A child's garden of waveforms

Ron Roberts (colonel_hack@yahoo.com)

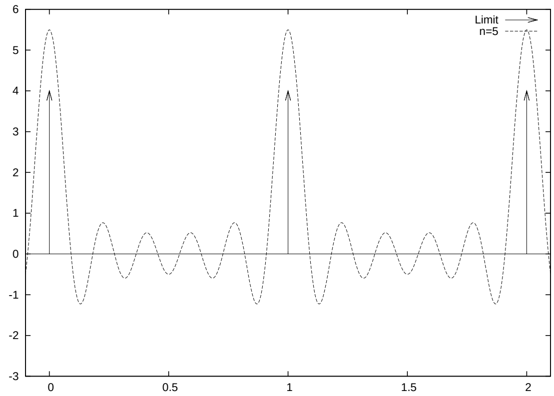
For those of you for whom an equation is worth a thousand pictures I'll first present a table of waveform equation so you can get started and later add a few graphs and comments. Or you can print first page and skip my commentary completely.

Name	Fourier series	Finite sum	
Impulse (δ)	$\frac{1}{2} + \sum_{n=1}^{\infty} \cos\big(n\omega x\big)$	$\frac{\sin(\frac{2n+1}{2}\omega x)}{2\sin(\frac{\omega x}{2})}$	(1)
Sawtooth	$\sum_{n=1}^{\infty} \frac{1}{n} \sin\big(n\omega x\big)$	integrated impulse	(2)
Parabolic #1	$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos\big(n\omega x\big)$	integrated sawtooth	(3)
Alternating impulse	$\sum_{n=1}^{\infty} \cos\big((2n-1)\omega x\big)$	$\frac{\sin(2n\omega x)}{2\sin(\omega x)}$	(4)
square wave	$\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\big((2n-1)\omega x\big)$	integrated alt. impulse	(5)
Triangle wave	$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\big((2n-1)\omega x\big)$	integrated square wave	(6)
Parabolic #2	$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin\big((2n-1)\omega x\big)$	integrated triangle wave	(7)
Every nth cosine	$\sum_{n=1}^{\infty} \cos\big((an-b)\omega x\big)$	$\frac{\sin\Big((\frac{(2n+1)}{2}a-b)\omega x\Big)-\sin\Big((\frac{a}{2}-b)\omega x\Big)}{2\sin(\frac{a}{2}\omega x)}$	(8)
Every nth sine	$\sum_{n=1}^{\infty} \sin\big((an-b)\omega x\big)$	$\frac{\cos\Big((\frac{a}{2}-b)\omega x\Big)-\cos\Big((\frac{2n+1}{2}a-b)\omega x\Big)}{2\sin(\frac{a}{2}\omega x)}$	(9)
$\pm\delta$ at $\mp\frac{p}{2}$	$\sum_{n=1}^{\infty} -\sin(n\pi p)\sin\big(n\omega x\big)$	$\frac{\sin(\frac{2n+1}{2}\omega(x+\frac{p}{2}))}{2\sin(\omega\frac{x+\frac{p}{2}}{2})} - \frac{\sin(\frac{2n+1}{2}\omega(x-\frac{p}{2}))}{2\sin(\omega\frac{x-\frac{p}{2}}{2})}$	(10)
Pulse, width p	$\sum_{n=1}^{\infty} \frac{1}{n}\sin(n\pi p)\cos\big(n\omega x\big)$	integrated δs above	(11)
Phase shift	$\sum_{n=1}^{\infty} \cos\big(\omega(nx+\varphi)\big)$	$\frac{\sin\Big(\omega(\frac{2n+1}{2}x+\varphi)\Big)-sin\Big(\omega(x/2+\varphi)\Big)}{2\sin(\frac{x}{2})}$	(12)
Every nth cosine with phase shift	$\sum_{n=1}^{\infty} \cos\big((an-b)\omega x+\varphi\big)$	$\frac{\sin\Big(\omega((\frac{(2n+1)}{2}a-b)x+\varphi)\Big)-\sin\Big(\omega((\frac{a}{2}-b)x+\varphi)\Big)}{2\sin(\omega\frac{a}{2}x)}$	(13)
Double sum	$\sum_{n,m=1}^{\infty} \cos\big((an+bm+c)\omega x\big)$	$\frac{\sin\Big((\frac{1}{2}a+\frac{1}{2}b+c)\omega x\Big)-\sin\Big((\frac{1}{2}a+\frac{2m+1}{2}b+c)\omega x\Big) - \sin\Big((\frac{2n+1}{2}a+\frac{1}{2}b+c)\omega x\Big)+\sin\Big((\frac{2n+1}{2}a+\frac{2n+1}{2}b+c)\omega x\Big)}{4\sin(\frac{a}{2}\omega x)\sin(\frac{b}{2}\omega x)}$	(14)
Exponential coefficient	$\sum_{n=1}^{\infty} a^n \cos\big((bn+c)x+\varphi\big)$	$\frac{\cos(cx)\Big(1-a^{n+1}\cos\Big((n+1)bx+\varphi\Big)-a\cos(bx+\varphi)+a^{n+2}\cos(nbx+\varphi)\Big) + \sin(cx)\Big(a^{n+1}\sin\Big((n+1)bx+\varphi\Big)-a\sin(bx+\varphi)-a^{n+2}\sin(nbx+\varphi)\Big)}{1-2acos(bx)+a^2}$	(15)
Clipped cosine $\max(\cos(\omega x)-c,0)$	$\sum_{n=0}^{\infty} a_n\cos(n\omega x)$ $a_0 = \sin(\alpha)-a\alpha$ $a_1 = a\alpha + \frac{\sin(2\alpha)}{2} - 2sin(\alpha)$ $a_n = \frac{\sin((n+1)\alpha)}{n+1} + \frac{\sin(n-1)\alpha)}{n-1}$ $-2\frac{\sin(n\alpha)}{n}$ where $\alpha = \cos^{-1}(c)$	see text	(16)

