2/10/2025

Monday, February 10, 2025 11:59 AM

Identifiability

- The Set up:
  - Let $y(\underline{x}_i) = f(\underline{x}_i; \vec{\theta})$ be the model output, &

$$y_i = f(\underline{x}_i; \vec{\theta}) + \varepsilon_i, \quad \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2), \; i = 1, \dots, N_y$$

- Given observations $\vec{y} = [y_1 \, y_2 \dots y_{N_y}]$, we have

$$\vec{\hat{\theta}}_{OLS} = \underset{\vec{\theta} \in \Gamma}{\operatorname{argmin}} \, SS(\vec{\theta}), \quad SS(\vec{\theta}) = (\vec{y} - f(\underline{\vec{x}}; \vec{\theta}))^T (\vec{y} - f(\underline{\vec{x}}; \vec{\theta}))$$

or

$$\vec{\hat{\theta}}_{MLE} = \underset{\vec{\theta} \in \Gamma}{\arg\max} \, LL(\vec{\theta}), \quad LL(\vec{\theta}) = -\frac{N_y}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} SS(\theta)$$

where, since $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

$$\vec{\hat{\theta}}_{OLS} = \vec{\hat{\theta}}_{MLE}$$

$\boxed{\text{Def}}$ Structural Identifiability

- We say that $\vec{\theta}$ are locally structurally identifiable if

$$f(\underline{x}; \vec{\theta}) = f(\underline{x}; \vec{\theta}^*) \implies \vec{\theta} = \vec{\theta}^*$$

holds for all $\vec{\theta}$ in a neighborhood of $\vec{\theta}^*$.

- $\vec{\theta}$ is globally structurally identifiable if this holds for

- $\vec{\theta}$ is globally structurally identifiable if this holds for almost all $\vec{\theta}^* \in \Gamma$.
- As IS if $f: \vec{\theta} \to \vec{y}$ is injective (one-to-one)

* This does not look at $C(f): f \to \vec{y}$ *

$\boxed{\text{Def}}$ Practical Identifiability

- We say $\vec{\theta}$ is <u>practically identifiable</u> if $\vec{\theta}$ can be uniquely determined by

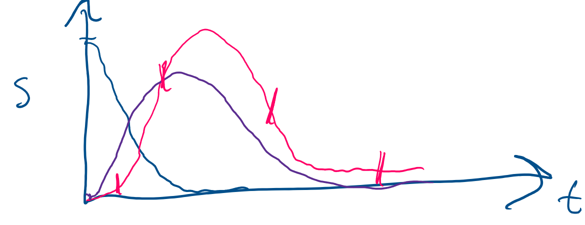
$$y_i = f(x_i; \vec{\theta}) + \varepsilon_i, \quad i = 1, \ldots, N_y$$

- This is a local property based on simulations, observations, + noise.

- $\boxed{\text{Note}}$ If $\vec{\theta}$ is <u>NOT</u> structurally identifiable, then $\vec{\theta}$ is <u>NOT</u> practically identifiable.
   - If $\vec{\theta}$ is structurally identifiable... still unsure.

SIR

Ex)

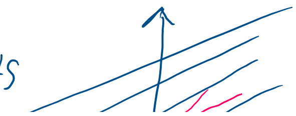


Ex) $y_i = \theta_1 t + \theta_2 + \varepsilon_i, \quad \varepsilon_i \overset{iid}{\sim} N(0, \sigma^2), \quad i = 1, \ldots, N_y$

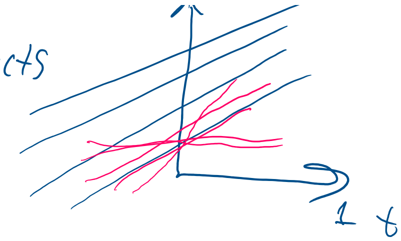
case i) $\theta_1 = \theta_2 = 2, \quad \sigma^2 = 0.01, \quad t \in [0, 1]$

case ii) $\theta_1 = \theta_2 = 2, \quad \sigma^2 = 0.01, \quad t \in [0, 0.01]$

- when $t \in [0, 1]$ θ_1 + θ_2 have distinct effects



- when $t \in [0,1]$, θ_1 + θ_2 have distinct effects



  - θ_1 is slope
  - θ_2 is intercept

- when $t \in [0, 0.01]$, $\theta_1 t << \theta_2$

★ So $\vec{\theta}$ is structurally identifiable, but t-domain dictates Practical identifiability

Ex) $y_i = \theta_1 \theta_2 t + \theta_3 + \varepsilon_i$, $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$

- Here, θ_1 + θ_2 are not structurally identifiable.
  - If $\theta_3 = 0$, then there is a manifold where

$$h(\vec{\theta}) = K - \theta_1 \theta_2 = 0, \quad K = \theta_1 \theta_2$$

  - Note: the OLS + MLE are satisfied on $h(\vec{\theta})$

★ we call this <u>parameter redundancy</u>.