Process- and object-based thinking in arithmetic

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Plan

- Process-to-object theories.
- Operationalising the constructs.
- Do all children follow a process-to-object route in early arithmetic?
- Remaining questions.

Process-to-object theories

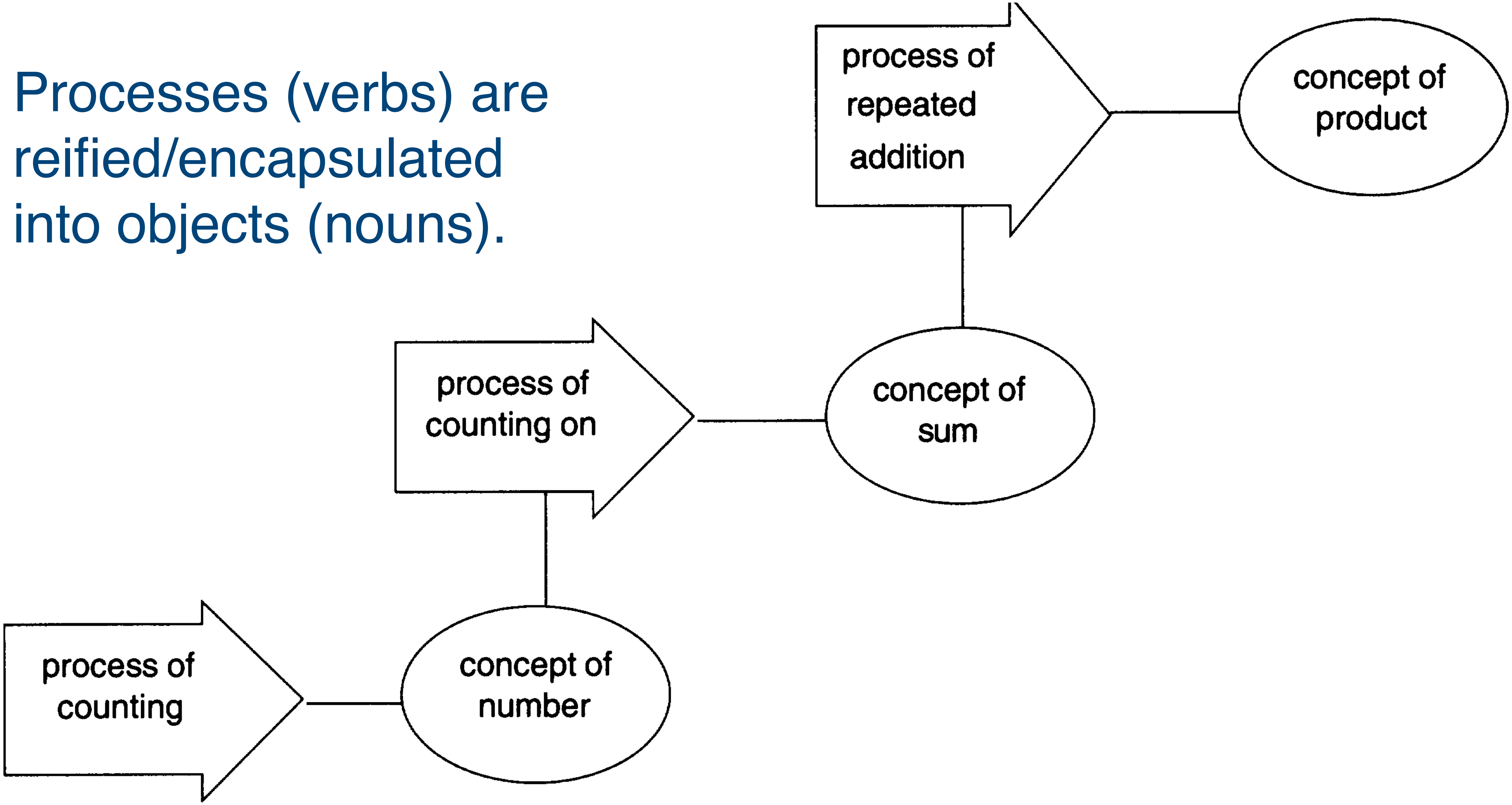
Many mathematics educators have proposed that children learn mathematical concepts by encapsulating, or reifying, processes into objects.

“The procedure, formerly only a thing to be done – a **verb** – has now become an object of scrutiny and analysis; it is now, in this sense, a **noun**.” (Davis, 1984, my emphasis).

(cf. Cottril et al., 1996; Dienes, 1960; Dubinsky, 1991; Gray, Pitta & Tall, 1999; Gray & Tall, 1994; Piaget, 1985; Sfard, 1991)

The development of different conceptions of number.

Processes (verbs) are reified/encapsulated into objects (nouns).