Note: sums 1, 8, 9, 10, 12, 13, 14 and 15 diverge as written and most or all grow as N so needs scaling by something like (Max allowable value)/N.

The first waveform we'll look at is the "impulse" (eqn. 1) with all harmonics present at equal amplitude:

Figure 1

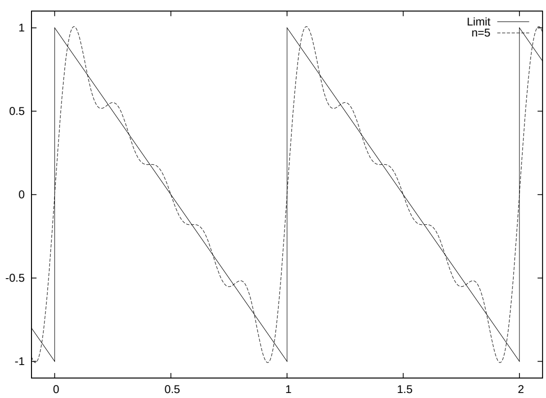


Theoretical & band-limited impulse
The arrows depict the δ function going to infinity

The oscillations in the finite sum near the discontinuity do not get smaller in the limit of large n (Gibb's phenomenon), they merely increase in frequency. In fact, the negative going spikes just to the right and left of the central positive peak get larger as n increases since they move closer to the divergence in their cosecant envelope, so make sure you leave a little negative headroom for them.

Since we are considering the functions as continuous at this point, this function is a chain of "Dirac δ (delta) functions" (plus a negative constant). The δ function has the funny property that it goes to infinity (not too much infinity or too little infinity. Just enough!) at a point in such a way that when they are integrated they cause a step or jump in the integral. So the integral of this waveform jumps up at each δ. To keep the integral finite, the $\frac{1}{2}$ needs to be subtracted off (or equivalently not included) and then, since the integrand is negative elsewhere, it decays until it hits another delta, producing a sawtooth wave, (fig. & eqn. 2) a staple in analog equipment, at least partially because it is easy to generate electronicly!

Figure 2

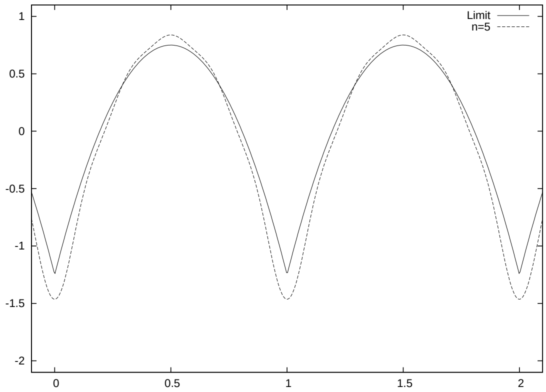


Sawtooth

Integrating again produces a piecewise parabolic wave (fig. & eqn 3) which is of the form $a(n-x)(n+1-x)-b$ on the interval $(n, n+1)$ $n \in \mathcal{Z}$. While the function is continuous, there is a discontinuity in the derivative at each integral value of x.

It is interesting that such a "spiky" wave has comparatively small high frequency content (which can be seen by the $1/n^2$ suppression of the higher harmonics).

Figure 3

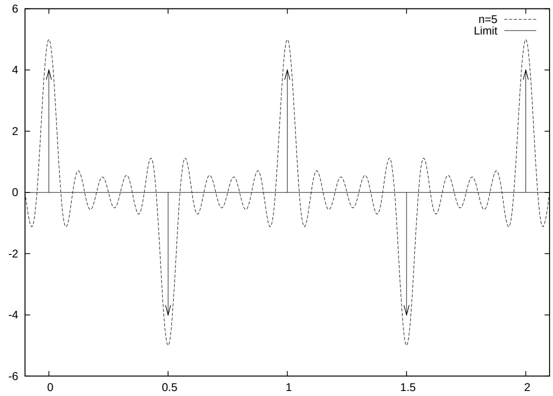


Parabolic wave, type #1
Splices at the negative peaks

Further integrations produce cubic, quartic etc. waves with less and less harmonic content and therefore typically less musical interest.

