Notes and derivations not given explicitly in referenced documents.

Derivatives of regular Laplace expansions, assuming normalized Legendre functions and unit normalization constants (α and β in G&D). Identities for Legendre functions come from CS-TR-4264.

$$\begin{aligned}
\partial_z &= \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}, && (1)\\
\partial_{xy} &= \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y} && (2)\\
&= \frac{\mathrm{e}^{\mathrm{i}\phi}}{r(1-\mu^2)^{1/2}}\left[(1-\mu^2)\left(r\frac{\partial}{\partial r} - \mu\frac{\partial}{\partial \mu}\right) + \mathrm{i}\frac{\partial}{\partial \phi}\right], && (3)\\
\overline{\partial_{xy}} &= \frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y} && (4)\\
&= \frac{\mathrm{e}^{-\mathrm{i}\phi}}{r(1-\mu^2)^{1/2}}\left[(1-\mu^2)\left(r\frac{\partial}{\partial r} - \mu\frac{\partial}{\partial \mu}\right) - \mathrm{i}\frac{\partial}{\partial \phi}\right], && (5)\\
\partial_x &= (\partial_{xy} + \overline{\partial_{xy}})/2, && (6)\\
\partial_y &= -\mathrm{i}(\partial_{xy} - \overline{\partial_{xy}})/2. && (7)
\end{aligned}$$

$$\begin{aligned}
R_n^m &= r^n N_n^m P_n^{|m|}\mathrm{e}^{\mathrm{i}m\phi}, && (8)\\
\frac{\partial R_n^m}{\partial z} &= \left[\mu n r^{n-1} P_n^{|m|} + r^{n-1}(1-\mu^2)\frac{\partial P_n^{|m|}}{\partial \mu}\right]\mathrm{e}^{\mathrm{i}m\phi} N_n^m, && (9)\\
&= r^{n-1} N_n^m P_{n-1}^{|m|}\mathrm{e}^{\mathrm{i}m\phi}, && (10)\\
&= \frac{N_n^m}{N_{n-1}^m} R_{n-1}^m, && (11)\\
&= \left[\frac{2n+1}{2n-1}(n^2-m^2)\right]^{1/2} R_{n-1}^m. && (12)
\end{aligned}$$

$$
\begin{aligned}
S_n^m &= r^{-n-1} N_n^m P_n^{|m|} \mathrm{e}^{\mathrm{i}m\phi}, && (13)\\
\partial_z S_n^m &= -\left[\mu(n+1) r^{-n-2} P_n^{|m|} + r^{-n-2}(1-\mu^2)\frac{\partial P_n^{|m|}}{\partial \mu}\right] \mathrm{e}^{\mathrm{i}m\phi} N_n^m, && (14)\\
&= -r^{-n-2} N_n^m (n-|m|+1) P_{n+1}^{|m|} \mathrm{e}^{\mathrm{i}m\phi}, && (15)\\
&= -(n-|m|+1)\frac{N_n^m}{N_{n+1}^m} S_{n+1}^m, && (16)\\
&= -\left\{\frac{2n+1}{2n+3}\left[(n+1)^2 - m^2\right]\right\}^{1/2} S_{n+1}^m, && (17)\\
\partial_{xy} S_n^m &= \left\{\left(1-\mu^2\right)^{1/2}\left[-(n+1)P_n^{|m|}(\mu) - \mu\frac{\partial P_n^{|m|}}{\partial \mu}\right] - \right.\\
&\left.\frac{m}{(1-\mu^2)^{1/2}} P_n^{|m|}(\mu)\right\} N_n^m \mathrm{e}^{\mathrm{i}(m+1)\phi} r^{-n-2}, && (18)\\
&= \left[\frac{2n+1}{2n+3}(n+m+1)(n+m+2)\right]^{1/2} S_{n+1}^{m+1}, && (19)\\
\overline{\partial_{xy}} S_n^m &= \left\{\left(1-\mu^2\right)^{1/2}\left[-(n+1)P_n^{|m|}(\mu) - \mu\frac{\partial P_n^{|m|}}{\partial \mu}\right] + \right.\\
&\left.\frac{m}{(1-\mu^2)^{1/2}} P_n^{|m|}(\mu)\right\} N_n^m \mathrm{e}^{\mathrm{i}(m-1)\phi} r^{-n-2}, && (20)\\
&= -\left[\frac{2n+1}{2n+3}(n-m+1)(n-m+2)\right]^{1/2} S_{n+1}^{m-1}. && (21)
\end{aligned}
$$

We store rotation coefficients $H_n^{\nu m}$ for $\nu = -n, \ldots, n$, $m = 0, \ldots, n$. By symmetry (G&D 5.5.2):

$$
H_n^{\nu,-m} = H_n^{-\nu,m}.
$$

References

[1] Nail A. Gumerov and Ramani Duraiswami. Recursions for the computation of multipole translation and rotation coefficients for the 3-D Helmholtz equation. *SIAM Journal on Scientific Computing*, 25(4):1344–1381, 2003.