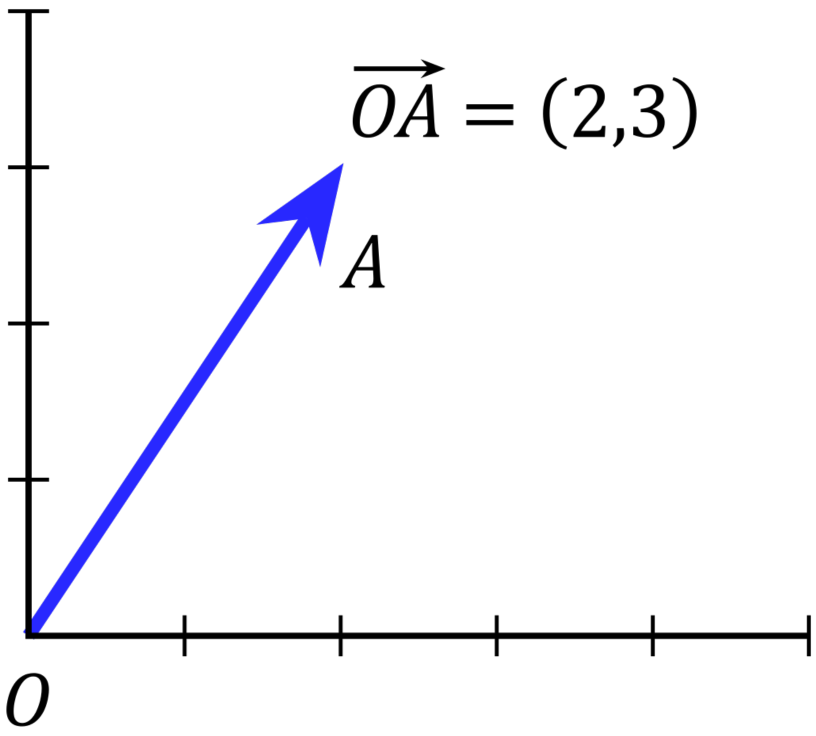
Vectors and Planes

Lecture 3b

CS3400 Machine Learning

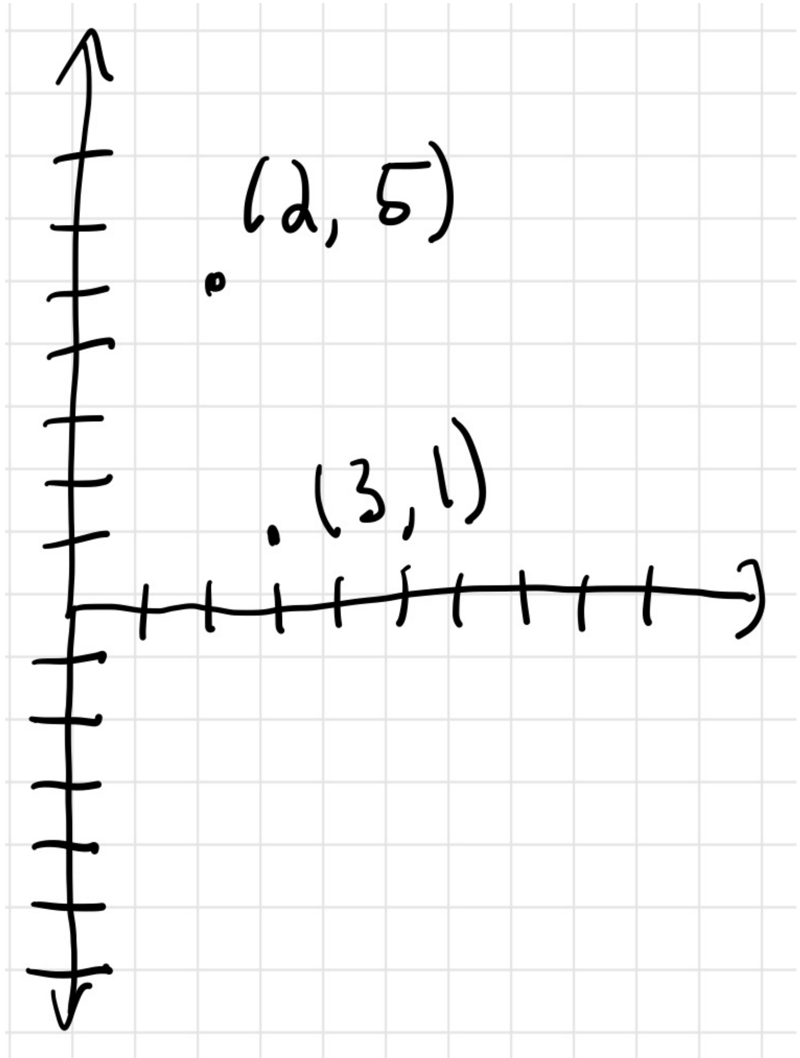
Vectors



- Directed line segment that goes from the origin (e.g., (0, 0)) to a specified point (e.g., (5, 5))
- Written as n numbers with angle brackets (e.g., $\langle 1, 2, 3 \rangle$)
- Variable names are usually lower case with an arrow

