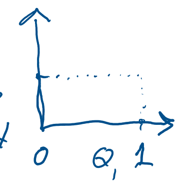
$$\boxed{\text{How to get } \int_\Omega P(\vec{y}\,|\,\vec{\theta})\,\pi_0(\vec{\theta})\,d\vec{\theta} \;?}$$

i) Quadrature

ii) Markov Chain Monte Carlo (MCMC)

Ex i) - Suppose $\vec{\theta} \in \mathbb{R}^2$. Also assume $\pi_0(\theta_1) = \pi_0(\theta_2) \sim U(0,1)$



- Assume $\varepsilon_i \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$ + let $SS(\vec{\theta})$ be the sum of squared error at $\vec{\theta}$

$$\Rightarrow \pi(\vec{\theta}\,|\,\vec{y}) = \frac{P(\vec{y}\,|\,\vec{\theta})\,\underline{\pi_0(\vec{\theta})}}{\int_\Omega P(\vec{y}\,|\,\vec{\theta})\,\underline{\pi_0(\vec{\theta})}\,d\vec{\theta}} = \frac{\exp\left(-\frac{1}{2\sigma_\varepsilon^2} SS(\vec{\theta})\right)}{\int_0^1\int_0^1 \exp\left(-\frac{1}{2\sigma_\varepsilon^2} SS(\vec{\omega})\right) d\vec{\omega}}$$

constant on $\vec{\theta}$

$$\Rightarrow \pi(\vec{\theta}\,|\,\vec{y}) = \frac{1}{\int_0^1\int_0^1 \exp\left(-\frac{1}{2\sigma_\varepsilon^2}\left(SS(\vec{\omega}) - SS(\vec{\theta})\right)\right) d\omega_2\, d\omega_1}$$

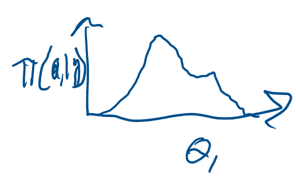
- If we discretize using quadrature

$$\pi(\vec{\theta}\,|\,\vec{y}) \approx \left[\sum_{r_1=1}^{R_1}\sum_{r_2=1}^{R_2} \exp\left(-\frac{1}{2\sigma_\varepsilon^2}\left(SS(\omega_{r_1}, \omega_{r_2}) - SS(\vec{\theta})\right)\right) W_{r_1} W_{r_2}\right]^{-1}$$

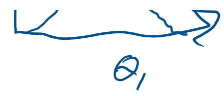
- Note that $SS(\vec{\theta})$ + $SS(\vec{\omega})$ both require nonlinear $f(x_i;\vec{\theta})$ solutions.

- The "marginal" posterior densities are defined as

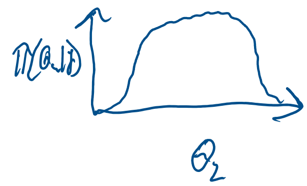
$$\pi(\theta_1\,|\,\vec{y}) = \int_0^1 \pi(\theta_1, \theta_2\,|\,\vec{y})\,d\theta_2 \Rightarrow$$



$$\pi(\theta_1 | \vec{y}) = \int_0^1 \pi(\theta_1, \theta_2 | y)\, d\theta_2 \Rightarrow$$



$$\pi(\theta_2 | \vec{y}) = \int_0^1 \pi(\theta_1, \theta_2 | \vec{y})\, d\theta_1$$



Markov Chain Monte Carlo

- Note that for $\vec{\theta} \in \mathbb{R}^p$, where p is "moderate" to "large", the marginal posterior for θ_i

$$\pi(\theta_i | \vec{y}) = \int_{\mathbb{R}^{p-1}} \pi(\vec{\theta} | \vec{y})\, d\theta_1\, d\theta_2 \cdots d\theta_{i-1}\, d\theta_{i+1} \cdots d\theta_p$$

is computationally expensive.

- Markov Chain Monte Carlo (MCMC) uses random Markov Chains to sample from $\pi(\vec{\theta} | \vec{y})$ & give estimates.

Markov Chains

- First, we need to define random processes (similar to KL-expansions)

Def Random Process

- A random process is defined as $X = \{X(t), t \in T\}$, which is an indexed set of R.V.'s all from the same probability space $(\Omega, \mathcal{F}, P)$

- Let $X_t(\omega)$ denote a realization of the process at time $t \in T$ with realization $\omega \in \Omega$.

· Let $X_t(\omega)$
with realization $\omega \in \Omega$.

<u>Def</u> Second-order Random process

- A Second-order Random process satisfies

$$E[X_t^2] < \infty, \quad \forall t \in T$$

· All Second-order Random processes have

$$E[X_t] = \mu_t, \quad t \in T$$

$$c(s,t) = Cov(X_s, X_t) = E[(X_s - \mu_s)(X_t - \mu_t)], \quad s,t \in T$$

<u>Def</u> Gaussian Process