The alternating impulse waveform (fig. & eqn. 4) can be made by subtracting two impulse waveform, one shifted by a half cycle. The even harmonics are then out of phase and cancel, leaving only the odd harmonics.

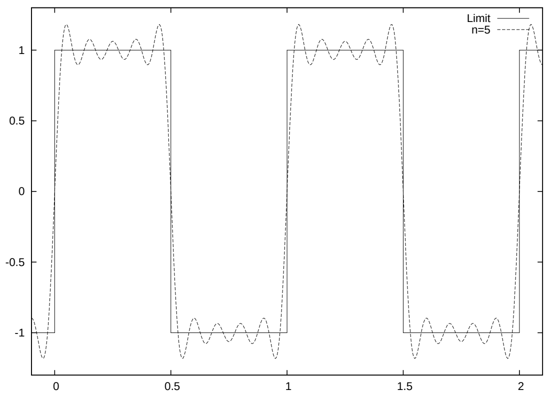
Figure 4



Alternating impulse

Integrating the alternating impulse will give the beloved square wave (eqn 5):

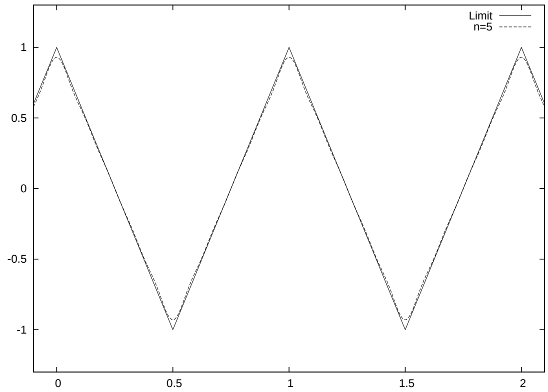
Figure 5



Square wave and BL approximation

Integrating the square wave gives a triangle wave (eqn 6):

Figure 6

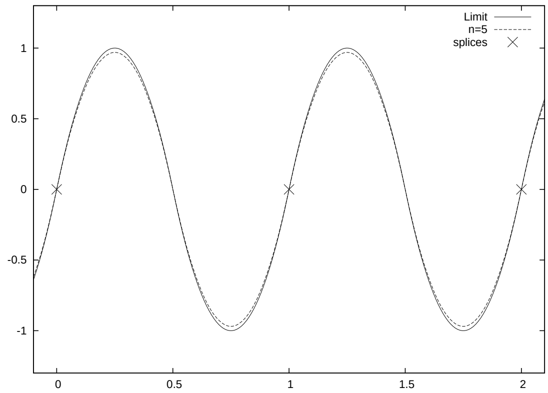


Triangle wave and BL approximation

Note how small the difference between the limit and finite sum wave are, indicating how little power is in the missing terms.

Integrating again give a much more sinusoidal looking parabolic wave (eqn 7):

Figure 7



Parabolic wave, type #2

The alternating impulse is the special case ($a = 2$, $b = 1$) of our next type of waveform where we take every nth harmonic (eqn. 8 & 9). It should be relatively clear that if you add all the waves for $b = 0 \ldots a-1$ you get the full impulse waveform, and that the case $b = 0$ gives an impulse waveform with $a\times$ the frequency.

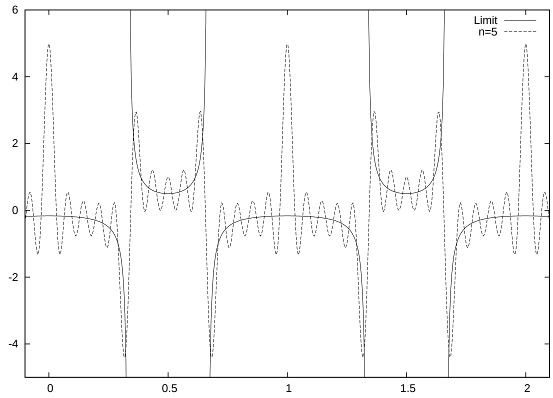
As examples, after integration, a $a = 2, b = 1$ is a square wave and $a = 2, b = 0$ is a $2\times$ frequency sawtooth and it is fairly easy to see they can add up to a $1\times$ sawtooth for the correct phase.

