Modelling a counterflow heat exchanger

TETB

The counterflow chiller (CFC) is a widely used type of light to medium duty heat exchanger. This paper develops an analytical model of a uniform device carrying ideal fluids that has reached a steady state. No consideration is given to geometry or fluid dynamics.

I. Device to be modelled

We consider a uniform device of length L with opposing flows a and b of specific heat capacity c_a, c_b and mass flow rate $\dot{m}_a, \dot{m}_b$.

Heat Q is transferred between the flows at a rate k_q w/°C per metre of the device.

Let the flow temperatures at point x be T_{a_x}, T_{b_x} such that

$$\begin{aligned} T_{a_0} &= T_{a_{\text{in}}} \\ T_{b_0} &= T_{b_{\text{out}}} \\ T_{a_L} &= T_{a_{\text{out}}} \\ T_{b_L} &= T_{b_{\text{in}}}\,. \end{aligned} \tag{1}$$

II. Energy balance

The thermal power delivered by each flow is

$$\begin{aligned} \dot{Q}_{\text{a}} &= (T_{a_{\text{in}}} - T_{a_{\text{out}}})\,\dot{m}_a c_a \\ \dot{Q}_{\text{b}} &= (T_{b_{\text{in}}} - T_{b_{\text{out}}})\,\dot{m}_b c_b \end{aligned} \tag{2}$$

so for energy to be conserved

$$(T_{a_{\text{in}}} - T_{a_{\text{out}}})\,\dot{m}_a c_a + (T_{b_{\text{in}}} - T_{b_{\text{out}}})\,\dot{m}_b c_b = \dot{Q}_{\text{stored}} + \dot{Q}_{\text{lost}}\,. \tag{3}$$

In the steady state $\dot{Q}_{\text{stored}}$ falls to zero, so if $\dot{Q}_{\text{lost}}$ is small

$$\frac{T_{a_{\text{in}}} - T_{a_{\text{out}}}}{T_{b_{\text{out}}} - T_{b_{\text{in}}}} \approx \frac{\dot{m}_b c_b}{\dot{m}_a c_a}\,. \tag{4}$$

III. Deriving the model

Neglecting heat loss to the environment, at equilibrium the rate of heat transfer in a small section δx is

$$\dot{Q}_{\text{ab}} = k_q\,(T_a - T_b)\,\delta x \tag{5}$$

and in time δt the changes in heat capacity are

$$\begin{aligned} \delta C_a &= \;\; \dot{m}_a c_a\,\delta t \\ \delta C_b &= -\dot{m}_b c_b\,\delta t \end{aligned} \tag{6}$$

therefore the flow temperatures change by

$$\begin{aligned} \delta T_a &= -\,\dot{Q}_{\text{ab}}/\dot{C}_a \\ &= -k_q\,\tfrac{1}{\dot{m}_a c_a}\,(T_a - T_b)\,\delta x \\ \delta T_b &= -k_q\,\tfrac{1}{\dot{m}_b c_b}\,(T_a - T_b)\,\delta x \end{aligned} \tag{7}$$

As the time derivatives have diminished to zero we can say

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}x}\big(T_{a_x}\big) &= -k_q\,\tfrac{1}{\dot{m}_a c_a}\,(T_a - T_b) \\ \tfrac{\mathrm{d}}{\mathrm{d}x}\big(T_{b_x}\big) &= -k_q\,\tfrac{1}{\dot{m}_b c_b}\,(T_a - T_b) \end{aligned} \tag{8}$$

and hence

$$\tfrac{\mathrm{d}}{\mathrm{d}x}\big(\Delta T_{\text{ab}}\big) = -k_q\,\big(\tfrac{1}{\dot{m}_a c_a} - \tfrac{1}{\dot{m}_b c_b}\big)\,\Delta T_{\text{ab}}\,. \tag{9}$$

As this is a first order ODE, let

$$\tfrac{\mathrm{d}}{\mathrm{d}x}\big(\Delta T_{\text{ab}}\big) = -\tfrac{1}{\xi}\,\alpha\,e^{-x/\xi} \tag{10}$$

which we integrate to obtain

$$\Delta T = \alpha\,e^{-x/\xi} + \gamma\,. \tag{11}$$

however for large x the flows converge to some common temperature T_∞ so the constant of integration is zero.

Substituting into (9)

$$-\tfrac{1}{\xi}\,\alpha\,e^{-x/\xi} = -k_q\,\big(\tfrac{1}{\dot{m}_a c_a} - \tfrac{1}{\dot{m}_b c_b}\big)\,\alpha\,e^{-x/\xi} \tag{12}$$

so

$$\boxed{\tfrac{1}{\xi} = k_q\,\big(\tfrac{1}{\dot{m}_a c_a} - \tfrac{1}{\dot{m}_b c_b}\big)}\,. \tag{13}$$

