AMAT 584 Homework 1

Mathew Goberdhan (mgoberdhan@gmail.com)

February 26, 2021

1. Problem 1

a) Show that two points $s, t \in S$ form an edge of $D(S)$ if and only if there is a circle with s and t on the boundary and no other points of S in the interior.

proof: Let S be a set of points of $\mathbb{R}^2$ in general position. Let $s, t \in S$.

$\Longrightarrow$ Suppose (s, t) form an edge in $D(S)$. Then, Voronoi cells V_s and V_t are adjacent and points s, t are closest points. So, $V_s \cap V_t$ is an edge of $V(S)$, which s, t are equidistant from. If we consider the circle, C, whose diameter is the line segment $\bar{st}$, then there cannot be any other points of S that fall in the interior of C.

$\Longleftarrow$ Suppose there is a circle, C, with s, t on the boundary and no other points in the interior of C. Then, there exists a Voronoi edge that passes into C. By definition, s, t are equidistant to this Voronoi edge and there are no closer points. Therefore, V_s and V_t are adjacent and there is an edge (s, t) that forms in $D(S)$.

b) No two edges of the Delaunay triangulation of a finite set of points in the plane cross each other. Give a convincing argument why this is the case.

proof: Let $T_{s,t}$ be the triangle formed by s, t and the circumcenter, x, of the circumcircle, $C_{s,t}$. It follows that $(s, x) \subset V_s$ and $(t, x) \subset V_t$ because of convexity of V_s and V_t.

Suppose by contradiction that two Delaunay edges intersect, (s, t) and (u,v). By part (a), u and v cannot be in the interior of $C_{s,t}$. Also, $T_{,s,t}$ is inside $C_{s,t}$. Thus, as (s, t) and (u, v) intersect, then (u, v) must exit $T_{s,t}$ through (s, x) or (t, x). Without loss of generality, say this edge is (s, x).

If you apply the same method above to (s, x), then it will intersect another edge of $T_{u,v}$, in addition to (u, v). But, these sides need to be contained by their Voronoi cell. $\Longrightarrow \Longleftarrow$. Therefore, no two Delaunay edges intersect.

2. Problem 2

Let S be a finite set of points in $\mathbb{R}^2$, s and t two points in S, and α a real number. Recall that $D_s(\alpha)$ is the α-disk centered at $s \in S$, V_s is the Voronoi region of s, and $R_s(\alpha) = D_s(\alpha) \cap V_s$.

a) Prove that every point $x \in V_s \cap D_t(\alpha)$ is contained in $D_s(\alpha)$

proof: Let $\alpha > 0$ be big enough so that for points $s, t \in S \subset \mathbb{R}^2$, $V_s \cap D_t(\alpha) \neq \varnothing$. Let $x \in V_s \cap D_t(\alpha)$.

<u>Case 1</u>: Suppose V_s and V_t are adjacent cells. Then, $D_s(\alpha)$ and $D_t(\alpha)$ meet at $V_s \cap V_t$ for the first time. Then, $x \in V_s \cap D_t(\alpha)$ implies $x \in V_s \cap D_s(\alpha)$. Therefore, $x \in D_s(\alpha)$.

Case 2: Suppose V_s and V_t are not adjacent cells. Then there there exists a Voronoi cell(s) such that s, t are not closest points. As $V_s \cap D_t(\alpha) \neq \varnothing$, then by how Voronoi cells are constructed, $V_s \subset D_s(\alpha)$. Thus, $x \in D_s(\alpha)$.

Therefore, every point $x \in V_s \cap D_t(\alpha)$ is contained in $D_s(\alpha)$.

b) Prove $\cup_{s\in S} D_s(\alpha) = \cup_{s\in S} R_s(\alpha)$

proof: "$\supseteq$" Let $x \in \cup_{s\in S} R_s(\alpha)$. Then for some $t \in S$, $x \in R_t(\alpha)$. As $R_t(\alpha) = V_t \cap D_t(\alpha) \subseteq D_t(\alpha)$, then $x \in D_s(\alpha)$. Thus, $x \in \cup_{s\in S} D_s(\alpha)$. Thus, "$\supseteq$".

