NE 427 Nuclear Instrumentation Laboratory

Lecture 3: Statistics and Error Analysis

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Fall 2019

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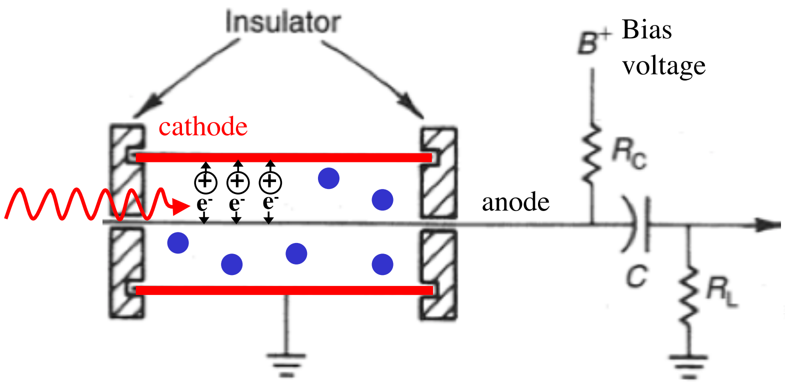
UNIVERSITY OF WISCONSIN–MADISON

Goals for today

- Discuss class logistics
  - Canvas: https://canvas.wisc.edu/courses/173270
  - Office hours changed to Wednesday 3:30 – 5 pm
  - Electronic lab notebook guidelines
- Geiger-Mueller counter operation
  - Detector dead time models
  - **Two-source method for measuring dead time**
- Counting statistics
  - Basic characterizations – **mean, standard deviation**
  - Probability distributions – binomial, **Poisson, Gaussian**
  - Derivation of Poisson statistics for nuclear events
  - Monte Carlo techniques
- Error estimates and statistical analyses in experiments
  - Definitions
  - $\boldsymbol{\chi^2}$ **tests**
  - **Error propagation**
  - **Counting time optimization**

Geiger-Mueller counter review



[Melissinos, Experiments in Modern Physics, Chapter 8]

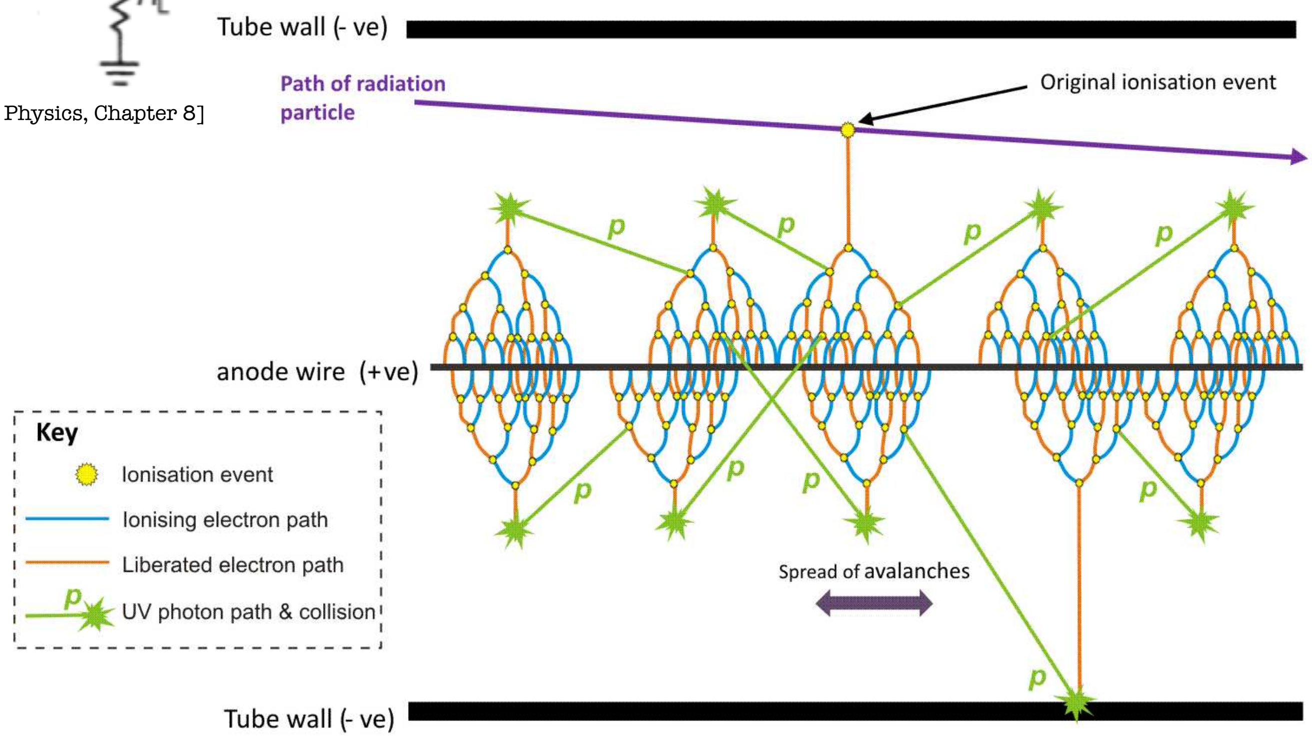
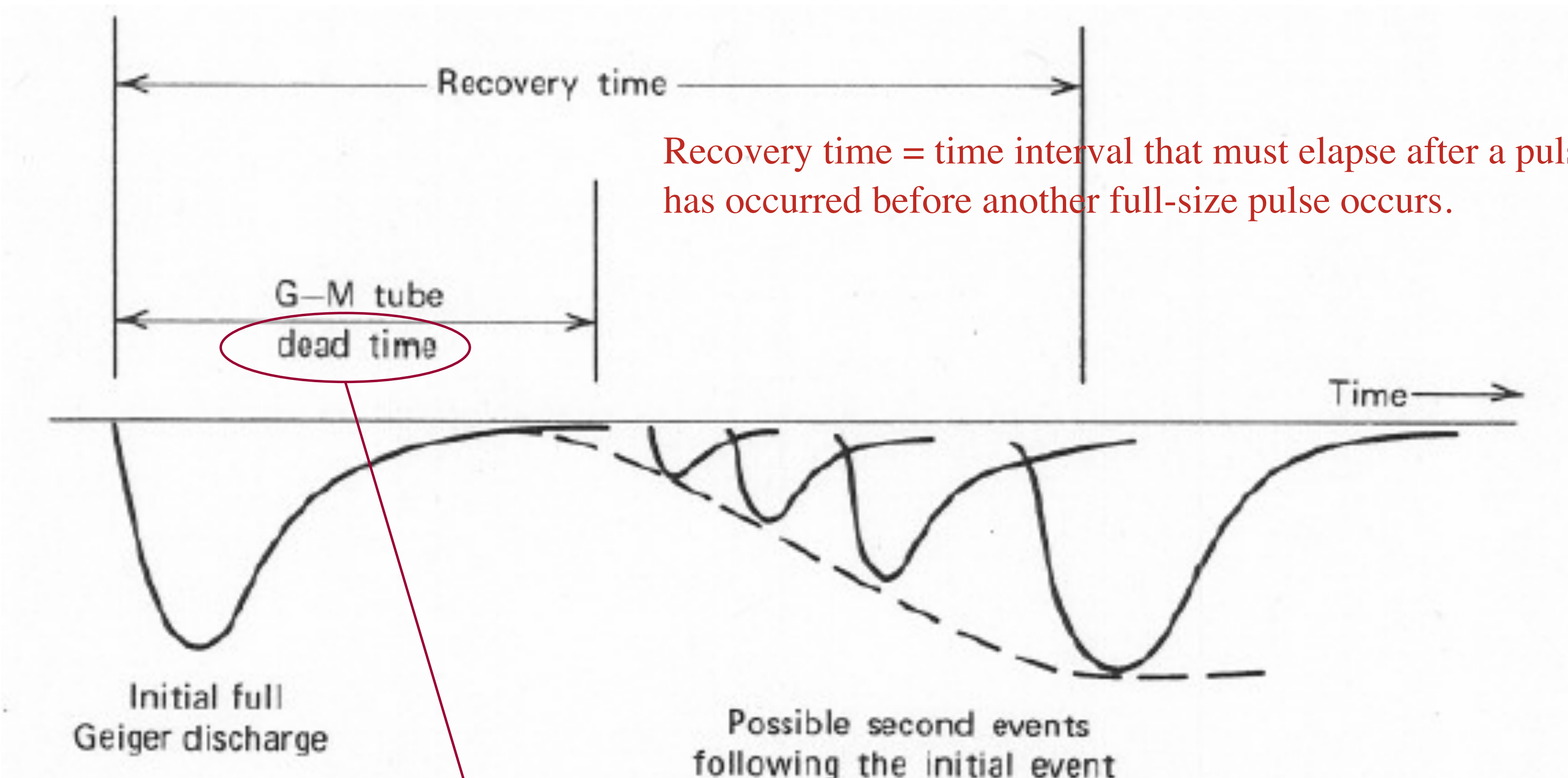


Figure from: https://en.wikipedia.org/wiki/Geiger–Müller_tube

Pulse time characteristics



Recovery time = time interval that must elapse after a pulse has occurred before another full-size pulse occurs.

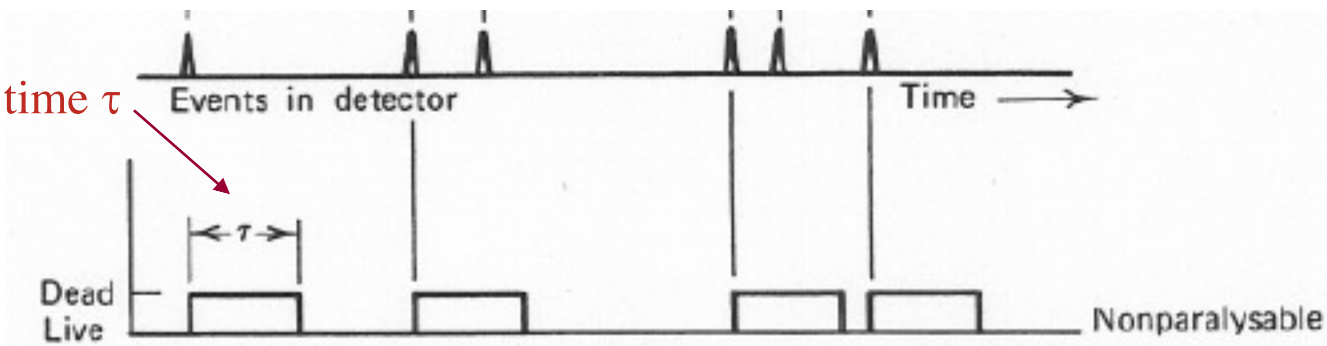
Dead time = the time interval required to detect two events as separate pulses

In a GM counter, main contribution to dead time is the slow movement of the positive charges away from the anode.

Idealized models of dead time behavior

Dead time can be caused by detector properties (*e.g.*, in GM counters) or signal processing system properties.