(Gray & Tall, 1994)

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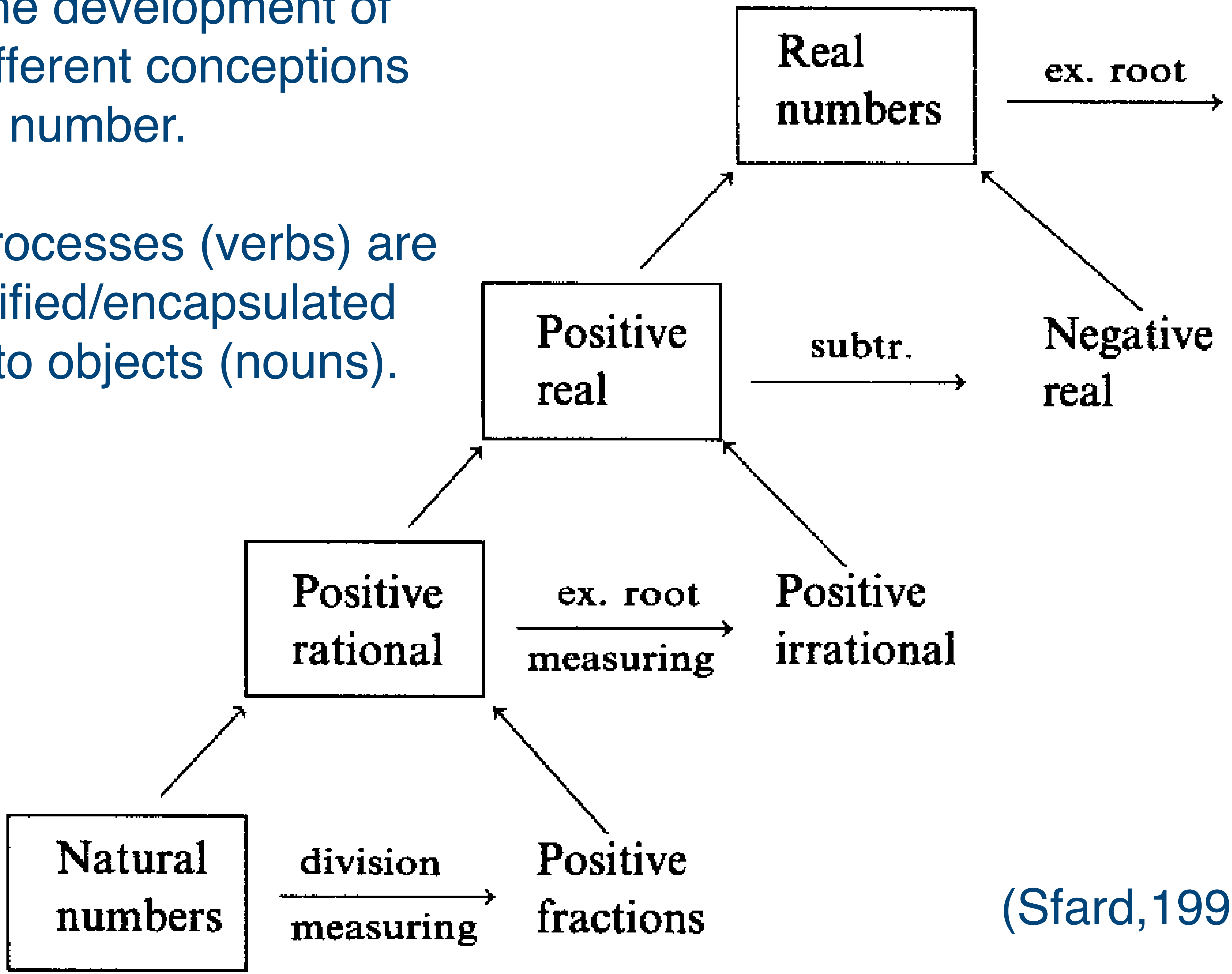


(Sfard,1991)

Process-to-object theories

"the whole of mathematics may therefore be thought of in terms of the construction of structures... mathematical entities move from one level to another; an operation on such 'entitities' becomes in turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures"

(Piaget, 1972)

Procepts

Gray & Tall (1994) noticed that often we use the same symbolism for both process and concept:

$$\frac{3}{4}$$

represents both the process of dividing 3 by 4, and the real number ¾

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

represents both the process of addition (½+¼+⅛+...), and the real number 1

Process-to-object theories

Key claims:

- There are two ways of thinking about (all/most/ some) mathematical concepts: as a process and as an object.
- Process-based thinking treats the concept as a verb, a thing to *do*. Object-based thinking treats the concept as a noun, a thing to *do things to*.
- Often we use ambiguous symbolism which allows for both ways of thinking (procepts).
- Process-based thinking is a prerequisite for object-based thinking.