These waveforms have a limit of the form

$$\frac{\sin\big((b-\frac{a}{2})\omega x\big)}{2\sin(\frac{a\omega x}{2})} \text{ if } ax \notin \mathcal{Z}, \text{ it diverges otherwise} \tag{20}$$

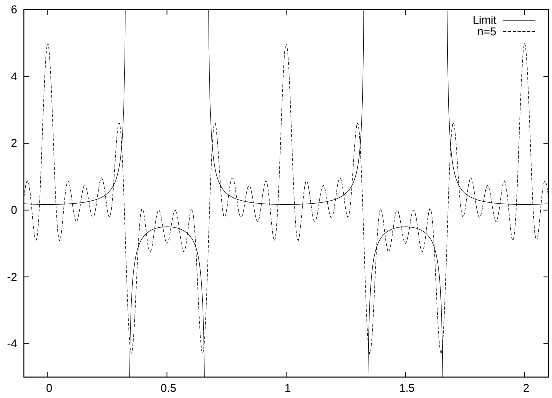
Interestingly, in this limit the $a = 3, b = 1$ and $a = 3, b = 2$ waveforms are almost everywhere negatives of each other but they do not *quite* add up to zero. At the divergences they are not negatives of each other. To help understand this, lets look at graphs for $a = 3$

Figure 8



$a = 3, b = 1$

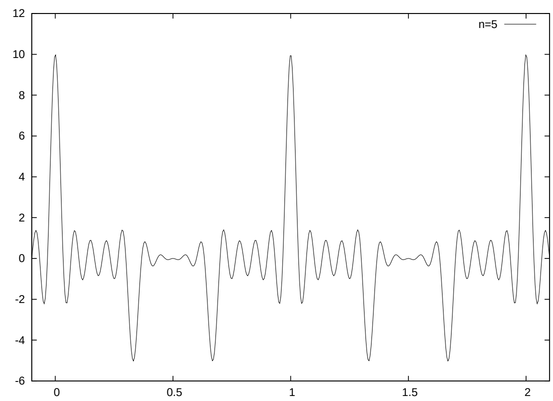
Figure 9



$a = 3, b = 2$

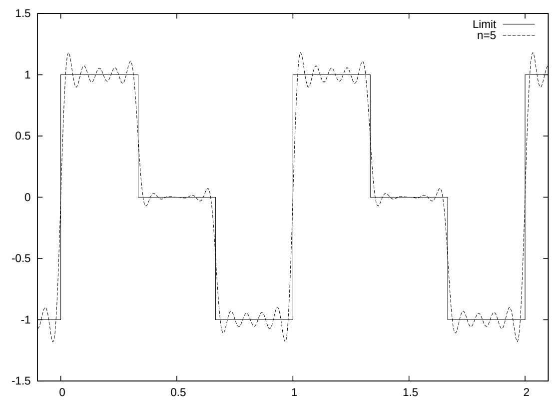
you can see in fig 8 & 9 where they diverge away from the limit for $x = \ldots -1, 0, 1, 2 \ldots$, also, there is a divergence in the sum and limit at multiples of $\frac{1}{3}$. It is not quite clear to me if these are δ functions in the limit or not.

Figure 10



Sum of the
$a = 3, b = 1 \ + \ a = 3, b = 2$
waveforms

Figure 11



Integral of the sum
of waveforms shown
in fig 10

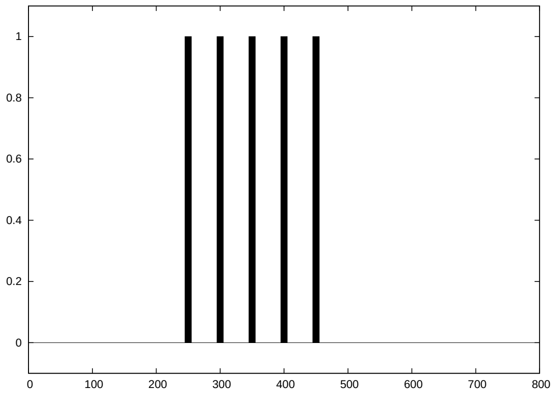
In the sum of the $b = 1$ and $b = 2$ waves, fig 10, you can see that spikes form at multiples of $\frac{1}{3}$. These do become δs and this sum is the Fourier series of the stairstep function shown in fig. 11 (and, in general, the integral of the sum of eqn. 8 for $b = 1 \ldots a - 1$, e.g. all but $b = 0$ produces an n step staircase). Since the case $a = 3$, $b = 0$ is just a $3\times$ frequency impulse wave, these negative δs are just what is needed to cancel the unwanted $+\delta$s in the $b = 0$ waveform to make the sum of all three a $\times 1$ impulse waveform.

Note that there is no assumption in the formula that a and b are integers and they do not need to be. Rational values may require a phase accumulator of more than 2π length and irrational may break using a single accumulator, requiring a separate a and a b accumulator.

a is just the difference (not a ratio!) between the frequencies and b adjusts the first frequency. In this case it is useful to set $a - b = 1$ so that the nominal frequency ω is the frequency of the 1st component and $a\omega$ is the size of the steps between frequencies.