Dead time τ



6 real events

4 of 6 counted

Non-paralyzable: True events which occur during the dead period are lost

[See G.F. Knoll, Chapter 4, VII]

Idealized models of dead time behavior

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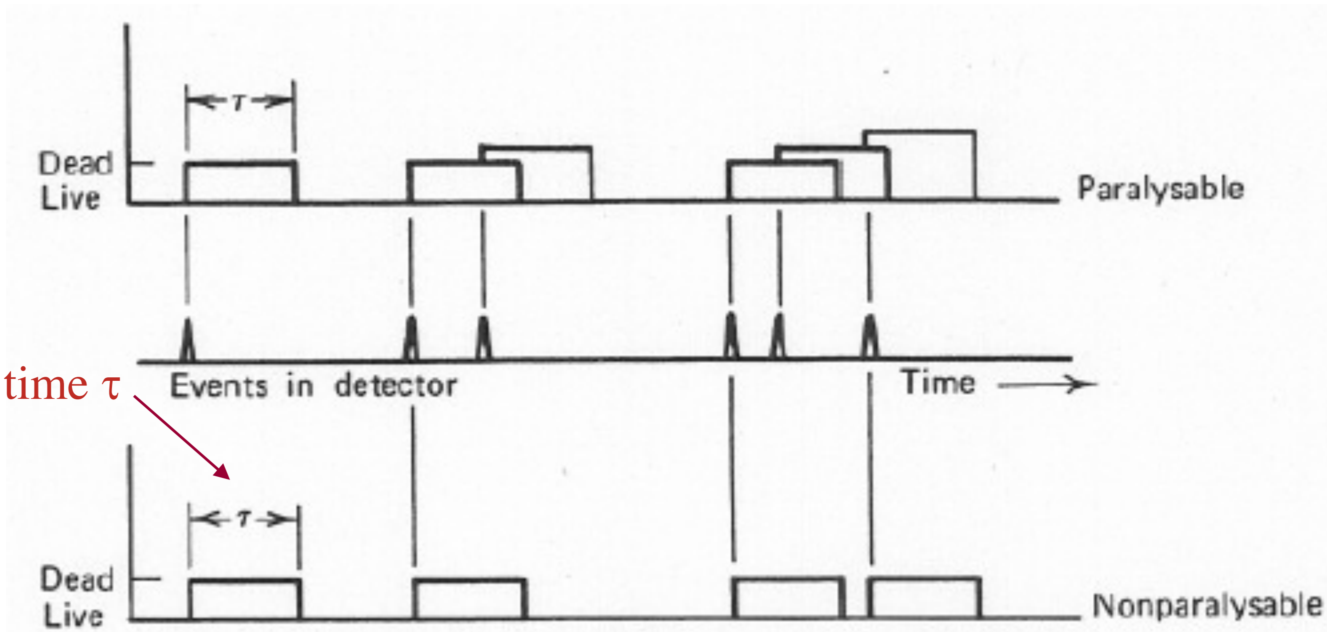
Dead time τ



3 of 6 counted

6 real events

4 of 6 counted

Non-paralyzable: True events which occur during the dead period are lost
Paralyzable: Dead time is extended by the new incoming pulse

Real systems operate in a mode between these two ideal explanations

[See G.F. Knoll, Chapter 4, VII]

Non-paralyzable model

[See G.F. Knoll, Chapter 4, VII]

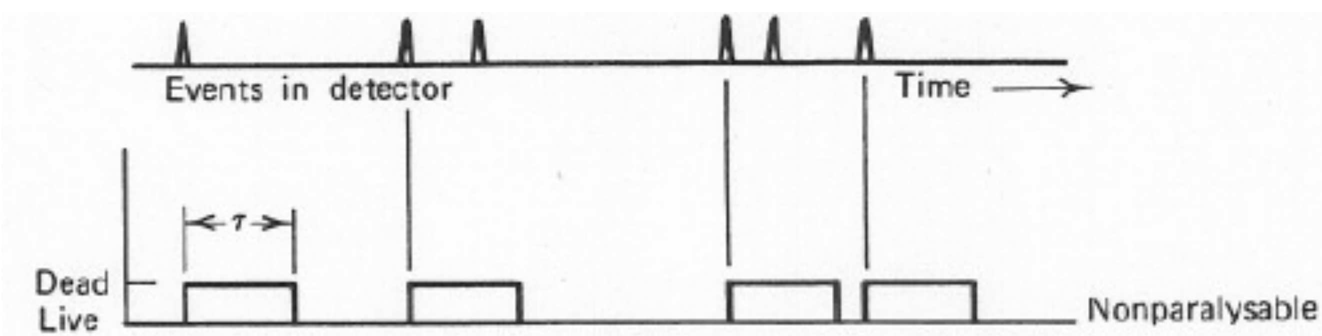
Definitions

- **n** = true interaction rate
- **m** = recorded interaction rate
- **τ** = system dead time

Fraction of time when the detector is dead is $m\tau$

Rate at which true events are lost = $nm\tau$ = $n-m$

$$\boxed{n = \frac{m}{1 - m\tau}}$$



Paralyzable model (1/2)

[See G.F. Knoll, Chapter 4, VII]

Definitions

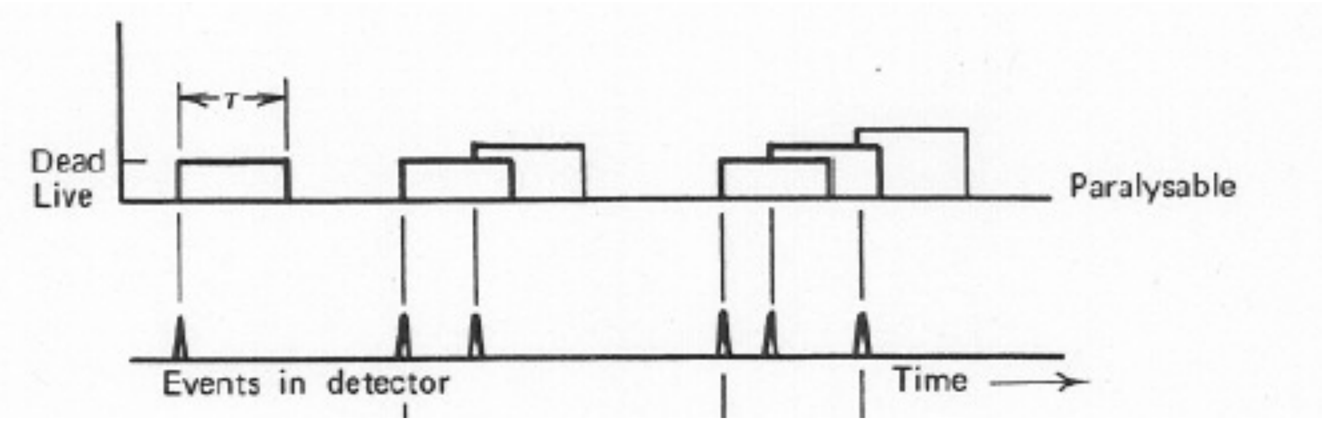
n = true interaction rate

m = recorded interaction rate = rate of occurrences of time intervals between true events which exceed τ

τ = system dead time

$$\boxed{m = ne^{-n\tau}}$$

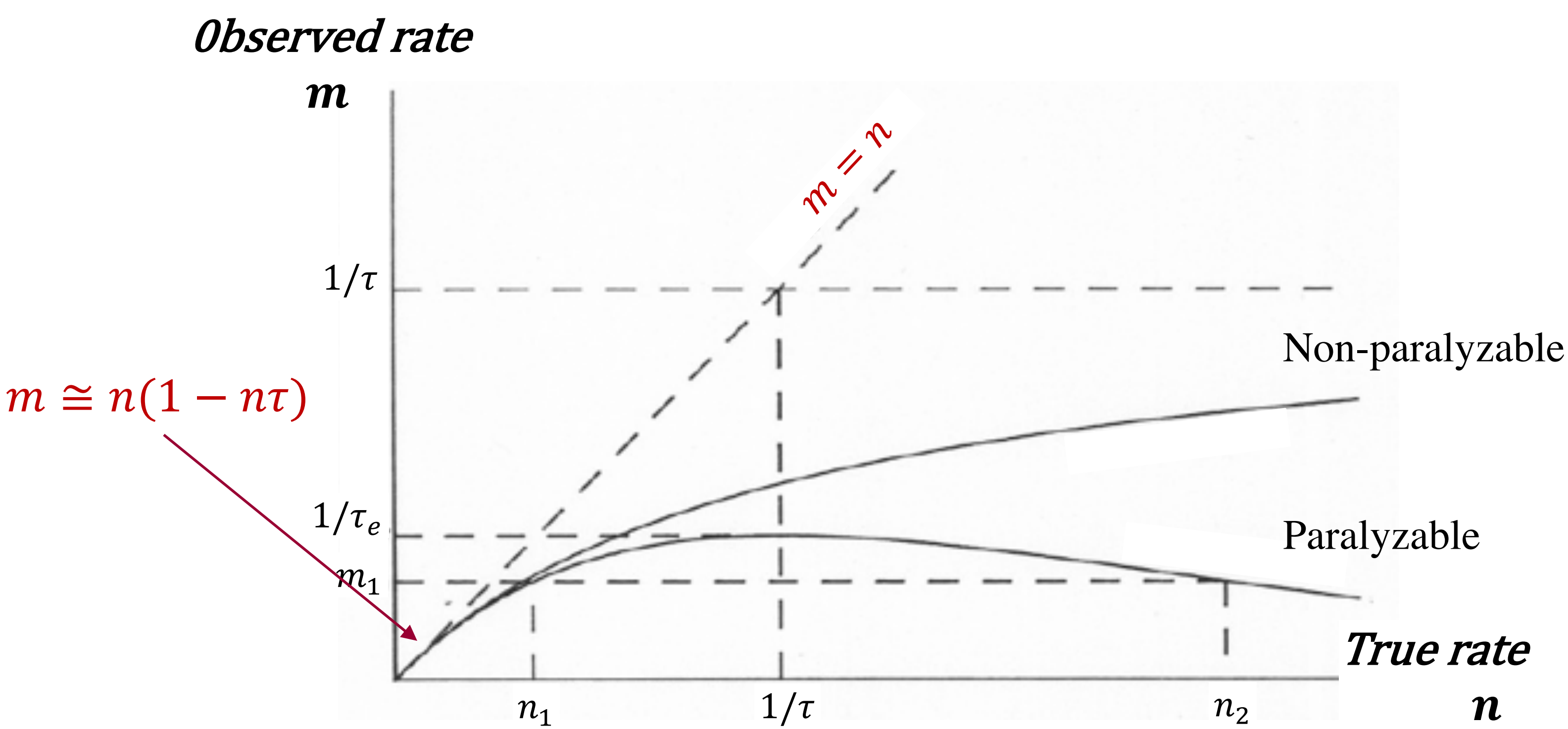
Equation must be solved iteratively given knowledge of m and τ



