

With reference to Gumerov & Duraiswami (CS-TR-4264), Laplace operators are:

$$\partial_{xy} S_n^m = - \left[ \frac{2n+1}{2n+3} (n+m+1)(n+m+2) \right]^{1/2} S_{n+1}^{m+1}, \quad m \geq 0, \quad (1)$$

$$= \left[ \frac{2n+1}{2n+3} (n+m+1)(n+m+2) \right]^{1/2} S_{n+1}^{m+1}, \quad m < 0, \quad (2)$$

$$\overline{\partial_{xy}} S_n^m = \left[ \frac{2n+1}{2n+3} (n-m+2)(n-m+1) \right]^{1/2} S_{n+1}^{m-1}, \quad m > 0, \quad (3)$$

$$= - \left[ \frac{2n+1}{2n+3} (n-m+2)(n-m+1) \right]^{1/2} S_{n+1}^{m-1}, \quad m \leq 0, \quad (4)$$

$$\partial_z S_n^m = - \left[ \frac{2n+1}{2n+3} (n+m+1)(n-m+1) \right]^{1/2} S_{n+1}^m, \quad \forall m. \quad (5)$$

For term  $cS_n^m + c^* S_n^{-m}$ ,  $m > 0$ ,

$$\partial_{xy} (cS_n^m + c^* S_n^{-m}) = -b_1 c \overline{S_{n+1}^{m+1}} e^{i(m+1)\phi} + b_2 c^* \overline{S_{n+1}^{m-1}} e^{-i(m-1)\phi}, \quad (6)$$

$$\overline{\partial_{xy}} (cS_n^m + c^* S_n^{-m}) = -b_1 c^* \overline{S_{n+1}^{m+1}} e^{-i(m+1)\phi} + b_2 c \overline{S_{n+1}^{m-1}} e^{i(m-1)\phi}, \quad (7)$$

$$b_1 = \left[ \frac{2n+1}{2n+3} (n+m+1)(n+m+2) \right]^{1/2}, \quad (8)$$

$$b_2 = \left[ \frac{2n+1}{2n+3} (n-m+1)(n-m+2) \right]^{1/2}, \quad (9)$$

$$\overline{S_n^m} = r^{-n-1} \overline{P_n^{|m|}} (\cos \theta). \quad (10)$$

$$\partial_{xy} R_n^m = - \left[ \frac{2n+1}{2n-1} (n-|m|)(n-|m|-1) \right]^{1/2} R_{n-1}^{m+1}, \quad m \geq 0, \quad (11)$$

$$= \left[ \frac{2n+1}{2n-1} (n+|m|)(n+|m|-1) \right]^{1/2} R_{n-1}^{m+1}, \quad m < 0, \quad (12)$$

$$\overline{\partial_{xy}} R_n^m = \left[ \frac{2n+1}{2n-1} (n+|m|)(n+|m|-1) \right]^{1/2} R_{n-1}^{m-1}, \quad m > 0, \quad (13)$$

$$= - \left[ \frac{2n+1}{2n-1} (n-|m|)(n-|m|-1) \right]^{1/2} R_{n-1}^{m-1}, \quad m \leq 0, \quad (14)$$

$$\frac{\partial}{\partial z} R_n^m = \left[ \frac{2n+1}{2n-1} (n-|m|)(n+|m|) \right]^{1/2} R_{n-1}^m. \quad (15)$$

For term  $cR_n^m + c^*R_n^{-m}$ ,  $m > 0$ ,

$$\partial_{xy} (cR_n^m + c^*R_n^{-m}) = -b_1 c \overline{R_{n-1}^{m+1}} e^{i(m+1)\phi} + b_2 c^* \overline{R_{n-1}^{m-1}} e^{-i(m-1)\phi}, \quad (16)$$

$$\overline{\partial_{xy}} (cR_n^m + c^*R_n^{-m}) = -b_1 c^* \overline{R_{n-1}^{m+1}} e^{-i(m+1)\phi} + b_2 c \overline{R_{n-1}^{m-1}} e^{i(m-1)\phi}, \quad (17)$$

$$b_1 = \left[ \frac{2n+1}{2n-1} (n-m)(n-m-1) \right]^{1/2}, \quad (18)$$

$$b_2 = \left[ \frac{2n+1}{2n-1} (n+m)(n+m-1) \right]^{1/2}, \quad (19)$$

$$\overline{R_n^m} = r^n \overline{P_n^{|m|}}(\cos \theta). \quad (20)$$