Matrices and Projections

Lecture 3c

CS3400 Machine Learning

Matrices

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix}$$

- Matrices are an ordered collection of vectors
- Either column vectors
- Or row vectors
- Used to store and perform operations on groups of vectors (e.g., records) in one operation

Matrices

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,3} \\ \vdots & \ddots & \vdots \\ a_{1,5} & \cdots & a_{5,3} \end{bmatrix}$$

- Variable names are capital with an arrow

$$\vec{A}$$

- Matrices are described by:
  - The number of rows
  - The number of columns
  - The type of number (e.g., real or complex)

$$\vec{A} \in \mathbb{R}^{5\times 3}$$

Matrices

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,3} \\ \vdots & \ddots & \vdots \\ a_{1,5} & \cdots & a_{5,3} \end{bmatrix}$$

Matrix elements can be described using indexing like so: $\overrightarrow{A_{i,j}}$. The variable i refers to the row, while the column j refers to the column.

- Variable names are capital with an arrow

$$\vec{A}$$

- Matrices are described by:
  - The number of rows
  - The number of columns
  - The type of number (real or complex)

$$\vec{A} \in \mathbb{R}^{5\times3}$$

Numpy

Create Matrix

```
A = np.array([[1,  3],
              [5,  7],
              [9, 11]])
```

```
A = np.ones((3, 2))
```

```
A = np.zeros((3, 2))
```

Get the dimensions

```
A.shape
```

```
(3, 2)
```

Operations on Matrices: Transpose

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,3} \\ \vdots & \ddots & \vdots \\ a_{1,5} & \cdots & a_{5,3} \end{bmatrix}^T =$$

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,5} \\ \vdots & \ddots & \vdots \\ a_{1,3} & \cdots & a_{3,5} \end{bmatrix}$$

- Transposing rotates the matrix around the diagonal
- so that the indices of the rows and columns are swapped

Numpy

Tranpose a Matrix

```
A = np.array([[1,  3],
              [5,  7],
              [9, 11]])
```

```
A.shape
(3, 2)
```

Matrix-Matrix Multiplication

```
B = A.T
```

```
array([[ 1,  5,  9],
       [ 3,  7, 11]])
```

```
B.shape
```

```
(2, 3)
```

Operations on Matrices: Multiplication

$$\vec{A} \cdot \vec{b} = \vec{c}$$

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$$

$$\vec{c}_1 = \begin{bmatrix} 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2$$

- Multiplying a matrix and a vector produces a new vector
- To multiple a matrix and a vector
- the vector is dotted with every row of the matrix
- to get each element of the new vector

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The length of the vector must match the number of columns in the matrix.

- Multiplying a matrix and a vector produces a new vector
- To multiple a matrix and a vector
- the vector is dotted with every row of the matrix
- to get each element of the new vector

Operations on Matrices: Multiplication

$$\vec{A} \cdot \vec{B} = \vec{C}$$

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -2 & 2 \\ -2 & 2 \end{bmatrix}$$

$$\vec{C}_{3,1} = \vec{A}_{,3} \cdot \vec{B}_{1,}$$
$$= \begin{bmatrix} 9 & 11 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= -2$$

- We can also multiple two matrices
- We dot each row of the first matrix
- with every column of the second matrix
- to get every element in the output matrix

Operations on Matrices: Multiplication

$$\vec{A} \cdot \vec{B} = \vec{C}$$

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$$= -2$$

- We can also multiple two matrices
- We dot each row of the first matrix
- with every column of the second matrix
- to get every element in the output matrix

The number of columns in the first matrix
must match
the number of rows in the second matrix.