Impact of dead time behavior on count rate

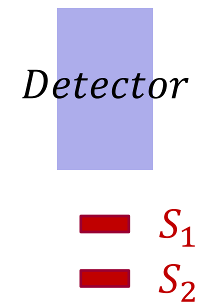
- Similar results at low counting rates: $m \cong n(1 - n\tau)$
- At high *n:*
  - Non-paralyzable model approaches *asymptote*
    - counter barely finishes one dead period before starting another
  - Paralyzable model yields *low count rate*

[See G.F. Knoll, Chapter 4, VII]



Measuring dead time with the "two-source method"

- Two sources S_1 and S_2 placed at distances D_1 and D_2 away from the counter, giving true rates n_1, n_2, and n_{12} (when combined)
- Due to dead time, the actual measured rates are $m_1 < n_1, m_2 < n_2$, and $m_{12} < n_{12}$
- n_b and m_b are the true and measured background rates



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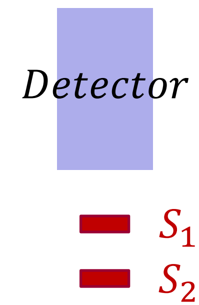

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$$n_{12} - n_b = (n_1 - n_b) + (n_2 - n_b)$$
$$n_{12} + n_b = n_1 + n_2$$

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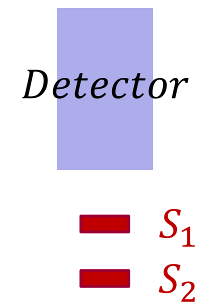
- Assume non-paralyzable model:

$$n_1 = n_1(1 - m_1\tau) \qquad n_b = n_b(1 - m_b\tau)$$
$$n_2 = n_2(1 - m_2\tau)$$
$$n_{12} = n_{12}(1 - m_{12}\tau)$$

- Solve for $n_1 + n_2$

$$\frac{m_{12}}{1-m_{12}\tau} + \frac{m_b}{1-m_b\tau} = \frac{m_1}{1-m_1\tau} + \frac{m_2}{1-m_2\tau}$$



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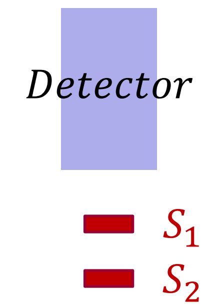

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- Solve for $n_1 + n_2$

$$\frac{m_{12}}{1-m_{12}\tau} + \frac{m_b}{1-m_b\tau} = \frac{m_1}{1-m_1\tau} + \frac{m_2}{1-m_2\tau}$$

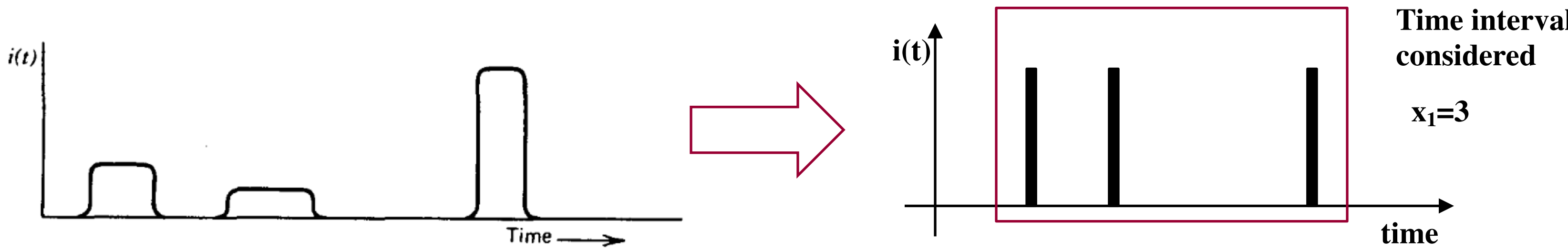
- Dead time τ after solving for quadratic equation:

$$\tau = \frac{X\left(1-\sqrt{1-Z}\right)}{Y}$$

$$X \equiv m_1 m_2 - m_b m_{12}$$
$$Y \equiv m_1 m_2 (m_{12} + m_b) - m_b m_{12}(m_1 + m_2)$$
$$Z \equiv \frac{Y(m_1 + m_2 - m_{12} - m_b)}{X^2}$$

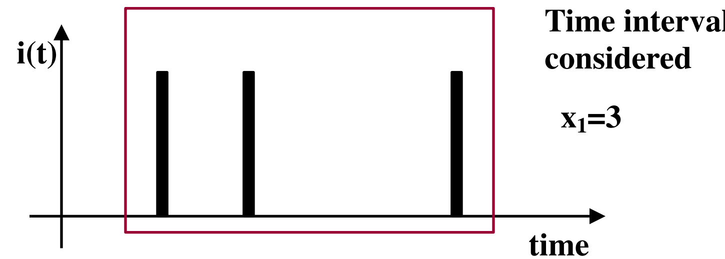
Counting statistics

Basic characterization of a signal



Need to apply statistical analysis on count occurrences to infer properties of a radiation process (*e.g.*, decay rate λ_d)

Basic characterization of a signal



Need to apply statistical analysis on count occurrences to infer properties of a radiation process (*e.g.*, decay rate λ_d)

Consider an ensemble of **N** independent measurements $\mathbf{x_i}$ (made during time T)

- **Sum** $\Sigma = \sum_{i=1}^{N} x_i$
- **Mean** $\bar{x}_e = \frac{\Sigma}{N}$ (Experimental mean; real mean would be obtained for $N \to \infty$)
- **Frequency distribution function** $F(x) = \frac{number\ of\ occurrences\ of\ value\ x}{number\ of\ measurements}$

[See G.F. Knoll, Chapter 3, I]

Basic characterization of a signal

Table 3.1 Example of Data Distribution Function

Data		Frequency Distribution Function	
8	14	$F(3) = 1/20$	$= 0.05$
5	8	$F(4)$	$= 0$
12	8	$F(5)$	$= 0.05$
10	3	$F(6)$	$= 0.10$
13	9	$F(7)$	$= 0.10$
7	12	$F(8)$	$= 0.20$
9	6	$F(9)$	$= 0.10$
10	10	$F(10)$	$= 0.15$
6	8	$F(11)$	$= 0.05$
11	7	$F(12)$	$= 0.10$
		$F(13)$	$= 0.05$
		$F(14)$	$= 0.05$
		$\sum_{x=0}^{\infty} F(x)$	$= 1.00$

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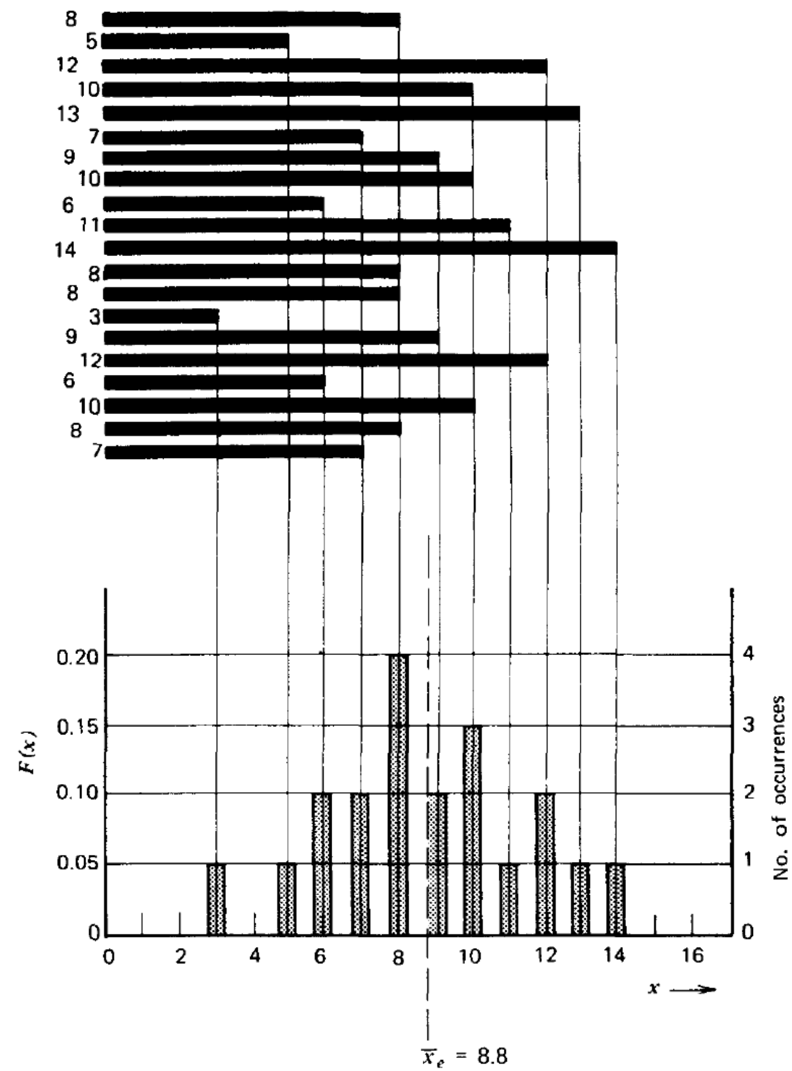
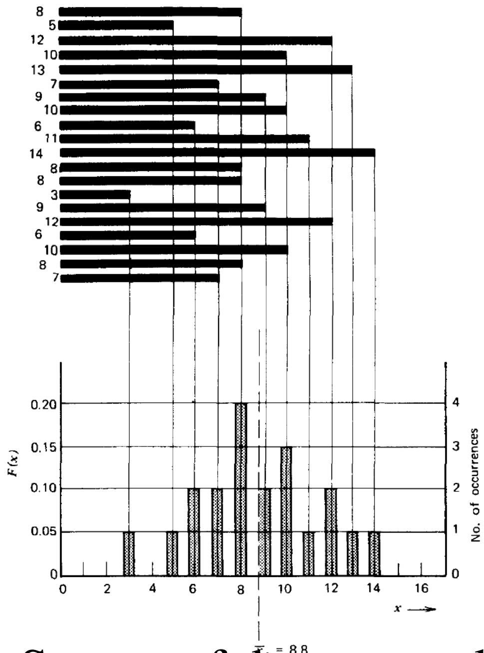
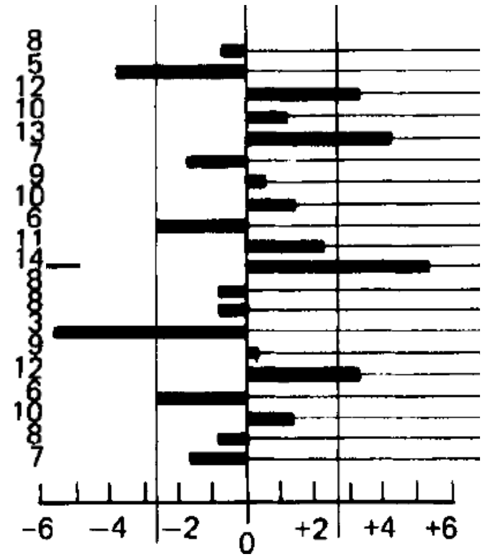
7	12	$F(8)$	$= 0.20$
9	6	$F(9)$	$= 0.10$
10	10	$F(10)$	$= 0.15$
6	8	$F(11)$	$= 0.05$
11	7	$F(12)$	$= 0.10$
		$F(13)$	$= 0.05$
		$F(14)$	$= 0.05$
		$\sum_{x=0}^{\infty} F(x)$	$= 1.00$