Using the initial conditions in (11)

$$\begin{aligned} \Delta T_0 &= (T_{a_{\text{in}}} - T_{b_{\text{out}}}) = \alpha \\ \Delta T_L &= (T_{a_{\text{out}}} - T_{b_{\text{in}}}) = \alpha\,e^{-L/\xi} \end{aligned} \tag{14}$$

so

$$e^{-L/\xi} = \tfrac{T_{a_{\text{out}}} - T b_{\text{in}}}{T_{a_{\text{in}}} - T b_{\text{out}}} \tag{15}$$

from which

$$\tfrac{1}{\xi} = \tfrac{1}{L}\,\ln\Big(\tfrac{T_{a_{\text{in}}} - T b_{\text{out}}}{T_{a_{\text{out}}} - T b_{\text{in}}}\Big) \tag{16}$$

hence from (13)

$$\boxed{k_q = \tfrac{1}{L}\,\tfrac{(\dot{m}_a c_a)\,(\dot{m}_b c_b)}{(\dot{m}_b c_b) - (\dot{m}_a c_a)}\,\ln\Big(\tfrac{T_{a_{\text{in}}} - T_{b_{\text{out}}}}{T_{a_{\text{out}}} - T_{b_{\text{in}}}}\Big)}\,. \tag{17}$$

We now observe that as $(T_a - T_b) = \alpha\,e^{-x/\xi}$, it follows that

$$\begin{aligned} T_a &= \alpha_a\,e^{-x/\xi} + T_\infty \\ T_b &= \alpha_b\,e^{-x/\xi} + T_\infty \end{aligned} \tag{18}$$

Assuming that the input temperatures are known we substitute them into (18) to obtain

$$\begin{aligned} T_{a_{\text{in}}} &= \alpha_a \;\; + T_\infty \\ T_{b_{\text{in}}} &= \alpha_b\,e^{-L/\xi} + T_\infty \end{aligned} \tag{19}$$

hence

$$T_{a_{\text{in}}} - T_{b_{\text{in}}} = \alpha_a - \alpha_b\,e^{-L/\xi} \tag{20}$$

and from (8)

$$\frac{\alpha_a}{\alpha_b} = \frac{\dot{m}_b c_b}{\dot{m}_a c_a} \tag{21}$$

so

$$T_{a_{\text{in}}} - T_{b_{\text{in}}} = \alpha_a\big(1 - \tfrac{\dot{m}_a c_a}{\dot{m}_b c_b}\, e^{-L/\xi}\big) \tag{22}$$

from which we obtain

$$\boxed{\begin{aligned} \alpha_a &= (T_{a_{\text{in}}} - T_{b_{\text{in}}})\, \frac{\dot{m}_b c_b}{\dot{m}_b c_b - \dot{m}_a c_a\, e^{-L/\xi}} \\ \alpha_b &= (T_{a_{\text{in}}} - T_{b_{\text{in}}})\, \frac{\dot{m}_a c_a}{\dot{m}_b c_b - \dot{m}_a c_a\, e^{-L/\xi}} \end{aligned}} \tag{23}$$

and

$$\boxed{T_\infty = T_{a_{\text{in}}} - \alpha_a}. \tag{24}$$

We can now calculate the output temperatures from (18) as

$$\boxed{\begin{aligned} T_{a_{\text{out}}} &= T_\infty + \alpha_a\, e^{-L/\xi} \\ T_{b_{\text{out}}} &= T_\infty + \alpha_b \end{aligned}}. \tag{25}$$

IV. Applying the model

To apply the model we collect a set of calibration measurements in the steady state, check the energy balance with (4) then use (17) to calculate k_q.

We now substitute the desired flow rates into (13) to obtain ξ, then the input temperatures in (23) and (24) to obtain α_a, α_b and T_∞.

The output temperatures can now be calculated from (25) or the full spatial temperature distributions from (18).

V. Validation with numerical simulation

To validate the model we constructed a finite element simulation (see appendix). Figure 1 and Tables I, II show good agreement between the simulated and calculated parameters over a wide range of operating conditions.

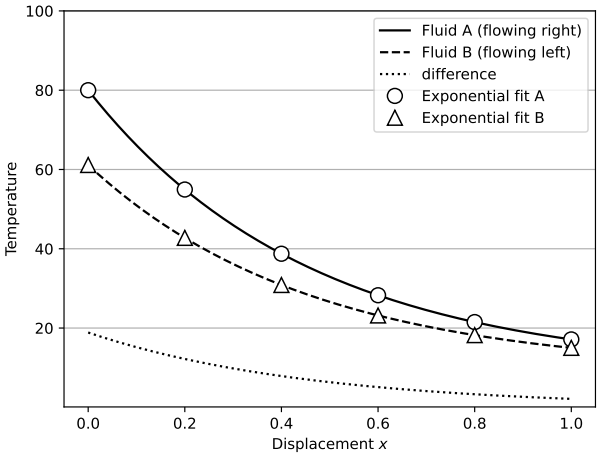


Fig. 1. Calculated temperature distributions versus simulation (steady state)

The calculated parameters are arguably more accurate than the simulated ones: indeed the results get closer if we use more elements in the simulation.