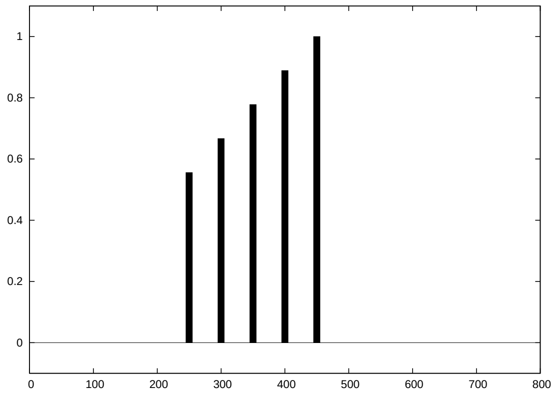
You may notice a similarity here to a ring modulator. If we take a sawtooth of frequency ω_0 and multiply it by a sine wave, frequency ω_c, we get overtones of frequencies $\omega_c \pm \omega_0$, $\omega_c \pm 2\omega_0$ etc. with amplitudes decaying as we move from ω_c. We can make a similar wave by taking $\omega = \omega_0$, $b = 1 - \frac{\omega_c}{\omega_0}$ and then integrating it. The harmonics will decay, but at a rate proportional to that components actual frequency, not like the ring modulator which decay as one over the unshifted frequency (fig. 12-15).

Figure 12



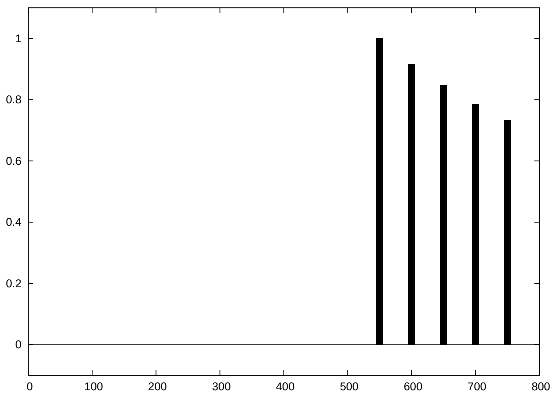
Spectrum of eqn. 8 for
$n = 5,\ a = 1,\ b = -4,\ f = 50$

Figure 13



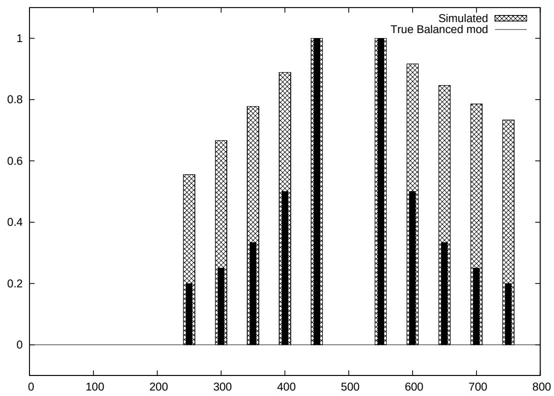
Spectrum of
Derivative of Fig. 11

Figure 14



Spectrum of integral of eqn. 8 for
$n - 5,\ a = 1,\ b = -11,\ f = 50$

Figure 15

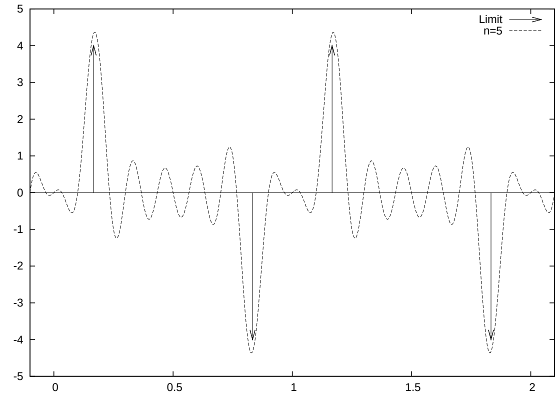


Spectrum of real
and simulated balanced modulator

The lower side band can be simulated by taking a & b to give a number of frequencies below ω_c such that the set ends at $\omega_c - \omega_0$ and then *differentiating* to produce increasing amplitude with increasing frequency, nd the upper side band by integrating a set that at $\omega_c + \omega_0$. The amplitudes are wrong enough that this is may not be a good simulation of a real balance modulator but may be more useful if higher harmonic content is desired.

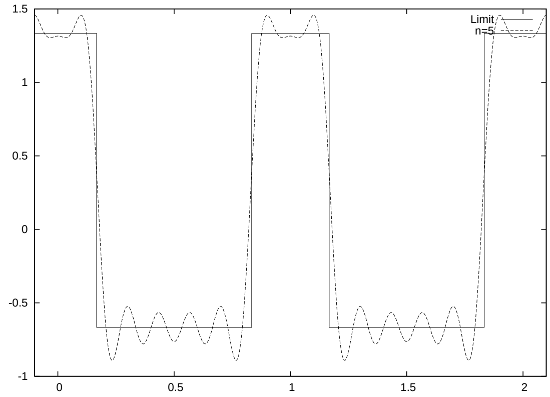
Next, we again consider a difference of two impulse waveforms, this time separated by p expressed as the fraction of the period (fig 16). When p is rational and n a multiple of the denominator, the corresponding harmonic is missing (so for $p = 1/2$ the even harmonics are missing and we have the alternating impulse back.) Also, for $p = 1/3$ it is amusing that the harmonic structure of the $\pm\delta$ series is $+1, +1, 0, -1, -1, 0, +1, +1, 0$. Integrating produces a square pulse of width $p\times$ the period (fig 17).

