Theorem Mercer's Theorem

- Let $C(x,x') = K(x,x')$ be a kernel s.t.

$$K : D \times D \to \mathbb{R}, \quad K(x,x') = K(x',x)$$

$\Rightarrow K$ is symmetric + pos. def.

- There exists an orthonormal basis, $\{\varphi_n\}$, in $L^2(D)$ consisting of eigenfunctions, T_K, s.t. $\{\lambda_i\} > 0$. Then $K(x,x')$ has form

$$K(x,x') = \sum_{i=1}^{\infty} \lambda_i \varphi_n(x) \varphi_n(x')$$

Karhunen-Loéve Expansion (KL-expansion)

- Given a random field, $\alpha(x,\omega)$, we can rewrite this field as a KL-Expansion

Deterministic Stochastic

$$\alpha(x,\omega) = \bar{\alpha}(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, \varphi_n(x)\, \Theta_n(\omega)$$

where λ_i + φ_n are eigenvalue-eigenfunction pair (φ is orthonormal) of a covariance function, $C(x,x')$, given by

$$\int_D C(x,x') \varphi_n(x)\, dx = \lambda_n \varphi_n(x'), \quad x \in D$$

+

$$\int_D \varphi_n(x) \varphi_m(x)\, dx = \delta_{mn} = \begin{cases} 1, & m=n \\ 0, & \text{else} \end{cases}$$

[illegible] $\Theta_n(\omega)$ which are centered mutually uncorrelated.

D

- The above definition uses $\theta_n(\omega)$, which are centered, mutually uncorrelated, random variables with unit variance

$$E[\theta_n] = 0, \quad E[\theta_n \theta_m] = \delta_{mn}, \quad Var[\theta_n] = 1$$

- we typically write

$$\alpha(x,\omega) = \bar{\alpha}(x) + \alpha_c(x,\omega), \quad \alpha_c = \sum \sqrt{\lambda_i}\, \phi_i\, \theta_i(\omega)$$

- Note that $E[\alpha_c(x,\omega)] = 0$
- the covariance of $\alpha_c(x,\omega)$ is

$$C(x,x') = E[\alpha_c(x,\omega)\,\alpha_c(x',\omega)] = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sqrt{\lambda_n \lambda_m}\, \phi_n(x)\, \phi_m(x')\, E[\theta_n \theta_m]$$

- Since ϕ_m & ϕ_n are orthonormal, we get

$$\int_D C(x,x')\, \phi_\ell(x)\, dx = \sum_{n=1}^{\infty} \sqrt{\lambda_\ell \lambda_n}\, \phi_n(x)\, E[\theta_\ell \theta_n]$$

- If we multiply by ϕ_L & integrate

$$\Rightarrow \int_D C(x,x')\, \phi_\ell(x)\, \phi_L(x)\, dx = \sum_{n=1}^{\infty} E[\theta_\ell \theta_n] \sqrt{\lambda_\ell \lambda_n}\, \delta_{nL}$$

since $\lambda_\ell \phi_\ell = \int_D C(x,x')\, \phi_\ell\, dx$, then

$$\int_D \lambda_\ell \phi_\ell \phi_L\, dx = \boxed{\lambda_\ell \delta_{\ell L} = \sqrt{\lambda_\ell \lambda_L}\; E[\theta_\ell \theta_L]}$$

$\therefore E[\theta_\ell \theta_\ell] = 1$, & $E[\theta_\ell \theta_L] = 0$,

$$\Rightarrow \underset{q=L}{E[\theta_q \theta_L]} = 1, \quad \& \quad \underset{q \neq L}{E[\theta_q \theta_L]} = 0,$$

* For KL-expansion, $\bar{\alpha}$ is mean, λ_q & ϕ_q are eigenpair for $C(x,x')$, & $\theta_n(\omega)$ are uncorrelated R.V.'s, $var[\theta_n] = 1$

Issues

i) we have to truncate KL-exp. to N terms (eigenvalues)

ii) $C(x,x')$ are typically not known

iii) To get λ_q & ϕ_q, we need to solve integral equations

Covariance

- Absolute exponential

$$C(x,x') = \sigma^2 \exp\left(-\frac{|x-x'|}{L}\right), \quad x, x' \in [-1,1]$$