1/15/2025

Wednesday, January 15, 2025 11:56 AM

Lecture 1 | Fundamentals of Prob./Stat.

Def) Consider the set Ω + collection of subsets $\mathcal{F}$.

- we say that $\mathcal{F}$ is a σ-field of Ω if:

i) $\emptyset \in \mathcal{F}$

ii) for $A_i \in \mathcal{F}$, $\bigcup_i A_i \in \mathcal{F}$

iii) if $A_i \in \mathcal{F}$, then $A_i' \in \mathcal{F}$

Def) A probability measure, P, on $(\Omega, \mathcal{F})$ is a function such that

i) $P: \mathcal{F} \to [0,1]$

ii) $P(\emptyset) = 0$, $P(\Omega) = 1$

iii) if $A_i \in \mathcal{F}$, $A_j \in \mathcal{F}$ + $A_i \cap A_j = \emptyset$, $i \neq j$

then $P\left(\bigcup_{i \neq j} A_i\right) = \sum_i P(A_i)$

- The probability space is denoted by the "triple" $(\Omega, \mathcal{F}, P)$

Ex) Two coins:

$\Omega = \{(H,H), (T,H), (H,T), (T,T)\}$

$\mathcal{F} = \Omega$ and all subsets of Ω

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- Choose $\mathcal{F} = \Omega$

· What $P(\{H,T\}) \overset{?}{=} 1/4$

· What is probability of at least one H?

$\Rightarrow P(\text{at least one } H) = 3/4$

Univariate Stats

$\boxed{\text{Def}}$ we say that a random variable (R.V.) is a function, $X : \Omega \to \mathbb{R}$ such that $\{\underset{\substack{\uparrow \\ \text{event}}}{\omega} \in \Omega \mid X(\omega) \le x\} \in \mathcal{F}$

· The value, $X(\omega) = x$, is a realization of the random variable

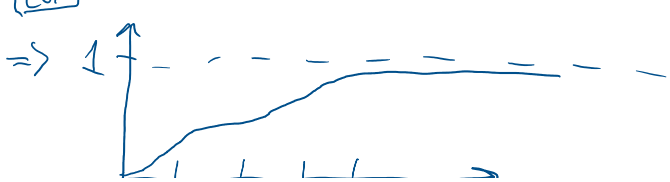
- ☐ Capital $\Rightarrow$ R.V.
- ☐ lowercase $\Rightarrow$ realization

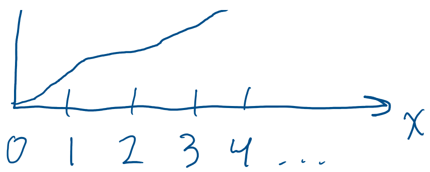
$\boxed{\text{Def}}$ For every R.V., the associated "Cumulative distribution function" (cdf) is a function, $F_X : \mathbb{R} \to [0,1]$, & is defined as

$$F_X(x) = P\{\omega \in \Omega \mid X(\omega) \le x\}$$

or $$F_X(x) = P\{X \le x\}$$

$\boxed{\text{cdf}}$





Ex) Coin problem, Let $X(\omega) = \#$ of heads, $\Omega =$ two coins, $x \equiv$ realization

$\boxed{\text{cdf}}$

$P(X \le x)$

$$F_X = \begin{cases} 0 & x < 0 \\ 1/4 & x \in [0,1) \\ 3/4 & x \in [1,2) \\ 1 & x \ge 2 \end{cases}$$

$\boxed{\text{Notes}}$ i) $\lim_{x \to -\infty} F_X(x) = 0$

ii) $x_1 \le x_2 \Rightarrow F_X(x_1) \le F_X(x_2)$

iii) $\lim_{x \to \infty} F_X(x) = 1$

<u>Def</u> The R.V. X is <u>continuous</u> R.V. if its cdf is absolutely continuous + can be written as

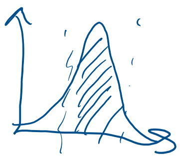
$$F_X(x) = \int_{-\infty}^{x} f_X(s)\, ds, \quad x \in \mathbb{R}$$

where $f_X(x) = \frac{dF}{dx}$, is the probability density function (pdf).

(note: $f_X(x) : \mathbb{R} \to [0, \infty)$)

- Note that $f_X(x) \ge 0$, $\int_{\mathbb{R}} f(x)\, dx = 1$, +

$\therefore$ $P(x_1 < X \le x_2) = F(x_2) - F(x_1) = \int^{x_2} f(x)\, dx$



$$P(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\,dx$$

- The nth <u>Moment of</u> a R.V. is provided

$$E(X^n) = \int_{\mathbb{R}} x^n f_X(x)\,dx$$

- We often look at first + Second moments:

$$E(X) = \mu = \int_{\mathbb{R}} x f_X(x)\,dx$$

$$Var(X) = \sigma^2 = E\left[(X-\mu)^2\right] = \int (x-\mu)^2 f(x)\,dx$$