"$\subseteq$" Let $x \in \cup_{s\in S} D_s(\alpha)$. Then for some $t \in S$, $x \in D_t(\alpha)$. Then, $x \in D_t(\alpha) \cap V_t$ or $x \in D_t(\alpha) \cap V_w$ for some $w \in V_w$.

Case 1: Suppose $x \in D_t(\alpha) \cap V_t$. Then, $x \in R_t(\alpha)$. Thus, $x \in \cup_{s\in S} R_s(\alpha)$.

Case 2: Suppose $x \in D_t(\alpha) \cap V_w$ for some $w \in V_w$. By part (a), $x \in D_w(\alpha)$. So, $x \in D_w(\alpha) \cap V_w$. Then, $x \in R_w(\alpha)$. Thus, $x \in \cup_{s\in S} R_s(\alpha)$.

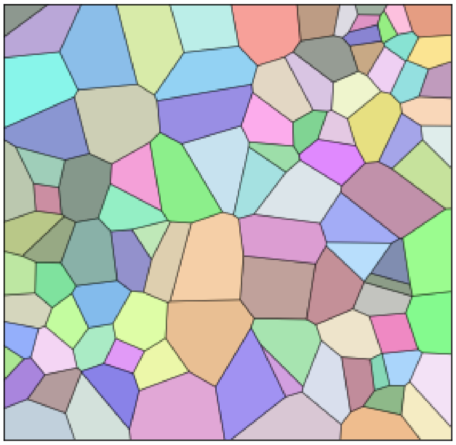
Thus, "$\subseteq$".

Therefore, $\cup_{s\in S} D_s(\alpha) = \cup_{s\in S} R_s(\alpha)$.

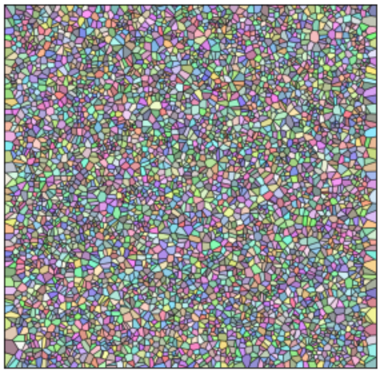
3. Problem 3

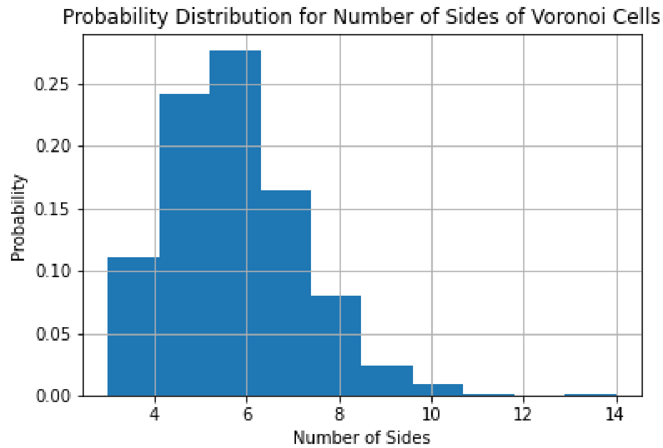
In this problem, you will compute statistics for Voronoi diagrams of uniformly distributed points in the square.

a) Make a figure similar to Figure 1 for a Voronoi diagram on 100 uniform random points in the unit square