Basic characterization of a signal





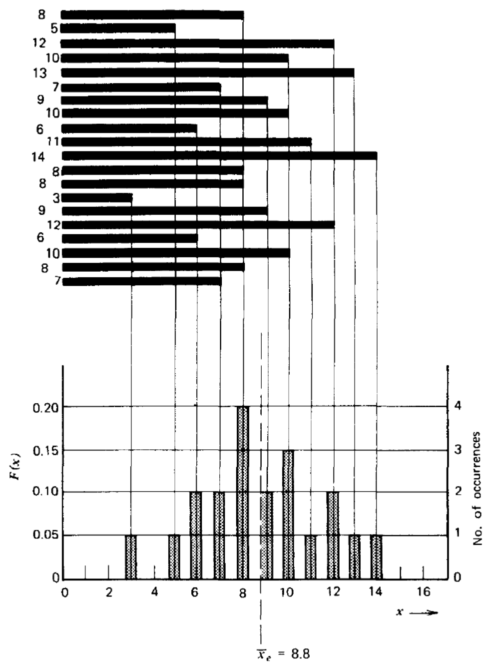
Residual, d

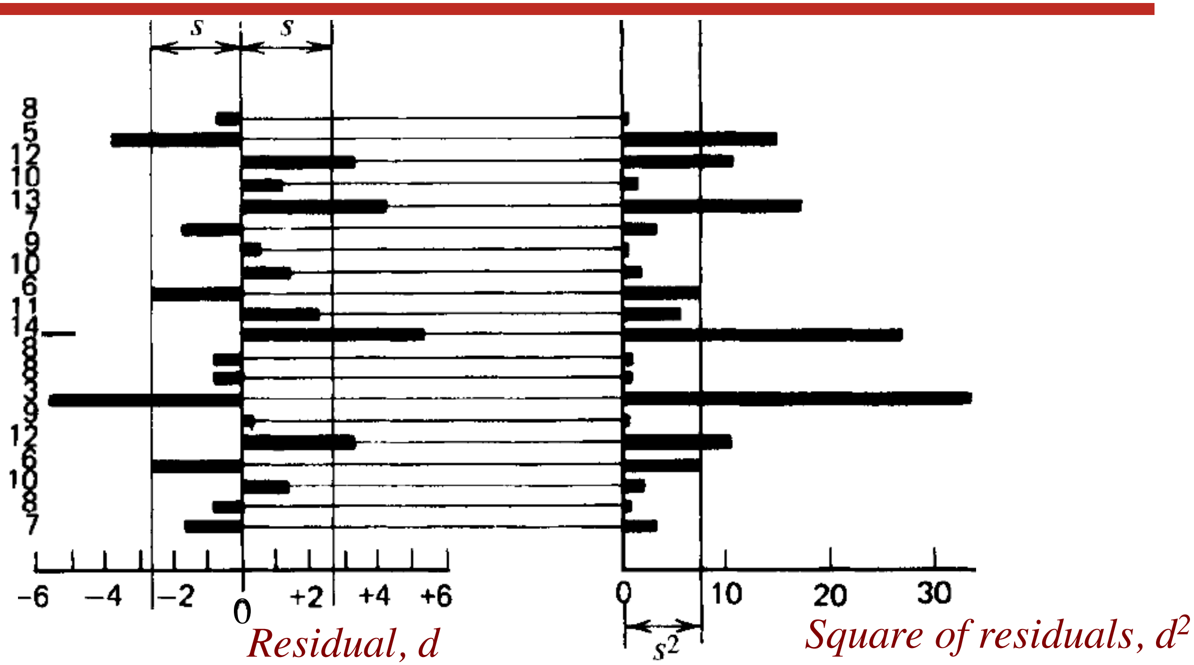
Scatter of data around mean is a measure for *randomness*

- **Residual** $d_i \equiv x_i - \boxed{\bar{x}_e}$
- **Deviation** $\epsilon_i \equiv x_i - \bar{x}$

[See G.F. Knoll, Chapter 3, I]

Basic characterization of a signal





Scatter of data around mean is a measure for *randomness*

- **Residual** $d_i \equiv x_i - \boxed{\bar{x}_e}$
- **Deviation** $\epsilon_i \equiv x_i - \bar{x}$
- **Variance** $s^2 \equiv \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \underline{\bar{x}_e})^2$ $\quad s^2 \equiv \bar{\epsilon^2} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \underline{\bar{x}})^2$
- *s* is the **standard deviation** and quantifies the amount of fluctuation in the data

[See G.F. Knoll, Chapter 3, I]

Models for frequency distribution functions

- Can we predict statistical behavior under certain assumptions?

  Consider a *binary process* with only a *true* or *false* result

Table 3.2 Examples of Binary Processes

Trial	Definition of Success	Probability of Success $\equiv p$
Tossing a coin	Heads	$1/2$
Rolling a die	A six	$1/6$
Observing a given radioactive nucleus for a time t	The nucleus decays during the observation	$1 - e^{-\lambda t}$

$$P(x) = \frac{\text{no. occurrences of } x}{\text{no. measurements } (= N)}$$

[See G.F. Knoll, Chapter 3, II]

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$$P(x) = \frac{\textit{no. occurrences of } x}{\textit{no. measurements } (= N)}$$

- Three common statistical models

Binomial Distribution

Generic, but calculation for nucleus decay difficult because of large n

Poisson Distribution

Simplification, under assumption of $p << 1$

Normal Distribution

Further simplification, under assumption of $\bar{x} >> 1$

Also known as *Gaussian Distribution*

[See G.F. Knoll, Chapter 3, II]

Binomial Distribution

Most general constant probability model for *binary processes*.

Define

n = number of trials of a specific event (coin toss, radioactive decay)
p = probability of success (head, decay event)
x = number of successes

- **Probability function**

$$P(x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

Joint probability of having success x times in a sequence

Number of permutations

$$\underbrace{pp\ldots p}_{x}\ \underbrace{(1\text{-}p)(1\text{-}p)\ldots(1\text{-}p)}_{n\text{-}x} = p^x\,(1\text{-}p)^{n\text{-}x}$$

[See G.F. Knoll, Chapter 3, II]

Binomial Distribution

Most general constant probability model for *binary processes*.

Define

n = number of trials of a specific event (coin toss, radioactive decay)
p = probability of success (head, decay event)
x = number of successes

- **Probability function** $$P(x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

$$\sum_{x=0}^{n} P(x) = 1$$ (normalized distribution)

- **Mean** $$\bar{x} = \sum_{x=0}^{n} x P(x) = pn$$

- **Variance** $$\sigma^2 \equiv \sum_{x=0}^{n} (x-\bar{x})^2 P(x) = np(1-p) = \bar{x}(1-p)$$

[See G.F. Knoll, Chapter 3, II]

Examples of binomial distributions

$$P(x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

$$\bar{x} = pn$$

$$\sigma^2 = \bar{x}(1-p)$$

Mean number of heads in 10 coin tosses?

Mean value of rolling 10 die?

[From Bevington and Robinson, Data Reduction and Error Analysis, Chapter 1]

Examples of binomial distributions

$$P(x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

$$\bar{x} = pn$$

$$\sigma^2 = \bar{x}(1-p)$$

Mean number of heads in 10 coin tosses?

$n = 10, p = \frac{1}{2}, np = 5$

Mean value of rolling 10 die?

$n = 10, p = \frac{1}{6}, np = 1.67$

[From Bevington and Robinson, Data Reduction and Error Analysis, Chapter 1]

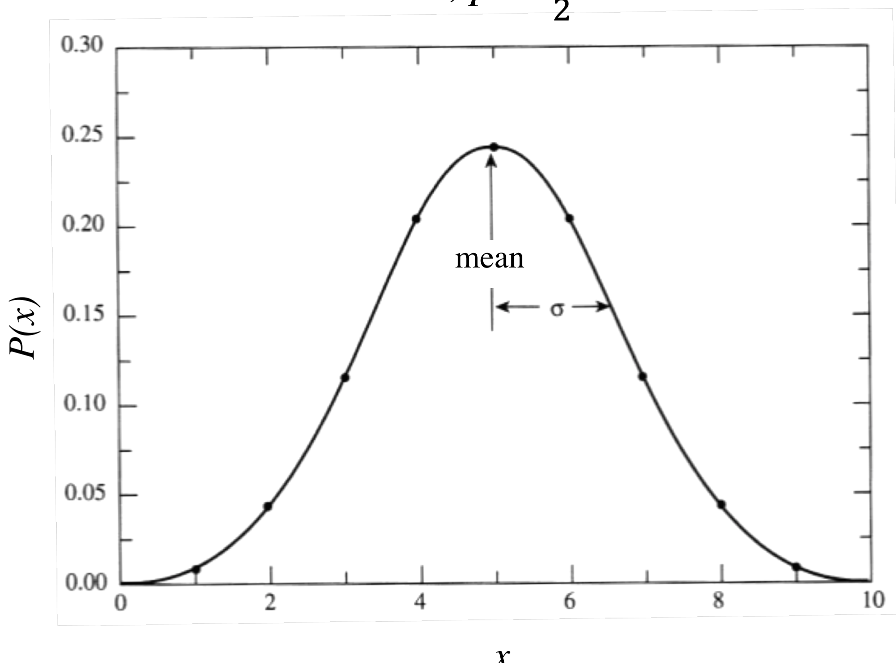
Examples of binomial distributions

$$P(x) = \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x}$$

$$\bar{x} = pn$$
$$\sigma^2 = \bar{x}(1-p)$$

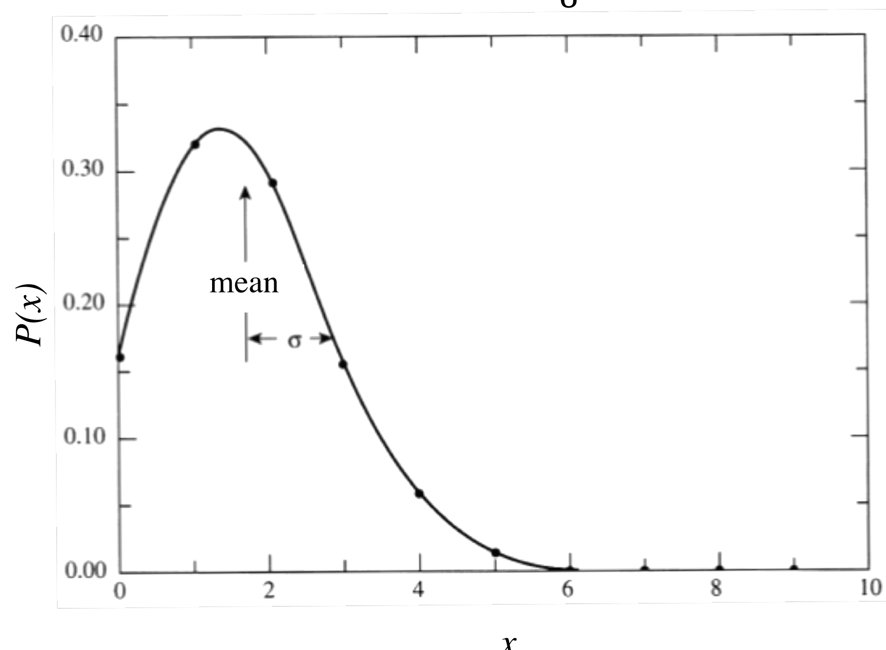
Probability distribution of 10 coin tosses:

$n = 10, p = \frac{1}{2}$



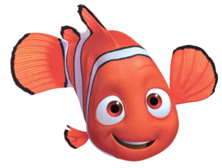
Probability distribution of rolling 10 die:

$n = 10, p = \frac{1}{6}$



The binomial distribution is computationally cumbersome to use when n is large

[From Bevington and Robinson, Data Reduction and Error Analysis, Chapter 1]

Poisson  Distribution

The binomial frequency distribution approaches the Poisson distribution for a finitely large *n* and *p << 1*

Radioactive decay with a large number of atoms each with a small probability of transition is a relevant example.