Numpy

Matrix-Vector Multiplication

```
A = np.array([[1,  3],
              [5,  7],
              [9, 11]])
b = np.array([1, -1])
c = np.dot(A, b)

array([-2, -2, -2])
```

Matrix-Matrix Multiplication

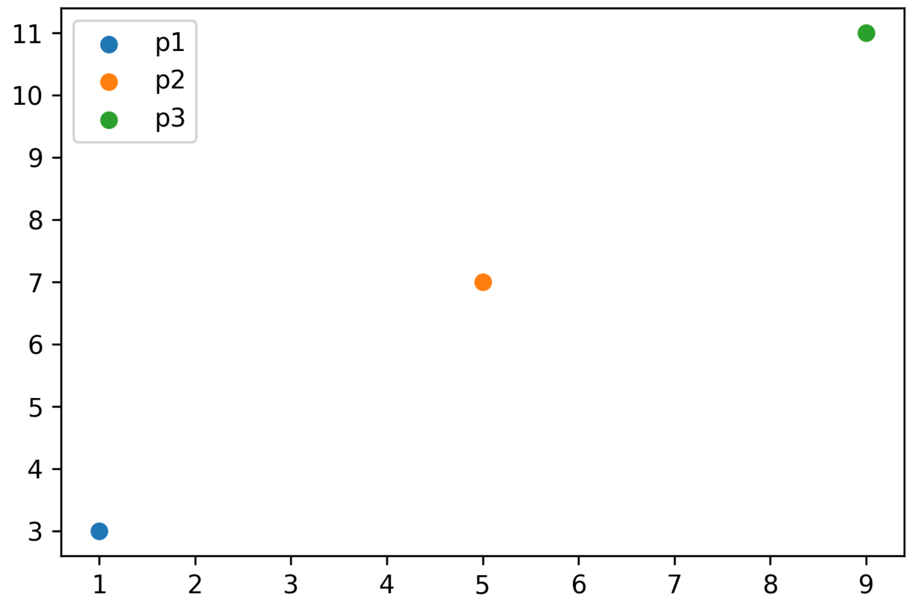
```
A = np.array([[1,  3],
              [5,  7],
              [9, 11]])
B = np.array([[1, -1],
              [-1, 1]])
C = np.dot(A, B)

array([[-2,  2],
       [-2,  2],
       [-2,  2]])
```

What is matrix multiplication?

- Conversion from one coordinate system to another
- Projection between spaces

Interpreting our Matrix

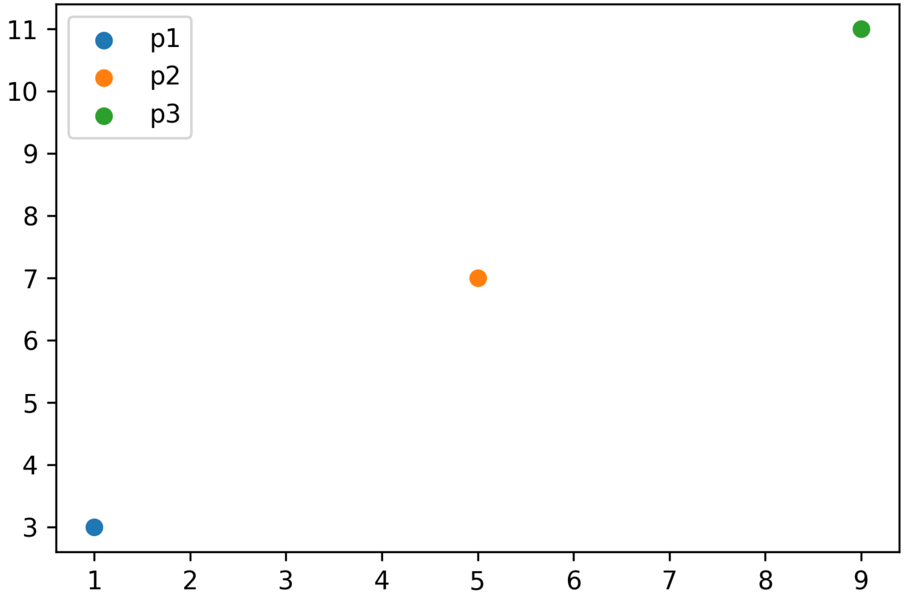
$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix}$$