Figure 16



alternating impulse
separation $p = \frac{1}{3}$

Figure 17



$p = \frac{1}{3}$ wide square impulse

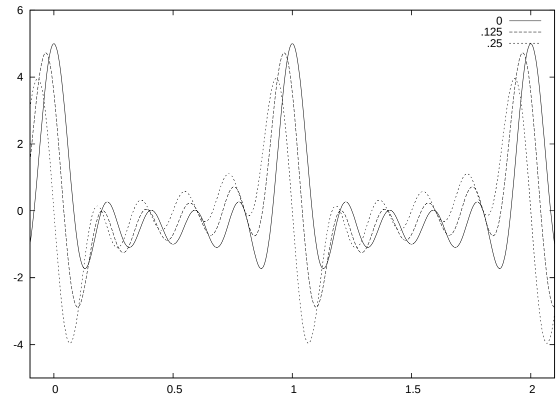
If the phase of each component of an impulse waveform is shifted by a constant amount, the wave changes shape (fig 18 & 19), but interestingly, the discontinuity doesn't move. By $\varphi = .25$, (equivalent to changing all the cosines in the sum to sines). the wave goes just as far negative as it does positive and therefore cannot be a simple δ function in the limit. (I suspect as soon as $\varphi \neq 0$ there is no δ any more, though it could be subtler and the δ may fade as φ moves away from zero.) The $\varphi = .25$ case has the lowest peak amplitude, but the highest peak-peak, so depending on application, one form or the other may be more useful numerically.

The limit of this sum is

$$\frac{-\sin\bigl(\omega(\frac{x}{2} + \varphi)\bigr)}{\sin\bigl(\omega\frac{x}{2}\bigr)} \tag{21}$$

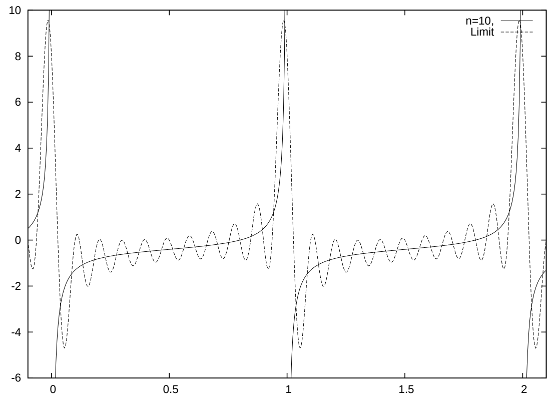
shown in figure 19.

Figure 18



Shifted impulses for
different phase φ

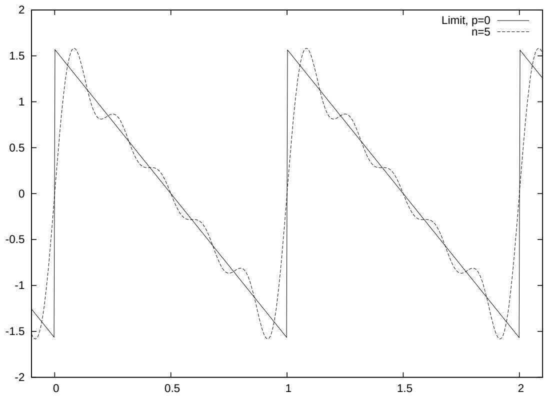
Figure 19



Higher order showing
approach to limit

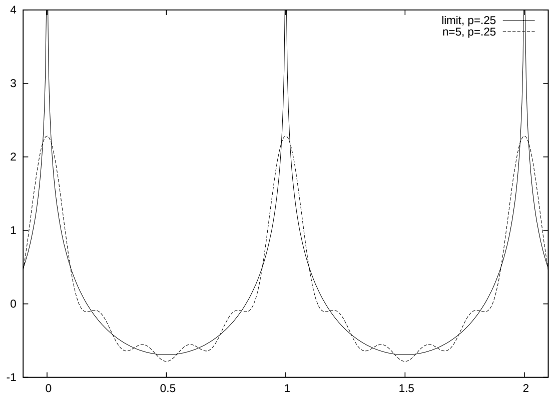
Integrating these waveforms gives our old friend the sawtooth (fig. 20) in the $\varphi = 0$ case (it better!) and a suspension bridge-looking wave (fig. 21) for $\varphi = .25$.

