From Cohl and Tohline:

$$\frac{1}{R} = \frac{1}{\pi\sqrt{rr_1}} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m(\theta-\theta_1)} Q_{m-1/2}(\chi), \tag{1}$$

$$R^2 = r^2 + r_1^2 - 2rr_1 \cos(\theta - \theta_1) + x^2,$$

$$\chi = \frac{r^2 + r_1^2 + x^2}{2rr_1}. \tag{2}$$

Field from an azimuthal mode:

$$G = \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}n\theta_1}}{4\pi R} \,\mathrm{d}\theta_1, \tag{3}$$

$$= \mathrm{e}^{\mathrm{i}n\theta} G_n, \tag{4}$$

$$G_n = \frac{Q_{n-1/2}(\chi)}{2\pi\sqrt{rr_1}}. \tag{5}$$

Recursion (Cohl and Tohline)

$$G_0 = \frac{1}{2\pi} \frac{\mu}{\sqrt{rr_1}} K(\mu), \tag{6}$$

$$G_1 = \frac{1}{2\pi} \frac{1}{\sqrt{rr_1}} \left(\chi\mu K(\mu) - (\chi+1)\mu E(\mu)\right), \tag{7}$$

$$G_n = 4\frac{n-1}{2n-1}\chi G_{n-1} - \frac{2n-3}{2n-1} G_{n-2}, \tag{8}$$

$$(2n-3)G_{n-2} = 4(n-1)\chi G_{n-1} - (2n-1)G_n. \tag{9}$$

(The descending recursion to find G_n is stable.)

Derivative w.r.t. x. From G&R, 8.832,

$$\frac{\partial G_n}{\partial x} = (n+1/2)\left(G_{n+1} - \chi G_n\right) \frac{1}{\chi^2 - 1} \frac{\partial \chi}{\partial x}, \tag{10}$$

$$= \frac{(n-1/2)(n+1/2)}{2n} \left(G_{n+1} - G_{n-1}\right) \frac{1}{\chi^2 - 1} \frac{\partial \chi}{\partial x}, \tag{11}$$

$$\frac{1}{\chi^2 - 1} \frac{\partial \chi}{\partial x} = \frac{x}{rr_1} \frac{1}{\chi - 1} \frac{1}{\chi + 1}. \tag{12}$$

$$\frac{\partial G_0}{\partial x} = -\frac{x}{8\pi} \frac{1}{(rr_1)^{3/2}} \frac{\mu^3}{1-\mu^2} E(\mu). \tag{13}$$

Derivative w.r.t. r_1. From G&R, 8.832,

$$\frac{\partial G_n}{\partial r_1} = -\frac{G_n}{2r_1} + (n+1/2)\left(G_{n+1} - \chi G_n\right) \frac{1}{\chi^2 - 1} \frac{\partial \chi}{\partial r_1}, \tag{14}$$

$$\tag{15}$$

Derivatives of $f = 1/(\chi \pm 1)$,

$$
\begin{aligned}
f &= \frac{1}{\chi \pm 1} = \frac{2rr_1}{(r \pm r_1)^2 + x^2}, \\
\left[(r \pm r_1)^2 + x^2\right] f &= 2rr_1, \\
\frac{\partial^m}{\partial x^m}\left[(r \pm r_1)^2 + x^2\right] f &= \sum_{q=0}^{m} \binom{m}{q} \frac{\partial^q}{\partial x^q}\left[(r \pm r_1)^2 + x^2\right] \frac{\partial^{m-q} f}{\partial x^{m-q}} = 0, \\
\frac{\partial^m f}{\partial x^m} &= -\sum_{q=1}^{m} \binom{m}{q} \frac{\partial^q}{\partial x^q}\left[(r \pm r_1)^2 + x^2\right] \frac{\partial^{m-q} f}{\partial x^{m-q}}, \\
&= -\frac{m}{2rr_1}\left[2x\frac{\partial^{m-1} f}{\partial x^{m-1}} + (m-1)\frac{\partial^{m-2} f}{\partial x^{m-2}}\right] f, \quad m \geq 2, \\
\frac{\partial f}{\partial x} &= -\frac{x}{rr_1} f^2.
\end{aligned}
$$