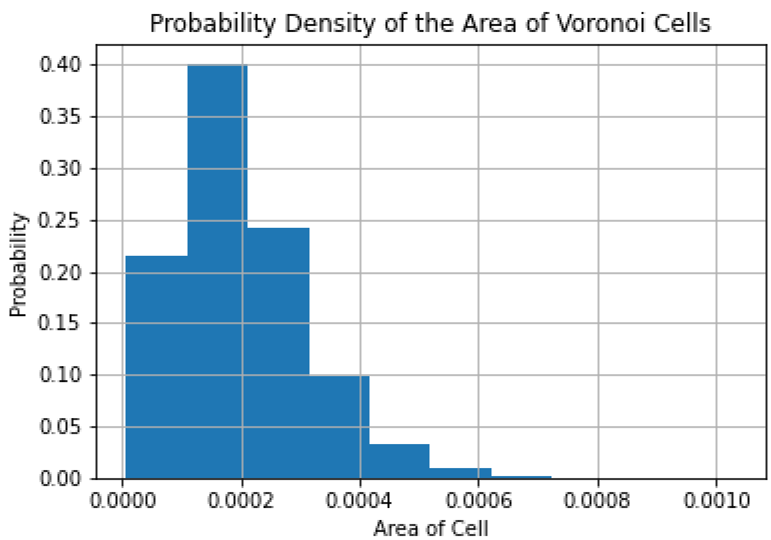


c) Sample a Voronoi diagram on 5000 random uniform points in the unit square. Compute the number of sides of each finite Voronoi face, and create a histogram showing the empirical probability distribution for the number of sides of a face. Normalize the histogram so that the sum of the probabilities equals 1.





d) Create a histogram illustrating the empirical probability density of the areas of the finite Voronoi faces for the sample of 5000 random uniform points, normalized so that the sum of the areas of the bars equals 1.

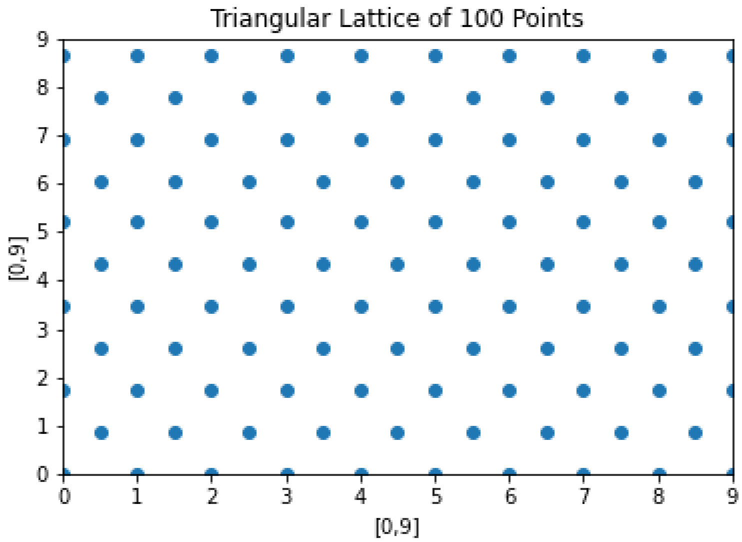


4. Problem 4

Perturbed lattice points are found by adding Gaussian noise to the points of the triangular lattice, where the triangular lattice is

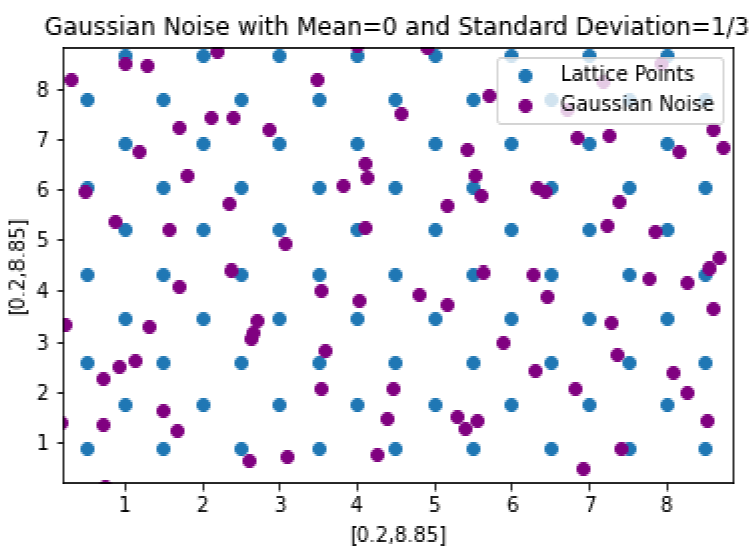
$$L = \{i \begin{bmatrix} 1 \\ 0 \end{bmatrix} + j \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} : i, j \in \mathbb{Z}\}$$

a) Find an N so that the square $[0, N] \times [0, N]$ contains approximately 100 points of L, and explain how you found N.