Figure 20



Sawtooth

Figure 21



Phase shifted sawtooth

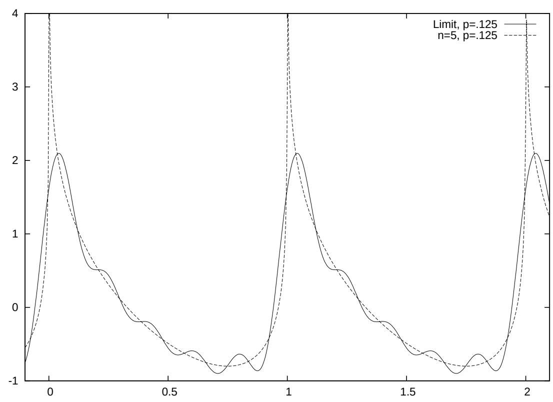
The limiting form for $\varphi = .25$ is

$$2\ln\bigl(\sin^2(\frac{x}{2})\bigr) \tag{22}$$

It is interesting (and in retrospect, obvious) that the limit of the case $4p \notin \mathcal{Z}$ is a linear combination of the two waveforms

$$\sum_{n=1}^{\infty} \cos\bigl(2\pi(nx+\varphi)\bigr) = cos(\varphi)\bigl(\text{floor}(x) - x + 12\bigr) + sin(\varphi)2\ln\bigl(\sin^2(\frac{x}{2})\bigr) \tag{23}$$

Figure 22

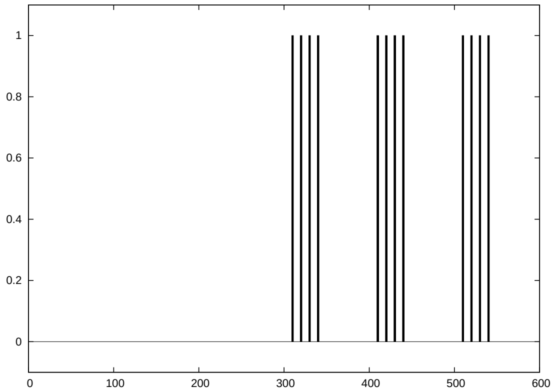


Impulse with phase shift of .125

These type of "phase shifted" waves can have more energy for a given peak or peak to peak amplitude, but also behave differently under wave shaping which may be their useful niche.

I'll leave pondering the "every nth" with phase shift to the reader and skip to the double sum. For 6 sine lookups/calculations you can have clusters of harmonics like fig. 23:

Figure 23

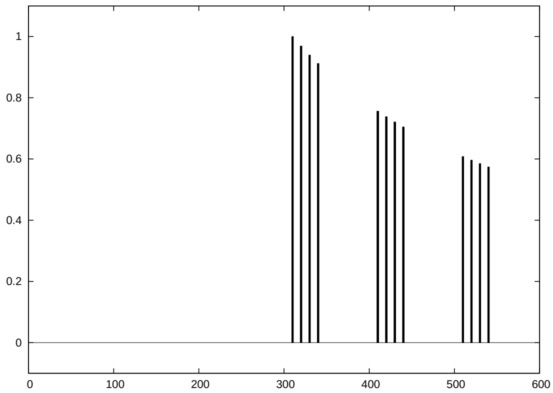


$a = 10,\ b = 200,\ c = -100$

Again, irrational values may require separate a, b and c phase accumulators. All this leads to temptation to modulate a, b and c (anybody brave enough to analyze letting n go non-integer? Or just code it up and give it a listen?)

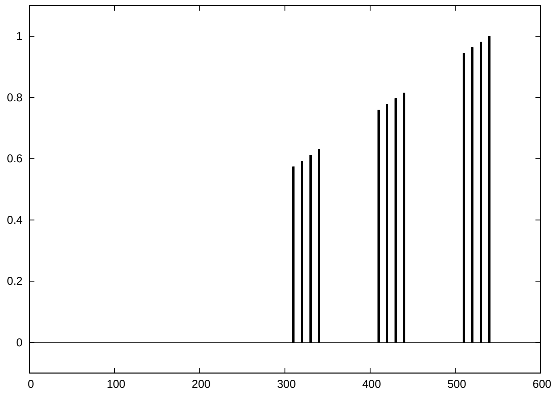
Integrating and differentiating these we get spectra like fig. 24 & 25

Figure 24



$a = 10,\ b = 200,\ c = -100$
integrated

Figure25



$a = 10,\ b = 200,\ c = -100$
differentiated

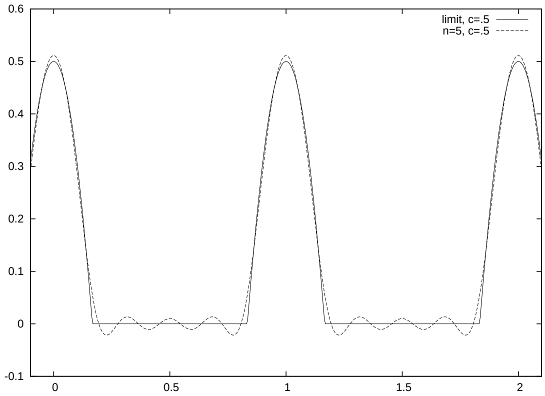
tripple & quadruple sums lead to forms analogous to eqn. 14 with 8 or 16 term in the numerator and three or four factors in the denominator. And, of course even with a double sum we can overlap bands, use other filters, wave shaping... $\Longrightarrow$?

