Markov Chains

- A random process, X_t, satisfies the Markov process if the probability of future events, X_{t+1}, only depends on current state, X_t

$$\Rightarrow P(X_{t+1} = x_{t+1} \mid X_t = x_t, X_{t-1} = x_{t-1}, \dots, X_0 = x_0) = P(X_{t+1} = x_{t+1} \mid X_t = x_t)$$

- A Markov chain is characterized by:
  - i) the state space, S $\quad (X_t \in S, \forall t)$
  - ii) the initial probability, $\vec{P}^0$
  - iii) a transition kernel

- Assume we have k finite states, i.e. $S = \{x_1, x_2, \dots, x_k\}$
- Let P_{ij} be defined as follows

$$P_{ij} = P(X_{N+1} = x_j \mid X_N = x_i) \quad (\text{i.e. } x_i \to x_j)$$

- This gives rise to a <u>transition matrix</u> $\underset{\sim}{P} = [P_{ij}]$

$\boxed{\text{Idea}}$

- Given some initial probability $\vec{P}^0 = [P_1^0, P_2^0, \dots, P_k^0]$, $\sum_{i=1}^{k} P_i^0 = 1$

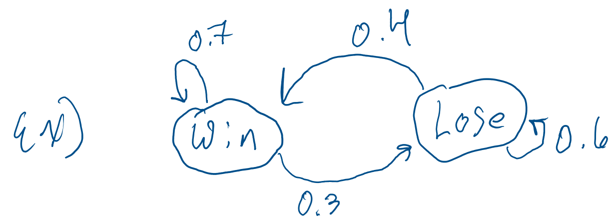
the Markov chain then transitions to a new state via the

$$\vec{P}^1 = \vec{P}^0 \underset{\sim}{P}$$

$$\vec{p}^{1} = \vec{p}^{0} \underset{\sim}{P}$$

- If Markov $\Rightarrow$ $\vec{p}^{2} = \vec{p}^{1} \underset{\sim}{P} = (\vec{p}^{0} \underset{\sim}{P}) \underset{\sim}{P} = \vec{p}^{0} \underset{\sim}{P}^{2}$

Ex)



$\Rightarrow$ $P(\text{win} \mid \text{lose}) = 0.4$

$P(\text{lose} \mid \text{win}) = 0.3$

$$\Rightarrow \underset{\sim}{P} = \begin{array}{c c} & \begin{array}{cc} \text{win} & \text{lose} \end{array} \\ \begin{array}{c} \text{win} \\ \text{lose} \end{array} & \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

- Assume $\vec{p}^{0} = [0.8, 0.2]$

$$\vec{p}^{1} = \vec{p}^{0} \underset{\sim}{P} = [0.64, 0.36]$$

- We can explicitly compute the <u>limiting distribution</u>, i.e. long term probability of the states as $N \to \infty$

- Let $\vec{\pi}$ be the limiting distribution.

. then as $N \to \infty$

$$\Rightarrow \quad \vec{\pi} = \vec{\pi} \underset{\sim}{P} \qquad \left(\vec{p}^{N} = \vec{p}^{0} \underset{\sim}{P}^{N} \right)$$

. we also know that $\sum_{i=1}^{q} \vec{\pi}_i = 1$

Ex) $[\pi_{win} \;\; \pi_{lose}] \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = [\pi_{win} \;\; \pi_{lose}]$ + $\pi_{win} + \pi_{lose} = 1$

$\Rightarrow \vec{\pi} = [0.5714, 0.4286]$ (MJC find Ref)

$$\Rightarrow \quad \vec{\pi} = [0.5714, 0.4286]$$ (MJC find Ref)

Def

- Given a Markov Chain with $\underset{\sim}{P}$, a distribution $\vec{\pi}$ satisfying

$$\text{i) } \pi_i \geq 0$$

$$\text{ii) } \sum_{i=1}^{q} \pi_i = 1$$

$$\text{iii) } \vec{\pi} = \vec{\pi}\underset{\sim}{P}$$

is called a "Stationary distribution" of the chain.

Def Detailed Balance

- A Markov Chain with $\underset{\sim}{P}$ + distribution $\vec{\pi}$ is "reversible" if

$$\pi_i P_{ij} = \pi_j P_{ji}, \quad \forall x_i, x_j \in S \quad (i, j \in [0, q])$$

- A Chain that Satisfies this satisfies "Detailed Balance"

$\boxed{\text{Markov Chain Monte Carlo}}$

- Consider $\vec{\theta} \in \mathbb{R}^P$ with observations $\vec{y} = f(\vec{x}; \vec{\theta})$, $\vec{y} \in \mathbb{R}^{N_y}$

- MCMC construct Markov chains whose stationary distributions

- MCMC construct Markov chains whose stationary distributions are the posterior distribution

Metropolis

- Consider some parameter value, $\vec{\theta}^{\,t-1} \in \mathbb{R}^p$, at iteration t.
- then
  i) Let the current chain state be $\vec{X}_{t-1} = \vec{\theta}^{\,t-1}$
  ii) Propose some new value, $\vec{\theta}^* \sim J(\vec{\theta}^* \mid \vec{\theta}^{\,t-1})$, where J is the proposal distribution
  iii) Select a probability of accepting $\vec{\theta}^*$