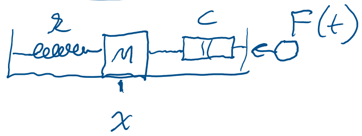
Model Lecture - 1/28/25

Tuesday, January 28, 2025 3:04 PM

Models used in this Course

Ex 1 Spring problem



- 2^{nd} order ODE: let $x(t)$ denote mass displacement. Then force balance gives

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t), \quad x(0) = x_0, \ \left.\frac{dx}{dt}\right|_{t=0} = v_0$$

- M: mass (kg)
- c: Damper coefficient (kg/s)
- k: Spring constant (kg/s^2)
- $F(t) = f_0 \cos(\omega t)$: oscillatory forcing function

- We rewrite system as two, first order equations

$$\frac{d}{dt}\vec{u} = \underset{\sim}{A}\vec{u} + \vec{F}, \quad \vec{u} = \begin{bmatrix} x \\ v \end{bmatrix}, \quad v = \frac{dx}{dt}$$

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 \\ -k/M & -c/M \end{bmatrix}, \quad \vec{F} = \begin{bmatrix} 0 \\ \frac{f_0}{m}\cos(\omega t) \end{bmatrix}$$

- M, c, k, f_0 + ω could be unknown or uncertain
- x_0 + v_0 could also be unknown

☐ we will use built in Runga-Kutta ODE solvers in MATLAB + Python

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Ex 2 1D Heat equation (Dirichlet BCs)

- Model describes change in temperature, $u(x,t)$, as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right), \quad x \in [0, L], \ t \in [0, T]$$

$$u(0,t) = u_0, \quad u(L,t) = u_L$$

$$u(x,0) = u_{IC}$$

- We define $\alpha(x)$ as the thermal diffusivity.
  - We can let α be scalar or a vector
  - We can treat α, u_0, u_L, + u_{IC} as potential unknown quantities to be informed by data.

- We numerically discretize using a forward time, centered space (FTCS) scheme.

Note: $\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right) = \frac{\partial \alpha}{\partial x}\frac{\partial u}{\partial x} + \alpha \frac{\partial^2 u}{\partial x}$

$$\frac{\partial \alpha}{\partial x} \approx \frac{\alpha_{i+1} - \alpha_{i-1}}{2\Delta x}, \quad \frac{\partial u}{\partial x} \approx \frac{u_{i+1}^{j} - u_{i-1}^{j}}{2\Delta x}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^{j} - 2u_i^{j} + u_{i-1}^{j}}{(\Delta x)^2}$$

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{j+1} - u_i^{j}}{\Delta t}, \quad \text{where } u_i^j = u(x_i, t_j)$$

$$+ \quad \alpha_i = \alpha(x_i)$$

- Note: Scheme is stable when

$$\boxed{\alpha \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2}}$$

Ex 3 | 2D Heat equation (Dirichlet BCs)

- Model describes change in temperature, $u(x,y,t)$ as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\alpha(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\alpha(x,y)\frac{\partial u}{\partial y}\right), \quad x \in [0, L_x], \; y \in [0, L_y], \; t \in [0, T]$$

$$u(0,y,t) = \vec{u}_{0,x}, \quad u(L_x,y,t) = \vec{u}_{L,x}$$

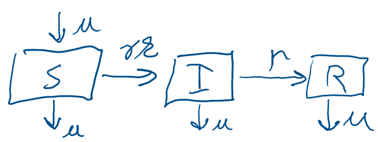
$$u(x,0,t) = \vec{u}_{0,y}, \quad u(x,L_y,t) = \vec{u}_{L,y}$$

$$u(x,y,0) = \vec{u}_{IC}$$

- Again, $\alpha(x,y)$ can be fixed or a matrix.
- α, BCs, + IC may be uncertain
- We also use FTCS scheme, but with stability criteria

$$\max(\alpha)\frac{\Delta t}{(\Delta x)^2} + \max(\alpha)\frac{\Delta t}{(\Delta y)^2} \leq \frac{1}{2}$$

Ex 4 | SIR model



- Assume fixed total population, N.
- We assume Susceptible (S), infected (I), + Recovered (R) individuals are described by:

$$\frac{dS}{dt} = \mu N - \mu S - \gamma \varepsilon \cdot S \cdot I$$

$$\frac{dI}{dt} = -\mu I + \gamma \cdot \varepsilon \cdot S \cdot I - r \cdot I = \gamma \varepsilon S I - (\mu + r) I$$

$$\frac{dR}{dt} = rI - \mu R$$

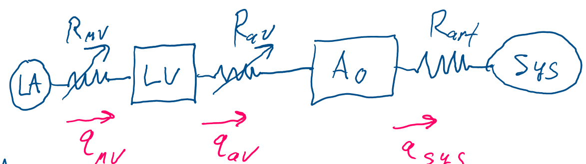
$S(0) = S_0$, $I(0) = I_0$, $R(0) = R_0$, $N = S + I + R = \text{constant}$

- μ: Birth + death rate ($1/\text{days}$)
- γ: infection rate ($1/\text{days}$)
- k: Contact rate ($1/\text{days}$)
- r: recovery rate ($1/\text{days}$)

- We assume people stay recovered once they arrive in that state.

- μ, γ, k, + r are assumed to be in $[0, 1]$

Ex 5) Cardiovascular Model



- Assume that LA + Sys are constant pressure sources.
- We assume fluid enters + exists LV + Ao compartments via mass balance, where blood volume, V, is

$$\frac{\partial V_j}{dt} = q_{in} - q_{out}, \quad j = LV, Ao$$

- In LV, we model heart contraction via function $E(t)$.
- Pressure in LV + Ao compartments are

$$P_{LV}(t) = E(t)\left(V_{LV}(t) - V_D\right), \quad P_{Ao}(t) = V_{ao}(t)/C_{ao}$$

whereas flow across resistors are defined by

$$q = \begin{cases} \max\left(0, \frac{\Delta P}{R}\right), & \text{if diode} \\ \frac{\Delta P}{R}, & \text{else.} \end{cases}$$

- Model has two volumes, V_{LV} + V_{Ao} + two

- Model has two volumes, V_{LV} + V_{AO}, + two pressures, P_{LV} + P_{AO}, that are quantities of interest.
- Includes 12 parameters + two IC.
  - We will focus on uncertainty in parameters