$$\vec{v} = \langle 5,5 \rangle$$

Creating a Vector

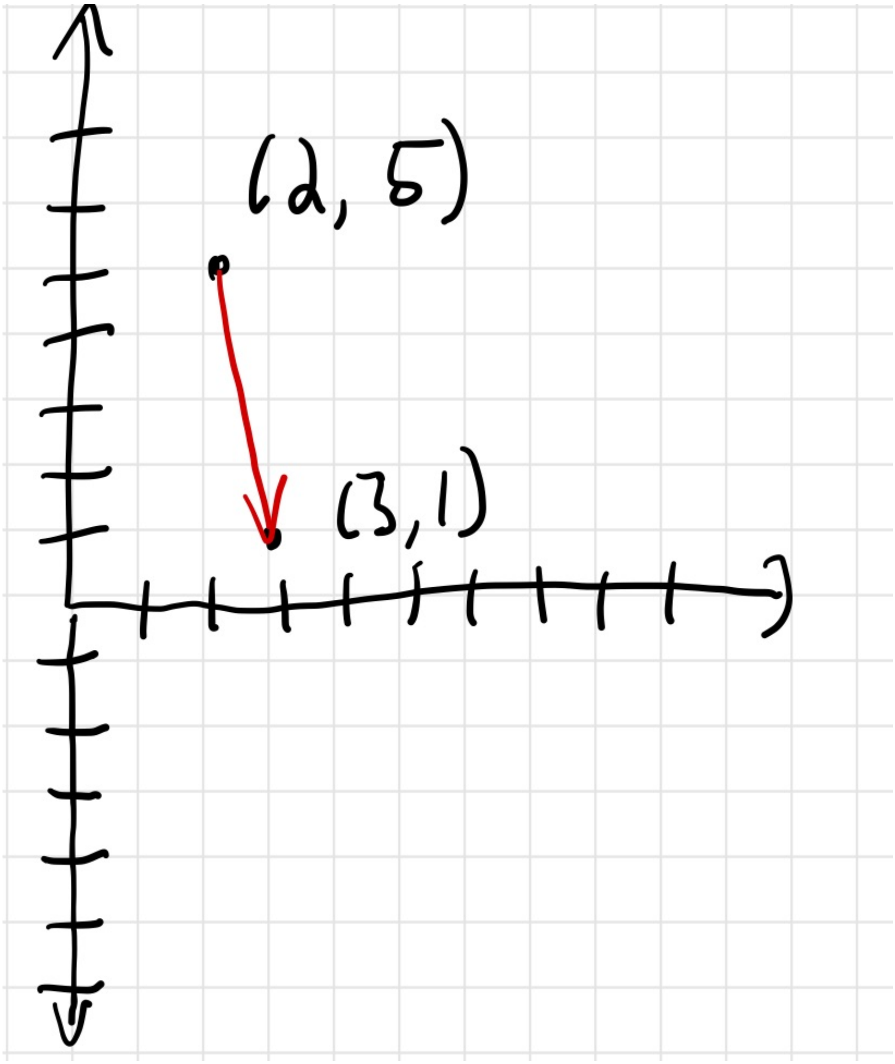


We create vectors from two points

$p_1 = (2, 5)$

$p_2 = (3, 1)$

Creating a Vector

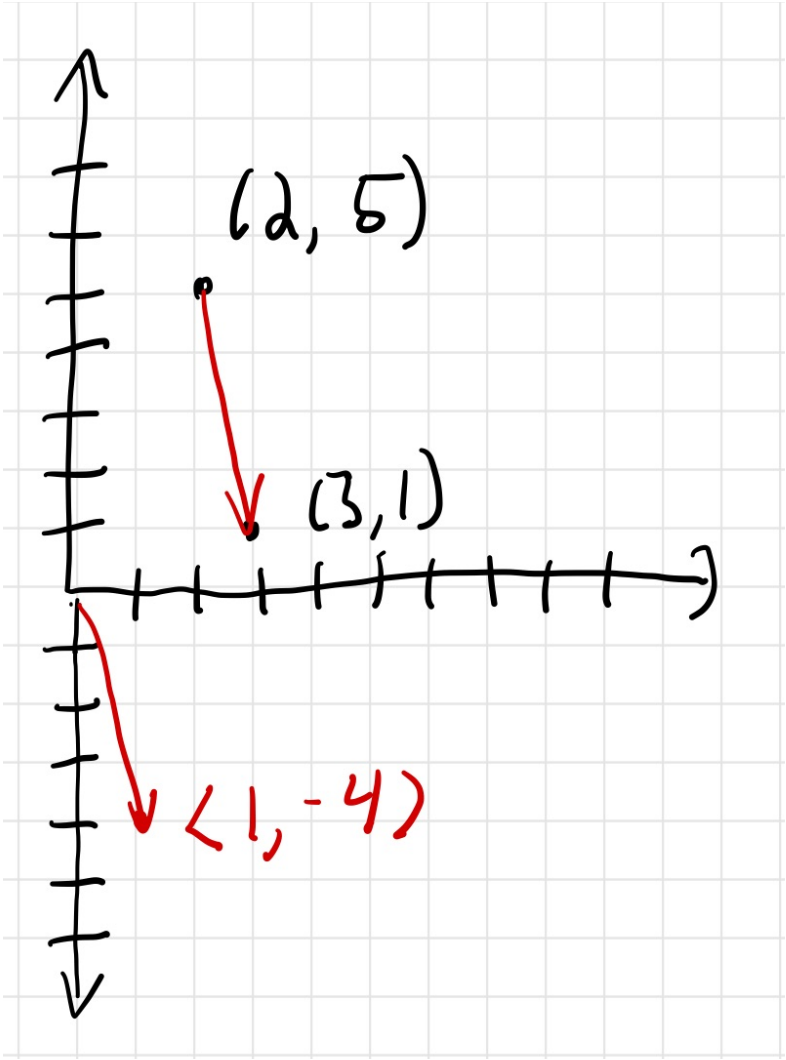


We create vectors from two points

$$p_1 = (2, 5)$$
$$p_2 = (3, 1)$$

$$\vec{v} = p_1 - p_2$$
$$= \langle 3 - 2, 1 - 5 \rangle$$
$$= \langle 1, -4 \rangle$$

Creating a Vector



We create vectors from two points

$$p_1 = (2, 5)$$

$$p_2 = (3, 1)$$

$$\vec{v} = p_1 - p_2$$

$$= <3 - 2, 1 - 5>$$

$$= <1, -4>$$

Vectors in Numpy

Vectors are represented as ndarrays in Numpy

- Created from Python lists

```
v = np.array([1, 2, 3,
4, 5])
```

- Create vector of all zeros:

```
v = np.zeros(length)
```

- Create vector of all ones:

```
v = np.ones(length)
```

Scalar Multiplication

Vectors can be multiplied by a scalar

$$\vec{v} = < 5, 3, 1 >$$

$$5\vec{v} = < 5 \cdot 5, 5 \cdot 3, 5 \cdot 1 > \\ = < 25, 15, 5 >$$

Numpy

```
v = np.array([5, 3, 1])

print(5 * v)

[25 15  5]
```

Vector Addition

Vectors can be added

$$\vec{u} = < 1, 2, 3 >$$
