Problem 1

Two exercises: use index notation to show that

(i) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

(ii) $(\vec{a}\times\vec{b})\cdot\Big[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\Big] = (abc)^2$. Here one defines $(abc)\equiv\vec{a}\cdot(\vec{b}\times\vec{c})$.

Problem 2

(i) Show that (in three dimensions), for arbitrary B_i one has

$$B_{ijk\ell} \equiv B_i\epsilon_{jk\ell} - B_j\epsilon_{ik\ell} - B_k\epsilon_{ji\ell} - B_\ell\epsilon_{jki} = 0\,.$$

(ii) Given three non-coplanar vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ establish the following decompositions of an arbitrary vector $\vec{d}$

$$\vec{d} = \frac{(dbc)}{(abc)}\vec{a} + \frac{(adc)}{(abc)}\vec{b} + \frac{(abd)}{(abc)}\vec{c} \tag{1}$$

$$\vec{d} = (\vec{d}\cdot\vec{a})\,\frac{\vec{b}\times\vec{c}}{(abc)} + (\vec{d}\cdot\vec{b})\,\frac{\vec{c}\times\vec{a}}{(abc)} + (\vec{d}\cdot\vec{c})\,\frac{\vec{a}\times\vec{b}}{(abc)} \tag{2}$$

Problem 3

A direct calculation of $\vec{A}(\vec{x})$ given $\vec{B} = \vec{\nabla}\times\vec{A}$. Prove that $\vec{A}(\vec{x}) = -\int_0^1 t\,dt\;\vec{x}\times\vec{B}(t\vec{x})$, by showing that $\vec{\nabla}\times\vec{A} = \vec{B}$ provided that $\nabla\cdot\vec{B} = 0$.

In order to solve this problem, it is useful to define $\vec{y} = t\vec{x}$, and then to prove that

$$\nabla_{\vec{x}} = t\nabla_{\vec{y}}, \quad \text{and} \quad \frac{d}{dt}\vec{B}(t\vec{x}) = \frac{1}{t}(\vec{x}\cdot\nabla_{\vec{x}})\vec{B}(t\vec{x}).$$