Process-to-object theories

- Process-to-object theories are very influential in mathematics education theory and practice.
- Teaching interventions are being developed based on these theories (Dubinsky et al., 2005; Tall, 2007; Weber, 2005).
- But how can process-to-object theories be tested?
- "We in mathematics education tend to invent theories, or at least theoretical ideas, at a pace faster than we produce data to possibly refute our theories" (Dreyfus, 2006)

How to operationalise?

- If a theory is to be testable, it is important that its constructs can be operationalised.
- i.e. How can we tell if a child is using process-based or object-based thinking?
- Difficult: Sfard (1991, p. 4) even suggested that “it is practically impossible” to do it!
- We attempted to do this in the domain of arithmetical missing number problems.

Missing number problem

$$15 + 13 - \square = 15$$

Missing number problem

Process-based thinking

$$15 + 13 - \square = 15$$

15 + 13 = 28
28 - 15 = 13
So, the answer is 13

Missing number problem

Object-based thinking

$$15 + 13 - \square = 15$$

Adding 13 and taking away 13 doesn't change anything;
So, the answer is 13

Missing number problem

Process-based thinking

verb: a process
to be performed

$$15 + 13 - \square = 15$$

15 + 13 = 28
28 - 15 = 13
So, the answer is 13

Missing number problem

Object-based thinking

$$15 + 13 - \square = 15$$

Adding 13 and taking away 13 doesn't change anything;
So, the answer is 13

noun: an object to be analysed and manipulated (i.e. its inverse found)

Two sorts of problems

The ambiguity of the +13 symbolism suggests an operationalisation:

Inverse Problem: $15 + 13 - \square = 15$

(both process- and object-based thinking possible)

Control Problem: $14 + 12 - \square = 17$

(only process-based thinking possible)

Indicators:

Process-based thinking will be slow and error-prone.
Object-based thinking will be quick and error-free.

Method

- Two groups of children: 26 from Year 4 (8-9 years old), 33 from Year 5 (9-10 years old).
- Solved 48 missing number problems: 24 inverse problems matched with 24 control problems (i.e. had the same missing number).
- Two sessions (plus first and minus-first)
- Session-order counterbalanced; problem-order randomised.
- Problems presented on-screen and read verbally.
- Children answered verbally, accuracy and RT recorded.

Example Problems

missing number position	Inverse		Control	
	plus first	minus first	plus first	minus first
position 1	$\square$+7-7=13	$\square$-9+9=12	$\square$+14-9=18	$\square$-8+12=16
position 2	13+$\square$-9=13	15-$\square$+13=15	15+$\square$-5=19	18-$\square$+8=13
position 3	16+14-$\square$=16	16-12+$\square$=16	18+9-$\square$=13	12-8+$\square$=16
position 4	15+12-12=$\square$	14-5+5=$\square$	11+11-7=$\square$	10-6+10=$\square$

Each child tackled 3 of each type, giving 48 problems in total. The numbers were chosen to be at the limit of what was expected for this age group: the first and fourth numbers were between 10 and 30, the second and third numbers were between 5 and 20.

Predictions

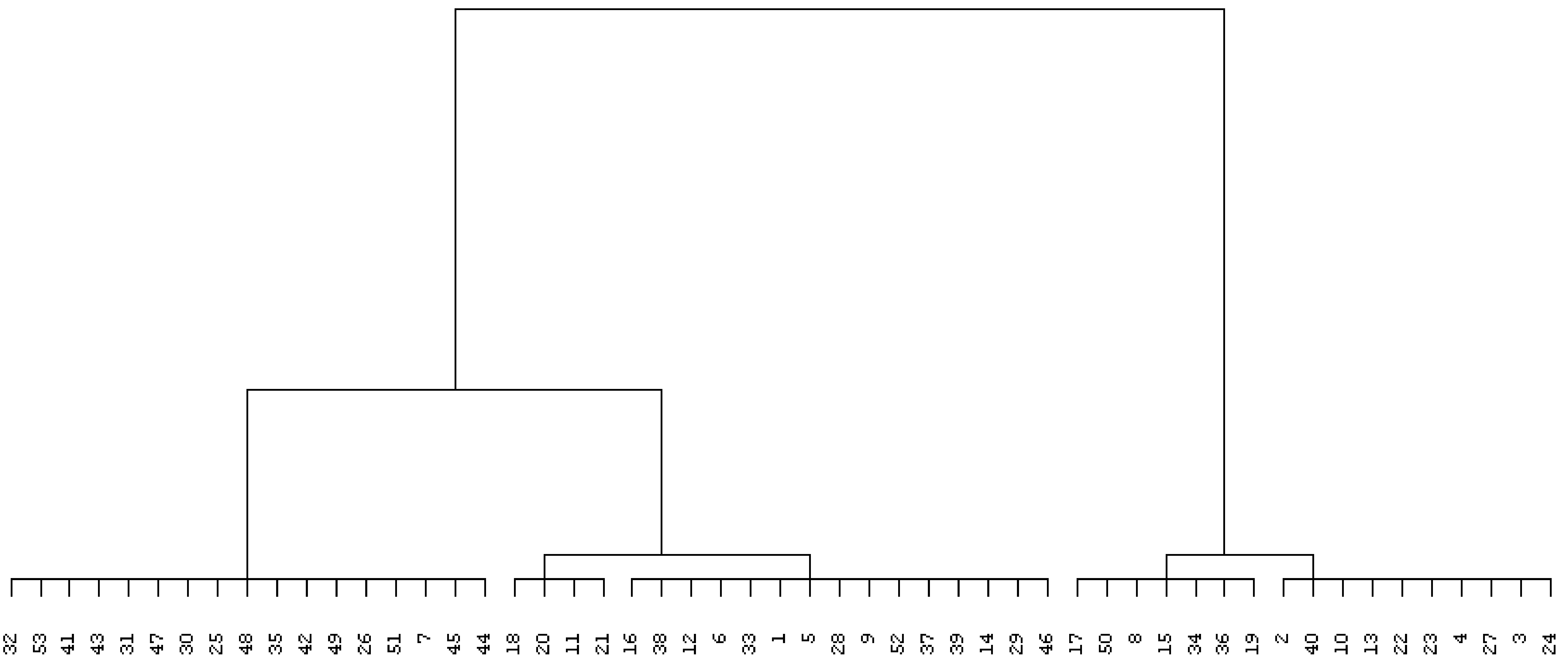
Process-to-object theories would predict:

1. There will be some children who are poor process-based thinkers.
2. There will be some children who are good process-based thinkers but poor object-based thinkers (i.e. who are on the verge of encapsulation).
3. There will be some children who are good process-based and good object-based thinkers (i.e. who have encapsulated).

Analysis

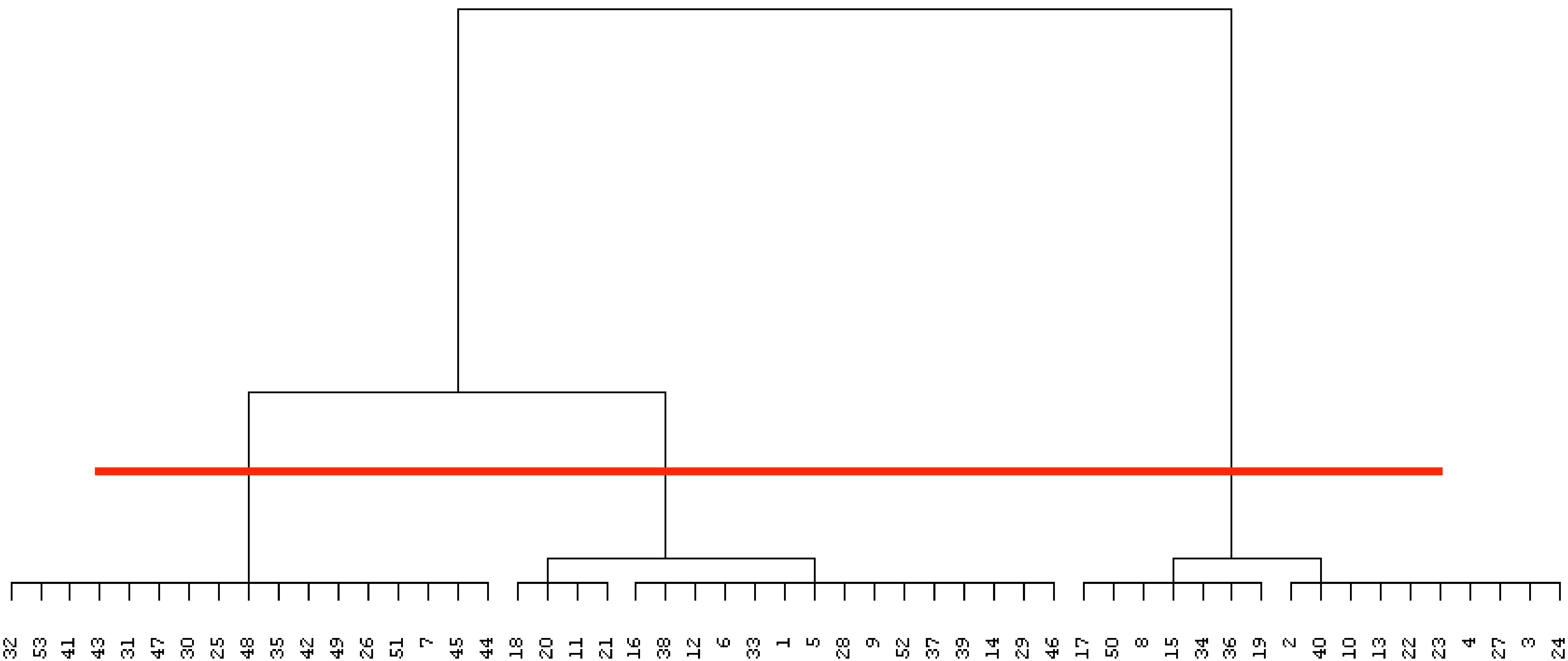
Children's two accuracy scores (for control and inverse problems) entered into a hierarachical cluster analysis.