- **Probability function** $$P(x) = \frac{(pn)^x e^{-pn}}{x!} = \frac{(\bar{x})^x e^{-\bar{x}}}{x!}$$

  n = number of trials
  p = probability of success
  x = number of successes

  $$\sum_{x=0}^{n} P(x) = 1$$

- **Mean** $$\bar{x} = \sum_{x=0}^{n} xP(x) = pn$$

- **Variance** $$\sigma^2 = \sum_{x=0}^{n} (x - \bar{x})^2 P(x) = pn = \bar{x}$$ $\Rightarrow$ Distribution is characterized by mean value (don't need to know *p* and *n* separately).

[See G.F. Knoll, Chapter 3, II]

Example of Poisson distribution: Radioactive Decay

- Assume that the decay of a nucleus is purely random and that the probability for decay in a time small interval (Δt) scales by some constant λ with Δt:

$$P_d = \lambda \, \Delta t$$

- If there is a sample of N nuclei and their decay processes are independent of one another, what is the probability distribution for obtaining n decays in time t, $P(n,t)$?

Example of Poisson distribution: Radioactive Decay

We start with considering the probability of having no decays in time *t+dt*, *P(0,t+dt):*

$$P(0,t+dt) = \underbrace{P(0,t)}_{\text{Probability of having no decays in time t}}\underbrace{(1-P(1,dt))}_{\text{Probability of having no decays in time dt = P(0,dt)}} \qquad [1]$$

Given a very short *dt*, we can assume that the possible outcomes are 0 or 1 decay, so that

$$P(1,dt) = \lambda dt, \qquad [2]$$

$$\text{or } P(0,dt) = 1 - \lambda dt \qquad [3]$$

Substituting this into equation [1], we get

$$\frac{P(0,t+dt) - P(0,t)}{dt} = -P(0,t)\lambda$$

$$-\frac{dP(0,t)}{dt} = P(0,t)\lambda$$

Note that when t = 0, P(0,0) = 1

$$\boxed{P(0,t) = e^{-\lambda t}} \qquad [4]$$

Example of Poisson distribution: Radioactive Decay

We can write $P(n, d + dt)$ similarly to [1], as a sum of two "either probabilities":

$$P(n, t + dt) = \underbrace{P(n, t)P(0, dt)}_{\text{Probability of having n decays in time t and no decays in time dt}} + \underbrace{P(n - 1, t)P(1, dt)}_{\text{Probability of having n-1 decays in time t and 1 decay in time dt}} \qquad [5]$$

Plugging in [2] and [3]:

$$P(n, t + dt) = P(n, t)(1 - \lambda dt) + P(n - 1, t)\lambda dt$$

$$\frac{P(n, t + dt) - P(n, t)}{dt} = \frac{dP(n, t)}{dt} = -\lambda P(n, t) + \lambda P(n - 1, t)$$

We multiply both sides by $e^{\lambda t}$ and rearrange the equation ahead of integration:

$$e^{\lambda t} \frac{dP(n,t)}{dt} = e^{\lambda t}(-\lambda P(n, t) + \lambda P(n - 1, t))$$

$$\underbrace{e^{\lambda t} \frac{dP(n, t)}{dt} + e^{\lambda t}\lambda P(n, t)}_{\frac{d\left(e^{\lambda t}P(n, t)\right)}{dt}} = e^{\lambda t}\lambda P(n - 1, t) \qquad [6]$$

Example of Poisson distribution: Radioactive Decay

Integrating both sides of the equation:

$$P(n,t) = e^{-\lambda t}\int_0^t e^{\lambda t}\lambda P(n-1,t)dt \qquad [6]$$

We can infer the solution to this recursive relationship by considering the first couple of cases:

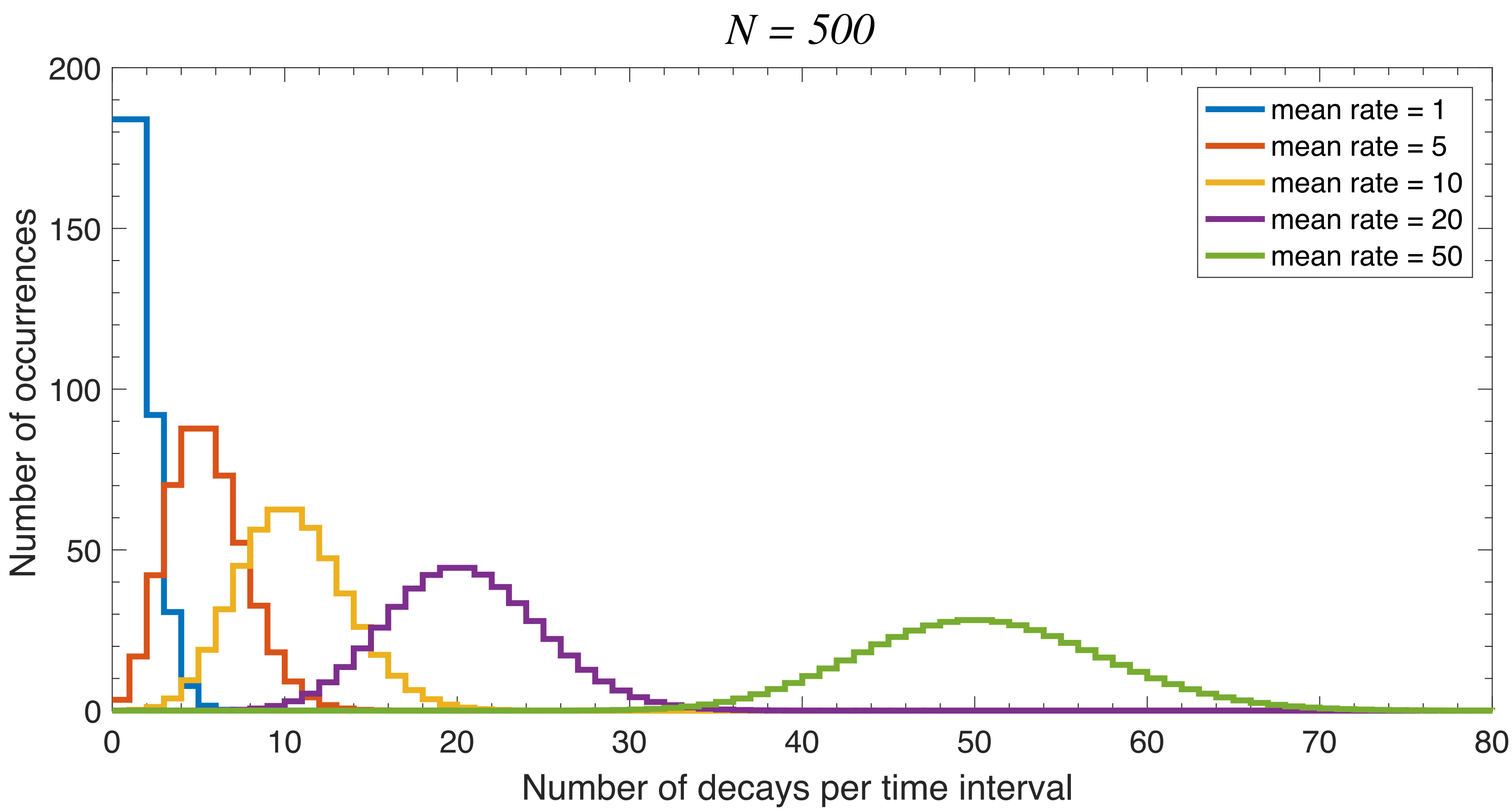
$$P(1,t) = e^{-\lambda t}\int_0^t e^{\lambda t}\lambda P(0,t)dt = e^{-\lambda t}\int_0^t e^{\lambda t}\lambda e^{-\lambda t}dt = \mathrm{e}^{-\lambda t}\lambda t$$

$$P(2,t) = e^{-\lambda t}\int_0^t e^{\lambda t}\lambda P(1,t)dt = e^{-\lambda t}\int_0^t e^{\lambda t}\lambda\left(\mathrm{e}^{-\lambda t}\lambda t\right)dt = e^{-\lambda t}\int_0^t \lambda^2 t dt$$