$$\vec{v} = < 5, 3, 1 >$$

$$\vec{u} + \vec{v} = < 1 + 5, 2 + 3, 3 + 1 >$$
$$= < 6, 5, 4 >$$

Numpy

```
u = np.array([1, 2, 3])
v = np.array([5, 3, 1])

print(u + v)

[6 5 4]
```

Dot Product

The dot product is a fundamental operation of linear algebra.

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$\vec{u} = < 1, 2, 3 >$$
$$\vec{v} = < 5, 3, 1 >$$

$$\vec{u} \cdot \vec{v} = 1 \cdot 5 + 2 \cdot 3 + 3 \cdot 1$$
$$= 14$$

Numpy

```
u = np.array([1, 2, 3])
v = np.array([5, 3, 1])

print(np.dot(u, v))
```

```
14
```

Magnitude

The magnitude, or length, of a vector can be computed using the dot product:

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$$

$$= \sqrt{u_1 u_1 + u_2 u_2 + \cdots + u_n u_n}$$

$$= \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Numpy

```
v = np.array([1, 1, 1, 1])

print(np.sqrt(np.dot(v,
      v)))

2
```

Euclidean Distance

We can also use magnitude to calculate Euclidean distance

$$\|\vec{u} - \vec{v}\| = \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})}$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

