CS 525: Deep Neural Networks
Homework 2 - Problem 1

We are given the following cost function to minimize:

$$J(w,b) = \frac{1}{4}\sum_{x\in\mathbb{X}}(f^*(x) - f(x;w,b))^2$$

where $\mathbb{X} = \{[0,0]^T, [0,1]^T, [1,0]^T, [1,1]^T\}$ and $f^*(x \in \mathbb{X}) = \{0,1,1,0\}$

To determine the optimal values of w and b, we find the gradient of $J(w,b)$ in terms of w_1, w_2, and b, and set them equal to zero:

$$\frac{\partial J}{\partial w_1} = \frac{1}{2}\sum_{x\in\mathbb{X}}(-x_1)(f^*(x) - x_1w_1 - x_2w_2 - b) = 0$$

$$\frac{\partial J}{\partial w_2} = \frac{1}{2}\sum_{x\in\mathbb{X}}(-x_2)(f^*(x) - x_1w_1 - x_2w_2 - b) = 0$$

$$\frac{\partial J}{\partial b} = -\frac{1}{2}\sum_{x\in\mathbb{X}}(f^*(x) - x_1w_1 - x_2w_2 - b) = 0$$

Next, we set $\frac{\partial J}{\partial w_1}$ equal to $\frac{\partial J}{\partial w_2}$ and plug in the given values for x_1 and x_2:

$$\frac{\partial J}{\partial w_1} = \frac{\partial J}{\partial w_2}$$

$$\frac{1}{2}\sum_{x\in\mathbb{X}}(-x_1)(f^*(x) - x_1w_1 - x_2w_2 - b) = \frac{1}{2}\sum_{x\in\mathbb{X}}(-x_2)(f^*(x) - x_1w_1 - x_2w_2 - b)$$

$$\begin{bmatrix}0\\0\\-1\\-1\end{bmatrix}\left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix} - w_1\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - w_2\begin{bmatrix}0\\1\\0\\1\end{bmatrix} - b\right) = \begin{bmatrix}0\\-1\\0\\-1\end{bmatrix}\left(\begin{bmatrix}0\\1\\1\\0\end{bmatrix} - w_1\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - w_2\begin{bmatrix}0\\1\\0\\1\end{bmatrix} - b\right)$$

$$\left(\begin{bmatrix}0\\0\\-1\\0\end{bmatrix} - w_1\begin{bmatrix}0\\0\\-1\\-1\end{bmatrix} - w_2\begin{bmatrix}0\\0\\0\\-1\end{bmatrix} - b\begin{bmatrix}0\\0\\-1\\-1\end{bmatrix}\right) = \left(\begin{bmatrix}0\\-1\\0\\0\end{bmatrix} - w_1\begin{bmatrix}0\\0\\0\\-1\end{bmatrix} - w_2\begin{bmatrix}0\\-1\\0\\-1\end{bmatrix} - b\begin{bmatrix}0\\-1\\0\\-1\end{bmatrix}\right)$$

$$-1 + 2w_1 + w_2 + 2b = -1 + w_1 + 2w_2 + 2b \tag{1}$$

$$w_1 = w_2$$

We now plug the given values of x_1 and x_2 into $\frac{\partial J}{\partial b}$, with the knowledge that $w_1 = w_2$:

$$\frac{\partial J}{\partial b} = -\frac{1}{2} \sum_{x \in \mathbb{X}} (f^*(x) - x_1 w_1 - x_2 w_1 - b)$$

$$= -\frac{1}{2} \left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - w_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - w_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - b \right)$$

$$= -\frac{1}{2}(2 - 2w_1 - 2w_1 - 4b)$$

$$= -1 + 2w_1 + 2b = 0$$

$$b = -w_1 + \frac{1}{2} \tag{2}$$

Next, plug this value of b into the left half of our expression from (1)[1] to find:

$$-1 + 2w_1 + w_2 + 2b = 0 \tag{3}$$

$$= -1 + 2w_1 + w_2 - 2w_1 + 1 = 0$$

$$w_2 = 0$$

We have found that $w_1 = w_2 = 0$. Finally, plug w_1 back into equation (2) to find:

$$b = -w_1 + \frac{1}{2}$$

$$b = \frac{1}{2}$$

[1]Recall that the left half of expression (1) was a simplified version of $\frac{dJ}{dw_1}$. This is why we set it equal to zero in equation (3).