Def | Local Sensitivity

- Let $\vec{y} = f(x; \vec{\theta})$ be the QoI with $\vec{\theta}^* \in \Gamma$.
  - we call $\vec{\theta}^*$ the <u>nominal</u> or <u>apriori</u> parameter value.
- The <u>local sensitivity</u> is

$$\Delta f_j \approx \left.\frac{\partial f}{\partial \theta_j}\right|_{\vec{\theta}^*} \cdot \Delta \theta_j$$

- we often approximate the above

$$\Delta f_j \approx \left(f(\theta^* + \vec{e}_j \Delta \theta_j) - f(\theta^*) \right) / \Delta \theta_j$$

Where $\vec{e}_j$ is a unit vector in the jth direction.

Properties |

- The above approx. is <u>linear</u>
  - highly nonlinear models may have sensitivities not well approx. by local sens.
- The results hinge on choice of $\vec{\theta}^*$.
- Choice of $\Delta \theta_j$ is subjective.
- the magnitudes of θ_i make $\frac{\partial f}{\partial \theta_i}$ hard to interpret

★ $S_n = \frac{\partial f}{\partial \vec{\theta}}$ gives insight into identifiability ★

Analytical Sensitivities

Ex) Consider the Spring problem

$$\frac{d\vec{u}}{dt} = \underset{\sim}{A}\vec{u}, \quad \vec{u} = \begin{bmatrix} dz/dt \\ z \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -K & -C \end{bmatrix}, \quad \vec{u}_0 = \begin{bmatrix} 0 \\ -C \end{bmatrix}$$

where $K = \frac{g}{m}$, $C = \frac{c}{m}$.

- Analytical Solution:

$$\boxed{z(t) = 2e^{-Ct/2} \cos\left(\sqrt{K - C^2/4}\; t\right)}$$

where $C^2 - 4K < 0$.

- We can derive the analytical sensitivities
- e.g.,

$$\frac{\partial z}{\partial K} = e^{-Ct/2} \cdot \frac{-2t}{\sqrt{4K - C^2}} \sin\left(\sqrt{K - C^2/4}\; t\right)$$

- Then $\underset{\sim}{S}(\vec{\theta}) \in \mathbb{R}^{N_t \times 2}$, where N_t is the number of time points.

Sensitivity Equations

- Sensitivity equations differentiate a system w.r.t. $\vec{\theta}$
- Consider an ODE System

$$\frac{d\vec{u}}{dt} = g(t, \vec{u}(t), \vec{\alpha}), \quad \vec{u}(t_0) = \vec{u}_0(\vec{\alpha})$$

where $\vec{u} \in \mathbb{R}^{N_u}$, + $\vec{\alpha} \in \mathbb{R}^p$ are model parameters.

- Let $\vec{\theta} = [\vec{\alpha}, \vec{u}_0]^T \in \mathbb{R}^{p+N_u}$.
- The sensitivities w.r.t. $\vec{\alpha}$ are

- the sensitivities w.r.t. $\vec{\alpha}$ are

$$\vec{S}_j(t) \equiv \frac{\partial \vec{u}}{\partial \alpha_j}, \quad j = 1, \ldots, P$$

- To get the $\vec{S}_j$, we differentiate the entire ODE System

$$\frac{d\vec{S}}{dt} = \underset{(i)}{\frac{\partial g}{\partial \vec{u}}} \cdot \underset{(ii)}{\frac{\partial \vec{u}}{\partial \vec{\alpha}}} + \underset{(iii)}{\frac{\partial g}{\partial \vec{\alpha}}}, \qquad \vec{S}(t_0) = \underset{(iv)}{\frac{\partial \vec{u}_0}{\partial \vec{\alpha}}}$$

(i) $\frac{\partial g}{\partial \vec{u}}$ is a $(N_u \cdot P) \times (N_u \cdot P)$ block diagonal matrix given by

$$\underset{\sim}{J}_{i\ell} = \frac{\partial g_i}{\partial u_\ell} \equiv \text{Jacobian}$$

(ii) $\frac{\partial \vec{u}}{\partial \vec{\alpha}}$ is the Sensitivity vector ($\mathbb{R}^{N_u \cdot P}$)

(iii) $\frac{\partial g}{\partial \vec{\alpha}}$ is an $N_u \cdot P$ vector with $\frac{\partial g}{\partial \vec{\alpha}}$

(iv) $\vec{S}(t_0)$ is the Jacobian at $t = t_0$.

$$\frac{\vec{S}}{d} = \underset{i}{\frac{\partial g}{\partial \vec{u}}} \cdot \underset{(ii)}{\frac{\partial \vec{u}}{\partial \vec{\alpha}}} + \underset{(iii)}{\frac{\partial g}{\partial \vec{\alpha}}}, \quad \vec{S}(t_0) = \underset{(iv)}{\frac{\partial \vec{u}_0}{\partial \vec{\alpha}}}$$

Ex) If $\vec{u} \in \mathbb{R}^2$, $\vec{\alpha} \in \mathbb{R}^2$, $\frac{d\vec{u}}{dt} = \begin{bmatrix} g_1(u_1, u_2, \vec{\alpha}) \\ g_2(u_1, u_2, \vec{\alpha}) \end{bmatrix}$

$\frac{d}{l}$

(i)

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} \frac{\partial u_1}{\partial \alpha_1} \\ \frac{\partial u_2}{\partial \alpha_1} \\ \frac{\partial u_1}{\partial \alpha_2} \\ \frac{\partial u_2}{\partial \alpha_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & 0 & 0 \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} & 0 & 0 \\ 0 & 0 & \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} \\ 0 & 0 & \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} \end{bmatrix} \begin{bmatrix} \frac{\partial u_1}{\partial \alpha_1} \\ \frac{\partial u_2}{\partial \alpha_1} \\ \frac{\partial u_1}{\partial \alpha_2} \\ \frac{\partial u_2}{\partial \alpha_2} \end{bmatrix} + \begin{bmatrix} \frac{\partial g_1}{\partial \alpha_1} \\ \frac{\partial g_2}{\partial \alpha_1} \\ \frac{\partial g_1}{\partial \alpha_2} \\ \frac{\partial g_2}{\partial \alpha_2} \end{bmatrix}, \quad \vec{S}(t_0) = \left.\begin{bmatrix} \frac{\partial \vec{u}_i}{\partial \alpha_j} \end{bmatrix}\right|_{t=t_0}$$

- We also have

$$\vec{S}_{o_j}(t) = \frac{\partial \vec{u}}{\partial \vec{u}_{o_j}}, \quad i = 1, \dots, N_u$$

Which gives ODE system

$$\frac{d\vec{S}_o}{dt} = \frac{\partial g}{\partial \vec{u}} \cdot \vec{S}_o(t), \quad \vec{S}(t_0) = 1$$