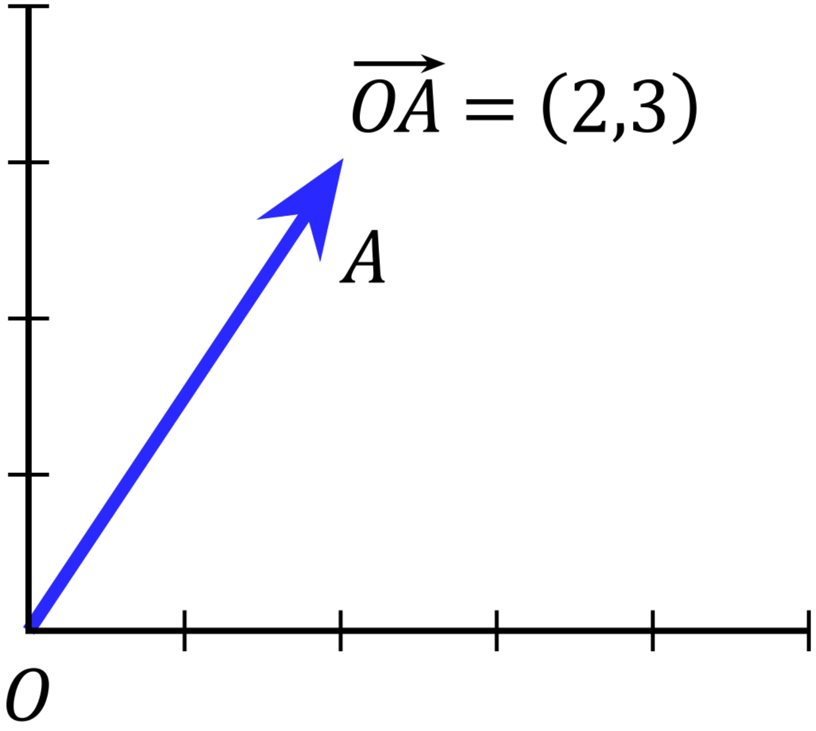
Numpy

```
u = np.array([1, 1, 1, 0])
v = np.array([1, 1, 1, 1])

diff = u - v
print(np.sqrt(np.dot(diff,
diff)))

1
```

Vectors: Magnitude and Direction



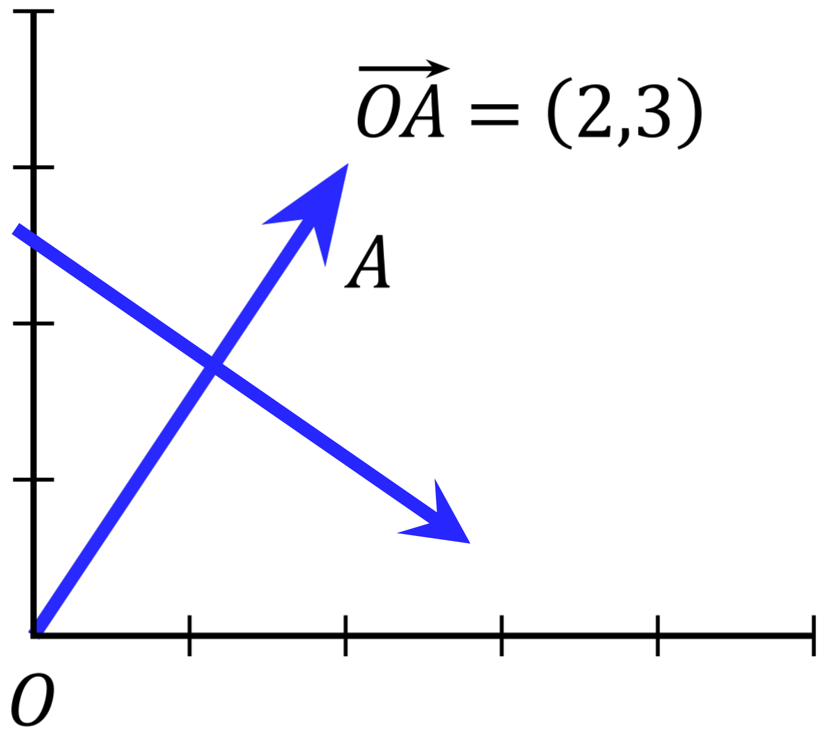
- Vectors have magnitudes (lengths) and direction
- Magnitude

$$\|\vec{v}\| = \vec{v} \cdot \vec{v}$$

- Direction (unit-length vector)

