Notes and derivations not given explicitly in referenced documents.

Derivatives of regular Laplace expansions, assuming normalized Legendre functions and unit normalization constants (α and β in G&D). Identities for Legendre functions come from CS-TR-4264.

$$\partial_z = \mu \frac{\partial}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial}{\partial \mu}, \tag{1}$$

$$\partial_{xy} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \tag{2}$$

$$= \frac{e^{i\phi}}{r(1 - \mu^2)^{1/2}} \left[(1 - \mu^2) \left(r \frac{\partial}{\partial r} - \mu \frac{\partial}{\partial \mu} \right) + i \frac{\partial}{\partial \phi} \right], \tag{3}$$

$$\overline{\partial_{xy}} = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \tag{4}$$

$$= \frac{e^{-i\phi}}{r(1 - \mu^2)^{1/2}} \left[(1 - \mu^2) \left(r \frac{\partial}{\partial r} - \mu \frac{\partial}{\partial \mu} \right) - i \frac{\partial}{\partial \phi} \right], \tag{5}$$

$$\partial_x = (\partial_{xy} + \overline{\partial_{xy}})/2, \tag{6}$$

$$\partial_y = -i(\partial_{xy} - \overline{\partial_{xy}})/2. \tag{7}$$

$$R_n^m = r^n N_n^m P_n^{|m|} e^{im\phi}, \tag{8}$$

$$\frac{\partial R_n^m}{\partial z} = \left[\mu n r^{n-1} P_n^{|m|} + r^{n-1}(1 - \mu^2) \frac{\partial P_n^{|m|}}{\partial \mu} \right] e^{im\phi} N_n^m, \tag{9}$$

$$= r^{n-1} N_n^m P_{n-1}^{|m|} e^{im\phi}, \tag{10}$$

$$= \frac{N_n^m}{N_{n-1}^m} R_{n-1}^m, \tag{11}$$

$$= \left[\frac{2n + 1}{2n - 1} (n^2 - m^2) \right]^{1/2} R_{n-1}^m. \tag{12}$$

$$S_n^m = r^{-n-1} N_n^m P_n^{|m|} \mathrm{e}^{\mathrm{i}m\phi}, \tag{13}$$

$$\partial_z S_n^m = -\left[\mu(n+1)r^{-n-2}P_n^{|m|} + r^{-n-2}(1-\mu^2)\frac{\partial P_n^{|m|}}{\partial \mu} \right] \mathrm{e}^{\mathrm{i}m\phi} N_n^m, \tag{14}$$

$$= -r^{-n-2}N_n^m(n-|m|+1)P_{n+1}^{|m|}\mathrm{e}^{\mathrm{i}m\phi}, \tag{15}$$

$$= -(n-|m|+1)\frac{N_n^m}{N_{n+1}^m}S_{n+1}^m, \tag{16}$$

$$= -\left\{ \frac{2n+1}{2n+3}\left[(n+1)^2 - m^2\right] \right\}^{1/2} S_{n+1}^m, \tag{17}$$

$$\partial_{xy} S_n^m = \left\{ \left(1-\mu^2\right)^{1/2}\left[-(n+1)P_n^{|m|}(\mu) - \mu\frac{\partial P_n^{|m|}}{\partial \mu} \right] - \right.$$
$$\left. \frac{m}{(1-\mu^2)^{1/2}}P_n^{|m|}(\mu) \right\} N_n^m \mathrm{e}^{\mathrm{i}(m+1)\phi} r^{-n-2}, \tag{18}$$

$$= \left[\frac{2n+1}{2n+3}(n+m+1)(n+m+2) \right]^{1/2} S_{n+1}^{m+1}, \tag{19}$$

$$\overline{\partial_{xy}} S_n^m = \left\{ \left(1-\mu^2\right)^{1/2}\left[-(n+1)P_n^{|m|}(\mu) - \mu\frac{\partial P_n^{|m|}}{\partial \mu} \right] + \right.$$
$$\left. \frac{m}{(1-\mu^2)^{1/2}}P_n^{|m|}(\mu) \right\} N_n^m \mathrm{e}^{\mathrm{i}(m-1)\phi} r^{-n-2}, \tag{20}$$

$$= -\left[\frac{2n+1}{2n+3}(n-m+1)(n-m+2) \right]^{1/2} S_{n+1}^{m-1}. \tag{21}$$