Snapshot Based Reduced Order Models

- Assume we have a expensive, high dimensional PDE Solution $u(\vec{x}, \vec{t})$

· Proper Orthogonal Decomposition (POD) to construct a set of orthonormal basis functions

$$\{\phi_j\}^R_{1,\dots,J_R}$$

that reduce our output dimensionality.

POD

- Consider a 1D PDE, $u = u(x,t)$, with "Snapshots"

$$u_m(\vec{x}) = u(\vec{x}, t_m), \quad m=1,\dots,M$$

· Define the Centered Snapshots

$$v_m(\vec{x}) = u_m(\vec{x}) - \bar{u}(x)$$

where $$\bar{u}(x) = \frac{1}{M}\sum_{m=0}^{M-1} u_m(x)$$

$m=0$

Goal: we want to find a structure for $\{v_m(\hat{x})\}_{m=1,\dots,M}$ which has the largest mean square projection onto observations.

. This gives

$$u(x,t) \approx \sum_{k=0}^{M-1} f_k(t)\, \phi_k(x)$$

Discrete Form

- Assume x is discretized into N points

$$\Rightarrow \vec{u}_m = [u_m(x_1), \dots, u_m(x_N)]$$

. we then construct the $N \times M$ snapshot matrix

$$\underset{\sim}{A} = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_M] \in \mathbb{R}^{N \times M}$$

. $\underset{\sim}{A}$ typically have rank $r \le \min\{N, M\}$

- We can construct a correlation matrix

$$\underset{\sim}{C} = \frac{1}{M} \underset{\sim}{A}^T \underset{\sim}{A} \in \mathbb{R}^{M \times M}$$

- The POD basis, $\{\phi_j^R\}^{J_R}$, $J_R \in \{1, \dots, r\}$, come from solving the $M \times M$ eigenvalue problem

Solving the $M \times M$ eigenvalue problem

$$\underset{\sim}{C}\, \vec{w}_j = \lambda_j \vec{w}_j \qquad (M < N)$$

+ then taking
$$\varphi_j^R = \frac{1}{\sqrt{M\lambda_j}} \underset{\sim}{A}\, \vec{w}_j, \quad j = 1, \dots, J_R$$

- If $M \geq N$, then we solve

$$\underset{\sim}{A}\,\underset{\sim}{A}^T \vec{b}_j = \eta_j \vec{b}_j, \quad + \quad \varphi_j^R = \vec{b}_j$$

SVD Representation

- We can instead write

$$\underset{\sim}{A} = \underset{\sim}{U}\, \underset{\sim}{\Sigma}\, \underset{\sim}{V}^T, \quad \underset{\sim}{U} \in \mathbb{R}^{N \times N}, \quad \underset{\sim}{V} \in \mathbb{R}^{M \times M}$$

$$+ \quad \underset{\sim}{\Sigma} = \begin{bmatrix} \underset{\sim}{S} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix}, \quad \underset{\sim}{S} = \mathrm{diag}(\sigma_1, \dots, \sigma_r) \in \mathbb{R}^{r \times r}$$

. Note that σ_i is related to eigenvalues of $\underset{\sim}{C}$

- Then the elements $\varphi_j^R = \vec{u}_j$, $j = 1, \dots, J_R$ are the POD basis functions

$$\Rightarrow \quad u(x,t) \approx \sum_{j=1}^{J_R} f_j(t)\, \varphi_j^R, \quad J_R \ll M$$

. Here, $f_j(t) = \sigma_j v_j^T$

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- The relative approximation error is

$$\varepsilon_{POD} = 1 - \left(\sum_{j=1}^{J_R} \sigma_j^2\right) \Big/ \left(\sum_{j=1}^{M} \sigma_j^2\right)$$

Algorithms (SVD)

i) Construct $\underset{\sim}{A} = [\vec{v}_1, \ldots, \vec{v}_M]$, $\vec{v}_i = \vec{u}_i(x) - \bar{u}$

+ Set $r = \text{rank}(\underset{\sim}{A})$.

ii) Compute SVD

$$\underset{\sim}{A} = \underset{\sim}{U}\, \underset{\sim}{\Sigma}\, \underset{\sim}{V}^T, \quad \underset{\sim}{\Sigma} = \text{diag}(\underset{\sim}{S}, \underset{\sim}{0}), \quad \underset{\sim}{S} = \text{diag}(\sigma_1, \ldots, \sigma_r)$$

iii) Determine J_R s.t. $\varepsilon_{POD} \geq 1 - \delta^2$

iv) Set POD basis functions to

$$\underline{\Phi}^R = [\phi_1^R, \ldots, \phi_{J_R}^R] = [\vec{u}_1, \ldots, \vec{u}_{J_R}]$$

+ Set coefficients to $f_i(t) = \sigma_i \vec{v}_i^T$

- Final result

$$u(x,t) \simeq \sum_{j=1}^{J_R} f_j(t)\, \phi_j^{J_R}(x)$$

Galerkin Projection

+ Galerkin

- Suppose you have a PDE governed by

$$\underset{\sim}{M} \frac{\partial \vec{u}}{\partial t} = \underset{\sim}{K} \vec{u} + g(t), \quad \vec{u}(0), \vec{u}_0$$

- If we can write

$$\vec{u}^{J_R} \simeq \sum_{i=1}^{J_R} f_i(t) \, \phi_i^R(x)$$

$$\Rightarrow \quad M_R \frac{\partial \vec{u}_{J_R}}{\partial t} = \underset{\sim}{K}_R \vec{u}_{J_R} + g_{J_R}(t)$$