With reference to Gumerov & Duraiswami (CS-TR-4264), Laplace operators are:

$$\partial_{xy} S_n^m = -\left[\frac{2n+1}{2n+3}(n+m+1)(n+m+2)\right]^{1/2} S_{n+1}^{m+1}, \quad m \geq 0, \tag{1}$$

$$= \left[\frac{2n+1}{2n+3}(n+m+1)(n+m+2)\right]^{1/2} S_{n+1}^{m+1}, \quad m < 0, \tag{2}$$

$$\overline{\partial_{xy}} S_n^m = \left[\frac{2n+1}{2n+3}(n-m+2)(n-m+1)\right]^{1/2} S_{n+1}^{m-1}, \quad m > 0, \tag{3}$$

$$= -\left[\frac{2n+1}{2n+3}(n-m+2)(n-m+1)\right]^{1/2} S_{n+1}^{m-1}, \quad m \leq 0, \tag{4}$$

$$\partial_z S_n^m = -\left[\frac{2n+1}{2n+3}(n+m+1)(n-m+1)\right]^{1/2} S_{n+1}^{m}, \quad \forall m. \tag{5}$$

For term $cS_n^m + c^* S_n^{-m}$, $m > 0$,

$$\partial_{xy}\left(cS_n^m + c^* S_n^{-m}\right) = -b_1 c \overline{S_{n+1}^{m+1}} \mathrm{e}^{\mathrm{i}(m+1)\phi} + b_2 c^* \overline{S_{n+1}^{m-1}} \mathrm{e}^{-\mathrm{i}(m-1)\phi}, \tag{6}$$

$$\overline{\partial_{xy}}\left(cS_n^m + c^* S_n^{-m}\right) = -b_1 c^* \overline{S_{n+1}^{m+1}} \mathrm{e}^{-\mathrm{i}(m+1)\phi} + b_2 c \overline{S_{n+1}^{m-1}} \mathrm{e}^{\mathrm{i}(m-1)\phi}, \tag{7}$$

$$b_1 = \left[\frac{2n+1}{2n+3}(n+m+1)(n+m+2)\right]^{1/2}, \tag{8}$$

$$b_2 = \left[\frac{2n+1}{2n+3}(n-m+1)(n-m+2)\right]^{1/2}, \tag{9}$$

$$\overline{S_n^m} = r^{-n-1} \overline{P}_n^{|m|}(\cos\theta). \tag{10}$$

$$\frac{\partial}{\partial z} R_n^m = \left[\frac{2n+1}{2n-1}(n-|m|)(n+|m|)\right]^{1/2} R_{n-1}^m, \tag{11}$$

$$\overline{R_n^m} = r^n \overline{P}_n^{|m|}(\cos\theta). \tag{12}$$