TABLE I
CALCULATED VERSUS SIMULATED k_q ($N = 256$)

k_q	$\dot{m}_a$	$\dot{m}_b$	$T_{a_{\text{in}}}$	$T_{b_{\text{in}}}$	calc ξ	calc k_q
0.5	0.25	0.75	80.0	20.0	0.75	0.499
0.5	0.50	0.75	80.0	20.0	2.99	0.499
0.5	1.00	0.75	80.0	20.0	−5.99	0.498
0.5	2.00	0.75	80.0	20.0	−2.40	0.498
3.0	1.5	10.0	100.0	50.0	0.59	2.990
3.0	1.5	10.0	75.0	50.0	0.59	2.990
3.0	1.5	10.0	25.0	50.0	0.59	2.990
3.0	1.5	10.0	0.0	50.0	0.59	2.990

TABLE II
CALCULATED VERSUS SIMULATED SPATIAL PARAMETERS (N=256)

$\dot{m}_a$	ref ξ	ref α_a	ref α_b	calc ξ	calc α_a	calc α_b
0.25	0.749	65.76	21.89	0.750	65.78	21.92
0.50	2.998	114.81	76.52	3.000	114.87	76.58
1.00	-5.997	-104.25	-139.02	-6.000	-104.32	-139.09
2.00	-2.399	-19.69	-52.52	-2.400	-19.70	-52.54

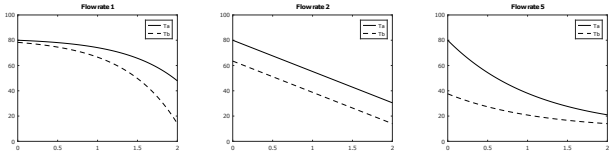
VI. Application to practical devices

Applying the model to practical devices is the subject of ongoing investigation.

As expected the quantitative accuracy is affected by factors excluded from the model, notably heat loss to the environment and increased turbulence at higher flow rates.

Even so the qualitative results of the model appear to match observed behaviour well, and provide good insight into the different operating modes of a real device.

For example the plots below show the result of increasing the mass flow rate $\dot{m}_b$ while holding the other parameters constant.



VII. Conclusion

We have developed an algebraic model for a heat exchanger based on simple temperature and flow rate measurements at a single operating point. The model predicts the behaviour at a wide range of operating conditions and closely matches the results of a finite element simulation.

The quantitative behaviour of real devices is affected by difficult to model effects such as flow-related turbulence and heat loss to the environment, but the model nonetheless provides a good basis for understanding how devices behave.

APPENDIX

APPENDIX I: NUMERICAL SIMULATION

Listing 1. Finite element simulation

```
import matplotlib.pyplot as plt
import matplotlib.animation as animation
import numpy as np

class CFC:
    def __init__(self, Ta_in, Tb_in, Fa, Fb, k, Ca, Cb, N:int):
        self.N = N
        self.Ta_in = Ta_in
        self.Tb_in = Tb_in
        self.Ta = np.array(N * [Ta_in], dtype='float')
        self.Tb = np.array(N * [Tb_in], dtype='float')
        self.Ca = Fa * Ca
        self.Cb = Fb * Cb
        self.k = k

    def update(self):
        for i in range(self.N):
            dQ = self.k * (self.Ta[i] - self.Tb[i]) / self.N
            self.Ta[i] -= dQ / self.Ca
            self.Tb[i] += dQ / self.Cb
        self.Ta = np.roll(self.Ta, 1)
        self.Tb = np.roll(self.Tb, -1)
        self.Ta[0] = self.Ta_in
        self.Tb[-1] = self.Tb_in

class Simulation:
    def __init__(self, Ta_in, Tb_in, Fa, Fb, k, Ca:float=1, Cb:float=1, N:int=250):
        self.cfc = CFC(Ta_in, Tb_in, Fa, Fb, k, Ca, Cb, N)
        self.x = np.linspace(0, 1, N)
        self.fig, ax = plt.subplots()
        plt.grid(axis='y')
        self.Ta_plot = ax.plot(self.x, self.cfc.Ta, '-r', label='Wort')[0]
        self.Tb_plot = ax.plot(self.x, self.cfc.Tb, '-g', label='Coolant')[0]
        self.Tdiff_plot = ax.plot(self.x, self.cfc.Ta - self.cfc.Tb, '-b',
                                  label='difference')[0]
        ax.set(xlim=[0, 1], ylim=[0, 100], ylabel='Temperature')
        ax.legend()

    def update(self, frame):
        self.cfc.update()
        self.Ta_plot.set_ydata(self.cfc.Ta)
        self.Tb_plot.set_ydata(self.cfc.Tb)
        self.Tdiff_plot.set_ydata(self.cfc.Ta - self.cfc.Tb)
        return (self.Ta_plot, self.Tb_plot, self.Tdiff_plot)

def main():
    sim = Simulation(N=254, Ta_in=80, Tb_in=14, Fa=2/60, Fb=3/60, k=0.001)
    ani = animation.FuncAnimation(
        fig=sim.fig,
        func=sim.update,
        save_count=1200,
        interval=10,
        blit=True
    )
    plt.show()
    print(sim.cfc.Ta[-1])
    print(sim.cfc.Tb[0])

if __name__ == '__main__':
    main()
```