The waveform with an exponential coefficient can be considered pre-filtered since the exponential will suppress or increase harmonics depending on if $|a| > 1$ or $|a| < 1$. In fact, a can be complex and will multiply the harmonic structure by a sinusoid (a cosine if you take the real part, sine for the imaginary, multiplying by an overall complex number can be used to select a phase at $f = 0$). This can also be easily done by a normal filter but using this method it can be done at the time of generation for frequencies that would be equivalent to a filter needing a fairly long delay and might be useful for minimal hardware (you are effectively regenerating the delayed wave instead of storing it, so there will be some differences if you are varying parameters).

The full equation is a bit unruly and not particularly computationally useful, so I won't show it here. To calculate use eqn. 15. Pre-calculate the powers of $a \times$ the phase selecting factor as real and imaginary parts, then for each time step (sample), calculate sine and cosines involving x and the real and imaginary parts of the numerator (call them N_R and N_I) and the real and imaginary parts of the denominator (D_R and D_I) and finally, calculating $\frac{N_R D_R - N_I D_i + i(N_R D_I + N_I D_R)}{D_R^2 + D_I^2}$ as the final sample value.

With waveform 16 we get away from waveforms where there is a nice formula for a finite sum known to me, but they may be of some interest anyway.

Figure 26



Clipping for $c = .5$

Clipping is, of course, simple to do electronically although softer clipping associated with tubes is often desired. Notice that clipping the top off of a wave instead of the bottom at the same level simply corresponds to subtracting the clipped waveform from the original, i.e. $\max(a, \sin(x)) = \sin(x) - \min(a, \sin(x))$

This waveform can be created by multiplying a 0-1 square pulse waveform $\times \sin(\omega x)$ so by using the finite approximation to waveform 11 plus a constant to produce an approximate 0-1 pulse waveform and multiplying by $\sin(\omega x)$ approximate hard clipping can be done.

Notes:

The basic tricks used to get the sums were for limits writing a series of (co)sines as complex exponentials and rearranging:

$$\sin(x) + sin(2x) + sin(3x) \ldots = \frac{1}{2i}\big(e^{ix} - e^{-ix} + e^{i2x} - e^{-i2x} + \ldots\big) = \frac{1}{2i}\big(e^{ix} + e^{i2x} + \ldots - e^{-ix} - e^{-i2x} - \ldots\big)$$

$$= \frac{1}{2i}\big(e^{ix} + (e^{ix})^2 + (e^{ix})^3 + \ldots - e^{-ix} - (e^{-ix})^2 + (e^{-ix})^3 - \ldots\big)$$

Recalling $x + x^2 + x^3 = \frac{x}{1-x}$ we have

$$\frac{1}{2i}\big(\frac{e^{ix}}{1-e^{ix}} - \frac{e^{-ix}}{1-e^{-ix}}\big) = \frac{1}{2i}\big(\frac{e^{ix} - 1 - e^{-ix} + 1}{(1-e^{ix})(1-e^{-ix})} = \frac{e^{ix} - e^{-ix}}{2 - e^{ix} - e^{-ix}}\big) = \frac{\sin(x)}{2 - 2cos(x)}$$

$$= \frac{\sin(x)}{4\sin^2(x/2)} = \frac{2\sin(x/2)\cos(x/2)}{4\sin^2(x/2)} = \frac{\cos(x/2)}{2\sin(x/2)}$$

and, for finite series with fixed frequency steps, multiplying by the sine or cosine of half of the step size over itself:

$$\cos(x) + cos(3x) + \ldots \cos\big((2n-1)x\big) = \frac{\sin(x)}{\sin(x)}\big(\cos(x) + cos(3x) + \ldots\big)$$

$$\frac{\csc(x)}{2}\big(\sin(x+x) + \sin(x-x) + \sin(x+3x) + \sin(x-3x) + \ldots + \sin(x+(2n-1)x) + \sin(x-(2n-1)x)\big)$$

$$= \frac{\sin(2nx)}{2\sin(x)}$$

and I usually do a quick graph for a few sets of parameters to confirm my algebra, assumptions and interpretation and/or proof by induction.

Equations in the table should list normalizations to unit peak-to-peak and unit power as well as to match the integrated waveforms. There is some inconstancy between using "sine" to mean a sinusoidal waveform and to mean $\sin(x)$ vs., say, $\cos(x)$ and similar sloppiness which needs to be cleaned up.