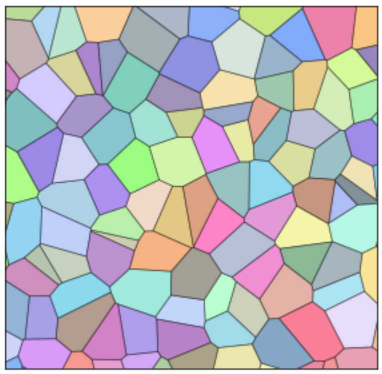


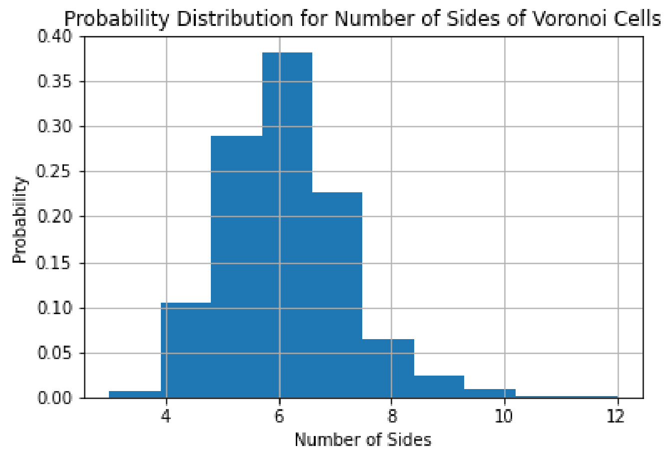
In python, I produced a scatter plot representing points of L generated by $[-15, 15] \in \mathbb{Z}$. Noticing the rough even distribution of the points throughout the lattice, I limited the lattice down to the square $[0, 10] \times [0, 10]$. Since vertical spacing is slightly less than horizontal spacing, there were more than 100 points in the lattice. Reducing to $[0, 9] \times [0, 9]$ showed to contain approximately 100 points. Therefore, $N = 9$.

b) Take a subset $L^{'}$ of L a bit bigger than to fit inside the square $[0, 9] \times [0, 9]$ and add Gaussian noise with mean 0 and standard deviation $\frac{1}{3}$. Let S be the collection of perturbed points that are contained in the square. Create a scatter plot showing the points of S.



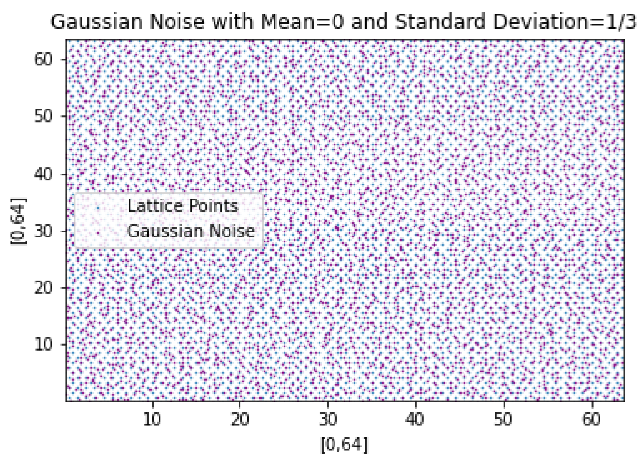
c) Compute the Voronoi diagram of S and create a figure analogous to the one you made in 3(a). The square $[0, 9] \times [0, 9]$ is bigger than before, so either adapt your code to this case or re-scale the coordinates of the points by a factor $\frac{1}{9}$ so that they fit in the unit square.

