With reference to Gumerov & Duraiswami (CS-TR-4264), Laplace operators are:

$$
\begin{aligned}
\partial_{xy} S_n^m &= -\left[\frac{2n+1}{2n+3}(n+m+1)(n+m+2)\right]^{1/2} S_{n+1}^{m+1}, \quad m \geq 0, && (1)\\
&= \left[\frac{2n+1}{2n+3}(n+m+1)(n+m+2)\right]^{1/2} S_{n+1}^{m+1}, \quad m < 0, && (2)\\
\overline{\partial_{xy}} S_n^m &= \left[\frac{2n+1}{2n+3}(n-m+2)(n-m+1)\right]^{1/2} S_{n+1}^{m-1}, \quad m > 0, && (3)\\
&= -\left[\frac{2n+1}{2n+3}(n-m+2)(n-m+1)\right]^{1/2} S_{n+1}^{m-1}, \quad m \leq 0, && (4)\\
\partial_z S_n^m &= -\left[\frac{2n+1}{2n+3}(n+m+1)(n-m+1)\right]^{1/2} S_{n+1}^{m}, \quad \forall m. && (5)
\end{aligned}
$$

For term $cS_n^m + c^* S_n^{-m}$, $m > 0$,

$$
\begin{aligned}
\partial_{xy}\left(cS_n^m + c^* S_n^{-m}\right) &= -b_1 c \overline{S_{n+1}^{m+1}} \mathrm{e}^{\mathrm{i}(m+1)\phi} + b_2 c^* \overline{S_{n+1}^{m-1}} \mathrm{e}^{-\mathrm{i}(m-1)\phi}, && (6)\\
\overline{\partial_{xy}}\left(cS_n^m + c^* S_n^{-m}\right) &= -b_1 c^* \overline{S_{n+1}^{m+1}} \mathrm{e}^{-\mathrm{i}(m+1)\phi} + b_2 c \overline{S_{n+1}^{m-1}} \mathrm{e}^{\mathrm{i}(m-1)\phi}, && (7)\\
b_1 &= \left[\frac{2n+1}{2n+3}(n+m+1)(n+m+2)\right]^{1/2}, && (8)\\
b_2 &= \left[\frac{2n+1}{2n+3}(n-m+1)(n-m+2)\right]^{1/2}, && (9)\\
\overline{S_n^m} &= r^{-n-1} \overline{P}_n^{|m|}(\cos\phi). && (10)
\end{aligned}
$$