i - \sum_j W_{ij} x_j \alpha|^2 =$$

$$= \alpha^2 \sum_i |x_i - \sum_j W_{ij} x_j|^2 =$$

$$= \alpha^2 E(x, W)$$

ii)
$$E(x+N, W) = \sum_i |x_i + N - \sum_j W_{ij} (x_j + N)|^2 =$$

$$= \sum_i |x_i + \cancel{N} - \sum_j W_{ij} x_j - \underbrace{\sum_j W_{ij} N}_{=1} \cancel{N}|^2 =$$

$$= \sum_i |x_i - \sum_j W_{ij} x_j|^2 = E(x, W)$$

iii)
$$E(Ux, W) = \sum_i |U x_i - \sum_j W_{ij} U x_j|^2 =$$

$$\left[= \sum_i |U \cdot (x_i - \sum_j W_{ij} x_j)|^2 = \right.$$

$$* \; \|U_z\|^2 =$$

$$= z^T U^T U z = z^T z = \|z\|^2 \left. \right] \quad \underbrace{z}_{\; *} $$

$$\underbrace{}_{I}$$

$$= \sum_i |x_i - \sum_j W_{ij} x_j|^2 = E(x, W)$$

Ex.2

$$\sum_j w_j = 1$$

i) $$\mathcal{E} = |x - \sum_k w_k \eta_k|^2 = |\sum_j w_j (x - \eta_j)|^2 =$$

$$= |w^T(\mathbb{1}x^T - \eta)|^2 = w^T(\mathbb{1}x^T - \eta)(\mathbb{1}x^T - \eta)w =$$

$$\underbrace{(\mathbb{1}x^T - \eta)(\mathbb{1}x^T - \eta)}_{C}$$

k $\quad k \cdot d$

$$= w^T C w$$

$$\hookrightarrow \quad \min_w \; w^T C w$$

$$\text{s.t} \quad \mathbb{1}^T w = 1$$

ii) $$\mathcal{L} = \tfrac{1}{2} w^T C w + l(1 - w^T \mathbb{1})$$

$$\frac{\partial \mathcal{L}}{\partial w} = C w - l\mathbb{1} = 0$$

$$\Rightarrow Cw = l\mathbb{1}$$

$$w = l\, C^{-1}\mathbb{1}$$

$$\underbrace{\mathbb{1}^T w}_{1} = l\, \mathbb{1}^T C^{-1}\mathbb{1} \iff l = \frac{1}{\mathbb{1}^T C^{-1}\mathbb{1}}$$

$$\Rightarrow w = \frac{C^{-1}\mathbb{1}}{\mathbb{1}^T C^{-1}\mathbb{1}}$$

iii)

$$C\tilde{w} = \mathbb{1} \iff \tilde{w} = C^{-1}\mathbb{1}$$

$$w = \frac{\tilde{w}}{\mathbb{1}^T \tilde{w}} \iff w = \frac{C^{-1}\mathbb{1}}{\mathbb{1}^T C^{-1}\mathbb{1}}$$