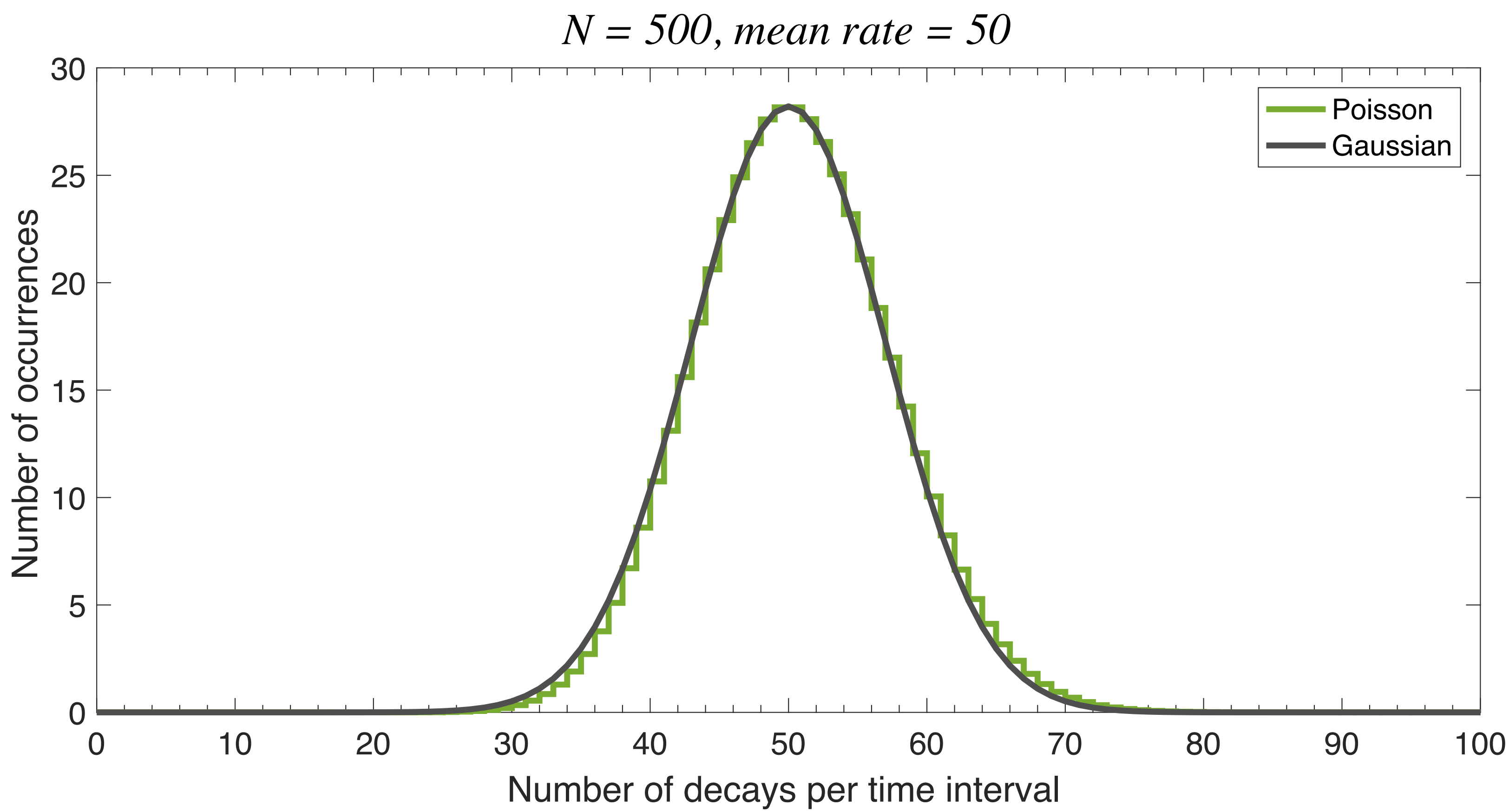
$$P(2,t) = \frac{e^{-\lambda t}(\lambda t)^2}{2}$$

The above can be generalized to: $\boxed{P(n,t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}}$ [7]

Example of Poisson distribution: Radioactive Decay



Large mean → convergence to Gaussian



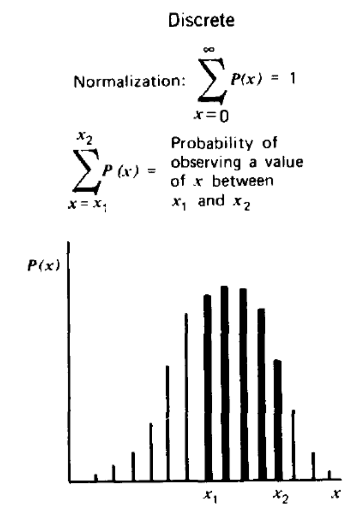
Gaussian Distribution

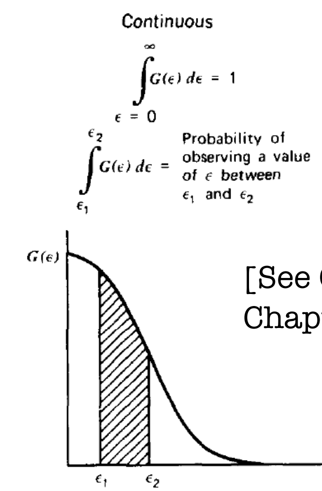
The Poisson distribution further simplifies to the Gaussian distribution when the mean is large ($np >> 1$)

- **Probability function** $$P(x) = \frac{1}{\sqrt{2\pi\bar{x}}} e^{-\frac{(x-\bar{x})^2}{2\bar{x}}}$$
  $$\sum_{x=0}^{n} P(x) = 1$$
- **Mean** $$\bar{x} = \sum_{x=0}^{n} xP(x) = pn$$
- **Variance** $$\sigma^2 = \sum_{x=0}^{n} (x-\bar{x})^2 P(x) = pn = \bar{x}$$

⇨ don't need to know p and n separately

- We can use discrete or continuous versions

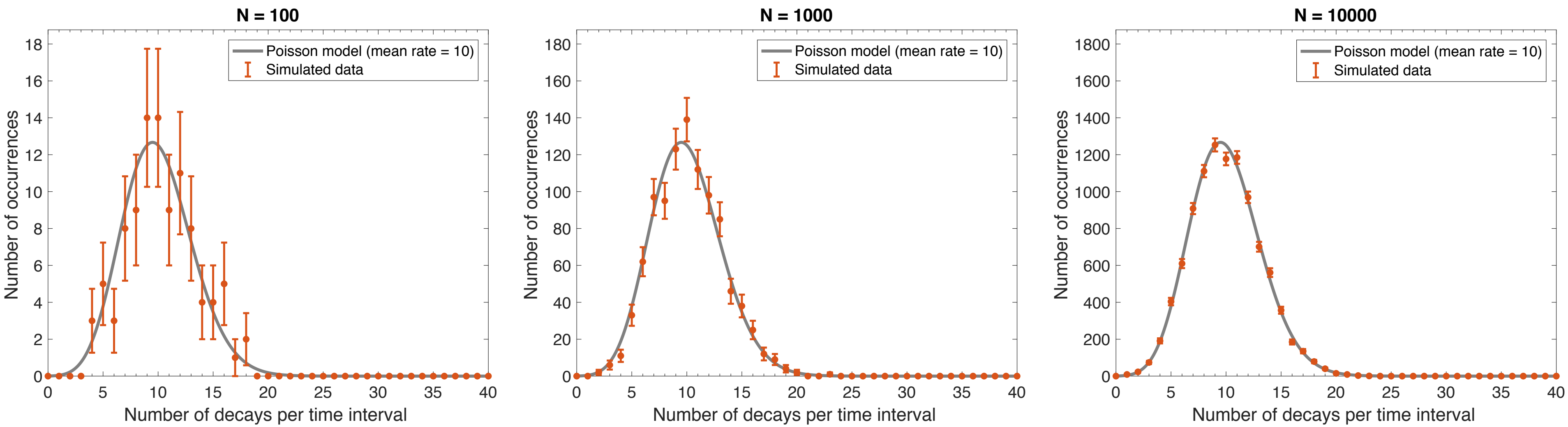




[See G.F. Knoll, Chapter 3, II]

Simulating probability distributions: Monte Carlo techniques

- Requires reliable set of random numbers; each generates a trial



- Large number of trials needed for good convergence to model
- The uncertainties (σ) are shown as error bars and inferred using

$$\boxed{\sigma = \sqrt{number\ of\ occurrences}}$$

Applications to experiments

Uncertainties in experimental data

- So far, we have covered *statistical uncertainties,* which arise from statistical fluctuations in the recording of finite number of events over finite intervals of time

$$\sigma^2 \cong \bar{x}$$

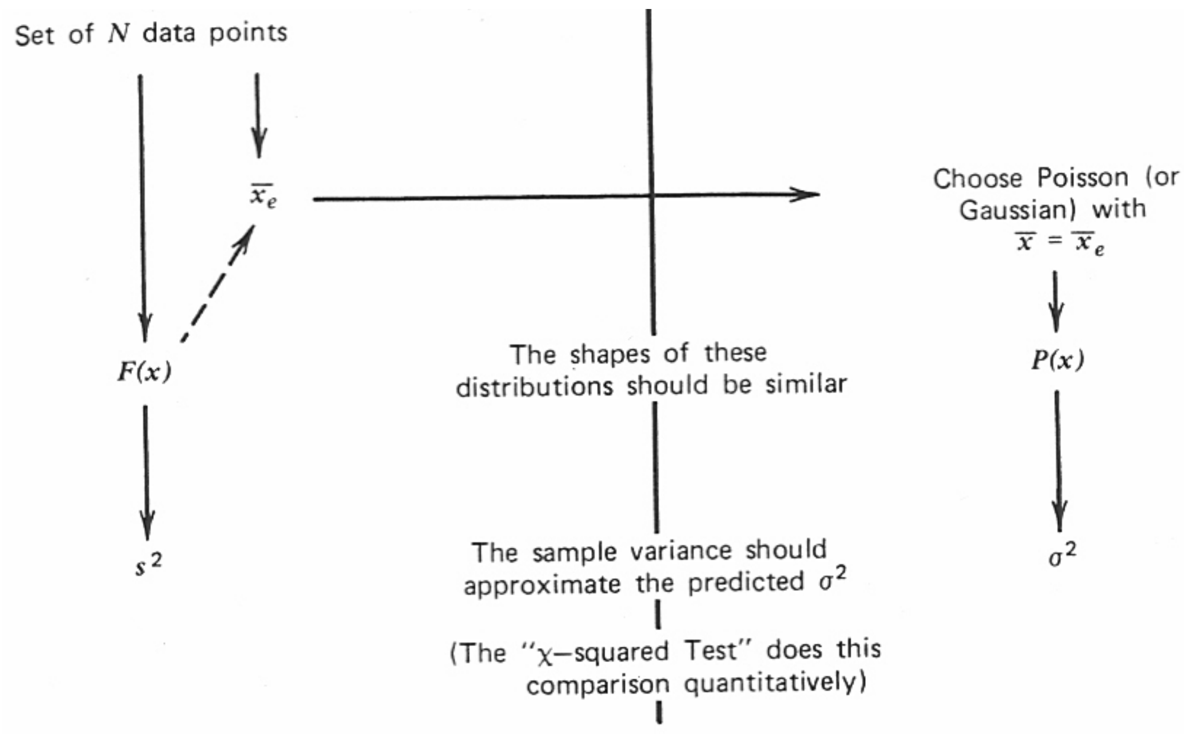
- *Instrumental uncertainties* come from lack of precision in the measuring instruments

$$\sigma^2 = s^2 = \frac{1}{N-1}\sum(x_i - \bar{x})^2$$

  - Can be estimated from repeated measurements of a known quantity
  - Or based on manufacturer's specifications
  - Typically, <u>precision cannot be better than half the last digit on the display</u>

Testing match by applying a statistical model

Experimental data	Statistical model



1. Assume a specific distribution (Poisson, Gaussian) based on measurement number and probability of occurrence
2. Apply distribution by assuming that the experimental mean equals the distribution mean
3. Compare measured variance (s^2)and predicted variance (σ^2) to determine if distribution is valid for the actual data set (χ^2 test)

[See G.F. Knoll, Chapter 3, III]

Chi-squared test

Define

$$\chi^2 = \frac{1}{\bar{x}_e}\sum_{i=1}^{N}(x_i - \bar{x}_e)^2 \qquad\qquad (x = \bar{x}_e)$$

Recall that variance is: $$s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2$$

$$\Rightarrow \quad \boxed{\chi^2 = \frac{(N-1)s^2}{\bar{x}_e} \cong \frac{(N-1)s^2}{\sigma^2}}$$