$$\frac{\vec{v}}{\|\vec{v}\|}$$

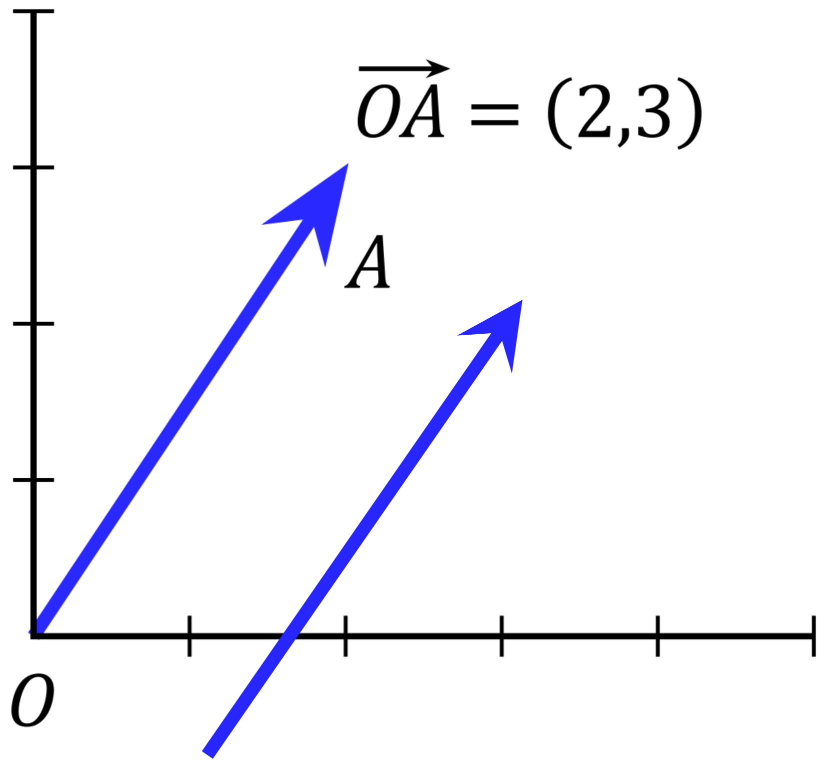
Vectors: Dot Product



If two vectors are perpendicular (orthogonal), then their dot product is 0

$$\vec{u} \cdot \vec{v} = 0$$

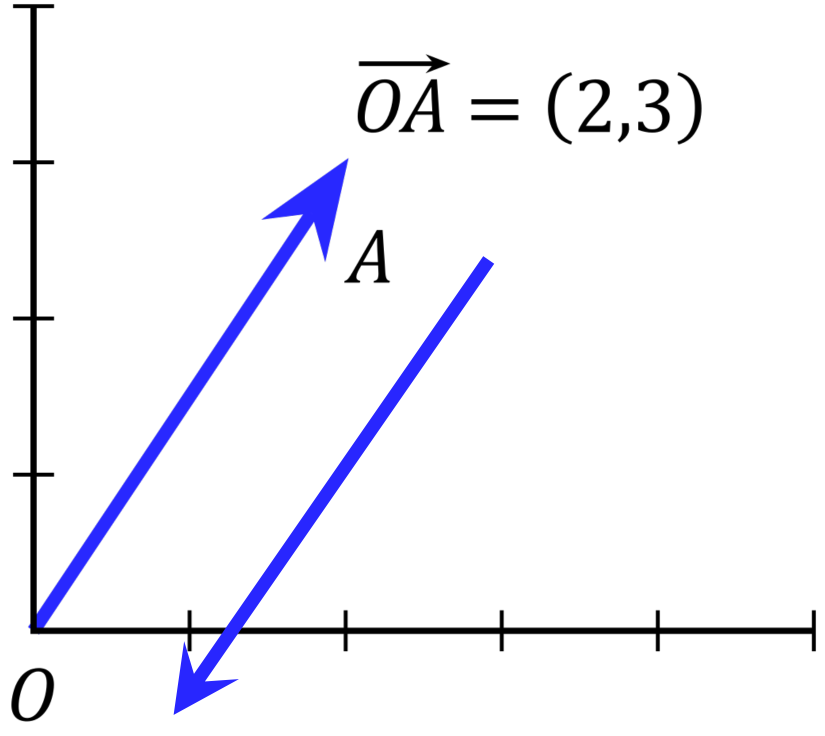
Vectors: Dot Product



If two vectors are parallel, then the dot product of their unit vectors is 1

$$\frac{\vec{u}}{||\vec{u}||} \cdot \frac{\vec{v}}{||\vec{v}||} = 1$$

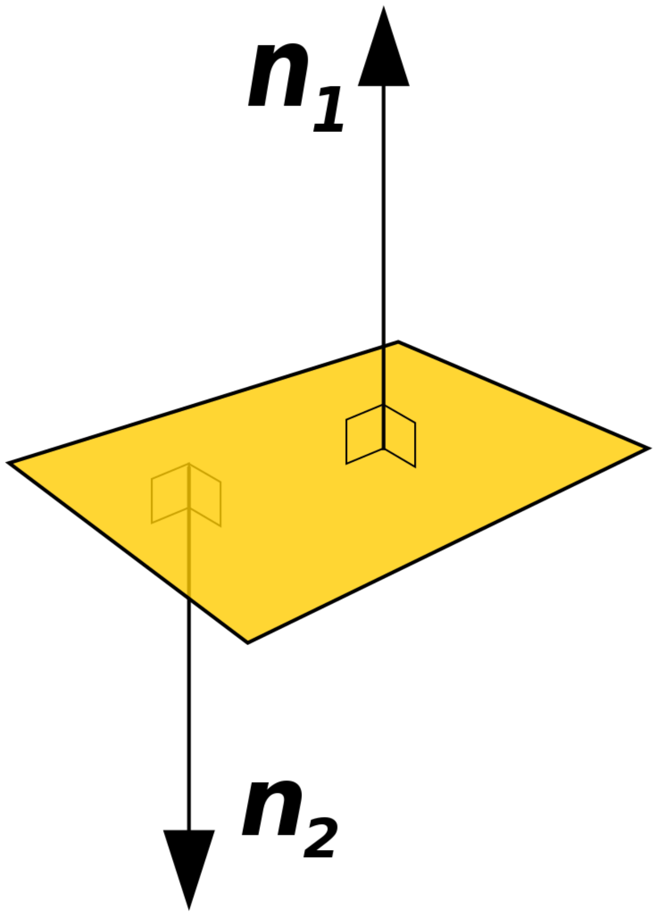
Vectors: Dot Product



If two vectors are anti-parallel, then the dot product of their unit vectors is -1