Interpreting our Matrix

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix}$$

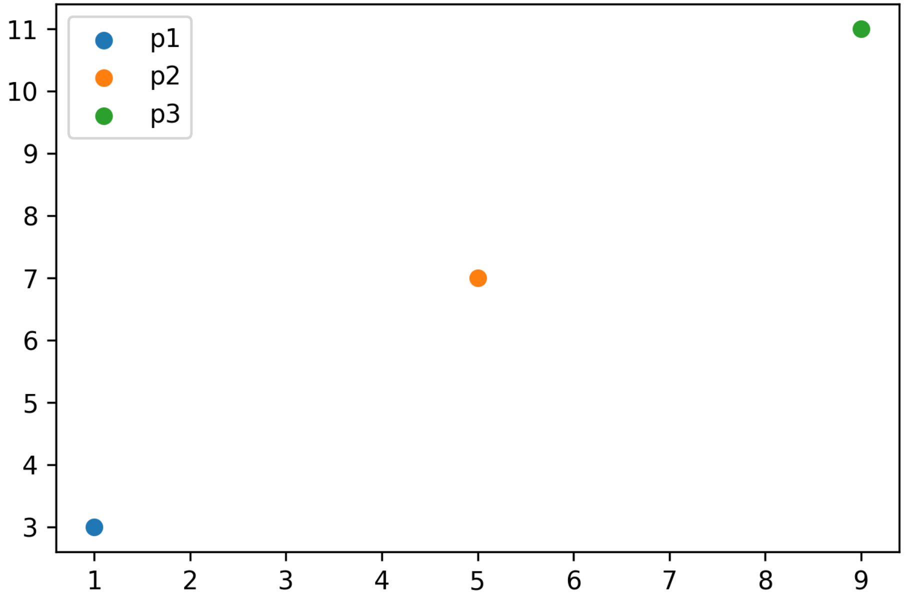
Each column is a coordinate (dimension).



Interpreting our Matrix

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix}$$

Each row represent is a single vector or point.



Identity Matrix

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix}$$

- An identity matrix is a:
  - square matrix
  - with 1s on the diagonal
  - and 0s everywhere else
- Multiplication by an identity matrix doesn't change the matrix
- An identity matrix projects a space into itself

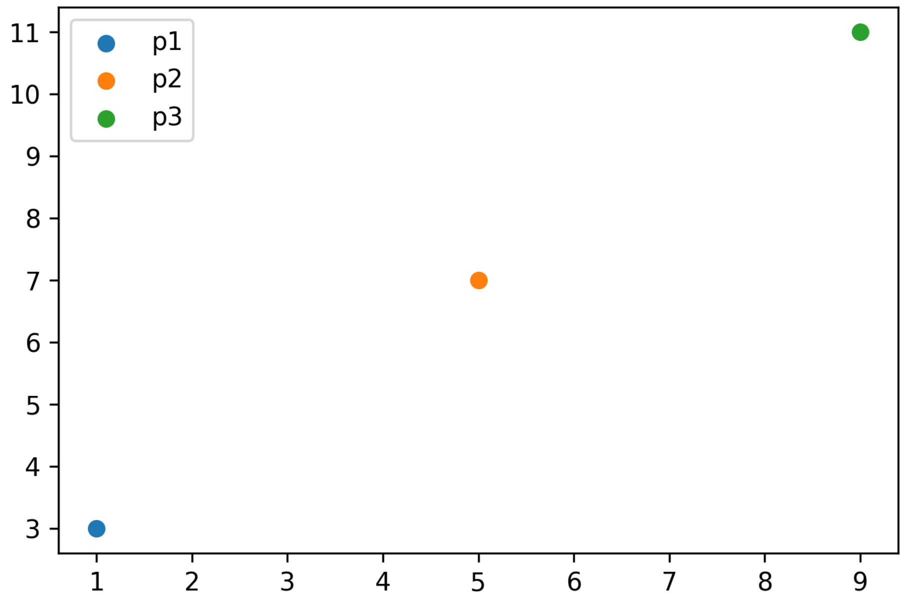
Identity Matrix -- Negation

- We can reverse all of the coordinates by using a matrix with -1s on the diagonals