Cluster Analysis



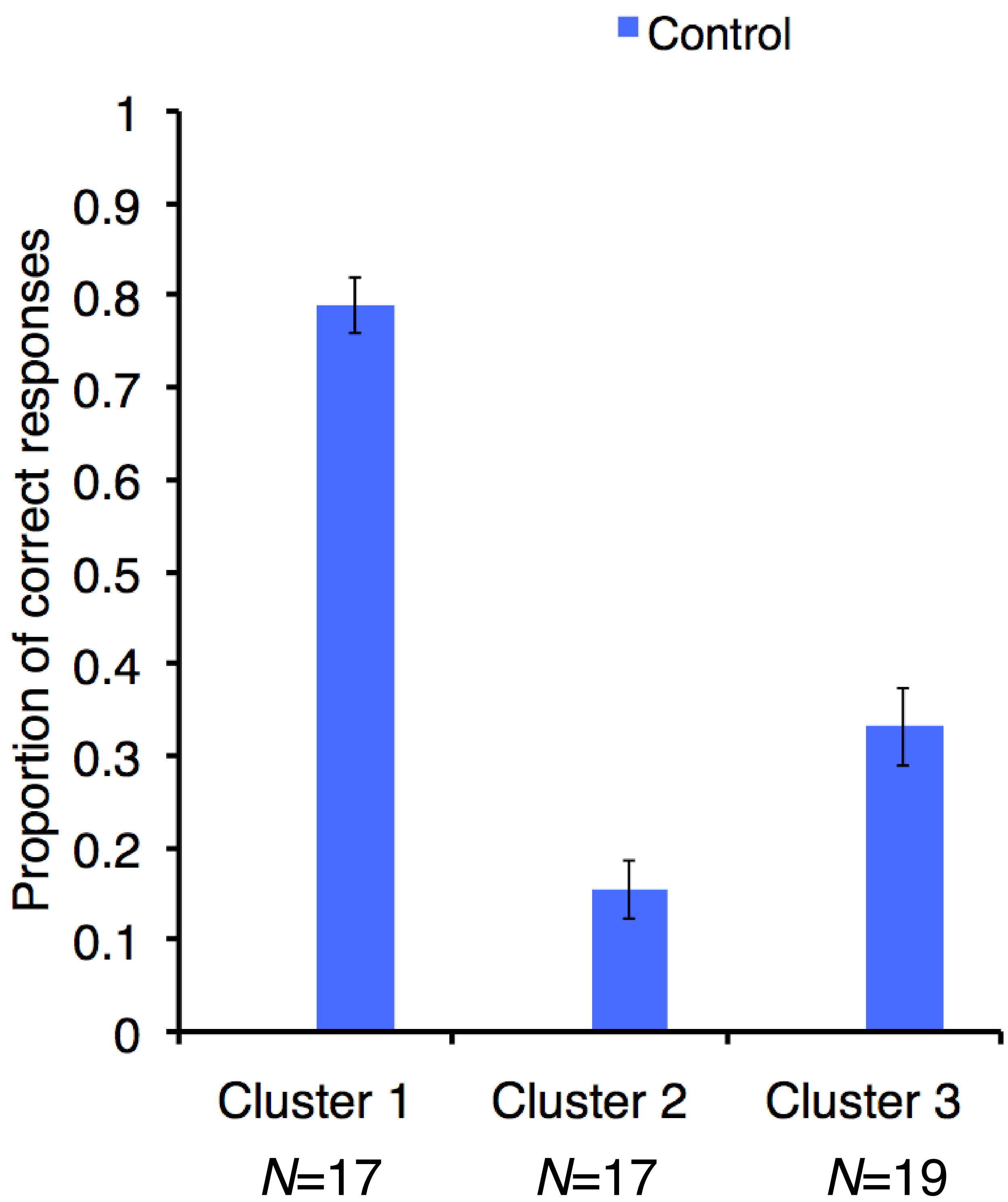
Cluster analysis on: inverse accuracy & control accuracy scores

