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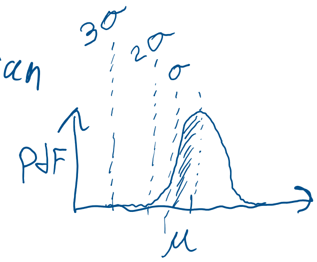
Friday, January 24, 2025 11:57 AM

- The univariate Gaussian

  $X_i \sim N(\mu, \sigma^2)$



- The multivariate normal (MVN) is given by:

$$\vec{X} \sim MVN(\vec{\mu}, \underset{\sim}{V}), \quad \vec{X} \in \mathbb{R}^n$$

where $\vec{\mu} = E[\vec{X}] = \int_{\mathbb{R}^n} \vec{x} f_{MVN}(\vec{x})\, d\vec{x}$

$$+ \quad \underset{\sim}{V} = Cov[\vec{X}] = \begin{bmatrix} Var(X_1) & Cov(X_2, X_1) & \cdots & \\ Cov(X_1, X_2) & Var(X_2) & & \\ \vdots & & \ddots & \\ & & & Var(X_p) \end{bmatrix}$$

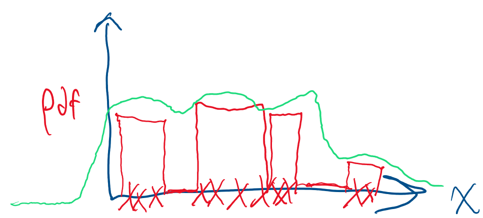
- Note: $\underset{\sim}{V}$ is almost always symmetric positive definite ($\vec{x}^T \underset{\sim}{V} \vec{x} > 0$)
  - Always symmetric
  - We want $Var(X_i) > 0$ (otherwise point estimate)

- The joint pdf of $\vec{X} \sim MVN(\vec{\mu}, \underset{\sim}{V})$ is

$$(\vec{X} \in \mathbb{R}^n) \quad f_{\vec{X}}(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n}} \cdot \det(\underset{\sim}{V})^{-1} \cdot \exp\left(-\frac{1}{2}(\vec{x}-\vec{\mu})^T \underset{\sim}{V}^{-1} (\vec{x}-\vec{\mu})\right)$$

$$(X \sim N(\mu, \sigma^2)) \quad f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\sigma^2}} \cdot \exp\left(-\frac{1}{2} \frac{(x-\mu)^T (x-\mu)}{\sigma^2}\right) \quad (x-\mu)^2$$

- How do we approximate f_X (pdf)?
  - Frequency of data ⇒ height of pdf



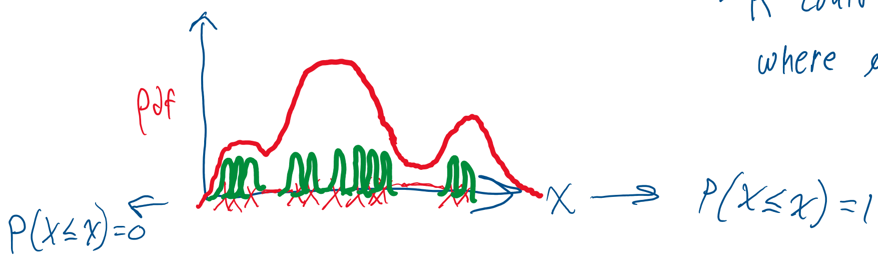
- We can get a continuous representation of pdf through

- We can get a continuous representation of pdf through kernel density estimation (kde)

$x_i,\ i=1,\dots,n$

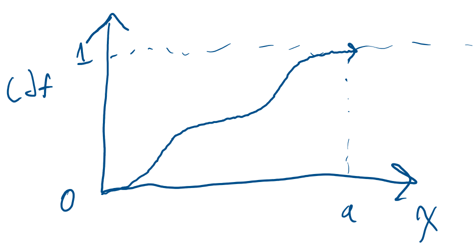
$$f_X(x) \approx \frac{1}{n}\cdot\frac{1}{h}\sum_{i=1}^{n} K\left(\frac{x-x_i}{h}\right)$$

• K could be $K=\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp\left(\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$ where $\mu = x_i$, σ is chosen based on data

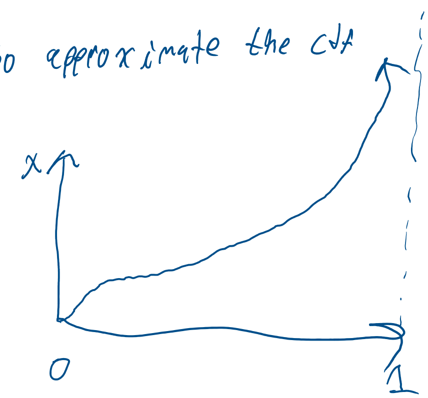


- If we can approximate the pdf, we can also approximate the cdf

cdf: $P(X\le x)$



$\Rightarrow$



- So if we have the cdf + inv. cdf, we can draw realizations from our process.

Theorem Let $\vec{y} \sim MVN(\vec{\mu}, \underset{\sim}{V})$, $\vec{y}\in\mathbb{R}^n$, + let $\underset{\sim}{V}$ be Sym. Pos. def.

• Let $\vec{Z} \sim N(\vec{0}, \underset{\sim}{I})$, $(MVN(\vec{0}, \underset{\sim}{I}))$

• We can decompose $\underset{\sim}{V} = \underset{\sim}{R}^T \underset{\sim}{R}$ using a Cholesky decomposition.

Then

$$\vec{y} = \vec{\mu} + \underset{\sim}{R}^T\vec{Z}$$

why?

$$E[\vec{\mu} + \underset{\sim}{R}^T\vec{Z}] = E[\vec{\mu}] + E[\underset{\sim}{R}^T\vec{Z}] = \vec{\mu}$$

$$Var[\vec{\mu} + \underset{\sim}{R}^T\underset{\sim}{\vec{Z}}] = 0 + Var[\underset{\sim}{R}^T\vec{Z}] = \underset{\sim}{V}$$

NOTE: The Cholesky decomposition matrix R is **unique**

V is symmetric, positive-definite. If V is one positive-semidefinite, R is **not unique.**