Derivatives of $f = 1/(\chi^2 - 1)$:

$$
\begin{aligned}
\rho_\pm^2 &= (r \pm r_1)^2 + x^2, \\
f(\chi) &= \frac{1}{\chi^2 - 1} = \frac{4r^2 r_1^2}{\rho_+^2 \rho_-^2}, \\
\rho_+^2 \rho_-^2 f(\chi) &= 4r^2 r_1^2, \\
\frac{\partial^k}{\partial x^k}\left(\rho_+^2 \rho_-^2 f(\chi)\right) &= \sum_{q=0}^{k} \binom{k}{q} \frac{\partial^q}{\partial x^q}\left(\rho_+^2 \rho_-^2\right) \frac{\partial^{k-q} f}{\partial x^{k-q}} = \frac{\partial^k}{\partial x^k}\left(4r^2 r_1^2\right), \\
\frac{\partial^k f}{\partial x^k} &= \frac{1}{\rho_+^2 \rho_-^2}\left[\frac{\partial^k}{\partial x^k}\left(4r^2 r_1^2\right) - \sum_{q=1}^{k} \binom{k}{q} \frac{\partial^q}{\partial x^q}\left(\rho_+^2 \rho_-^2\right) \frac{\partial^{k-q} f}{\partial x^{k-q}}\right]. \\
\frac{\partial^{j+k}}{\partial r_1^j \partial x^k}\left[\rho_+^2 \rho_-^2 f(\chi)\right] &= \sum_{j,k} \binom{k}{q} \binom{j}{u} \frac{\partial^{u+q}}{\partial r_1^u \partial x^q}\left(\rho_+^2 \rho_-^2\right) \frac{\partial^{k+j-u-q} f}{\partial r_1^{j-u} \partial x^{k-q}} = \frac{\partial^{j+k}}{\partial r_1^j \partial x^k}\left(4r^2 r_1^2\right).
\end{aligned}
$$

From Laplace equation:

$$
\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial G_n}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 G_n}{\partial \theta^2} + \frac{\partial^2 G_n}{\partial x^2} = 0, \tag{16}
$$

$$
\frac{\partial^2 G_n}{\partial r^2} + \frac{1}{r}\frac{\partial G_n}{\partial r} - \frac{n^2}{r^2} G_n + \frac{\partial^2 G_n}{\partial x^2} = 0. \tag{17}
$$

From Cohl, Rau, Tohline, et al.

$$
\begin{aligned}
G_n &= \frac{Q_{n-1/2}(\chi)}{2\pi\sqrt{rr_1}}, && (18)\\
&= \frac{(-1)^n}{2\sqrt{\rho\rho_1}} \sum_{i=0}^{\infty} \left(\frac{-1}{2}\right)^i \frac{(2i)!}{i!(2n+2i)!!} \left(\frac{\rho_1}{\rho}\right)^{n+2i+1/2} P_{n+2i}^{n}(\cos\phi), && (19)\\
x &= \rho\cos\phi, \quad r = \rho\sin\phi, && (20)\\
\rho_1 &= r_1. && (21)
\end{aligned}
$$

$$
\begin{aligned}
G_n &= \frac{Q_{n-1/2}(\chi)}{2\pi\sqrt{rr_1}}, && (22)\\
&= \sum_{q=0}^{\infty} \frac{1}{q!} \frac{\mathrm{d}^q Q_{n-1/2}(\chi_0)}{\mathrm{d}\chi^q} \frac{(\chi-\chi_0)^q}{\sqrt{rr_1}}, && (23)\\
\frac{(\chi-\chi_0)^q}{\sqrt{rr_1}} &= \left[\frac{f(\delta r, \delta r_1, \delta x)}{4rr_1(r+\delta r)^{1/2}(r_1+\delta r_1)^{1/2}}\right]^q, && (24)\\
f(\delta r, \delta r_1, \delta x) &= 2rr_1\left[(r+\delta r)^2 + (r_1+\delta r_1)^2 + (x+\delta x)^2\right]\\
&\quad - 2(r+\delta r)(r_1+\delta r_1)(r^2+r_1^2+x^2), && (25)\\
f^q &= \sum \frac{1}{m!} \left.\frac{\partial^m f^q}{\partial(\delta r)^m}\right|_{\delta r=0} (\delta r)^m \text{ (polynomial)}, && (26)\\
\frac{(\chi-\chi_0)^q}{\sqrt{rr_1}} &= \sqrt{2}\frac{f^q}{(2rr_1)^{2q+1/2}} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i^{(q+1/2)} a_j^{(q+1/2)} \left(\frac{\delta r}{r}\right)^i \left(\frac{\delta r_1}{r_1}\right)^j && (27)
\end{aligned}
$$

