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- Given a random field, $\alpha(x,\omega)$, we can write

$$\alpha(x,\omega) = \bar{\alpha}(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, \phi_i(x)\, \Theta_i(\omega)$$

- Note $\bar{\alpha}$, λ_i, ϕ_i are deterministic
- $\Theta_i(\omega)$ is stochastic, i.e., if we assume α is a Gaussian random field, $\Theta_i \sim N(0,1)$

- λ_i + ϕ_i are eigenpairs corresponding $C(x,x')$

Ex) Absolute exponential

$$C(x,x') = \sigma^2 \exp\left(\frac{-|x-x'|}{L}\right), \quad x \in [-1,1]$$

- This works well when dealing with non differentiable random field
- L is the correlation length, σ^2 is general variance

- Note, the eigenvalues + eigenfunctions

$$\lambda_n = \sigma^2 \frac{2L}{1+L^2 v_n^2}, \quad \phi_n = \begin{cases} \dfrac{\sin(v_n x)}{\left(1 - \sin(2v_n)/2v_n\right)^{1/2}}, & n \text{ is even} \\ \dfrac{\cos(v_n x)}{\left(1 + \dfrac{\sin(2v_n)}{2v_n}\right)^{1/2}}, & n \text{ is odd} \end{cases}$$

where v_n solve system of transcendal equations

Note: If $L \to \infty$, $C(x,x') = \sigma^2$

$\Rightarrow$ "Full correlation"

$\Rightarrow$ you get one eigenvalue > 0

- If $L \to 0$, no correlation
  - To analyze, we set $\sigma^2 = \dfrac{1}{2L\left(1 - \exp(-1/L)\right)}$

$$\Rightarrow C(x, x') = \frac{1}{2L\left(1 - \exp\left(\frac{-1}{L}\right)\right)} \cdot \exp\left(\frac{-|x - x'|}{L}\right)$$

$$\Rightarrow \lim_{L \to 0} C(x, x') = \delta(x - x')$$

- The eigenvalue problem becomes

$$\varphi_n(x) = \lambda_n \varphi_n(x) \Rightarrow \lambda_n = 1$$

$\ast$ So no information is communicated through $C(x, x')$

Ex) Squared exponential (radial basis function)

$$C(x, x') = \sigma^2 \exp\left(\frac{-(x - x')^2}{2L^2}\right), \quad x \in \mathbb{R}$$

- eigenvalues come from solving eigenvectors that are Hermitian Polynomials

$$\varphi_n(x) = H_n\left(\sqrt{2d}\; x\right)$$

- Consider the truncated random field

$$\alpha(x, \omega) \approx \underbrace{\bar{\alpha}(x) + \sum_{i=1}^{N} \sqrt{\lambda_i}\, \varphi_i(x)\, \theta_i(\omega)}_{\alpha_N(x, \omega)}$$

- note that $E\left[\|\alpha \ldots\|^2\right]^{N \to \infty}$ [illegible]

- Note that $E\left[\|\alpha_N\|^2\right]^{N\to\infty} = \sum_{i=1}^{\infty} \lambda_n$

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- This definition was not so clear. Let's instead write the following:
  - Assume $\alpha(x,\omega)$ is zero mean (or assume we are working with $\alpha_c(x,\omega)$)
  - By definition, orthogonality of ϕ_i gives us

$$E\left[\|\alpha\|^2\right] = \int_\Omega \left[\int_D |\alpha(x,\omega)|^2 dx\right] d\underbrace{P(\omega)}_{\text{Probability measure}}$$

$$= \int_\Omega \left[\int_D \left|\sum_{i=1}^{\infty} \sqrt{\lambda_i}\, \phi_i\, \theta_i\right|^2\right] dP(\omega)$$

$$= \int_\Omega \left[\int_D \sum_{i=1}^{\infty} \lambda_i |\phi_i|^2 |\theta_i|^2\right] dP(\omega)$$

(- Recall that $E[\theta_i \theta_j] = \delta_{ij}$ + ϕ_i are orthonormal)

$= \sum_{i=1}^{\infty} \lambda_i$. we then assume $\exists\, \varepsilon > 0$ s.t. for $N < \infty$

$$D_N = \frac{E\left[\|\alpha^N - \alpha\|^2\right]}{E\left[\|\alpha\|^2\right]} = \frac{\sum_{i=N+1}^{\infty} \lambda_i}{\sum_{i=1}^{\infty} \lambda_i} \le \varepsilon^2$$

where N dictates our tolerance

- For sufficiently large N, $\alpha_N \to \alpha - \bar{\alpha}$, based on λ_i decay. (Note: If $\bar{\alpha} = 0$, $\alpha_N \to \alpha$)

- If we assume $\alpha(x,\omega)$ is a Gaussian Random field then uncorrelated $\Rightarrow$ independent

$$\Rightarrow \quad \Theta_i(\omega) \sim N(0, I)$$

- Throughout, we have assumed a $C(x,x')$ for $\alpha(x,\omega)$.
  - The true $C(x,x')$ is typically unknown
  - One alternative is to construct an estimate of $C(x,x')$ from samples of $\alpha(x,\omega)$

$$\tilde{C}_{est} = \frac{1}{N-1} \sum (x - \bar{x})(x - \bar{x})^T$$