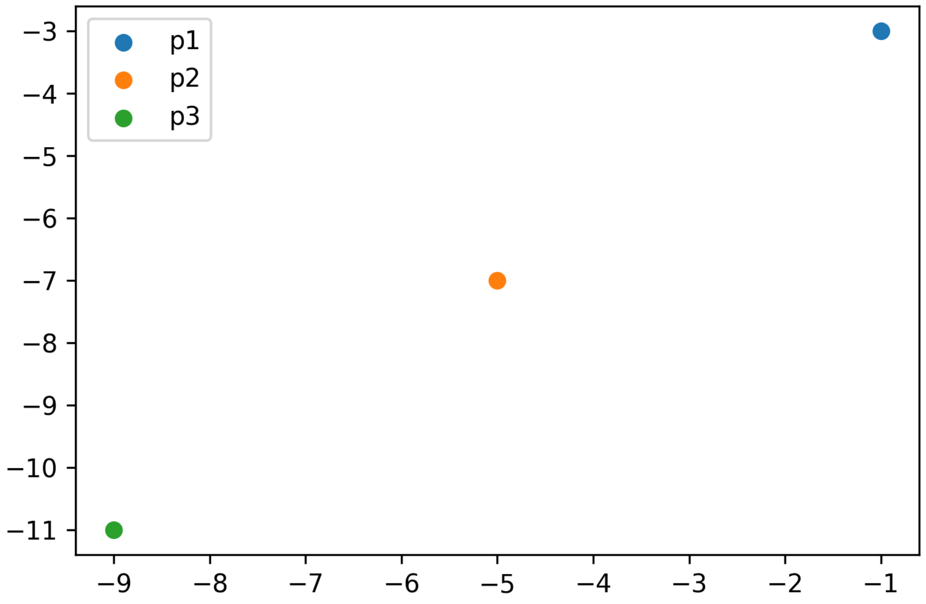
$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ -5 & -7 \\ -9 & -11 \end{bmatrix}$$

Identity Matrix -- Negation

Before



After



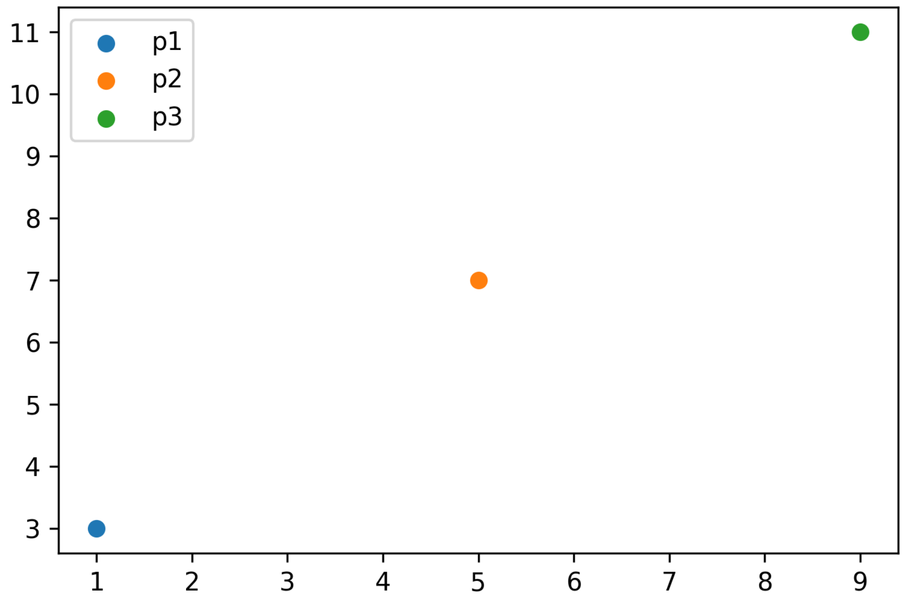
Permutation Matrix

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 7 & 5 \\ 11 & 9 \end{bmatrix}$$