Derivatives of f:

$$
\begin{aligned}
\frac{\partial f}{\partial \delta r} &= 4rr_1(r+\delta r) - 2(r_1+\delta r_1)(r^2+r_1^2+x^2), && (28)\\
&= 4r^2 r_1 - 2r_1(r^2+r_1^2+x^2),\ \delta = 0, && (29)\\
\frac{\partial^2 f}{\partial(\delta r)^2} &= 4rr_1, && (30)\\
\frac{\partial f^q}{\partial \delta r} &= q f^{q-1}\frac{\partial f}{\partial \delta r}, && (31)\\
\frac{\partial^M f^q}{\partial(\delta r)^M} &= q\sum_{m=0}^{M-1} \binom{M-1}{m} \frac{\partial^{m+1} f}{\partial(\delta r)^{m+1}} \frac{\partial^{M-m-1} f^{q-1}}{\partial(\delta r)^{M-m-1}} && (32)
\end{aligned}
$$

$$\chi - \chi_0 = \frac{1}{2}\frac{f(\delta r, \delta r_1, \delta x)}{(r+\delta r)(r_1+\delta r_1)}, \tag{33}$$

$$f(\delta r, \delta r_1, \delta x) = A(\delta r) + B(\delta r_1) + C(\delta r, \delta r_1) + D(\delta x), \tag{34}$$

$$A(\delta r) = (\delta r)^2 - \left(r_1^2 - r^2 + x^2\right)\frac{\delta r}{r} = (\delta r)^2 + a\delta r, \tag{35}$$

$$B(\delta r_1) = (\delta r_1)^2 - \left(r^2 - r_1^2 + x^2\right)\frac{\delta r_1}{r_1} = (\delta r_1)^2 + b\delta r_1, \tag{36}$$

$$C(\delta r, \delta r_1) = -\frac{(r^2 + r_1^2 + x^2)}{r r_1}\delta r \delta r_1 = c\delta r \delta r_1, \tag{37}$$

$$D(\delta x) = (\delta x)^2 + 2x\delta x = d_2\,(\delta x)^2 + d\delta x. \tag{38}$$

$$\begin{aligned}
[f(\delta r, \delta r_1, \delta x)]^q &= \sum_{|\boldsymbol{\alpha}|=q}\binom{q}{\boldsymbol{\alpha}}c^{\alpha_3} \\
&\times \left[\sum_{\beta=0}^{\alpha_1}\binom{\alpha_1}{\beta_1}a^{\alpha_1-\beta_1}\,(\delta r)^{\alpha_1+\alpha_3+\beta_1}\right] \\
&\times \left[\sum_{\beta_2=0}^{\alpha_2}\binom{\alpha_2}{\beta_2}b^{\alpha_2-\beta_2}\,(\delta r_1)^{\alpha_2+\alpha_3+\beta_2}\right] \\
&\times \left[\sum_{\beta_4=0}^{\alpha_4}\binom{\alpha_4}{\beta_4}d^{\alpha_4-\beta_4}\,(\delta x)^{\alpha_4+\beta_4}\right], \qquad (39) \\
&= \sum_{|\boldsymbol{\alpha}|=q}\frac{q!}{\alpha_3!}c^{\alpha_3} \\
&\times \left[\sum_{\beta_1=0}^{\alpha_1}\frac{a^{\alpha_1-\beta_1}}{\beta_1!(\alpha_1-\beta_1)!}\,(\delta r)^{\alpha_1+\alpha_3+\beta_1}\right] \\
&\times \left[\sum_{\beta_2=0}^{\alpha_2}\frac{b^{\alpha_2-\beta_2}}{\beta_2!(\alpha_2-\beta_2)!}\,(\delta r_1)^{\alpha_2+\alpha_3+\beta_2}\right] \\
&\times \left[\sum_{\beta_4=0}^{\alpha_4}\frac{d^{\alpha_4-\beta_4}}{\beta_4!(\alpha_4-\beta_4)!}\,(\delta x)^{\alpha_4+\beta_4}\right]. \qquad (40)
\end{aligned}$$