For Gaussian or Poisson distributions:
$\sigma^2 = \bar{x} = \bar{x}_e$

Chi-squared test

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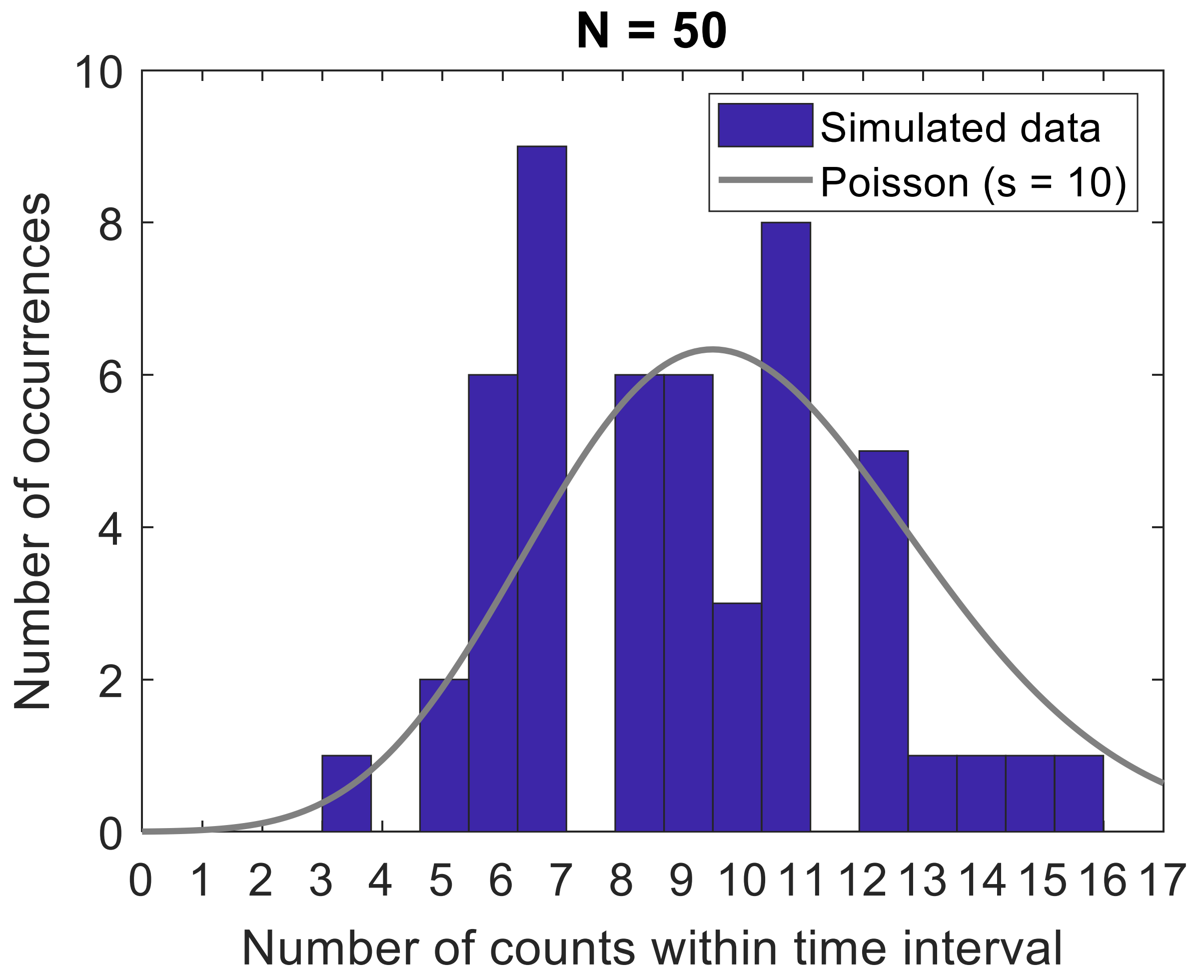
For Gaussian or Poisson distributions:
$\sigma^2 = \bar{x} = \bar{x}_e$

The degree to which the ratio $\frac{s^2}{\bar{x}_e}$ deviates from unity is a measure for how the measured variance differs from the predicted variance

⇒ How χ^2 differs from N-1 is a measure of the departure of the data from predictions of the distribution.

Example

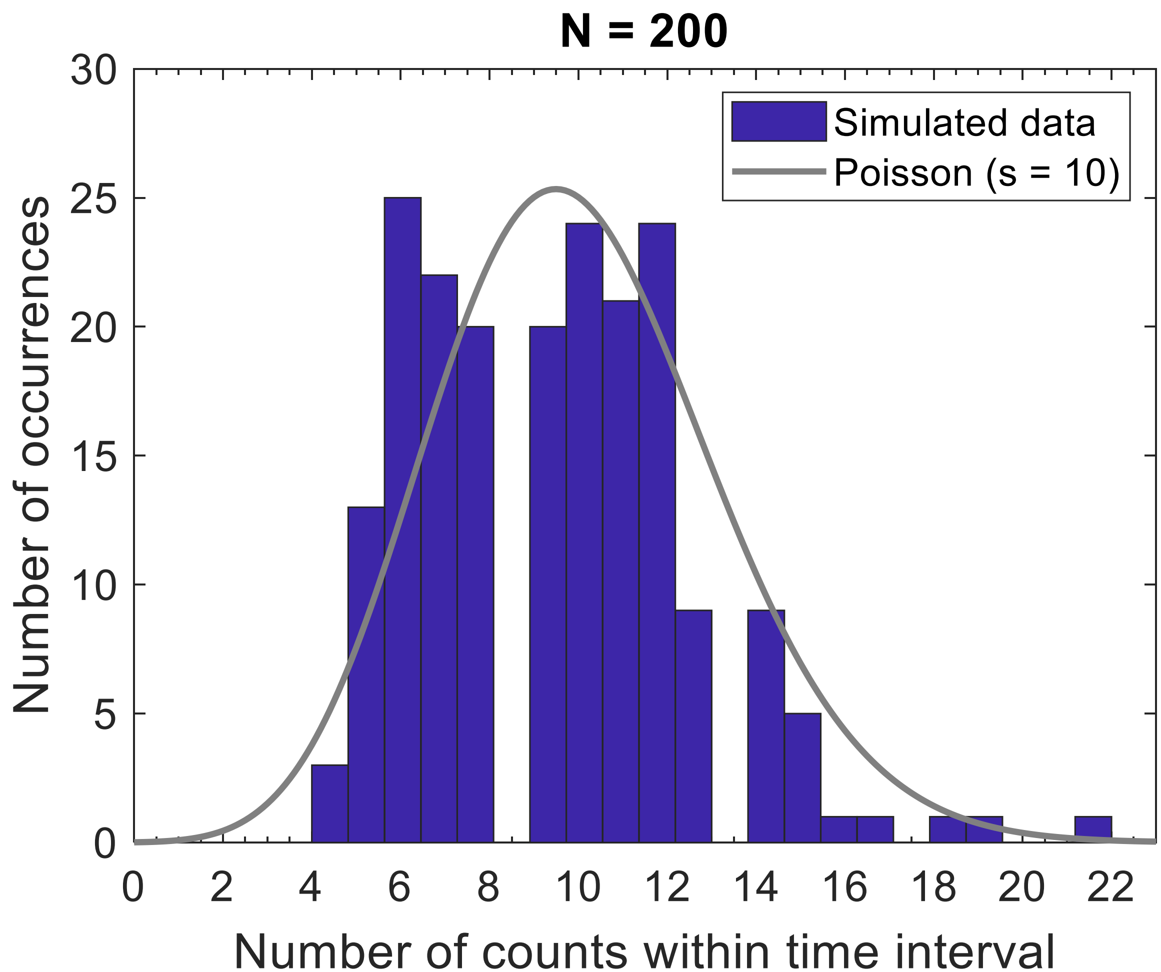


$$\bar{x}_e = 9$$

$$\frac{s^2}{\bar{x}_e} \sim 1.1$$

$$\chi^2 \sim 54.4$$

Example



$$\bar{x}_e = 9.45$$

$$\frac{s^2}{\bar{x}_e} \sim 1.05$$

$$\chi^2 \sim 210.6$$

Chi-squared distribution table

TABLE 2.3 Probability Table for χ^2 Criterion†

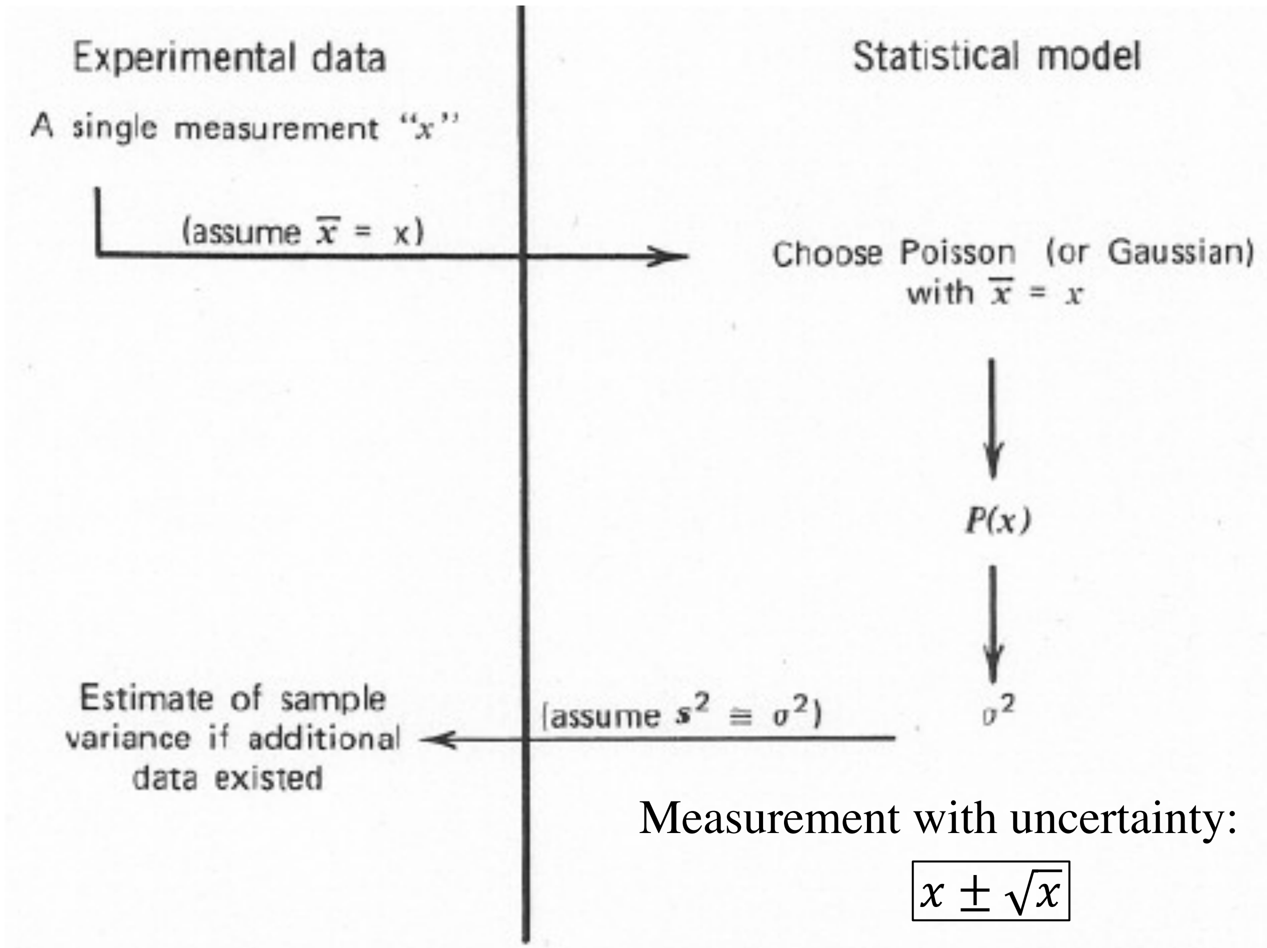
Degrees of freedom‡ $(N-1)$	Probability						
	0.99	0.95	0.90	0.50	0.10	0.05	0.01
2	0.020	0.103	0.211	1.386	4.605	5.991	9.210
3	0.115	0.352	0.584	2.366	6.251	7.815	11.345
4	0.297	0.711	1.064	3.357	7.779	9.488	13.277
5	0.554	1.145	1.610	4.351	9.236	11.070	15.086
6	0.872	1.635	2.204	5.348	10.645	12.592	16.812
7	1.239	2.167	2.833	6.346	12.017	14.067	18.475
8	1.646	2.733	3.490	7.344	13.362	15.507	20.090
9	2.088	3.325	4.168	8.343	14.684	16.919	21.666
10	2.558	3.940	4.865	9.342	15.987	18.307	23.209
11	3.053	4.575	5.578	10.341	17.275	19.675	24.725
12	2.571	5.226	6.304	11.340	18.549	21.026	26.217
13	4.107	5.892	7.042	12.340	19.812	22.363	27.688
14	4.660	6.571	7.790	13.339	21.064	23.685	29.141
15	5.229	7.261	8.547	14.339	22.307	24.996	30.578
16	5.812	7.962	9.312	15.338	23.542	26.296	32.000
17	6.408	8.672	10.085	16.338	24.769	27.587	33.409
18	7.015	9.390	10.865	17.338	25.989	28.869	34.805
19	7.633	10.117	11.651	18.338	27.204	30.144	36.191
20	8.260	10.851	12.443	19.337	28.412	31.410	37.566
21	8.897	11.591	13.240	20.337	29.615	32.671	38.932
22	9.542	12.338	14.041	21.337	30.813	33.924	40.289
23	10.196	13.091	14.848	22.337	32.007	35.172	41.638
24	10.856	13.848	15.659	23.337	33.196	36.415	42.980
25	11.534	14.611	16.473	24.337	34.382	37.382	44.314
26	12.198	15.379	17.292	25.336	35.563	38.885	45.642
27	12.879	16.151	18.114	26.336	36.741	40.113	46.963
28	13.565	16.928	18.939	27.336	37.916	41.337	48.278
29	14.256	17.708	19.768	28.336	39.087	42.557	49.588

