Koornwinder polynomials from Greengard et al:

$$K_{nm}(u,v) = c_{nm}(1-v)^m P_{n-m}^{(0,2m+1)}(1-2v) P_m\left(\frac{2u+v-1}{1-v}\right), \quad m \le n, \tag{1}$$
$$c_{nm} = [2(1+2m)(n+1)]^{1/2}. \tag{2}$$

Recursions (DLMF):

$$P_{n-m+1}^{(0,2m+1)}(x) = (A_n x + B_n) P_{n-m}^{(0,2m+1)}(x) - C_n P_{n-m-1}^{(0,2m+1)}(x), \tag{3}$$
$$A_n = \frac{(n+1)(2n+3)}{(n-m+1)(n+m+2)}, \tag{4}$$
$$B_n = -\frac{(2m+1)^2(n+1)}{(n-m+1)(n+m+2)(2n+1)}, \tag{5}$$
$$C_n = \frac{(n-m)(n+m+1)(2n+3)}{(n-m+1)(n+m+2)(2n+1)}, \tag{6}$$
$$\frac{\mathrm{d}P_{n-m}^{0,2m+1}(x)}{\mathrm{d}x} = 2(n-m)(n+m+1)P_{n-m-1}^{0,2m+1}(x) - (n-m)((2n+1)x+2m+1)P_{n-m}^{0,2m+1}. \tag{7}$$

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x), \tag{8}$$
$$\frac{\mathrm{d}P_{m+1}}{\mathrm{d}x} = (m+1)\frac{P_m(x) - xP_{m+1}(x)}{1-x^2}. \tag{9}$$

Spherical patch test integrals, on triangular patch (θ_0, ϕ_0), (θ_1, ϕ_0), (θ_0, ϕ_1) of radius ρ:

$$\int_\theta \int_\phi f(\rho)\mathrm{d}S = \rho^2 f(\rho)\left\{\frac{1}{a}\left[\sin(\gamma - a\theta_1) - \sin(\gamma - a\theta_0)\right] + (\theta_1 - \theta_0)\cos\phi_0\right\}, \tag{10}$$
$$\gamma = \frac{\theta_1\phi_1 - \theta_0\phi_0}{\theta_1 - \theta_0}, \tag{11}$$
$$a = \frac{\phi_1 - \phi_0}{\theta_1 - \theta_0}. \tag{12}$$