From Cohl, Rau, Tohline, Browne, Cazes, Barnes, 2001:

$$G_n = \frac{1}{2\rho_>} \sum_{\ell=n}^{\infty} \frac{(\ell-n)!}{(\ell+n)!} \left(\frac{\rho_<}{\rho_>}\right)^{\ell} P_\ell^n(\cos\phi) P_\ell^n(\cos\phi_1), \tag{41}$$

$$r = \rho\sin\phi,\ z = \rho\cos\phi. \tag{42}$$

$$G_n^m = \int_0^{2\pi} \frac{\cos n\theta_1}{4\pi R^m}\, \mathrm{d}\theta_1, \tag{43}$$

$$R^2 = r^2 + r_1^2 + z^2 - 2rr_1\cos\theta_1, \tag{44}$$

$$G_n^m = \frac{1}{2\pi\rho^m} \int_0^{\pi} \frac{\cos 2n\theta_1}{(1-\lambda^2\cos^2\theta_1)^{m/2}}\, \mathrm{d}\theta_1, \tag{45}$$

$$\rho^2 = (r+r_1)^2 + z^2,\ \lambda^2 = \frac{4rr_1}{\rho^2}, \tag{46}$$

$$G_n^m = \frac{(-1)^n}{4\pi(\rho_+\rho_-)^{m/2}} \int_0^{2\pi} \frac{\cos n\theta_1}{(\chi + \sqrt{\chi^2-1}\cos\theta_1)^{m/2}}\, \mathrm{d}\theta_1, \tag{47}$$

$$\chi = \frac{1}{2}\frac{\rho_+^2 + \rho_-^2}{\rho_+\rho_-}, \tag{48}$$

$$G_n^m = \frac{1}{2(\rho_+\rho_-)^{m/2}} \frac{\Gamma(m/2-n)}{\Gamma(m/2)} P_{m/2-1}^n(\chi). \tag{49}$$

$$G_0^1 = \frac{K(\lambda)}{\pi\rho}, \tag{50}$$

$$G_1^1 = \frac{1}{\pi\rho}\left[2\frac{K(\lambda)-E(\lambda)}{\lambda^2} - K(\lambda)\right]. \tag{51}$$

From DLMF 14.10.6 and 14.10.3

$$(m-2n-4)G_{n+2}^m + 4(n+1)\frac{\chi}{\sqrt{\chi^2-1}}G_{n+1}^m - (m+2n)G_n^m = 0, \tag{52}$$

$$m(m+2)(\rho_+\rho_-)^2 G_n^{m+4} - 2m(m+1)(\rho_+\rho_-)\chi G_n^{m+2} + (m+2n)(m-2n)G_n^m = 0. \tag{53}$$

Using G&R 8.732,

$$\frac{\partial G_n}{\partial z} = \frac{n+1/2}{\chi^2-1}\left(G_{n+1} - \chi G_n\right)\frac{z}{rr_1}, \tag{54}$$

$$= (n+1/2)\left(F_1 G_{n+1} - F_2 G_n\right), \tag{55}$$

$$F_1 = \frac{4rr_1 z}{\rho_+^2\rho_-^2},\ F_2 = 2z\frac{r^2+r_1^2+z^2}{\rho_+^2\rho_-^2}. \tag{56}$$

