Uncertainty prop. using Monte Carlo estimators

- We can get global approximations for $E[f]$ + $Var[f]$ by

$$E[f]_{MC} \approx \frac{1}{M} \sum_{i=1}^{M} f(\vec{q}; \vec{\theta}^i) = \bar{f}$$

$$Var[f]_{MC} \approx \frac{1}{M-1} \sum_{i=1}^{M} \left(f(\vec{q}; \vec{\theta}^i) - \bar{f} \right)^2$$

$$= \underbrace{E[f^2]}_{\text{approx by MC}} - \bar{f}^2$$

where $\vec{\theta}^i \sim \rho(\vec{\theta})$ come from some PDF on $\vec{\theta}$.

Issue: This converges with rate $O(1/\sqrt{M})$

Bayesian Framework

- If we solve a Bayesian inverse problem first, we can use $\pi(\vec{\theta} \mid \vec{y})$ to generate credible + prediction intervals for $\vec{y}$.

- Let $Y_i = f(q_i; \vec{\theta}) + \varepsilon_i$, $\varepsilon_i \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$, $i = 1, \dots, N_y$, $Y_i \in \mathbb{R}$

- We have likelihood

$$P(\vec{y} \mid \vec{\theta}, \sigma_\varepsilon^2) = \frac{1}{(2\pi\sigma_\varepsilon^2)^{N_y/2}} \exp\left(-\frac{1}{2\sigma_\varepsilon^2} \sum_{i=1}^{N_y} \left(y_i - f(\lambda_i; \vec{\theta})\right)^2\right)$$

- The posterior predictive distribution is

$$\pi^*(y^* \mid \vec{y}_{prev}) = \int_{\mathbb{R}} \int_{\mathbb{R}^p} P(y^* \mid \vec{\theta}, \sigma_\varepsilon^2) \underbrace{\pi(\vec{\theta}, \sigma_\varepsilon^2 \mid \vec{y}_{prev})}_{\text{Posterior}} \, d\vec{\theta} \, d\sigma_\varepsilon^2$$

$\boxed{\text{Algorithm}}$

i) Specify M_{pred} Samples

ii) For $i=1, \dots, M_{pred}$

1) Draw $\vec{\theta}^i$ from Posterior Chain (use previous Chain)

2) Sample $\varepsilon^i \sim N(0, \sigma_\varepsilon^2)$, where

a) σ_ε^2 is fixed

b) Draw σ_ε^2 from Chain

3) Generate $\vec{y}^* = f(\lambda; \vec{\theta}^i) + \varepsilon^i$

iii) Sort $\vec{y}^*$ + construct quantile