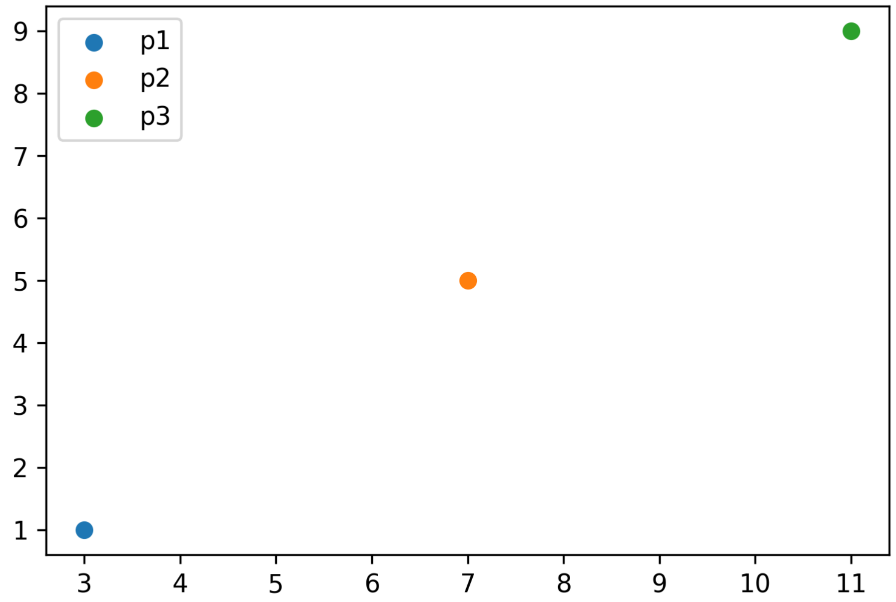
- A permutation matrix is an identity matrix with the columns swapped.
- In this case, we swapped the two coordinates

Permutation Matrix

Before



After



Composing Operations

$$\begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

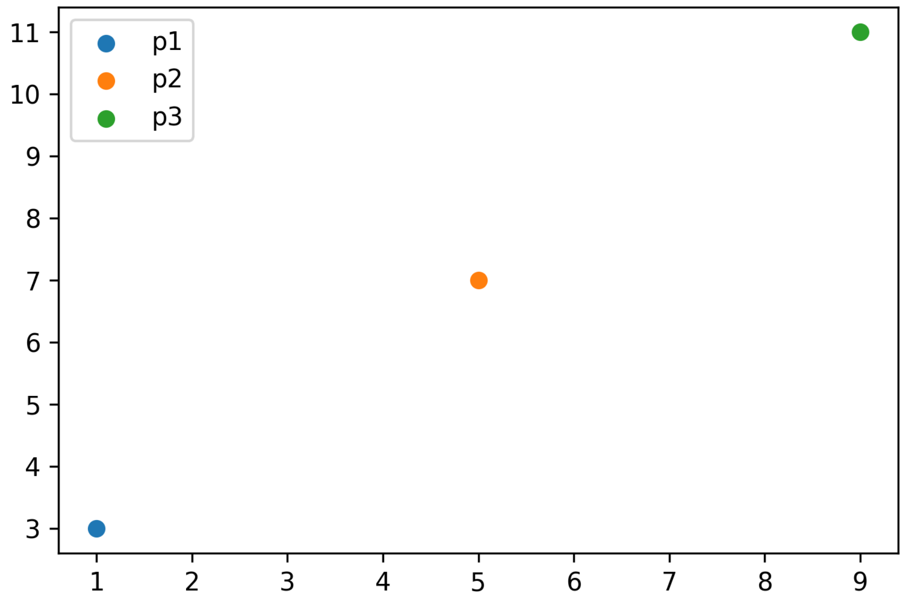
$$= \begin{bmatrix} 1 & 3 \\ 5 & 7 \\ 9 & 11 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -1 \\ -7 & -5 \\ -11 & -9 \end{bmatrix}$$