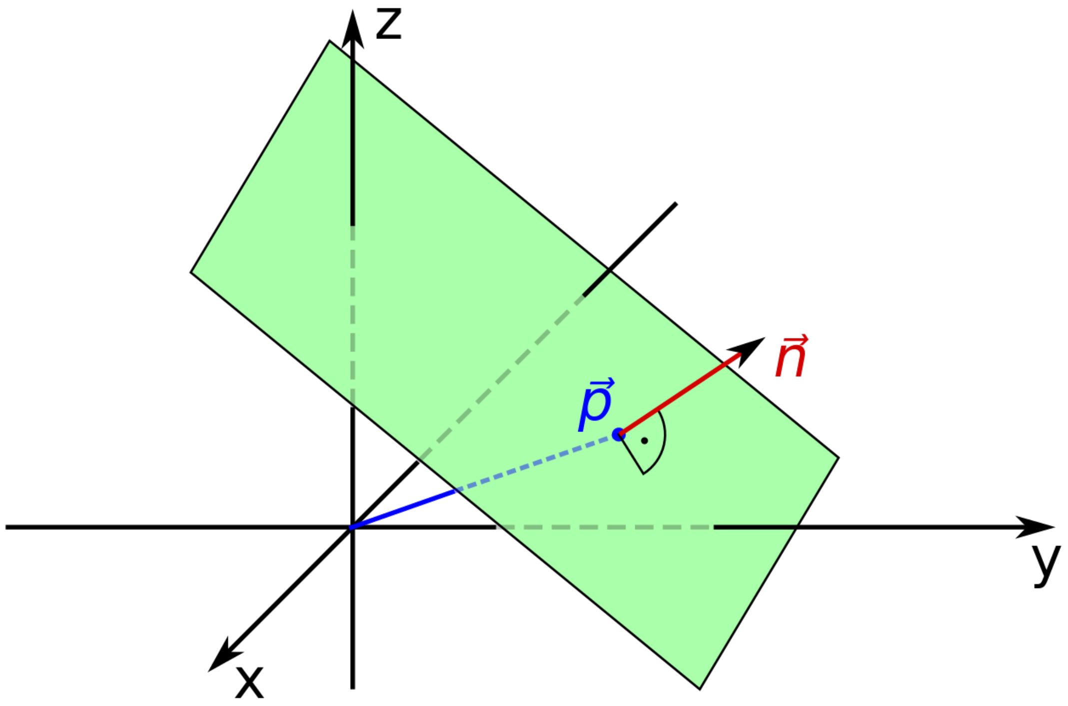
$$\frac{\vec{u}}{||\vec{u}||} \cdot \frac{\vec{v}}{||\vec{v}||} = -1$$

Normal Vectors



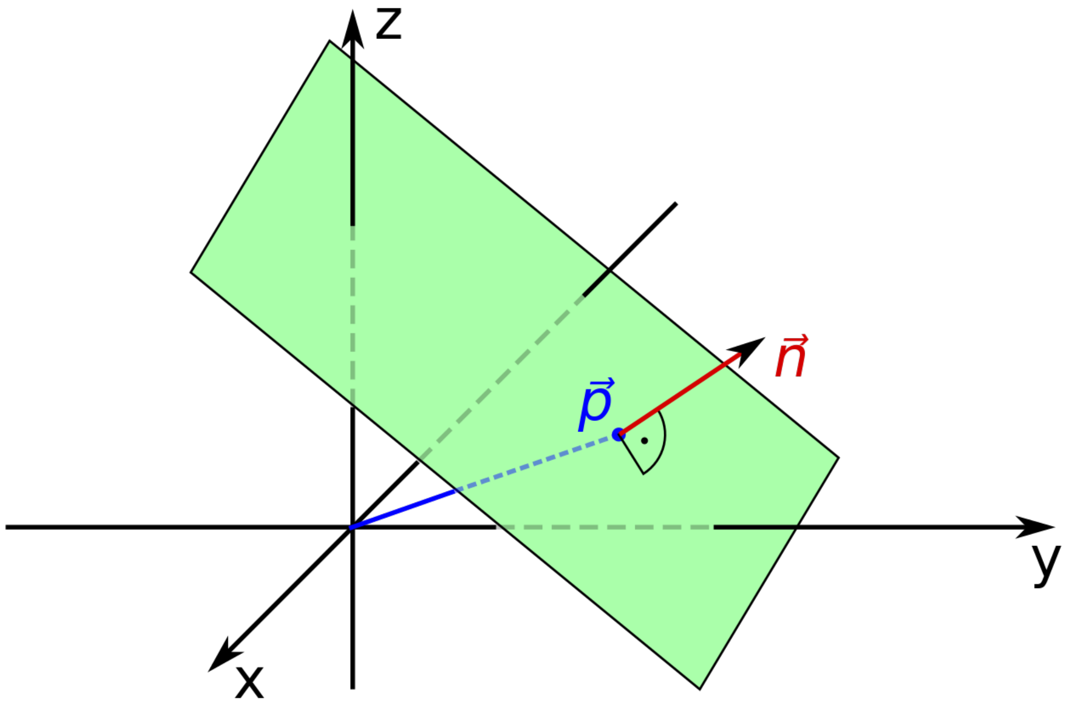
- Normal vectors are perpendicular (orthogonal) to surfaces (e.g., planes, lines)
- Normal vectors are often used to:
  - Define a geometric object
  - Indicate its orientation