Cluster Analysis

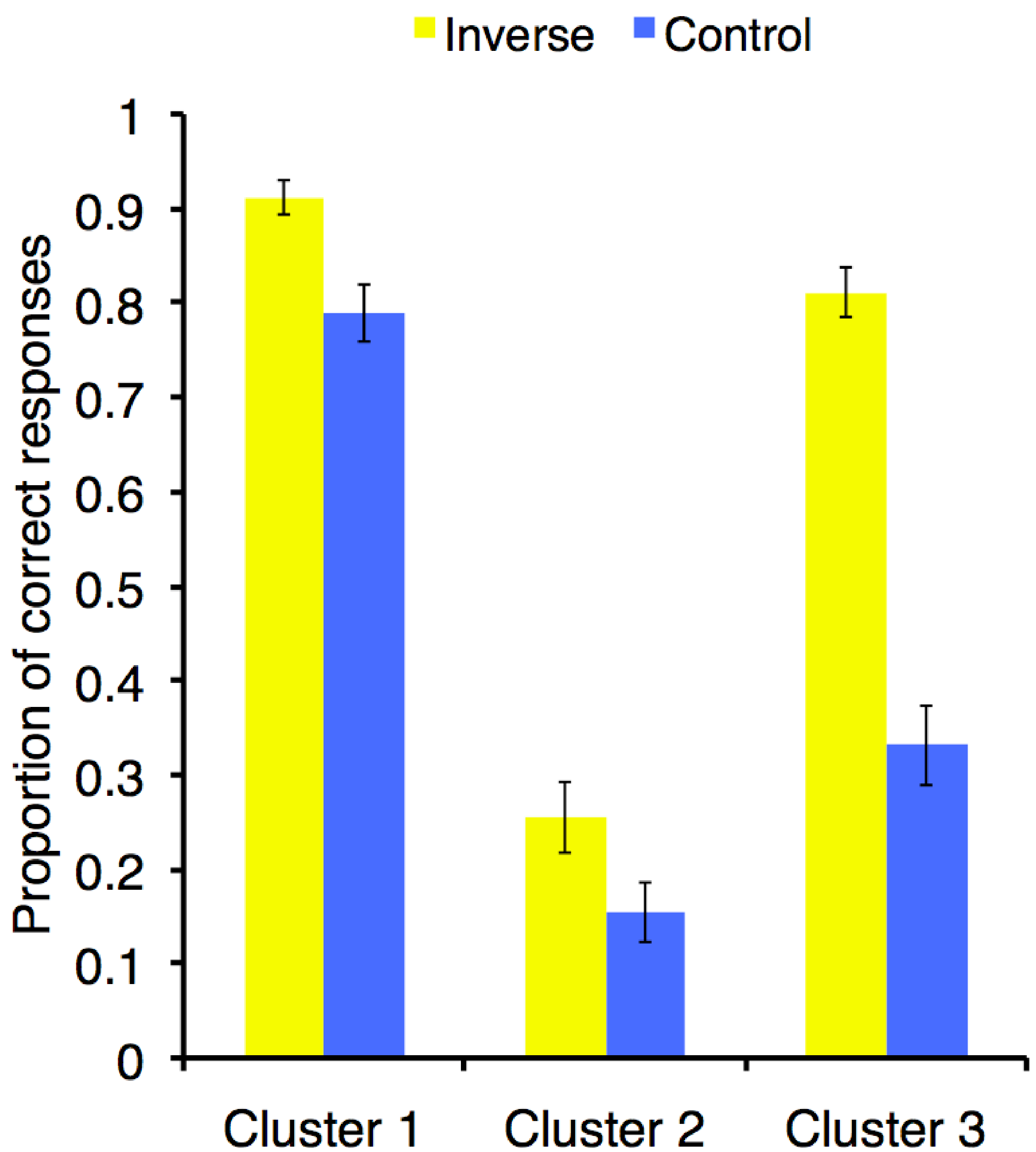


Three cluster solution: 81% of the variance explained

Cluster Analysis: Accuracy

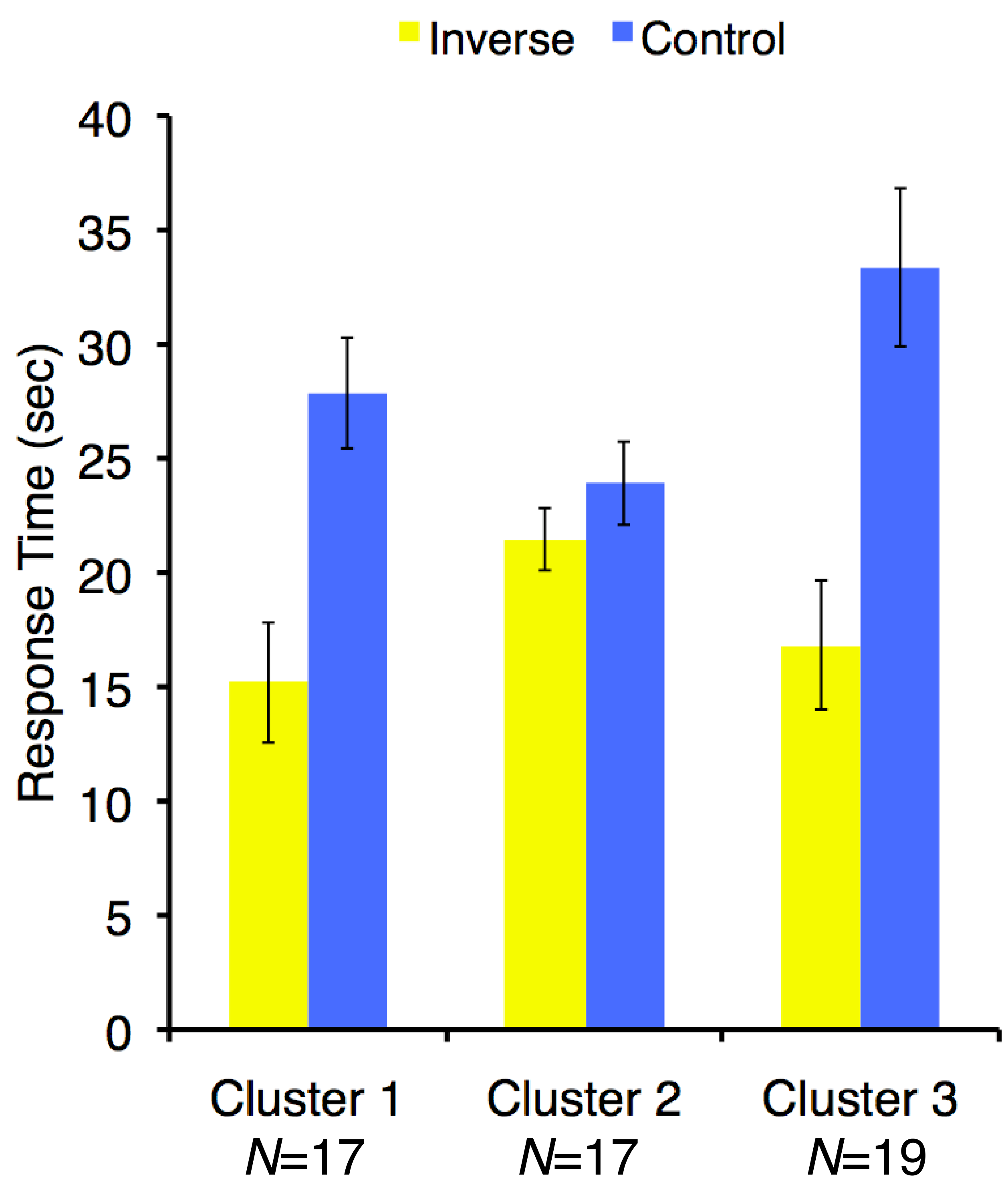


Cluster Analysis: Accuracy

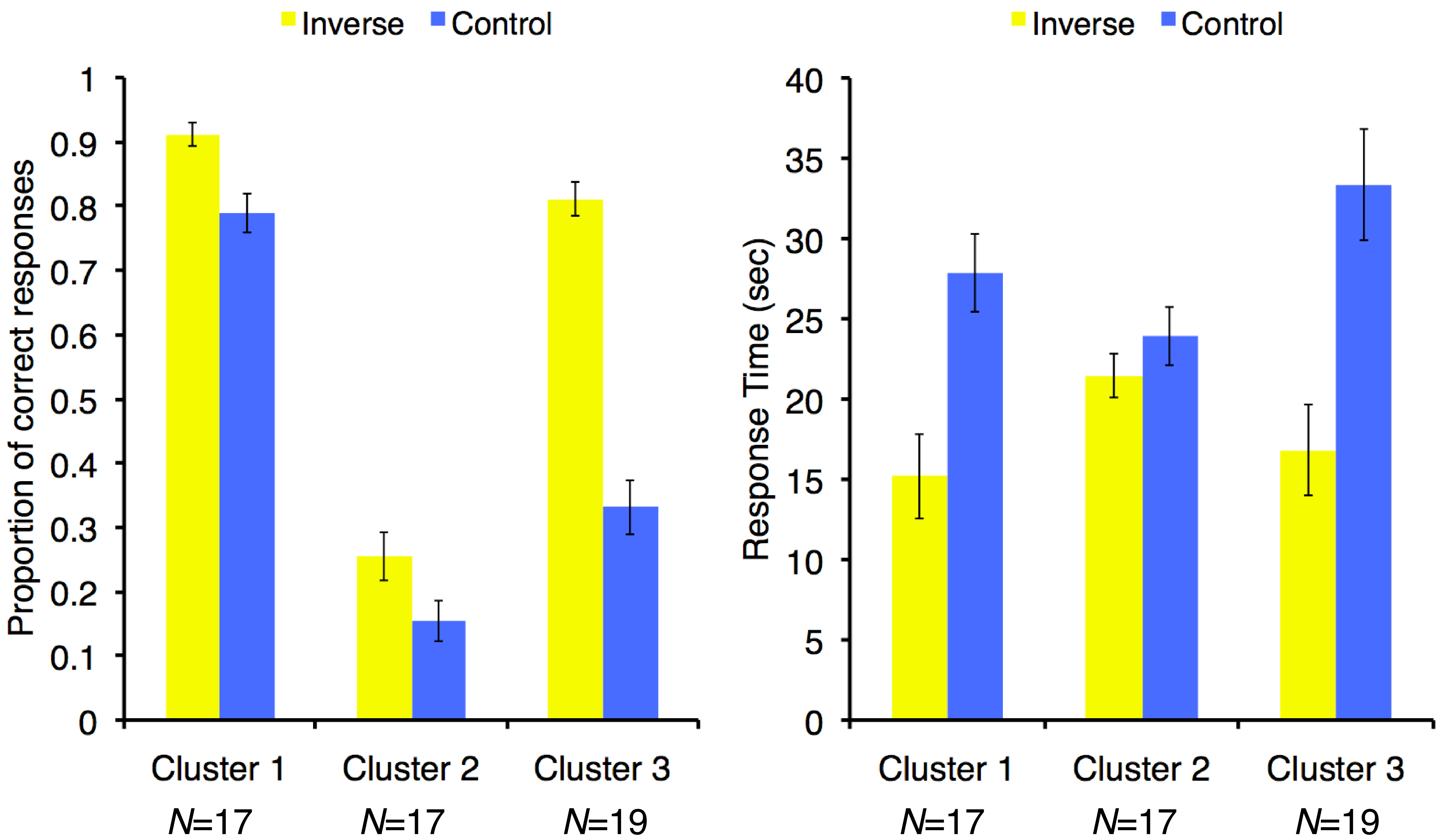


N=17 *N*=17 *N*=19

Cluster Analysis: RTs



Cluster Analysis



Cluster Summaries:

Cluster 1:

- Were successful at both inverse and control problems. Took longer to complete control problems.
- Successfully used object-based thinking on inverse problems, and process-based thinking on control problems.

Cluster 2:

- Were unsuccessful at both inverse and control problems. No difference in RT between problem types.
- Unsuccessfully used process-based thinking throughout.

Cluster 3:

- Were unsuccessful at control problems, but successful at inverse problems. Took longer to tackle control problems.
- *Successfully* used object-based thinking on inverse problems, but *unsuccessfully* used process-based thinking on control problems.

Predictions

Process-to-object theories would predict:

1. There will be some children who are poor process-based thinkers. } Cluster 2