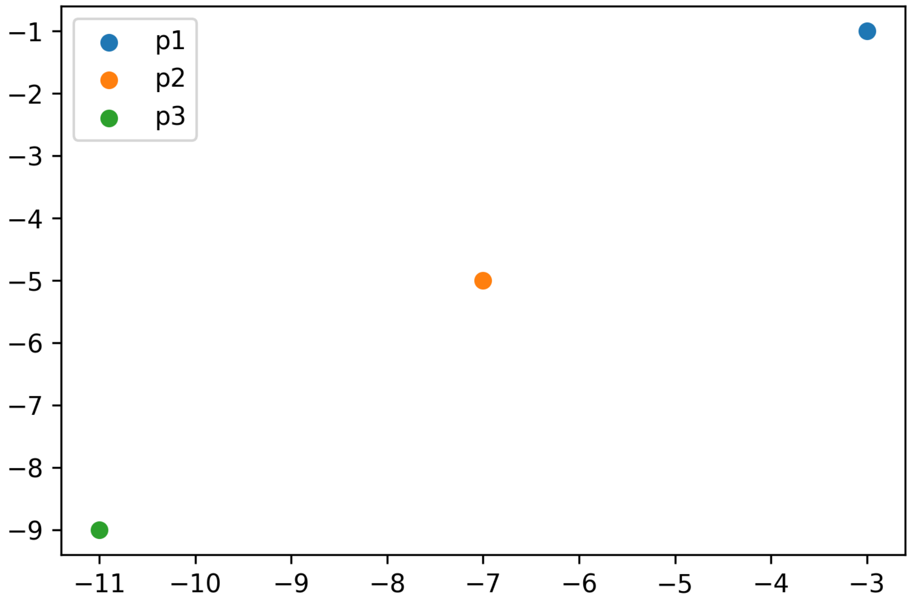
- We can also combine operations by multiplying multiple matrices
- In this case, we use the negative identity to negate the coordinates and the permutation matrix to swap coordinates