Higher derivatives:

$$\frac{\partial^{k+1} G_n}{\partial z^{k+1}} = (n+1/2) \sum_{q=0}^{k} \binom{k}{q} \left(\frac{\partial^{k-q} F_1}{\partial z^{k-q}} \frac{\partial^q G_{n+1}}{\partial z^q} - \frac{\partial^{k-q} F_2}{\partial z^{k-q}} \frac{\partial^q G_n}{\partial z^q} \right). \quad (57)$$

$$\frac{\partial G_n}{\partial r} = -\frac{G_n}{2r} + (n+1/2)\left(A_1 G_{n+1} - A_2 G_n\right), \quad (58)$$

$$A_1 = \frac{2r_1}{\rho_+^2 \rho_-^2}\left(r^2 - r_1^2 - z^2\right),\ A_2 = \frac{1}{r}\frac{1}{\rho_+^2 \rho_-^2}\left[r^4 - (r_1^2 + z^2)^2\right]. \quad (59)$$

Generation of derivatives from Laplace equation:

$$\frac{\partial^{i+k+2} G_n}{\partial r^{i+2} \partial x^k} = -\frac{\partial^{i+k+2} G_n}{\partial r^i \partial x^{k+2}} + n^2 \sum_{u=0}^{i} (-1)^u \binom{i}{u} \frac{(u+1)!}{r^{u+2}} \frac{\partial^{i+k-u} G_n}{\partial r^{i-u} \partial x^k}$$
$$- \sum_{u=0}^{i} (-1)^u \binom{i}{u} \frac{u!}{r^{u+1}} \frac{\partial^{i+k-u+1} G_n}{\partial r^{i-u+1} \partial x^k}. \quad (60)$$

$$\frac{\partial G_n}{\partial x} = (n-1/2)\left[\frac{x}{\rho_+^2}(G_n + G_{n-1}) + \frac{x}{\rho_-^2}(G_n - G_{n-1})\right], \quad (61)$$

$$\frac{\partial^{k+1} G_n}{\partial x^{k+1}} = (n-1/2) \sum_{q=0}^{k} \binom{k}{q} \left[\frac{\partial^q}{\partial x^q}\left(\frac{x}{\rho_+^2}\right) \frac{\partial^{k-q}}{\partial x^{k-q}} (G_n + G_{n-1}) + \frac{\partial^q}{\partial x^q}\left(\frac{x}{\rho_-^2}\right) \frac{\partial^{k-q}}{\partial x^{k-q}} (G_n - G_{n-1}) \right]. \quad (62)$$

$$\frac{\partial G_n}{\partial r} = -\frac{G_n}{2r} + (n-1/2)\frac{r^2 - r_1^2 - x^2}{2r}\left[\frac{1}{\rho_+^2}(G_n + G_{n-1}) + \frac{1}{\rho_-^2}(G_n - G_{n-1})\right],$$

(63)

$$g_{1,0,k}^{(n)} = -\frac{g_{0,0,k}^{(n)}}{2r} + \frac{n-1/2}{2r} \sum_{q=0}^{k} \binom{k}{q} \left[\left(g_{0,0,k-q}^{(n)} + g_{0,0,k-q}^{(n-1)}\right) \frac{\partial^q}{\partial x^q} \frac{f(r,r_1,x)}{\rho_+^2} + \left(g_{0,0,k-q}^{(n)} - g_{0,0,k-q}^{(n-1)}\right) \frac{\partial^q}{\partial x^q} \frac{f(r,r_1,x)}{\rho_-^2} \right], \quad (64)$$

$$f(r, r_1, x) = r^2 - r_1^2 - x^2. \quad (65)$$

$$
\begin{aligned}
g^{(n)}_{1,1,k} = -\frac{g^{(n)}_{0,1,k}}{2r} + \frac{n-1/2}{2r}\sum_{q=0}^{k}\binom{k}{q}\Bigg[& \left(g^{(n)}_{0,0,k-q} + g^{(n-1)}_{0,0,k-q}\right)\frac{\partial^{q+1}}{\partial r_1 \partial x^q}\frac{f(r,r_1,x)}{\rho_+^2} + \\
& \left(g^{(n)}_{0,1,k-q} + g^{(n-1)}_{0,1,k-q}\right)\frac{\partial^{q}}{\partial x^q}\frac{f(r,r_1,x)}{\rho_+^2} + \\
& \left(g^{(n)}_{0,0,k-q} - g^{(n-1)}_{0,0,k-q}\right)\frac{\partial^{q+1}}{\partial r_1 \partial x^q}\frac{f(r,r_1,x)}{\rho_-^2} + \\
& \left(g^{(n)}_{0,1,k-q} - g^{(n-1)}_{0,1,k-q}\right)\frac{\partial^{q}}{\partial x^q}\frac{f(r,r_1,x)}{\rho_-^2}\Bigg]
\end{aligned} \tag{66}
$$

