CS 525: Deep Neural Networks
Homework 2 - Problem 1

We are given the following cost function to minimize:

$$J(w,b) = \frac{1}{4}\sum_{x\in\mathbb{X}}(f^*(x) - f(x;w,b))^2$$

where $\mathbb{X} = \{x_1, x_2\} = \{[0,0,1,1]^T, [0,1,0,1]^T\}$ and $f^*(\mathbb{X}) = [0,1,1,0]^T$

To determine the optimal values of w and b, we find the gradient of $J(w,b)$ in terms of w_1, w_2, and b, and set them equal to zero:

$$\begin{aligned}
\frac{dJ}{dw_1} &= \frac{1}{2}\sum_{x\in\mathbb{X}}(-x_1)(f^*(x) - x_1w_1 - x_2w_2 - b) = 0 \\
\frac{dJ}{dw_2} &= \frac{1}{2}\sum_{x\in\mathbb{X}}(-x_2)(f^*(x) - x_1w_1 - x_2w_2 - b) = 0 \\
\frac{dJ}{db} &= -\frac{1}{2}\sum_{x\in\mathbb{X}}(f^*(x) - x_1w_1 - x_2w_2 - b) = 0
\end{aligned}$$

Next, we set $\frac{dJ}{dw_1}$ equal to $\frac{dJ}{dw_2}$ and plug in the given values for x_1 and x_2:

$$\frac{dJ}{dw_1} = \frac{dJ}{dw_2}$$

$$\frac{1}{2}\sum_{x\in\mathbb{X}}(-x_1)(f^*(x) - x_1w_1 - x_2w_2 - b) = \frac{1}{2}\sum_{x\in\mathbb{X}}(-x_2)(f^*(x) - x_1w_1 - x_2w_2 - b)$$

$$\sum_{x\in\mathbb{X}}\begin{bmatrix}0\\0\\-1\\-1\end{bmatrix}\left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix} - w_1\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - w_2\begin{bmatrix}0\\1\\0\\1\end{bmatrix} - b\right) = \sum_{x\in\mathbb{X}}\begin{bmatrix}0\\-1\\0\\-1\end{bmatrix}\left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix} - w_1\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - w_2\begin{bmatrix}0\\1\\0\\1\end{bmatrix} - b\right)$$

$$\sum_{x\in\mathbb{X}}\left(\begin{bmatrix}0\\0\\-1\\0\end{bmatrix} - w_1\begin{bmatrix}0\\0\\-1\\-1\end{bmatrix} - w_2\begin{bmatrix}0\\0\\0\\-1\end{bmatrix} - b\begin{bmatrix}0\\0\\-1\\-1\end{bmatrix}\right) = \sum_{x\in\mathbb{X}}\left(\begin{bmatrix}0\\-1\\0\\0\end{bmatrix} - w_1\begin{bmatrix}0\\0\\0\\-1\end{bmatrix} - w_2\begin{bmatrix}0\\-1\\0\\-1\end{bmatrix} - b\begin{bmatrix}0\\-1\\0\\-1\end{bmatrix}\right)$$

$$-1 + 2w_1 + w_2 + 2b = -1 + w_1 + 2w_2 + 2b \tag{1}$$

$$w_1 = w_2$$

We now plug the given values of x_1 and x_2 into $\frac{dJ}{db}$, with the knowledge that $w_1 = w_2$:

$$\frac{dJ}{db} = -\frac{1}{2}\sum_{x\in\mathbb{X}}(f^*(x) - x_1 w_1 - x_2 w_1 - b)$$

$$= -\frac{1}{2}\sum_{x\in\mathbb{X}}\left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix} - w_1\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - w_1\begin{bmatrix}0\\1\\0\\1\end{bmatrix} - b\right)$$

$$= -\frac{1}{2}(2 - 2w_1 - 2w_1 - 4b)$$

$$= -1 + 2w_1 + 2b = 0$$

$$b = -w_1 + \frac{1}{2} \tag{2}$$

Next, plug this value of b into the left half of our expression from (1)[1] to find:

$$-1 + 2w_1 + w_2 + 2b = 0 \tag{3}$$

$$= -1 + 2w_1 + w_2 - 2w_1 + 1 = 0$$

$$w_2 = 0$$

We have found that $w_1 = w_2 = 0$. Finally, plug w_1 back into equation (2) to find:

$$b = -w_1 + \frac{1}{2}$$

$$b = \frac{1}{2}$$

[1]Recall that the left half of expression (1) was a simplified version of $\frac{dJ}{dw_1}$. This is why we set it equal to zero in equation (3).