Composing Operations

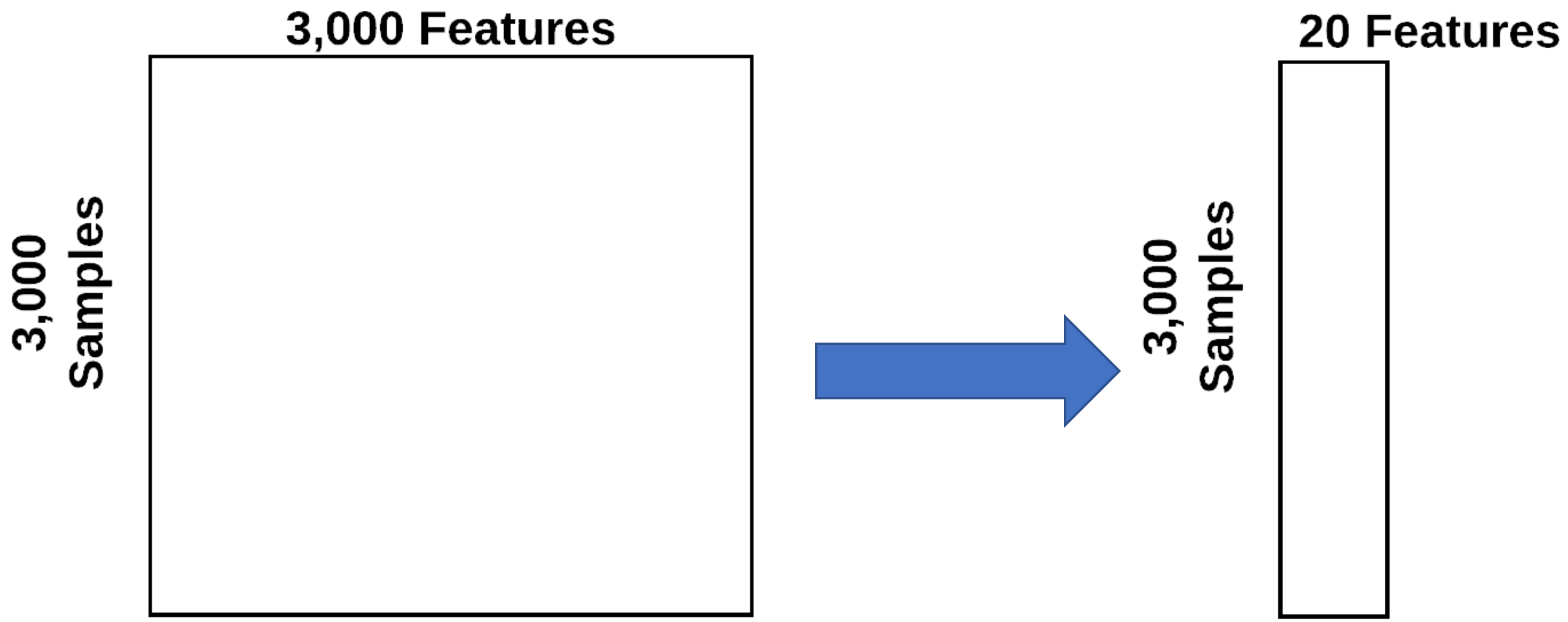
Before



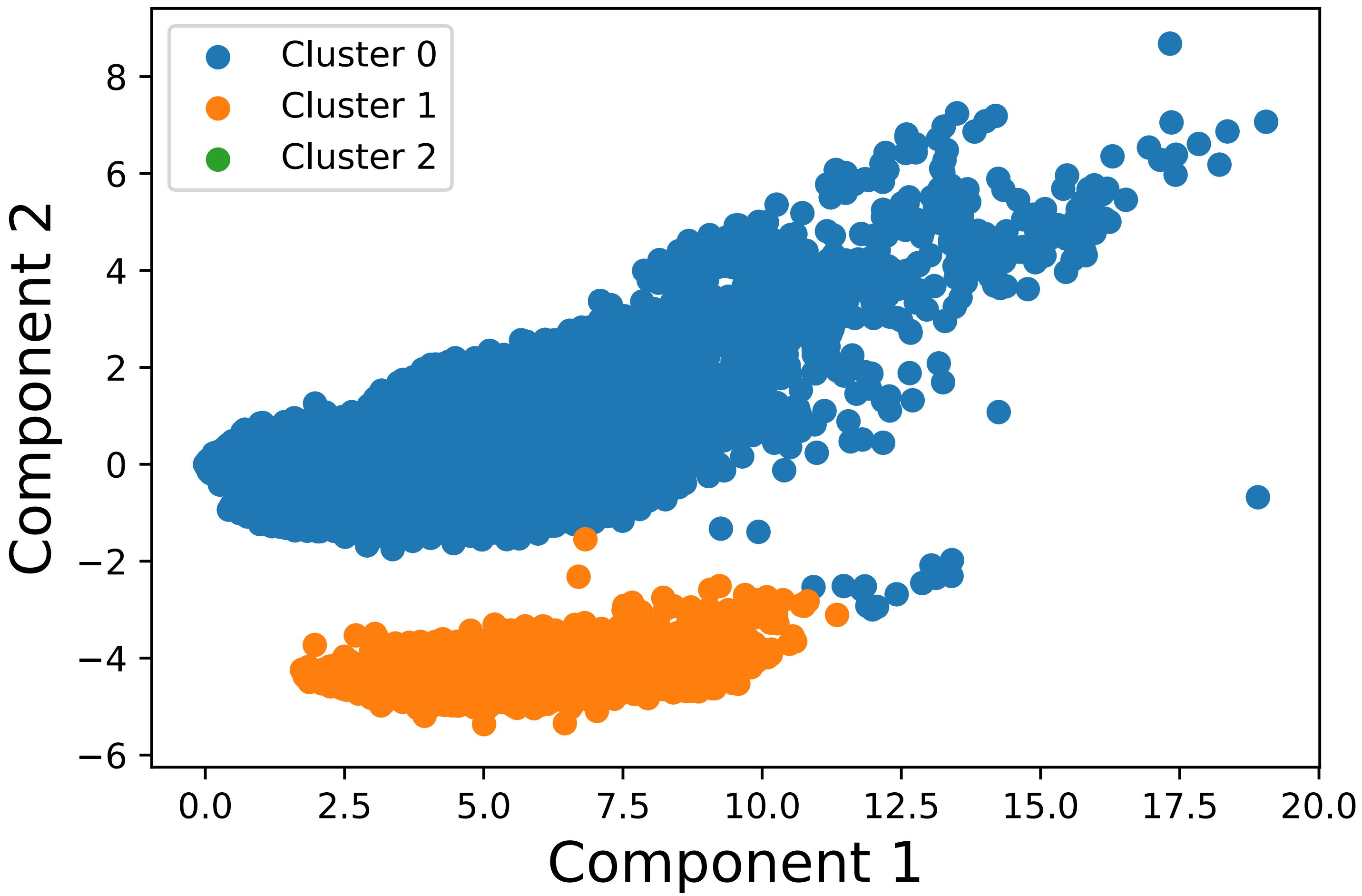
After



Principle Component Analysis



Principal Component Analysis



Discrete Fourier Transform