$$
\frac{\partial}{\partial x}\frac{x}{\rho_\pm^2} = \left[1 - 2x\frac{x}{\rho_\pm^2}\right]\frac{1}{\rho_\pm^2}, \tag{67}
$$

$$
\frac{\partial^n}{\partial x^n}\frac{x}{\rho_\pm^2} = -\frac{n}{\rho_\pm^2}\left[2x\frac{\partial^{n-1}}{\partial x^{n-1}}\frac{x}{\rho_\pm^2} + (n-1)\frac{\partial^{n-2}}{\partial x^{n-2}}\frac{x}{\rho_\pm^2}\right], \; n \geq 2. \tag{68}
$$

This bit can go in a paper.
Transfer operator:

$$
G_n(r,r_1,x) = \sum_{m=0}^{\infty}\sum_{i+j+k=m}\overline{g}^{(n)}_{i,j,k}(\Delta r)^i(\Delta r_1)^j(\Delta x)^k, \tag{69}
$$

$$
\overline{g}^{(n)}_{i,j,k}(r,r_1,x) = \frac{1}{i!j!k!}\frac{\partial^{i+j+k}}{\partial r^i \partial r_1^j \partial x^k}G_n(r,r_1,x). \tag{70}
$$

Source expansion, for M point sources:

$$
S_{ij} = \sum_{m=1}^{M}\sigma^{(m)}(\Delta r_1^{(m)})^i(\Delta z_1^{(m)})^j \tag{71}
$$

$$
\phi(r,z) = \sum_{i,j}(-1)^j S_{ij}\overline{g}^{(n)}_{0,i,j}(r,r_1,z-z_1), \tag{72}
$$

$$
\phi(r+\Delta r, z+\Delta z) = \sum_{k,\ell}\Phi_{k\ell}(\Delta r)^k(\Delta z)^\ell, \tag{73}
$$

$$
\Phi_{k\ell} = \sum_{i,j}(-1)^j\binom{j+\ell}{j}s_{ij}\overline{g}^{(n)}_{k,i,j+\ell}(r,r_1,z-z_1) \tag{74}
$$

Moment and expansion shifting from (r, z) to $(r+\Delta r, z+\Delta r)$ (primes indicate original centre of moment or expansion),

$$S_{ij} = \sum_{q=0}^{i} \sum_{u=0}^{j} \binom{i}{q} \binom{j}{u} (-\Delta r_1)^q (-\Delta z_1)^u S'_{i-q,j-u}, \tag{75}$$

$$\Phi_{ij} = \sum_{q=0}^{\infty} \sum_{u=0}^{\infty} \binom{i+q}{q} \binom{j+u}{u} (-\Delta r)^q (-\Delta z)^u \Phi'_{i+q,j+u}. \tag{76}$$

For "backward" translation (switching z and z_1), use the same derivatives of G and change the signs of Δz and Δz_1:

$$\phi(r + \Delta r, z_1 + \Delta z) = \sum_{k,\ell} (-1)^\ell (\Delta r)^k (\Delta z)^\ell \sum_{i,j} S_{ij} \overline{g}^{(n)}_{k,i,j+l}(r, r_1, z - z_1). \tag{77}$$

For "inward" translation (switching r and r_1), switch indices in derivatives of G,

$$\Phi_{k\ell} = \sum_{i,j} (-1)^j s_{ij} \overline{g}^{(n)}_{i,k,j+\ell}(r, r_1, z - z_1) \tag{78}$$

Separation of boxes. For box at radial station i, boxes at radial stations j are well separated, if:

$$j \geq \left[\chi + \left(\chi^2 - 1\right)^{1/2}\right] (i + 1).$$

Boxes with axial displacement j are well separated if

$$j \geq \sqrt{2}\, (\chi - 1)^{1/2}\, (i + 1).$$