Hyper Planes



- A plane is a flat geometric object that divides a space in half
- In a space with n dimensions, a plane is an n – 1 dimensional object (it has no depth)

Defining Hyper Planes

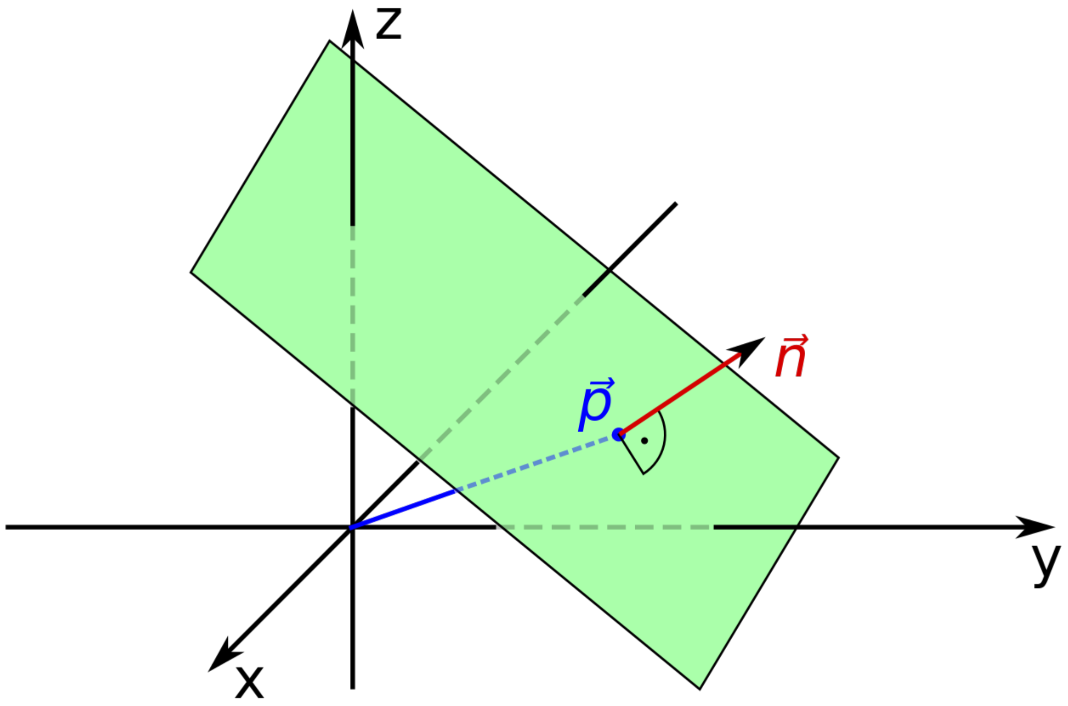


- Planes are defined using "normal form" using a normal vector $\vec{n}$ and a point r_0 that lies on the plane:

$$\vec{n} \cdot (r - r_0) = 0$$

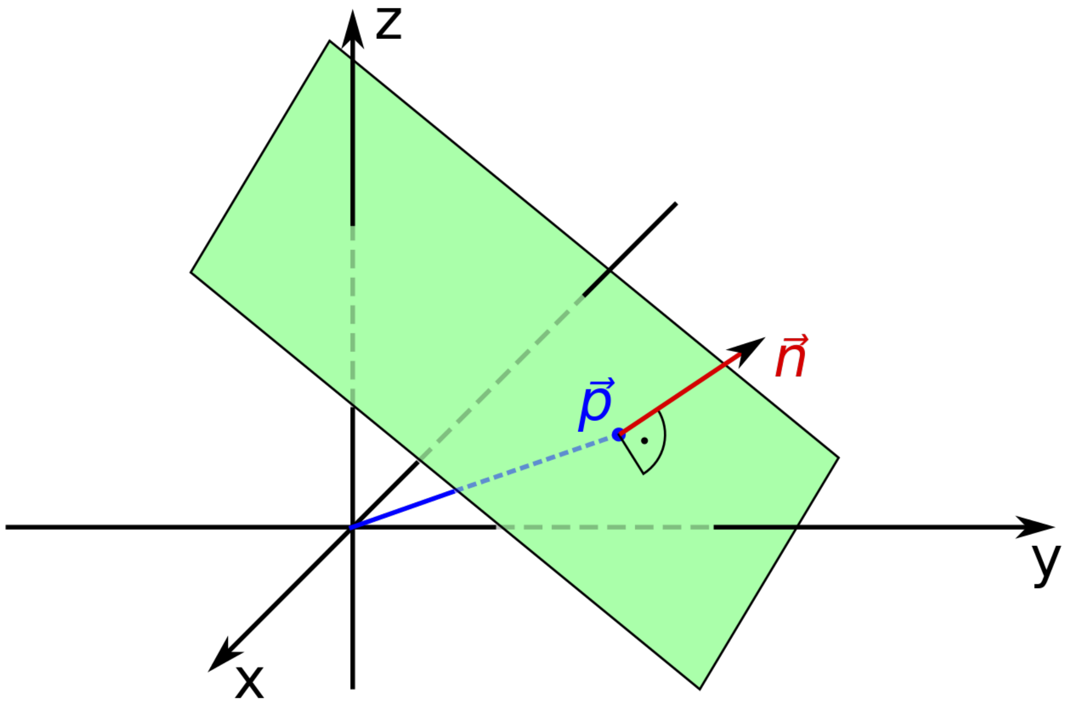
- All points r for which the equation is true lie on the plane
- All other points are not on the plane

Defining Hyper Planes



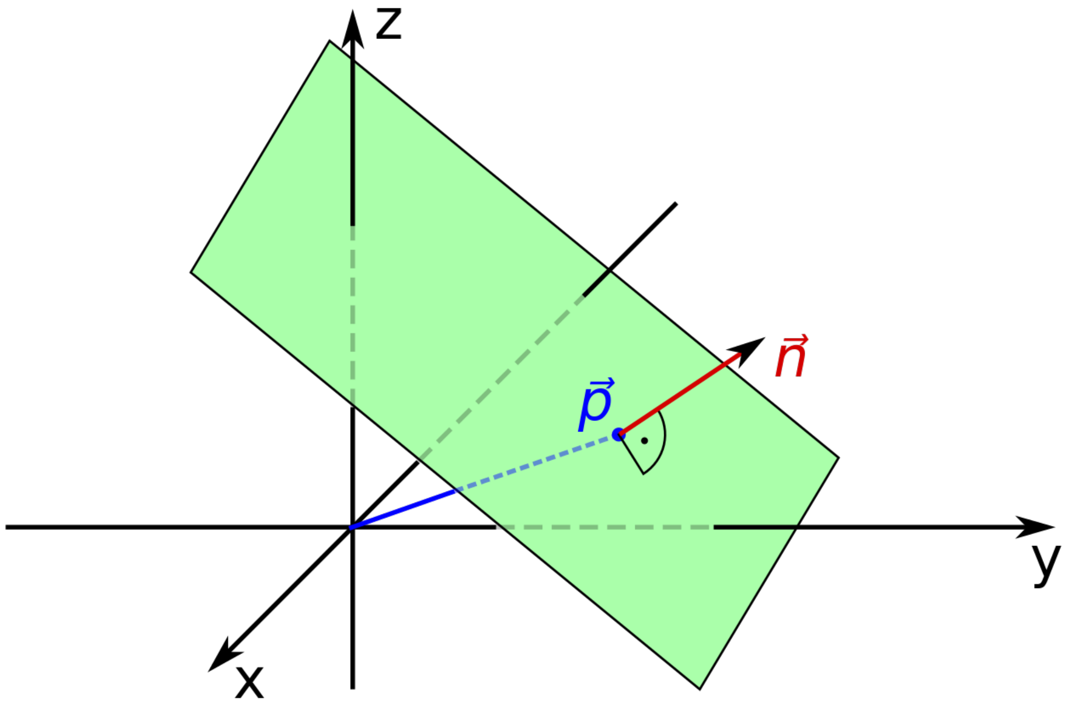
- Note the dot product in the equation:
$$\vec{n} \cdot (r - r_0) = 0$$
- The equation measures the angle between the normal vector and a vector between the two points
- If two vectors are orthogonal, then their dot product is 0

Checking where a point lies relative to a plane



- We can use a hyper plane to divide a space in half
- Knowing that the dot product measures angles, we can use the the definition of a hyper plane to determine on which "side" of the plane a point lies

Checking where a point lies relative to a plane



- If $\vec{n} \cdot (r - r_0) == 0$:
  - Point lies on the plane
- else if $\vec{n} \cdot (r - r_0) > 0$:
  - Point lies on the same side of the plane as the normal vector
- else if $\vec{n} \cdot (r - r_0) < 0$:
  - Point lies on the opposite side of the plane as the normal vector