2. There will be some children who are good process-based thinkers but poor object-based thinkers (i.e. who are on the verge of encapsulation). } not found

3. There will be some children who are good process-based and good object-based thinkers (i.e. who have encapsulated). } Cluster 1

Cluster 3

- The third cluster was *the opposite* to that predicted by process-to-object theories.
- It contained children who, on this task, were object-based thinkers but not process-based thinkers.
- Possible explanation: these children follow an object-to-process developmental route.

Alternative Explanations

Had the children in Cluster 3 been taught the inversion strategy?

Both class teachers reported that they had not taught the inversion short-cut strategy. All three clusters were present in both classes (i.e. different teaching practices cannot account for the different clusters).

Reviewer's question: Did Cluster 3 use a pattern matching strategy? (i.e. by 'crossing out'?)

There was no evidence of this. A subset of the control problems were of the form $11 + 11 - \square = 5$. Children's scores on this subset were examined, and no evidence of a pattern matching strategy was found.

Was our operationalisation 'unfair' to PtO theories?

We don't think so. But the onus should be on theorists to provide an operationalisation of their theories.

Conclusions

- Process-to-object theories are very influential in mathematics education theory and practice.
- Teaching interventions are being developed based on these theories (Dubinsky et al., 2005; Tall, 2007; Weber, 2005).
- The evidence from this study suggests that at least some children follow an object-to-process route in early arithmetic.

Open questions

- Can these findings be replicated in the context of arithmetic by other researchers?
- Can these findings be generalised to other areas of mathematics where process-to-object theories have been applied? (e.g. functions, limits, cardinal numbers, etc).
- Do *all* students follow an object-to-process route? Or only *some*? Does this depend on the area of mathematics?
- How, if they exist, do children on the object-to-process route cope with pedagogy designed with process-to-object theories in mind?

thank you

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