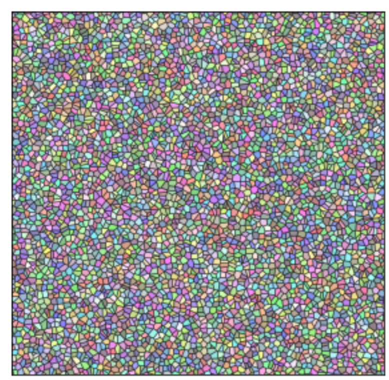




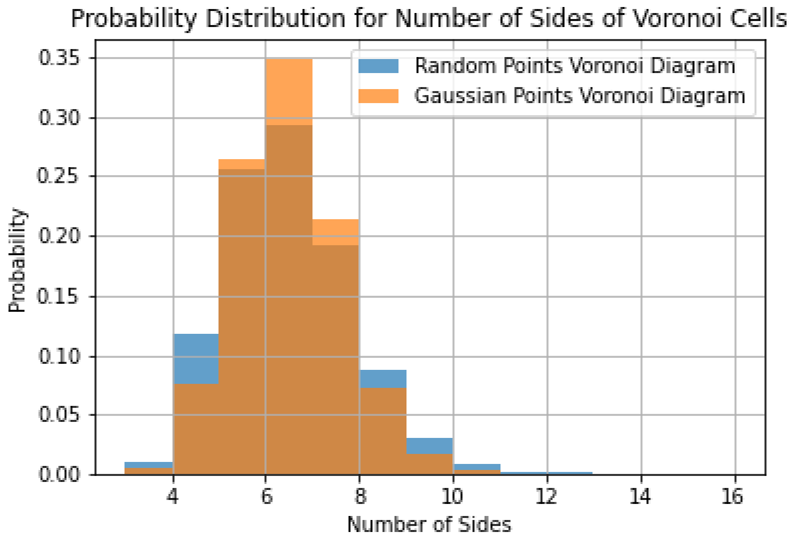
d) Find an M so that the square $[0, M] \times [0, M]$ contains about 5000 points of L, and continue as before to find a set of about 5000 perturbed lattice points in $[0, M] \times [0, M]$. Call this point set T.

Shown is the scatter plot of 5000 lattice points, T, with $M = 9\sqrt{50}$, and its corresponding Voronoi diagram.

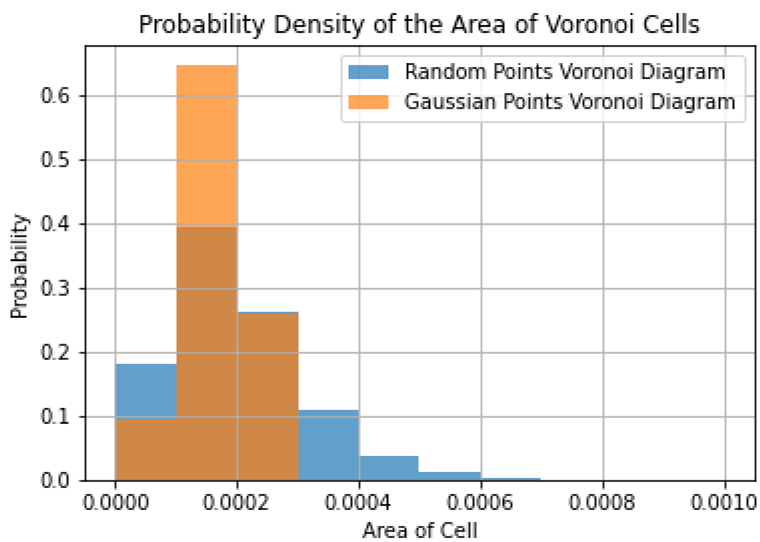




e) Create a single figure showing your histogram from 3(c), and an analogous one showing the distribution of number of sides for Voronoi diagram on T. Include a legend, make sure the bins are the same for the two histograms, and normalize each histogram so the sum of the probabilities equals 1.



f) Create a single figure showing your histogram from 3(d), and an analogous one approximating the probability density of the area distribution of the Voronoi diagram on T.



5. Problem 5

Write a paragraph explaining and interpreting the figures you made in 4(e) and 4(f).

Voronoi diagrams were produced of random uniform points and perturbed lattice points, both consisting of a 5000 point sample space. From observation and statistical study, perturbed lattice points appear to be more ordered. Comparing their empirical distributions, perturbed lattice points had more constraint, with a truer mean and smaller standard deviation. When observing the empirical distribution of the point cloud consisting of random uniform points, standard deviation exceeded that of the standard deviation of perturbed lattice point cloud, showing wider distribution, or more disorder. Therefore, in theory, the empirical distribution of the perturbed lattice point cloud is a better fitting model of the behavior exhibited in Voronoi diagrams.