Probability that a random sample from a true Poisson distribution would have a larger value of χ^2:
High probability = small fluctuations
Low probability = large fluctuations
Probability of 0.5 ~ perfect Poisson fit

χ^2 values for a given probability and number of measurements N

Precision Estimation and Sample Prediction

Single measurement of a random event (*e.g.*, count rate from radioactive decay)



Measurement with uncertainty:

$$\boxed{x \pm \sqrt{x}}$$

Error Propagation

Gaussian Error propagation law

Let $u(x,y,z)$ be a statistical quantity with x, y, and z as statistically independent variables. Each variable has the measurement uncertainty σ_i.

Then the combined uncertainty of σ_u of $u(x,y,z)$ is

$$\sigma_u^2 = \left(\frac{\partial u}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial u}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial u}{\partial z}\right)^2 \sigma_z^2 + \ldots$$

$$u = u(x,y,z,\ldots)$$

[See G.F. Knoll, Chapter 3, IV]

Error Propagation: Sums and Differences

Simple sum or difference with a constant:

$$u = x \pm a$$

$$\frac{\partial u}{\partial x} = 1$$

$$\boxed{\sigma_u = \sigma_x}$$

Error Propagation: Sum and Difference

Simple sum or difference with a constant:

$$u = x \pm a$$

$$\frac{\partial u}{\partial x} = 1$$

$$\boxed{\sigma_u = \sigma_x}$$

Involving two or more independent variables:

$$u = x \pm y$$

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = \pm 1$$

$$\sigma_u^2 = (1)^2 \sigma_x^2 + (\pm 1)^2 \sigma_y^2$$

$$\boxed{\sigma_u = \sqrt{\sigma_x^2 + \sigma_y^2}}$$

Error Propagation: Sums and Differences

Simple sum or difference with a constant:

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$$\sigma_u^2 = (1)^2\sigma_x^2 + (\pm 1)^2\sigma_y^2$$

$$\boxed{\sigma_u = \sqrt{\sigma_x^2 + \sigma_y^2}}$$

Weighted:

$$u = ax \pm by$$

$$\frac{\partial u}{\partial x} = a, \frac{\partial u}{\partial y} = \pm b$$

$$\boxed{\sigma_u^2 = a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_{xy}^2}$$

0 if x and y are uncorrelated

Example: Background subtraction

N = number of radiation events from a source, with uncertainty σ_N
B = number of background radiation counts, with uncertainty σ_B
C = number of counts collected by the detector, with uncertainty σ_C

What is the uncertainty in estimating the true number of radiation events?

Example: Background subtraction

N = number of radiation events from a source, with uncertainty σ_N
B = number of background radiation counts, with uncertainty σ_B
C = number of counts collected by the detector, with uncertainty σ_C

What is the uncertainty in estimating the true number of radiation events?

$$C = N + B$$

$$N = C - B$$

$$\frac{\partial N}{\partial C} = 1, \frac{\partial N}{\partial B} = 1$$

$$\sigma_N = \sqrt{\sigma_C^2 + \sigma_B^2}$$

For C = 1000 counts per second,
B = 10 counts per second:

$N = 900 \pm 32$ counts per second

Error Propagation: Multiplication and Division

$$u = axy$$

$$\frac{\partial u}{\partial x} = ay, \frac{\partial u}{\partial y} = ax$$

$$\sigma_u^2 = (ay\sigma_x)^2 + (ax\sigma_y)^2 + 2a^2xy\sigma_{xy}^2$$

$$\boxed{\frac{\sigma_u^2}{u^2} = \frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2} + 2\frac{\sigma_{xy}^2}{xy}}$$

$$u = ax^b$$

$$\frac{\partial u}{\partial x} = abx^{b-1} = \frac{bu}{x}$$

$$\boxed{\frac{\sigma_u}{u} = b\frac{\sigma_x}{x}}$$

Mean value of multiple independent counts

Assume we record N repeated counts from a single source for equal counting times

$$\sum = x_1 + x_2 + \ldots + x_N$$

$$\sigma_{\Sigma}^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \ldots \sigma_{x_N}^2$$

For Poisson or Gaussian distributions

$$\sigma i = \sqrt{x_i}$$

$$\sigma_{\Sigma}^2 = x_1 + x_2 + \ldots + x_N = \sum$$

$$\boxed{\sigma_{\Sigma} = \sqrt{\sum}}$$

Uncertainty is the same as a single extended count

Mean value of multiple independent counts

$$\overline{x} = \frac{\Sigma}{N}$$

Uncertainty propagation

$$\sigma_{\overline{x}} = \frac{\sigma_\Sigma}{N} = \frac{\sqrt{\Sigma}}{N} = \frac{\sqrt{N\overline{x}}}{N}$$

$$\boxed{\sigma_{\overline{x}} = \sqrt{\frac{\overline{x}}{N}}}$$

Standard error of the mean

- The mean value based on N independent counts will have an expected error which is smaller by a factor of $\sqrt{N}$ compared to any single measurement on which the mean is based
- For a given measurement, how much longer should I count to improve the statistical precision by a factor of 2?

Counting time optimization

Assume

S = source count rate
B = background count rate
T_{S+B} = time to count source + background (resulting in N_1 total counts)
T_B = time to count background (resulting N_2 total counts)

$$S = \frac{N_1}{T_{S+B}} - \frac{N_2}{T_B} \qquad T = T_{S+B} + T_B$$

How can the measurement time be optimized?

Counting time optimization

Assume

S = source count rate

B = background count rate

T_{S+B} = time to count source + background (resulting in N_1 total counts)

T_B = time to count background (resulting N_2 total counts)

$$S = \frac{N_1}{T_{S+B}} - \frac{N_2}{T_B} \qquad T = T_{S+B} + T_B$$

$$\sigma_s^2 = \left[\left(\frac{\sigma_{N_1}}{T_{S+B}} \right)^2 + \left(\frac{\sigma_{N_2}}{T_B} \right)^2 \right]$$

Counting time optimization

$$\sigma_s^2 = \left[\left(\frac{\sigma_{N_1}}{T_{S+B}}\right)^2 + \left(\frac{\sigma_{N_2}}{T_B}\right)^2\right] \qquad \sigma_{N_1} = \sqrt{N_1} \qquad \sigma_{N_2} = \sqrt{N_2}$$

$$\sigma_s^2 = \left[\frac{N_1}{T_{S+B}{}^2} + \frac{N_2}{T_B{}^2}\right] \qquad N_1 = (S+B)T_{S+B} \qquad N_2 = BT_B$$

$$\sigma_s^2 = \left[\frac{S+B}{T_{S+B}} + \frac{B}{T_B}\right]$$

Find optimum by taking derivative and setting equal to zero

Optimizing Counting Time

Find optimum by taking derivative and setting equal to zero gives

$$2\sigma_S d\sigma_S = -\frac{S+B}{T_{S+B}^2} dT_{S+B} - \frac{B}{T_B^2} dT_B$$

Set $d\sigma_S = 0$

$$-\frac{B}{T_B^2} dT_B = \frac{S+B}{T_{S+B}^2} dT_{S+B}$$

$$-\frac{T_{S+B}^2}{T_B^2} dT_B = \frac{S+B}{B} dT_{S+B}$$

$$-\frac{T_{S+B}^2}{T_B^2} \frac{dT_B}{dT_{S+B}} = \frac{S+B}{B}$$

$$\frac{T_{S+B}^2}{T_B^2} = \frac{S+B}{B}$$

$$\frac{T_{S+B}}{T_B} = \sqrt{\frac{S+B}{B}}$$

$$T = T_{S+B} + T_B$$

$$dT = dT_{S+B} + dT_B$$

$$0 = dT_{S+B} + dT_B$$

$$0 = \frac{dT_{S+B}}{dT_{S+B}} + \frac{dT_B}{dT_{S+B}}$$

$$\frac{dT_B}{dT_{S+B}} = -1$$

$$\boxed{\left.\frac{\mathrm{T_{S+B}}}{\mathrm{T_B}}\right|_{\mathrm{optimum}} = \sqrt{\frac{\mathrm{S+B}}{\mathrm{B}}}}$$