Problem 1. A problem with spherical harmonics.

A set of six charges sit at the corners of a regular hexagon lying on the xy plane and centered at the origin. The charges alternate in value between $+q$ and $-q$. Using spherical coordinates all charges are at $r = R$ and $\theta = \pi/2$. At $\phi = 0$ there is a charge $+q$.

(a) The charge distribution has quite a bit of symmetry and the resulting potential will have that symmetry. In particular as a funtion of ϕ the charge distribution ρ is even, and it has period $2\pi/3$. Moreover $\rho(\pi/6 + \phi) = -\rho(\pi/6 - \phi)$ (explain why). Using these facts show that the Fourier expansion of $\Phi(r, \theta, \phi)$ as a function of ϕ goes like

$$\Phi(r, \theta, \phi) = \sum_m f_m(r, \theta) \cos m\phi$$

What are the first few values of m that correspond to nonvanishing terms in this expansion?

(b) Compare the above expansion with the spherical harmonic expansion of Φ for $r < R$. What is the lowest power of r to appear ? Find the form of the leading term in the potential (up to a constant of proportionality) near $r = 0$ as a polynomial in x, y, and z.

(c) To leading approximation the charge density is expanded as

$$\rho(r, \theta, \phi) = \rho_{3,3}(r) Y_{3,3} + \rho_{3,-3}(r) Y_{3,-3},$$

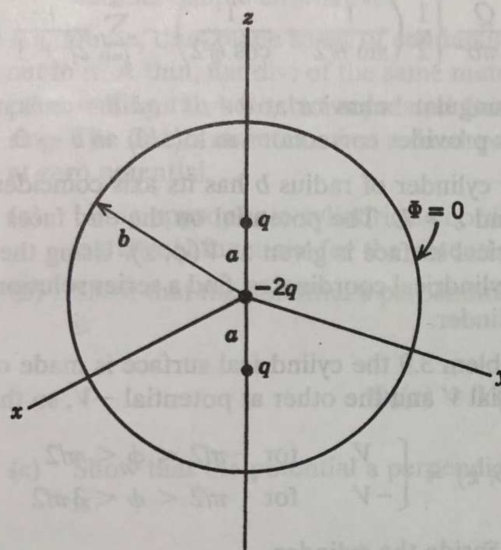
where the Y's are spherical harmonics. A similar expansion holds for Φ. Show that the reality condition for ρ is satisfied in the above expansion (you will need to use the conjugation property of the Y's). Show that

$$\rho_{3,3}(r) = \frac{6qa}{R^2} \delta(r - R), \quad a = -\frac{1}{4}\sqrt{\frac{35}{4\pi}}.$$

(d) Use the radial equation (as discussed in class) to find $\Phi_{3,3}(r)$. Finally, write the first nonvanishing term in the $r \sim 0$ expansion of $\Phi(r, \theta, \phi)$.

Problem 2. Jackson 3.7

3.7 Three point charges (q, $-2q$, q) are located in a straight line with separation a and with the middle charge ($-2q$) at the origin of a grounded conducting spherical shell of radius b, as indicated in the sketch.



Problem 3.7

(a) Write down the potential of the three charges in the absence of the grounded sphere. Find the limiting form of the potential as $a \to 0$, but the product $qa^2 = Q$ remains finite. Write this latter answer in spherical coordinates.

(b) The presence of the grounded sphere of radius b alters the potential for $r < b$. The added potential can be viewed as caused by the surface-charge density induced on the inner surface at $r = b$ or by image charges located at $r > b$. Use linear superposition to satisfy the boundary conditions and find the potential everywhere inside the sphere for $r < a$ and $r > a$. Show that in the limit $a \to 0$,

$$\Phi(r, \theta, \phi) \to \frac{Q}{2\pi\epsilon_0 r^3}\left(1 - \frac{r^5}{b^5}\right)P_2(\cos\theta)$$