PROCESS- AND OBJECT-BASED THINKING IN ARITHMETIC

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Many influential theorists have proposed that learners construct mathematical objects via the encapsulation (or reification) of processes into objects. These process-to-object theories posit that object-based thinking comes later in the developmental path than process-based thinking. In this paper we directly test this hypothesis in the field of early arithmetic. An experiment is reported which studied 8 and 9 year-old children's use of the inverse relationship between addition and subtraction. We demonstrate that a subset of children were unable to solve arithmetic problems using process-based thinking, but that, nevertheless, they were able to use the inverse relationship between addition and subtraction to solve problems where appropriate. The implications of these findings for process-to-object theories are discussed.

PROCESS-OBJECT THEORIES IN MATHEMATICS EDUCATION

Throughout the history of mathematics education research, many theorists have proposed that a key component of coming to know mathematics is related to the encapsulation, or reification, of processes into objects (e.g. Davis, 1984; Dienes, 1960; Dubinsky, 1991; Gray, Pitta, & Tall, 1999; Gray & Tall, 1994; Piaget, 1985; Sfard, 1991). During the 1990s several influential theorists developed specific theories which state that students come to learn mathematics, and in particular arithmetic, in this way. These theories are in wide use for analysing mathematics learning, and designing instruction (e.g. Dubinsky, Weller, McDonald, & Brown, 2005; Tall, 2007; Weber, 2005). During the 1990s, three major flavours of process-object theories became influential in the field.

Sfard (1991) spoke of a three stage process to concept development. First, she claimed, comes the interiorization stage: a process or operation is performed on a familiar mental object. If a learner is able to consider the process without actually performing it, they are said to have interiorized it. Later, the learner may 'reify' the process. Reification was described as a sudden "ontological shift" when the learner sees a familiar object in a new light: the process becomes a static structural object, and can serve as the base object for further, more advanced, processes.

Dubinsky (1991; Cottril et al, 1996) proposed an essentially identical developmental path in what became known as APOS theory. The APOS theorists suggested that objects (O) are encapsulated processes (P), which in turn are interiorised actions (A). In some cases learners may coordinate a series of objects, processes and actions into schemas (S). The developmental path was claimed to follow the acronym of the theory's name: A first, followed by P, O and S.

Gray and Tall (1994) agreed with the process/object dichotomy proposed by Sfard (1991) and Dubinsky (1991), but extended it by emphasising the importance of mathematical symbolism. Defining a procept as a symbol which flexibly and ambiguously represents both process and object, they claimed that a key barrier to success in mathematics learning was bridging the proceptual divide. That is to say that children must successfully be able to encapsulate processes into objects and flexibly move between these two conceptualisations using ambiguous symbolism.

Although the process-object theorists agreed that once encapsulation has been achieved learners can flexibly move between process and object conceptualisations, they differed somewhat about the ordering of the two parts to concept understanding. Gray and Tall (1994) made no strong claims about the ordering of process- and object-based thinking. But both Sfard (1991) and Cottril et al (1996) made explicit predictions, claiming a distinct ordering of the process and object conceptions (operational and structural conceptions in Sfard's terms). They suggested that the process came first, followed by the object:

> We have good reasons to expect that in the process of concept formation, operational conceptions would precede the structural (Sfard, 1991, p. 10).

> An object is constructed through the encapsulation of a process (Cottril et al, 1996, p. 171).

In this sense both APOS and Sfard's reification can be said to be process-to-object theories of mathematical development. Our goal in this paper is to question the universality of this ordering in children's arithmetic development. We will argue that a subset of children actually exhibit a different developmental path in the domain of arithmetic. To proceed with this argument we first consider the nature of object-based thinking (also called structural or conceptual thinking), and how it can be operationalised for the empirical researcher.

WHAT IS OBJECT-BASED THINKING?

A key component to testing the developmental route proposed by process-object theorists is operationalising the notion of object-based thinking. Davis (1984) clearly described the distinction:

> The procedure, formerly only a thing to be done - a verb - has now become an object of scrutiny and analysis; it is now, in this sense, a noun (p. 30).

But how can the researcher determine whether a student is using process- or object-based thinking? Clearly an operationalisation which merely attributes object-based thinking to those who succeed and process-based thinking to those who fail will lead to circularity. Sfard (1991) recognised this problem, but failed to offer a solution, instead she claimed that "it is practically impossible to […] formulate exact definitions of the structural and operational ways of thinking" (p. 4). But this is an unsatisfactory situation: without exact definitions of these terms, or at least exact operationalisations within restricted domains, the process-to-object theories remain unfalsifiable.

Others have offered more precise characterisations than Sfard (1991). Cotrill et al (1996) emphasised the importance of whether or not the learner could recognise and construct transformations of the new object. They wrote that a process becomes an object when "the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations" (p. 171). In the context of early arithmetic, Gray and Tall (1994) agreed, writing that proceptual thinking included the ability to see symbolism "as the representation of a mental object that may be decomposed, recomposed, and manipulated at a higher level" (p. 125). In a later paper Tall, Thomas, Davis, Gray and Simpson (2000) attempted a direct characterisation of object-based thinking. They asked "what is the object of the encapsulation of a process?", and answered

> [It] is a way of thinking which uses a rich concept image to allow it to be a manipulable entity, in part, by using mental processes and relationships to *do* mathematics (Tall et al, 2000, p. 239).

Essentially Tall et al. suggested that object-based thinking revolves around seeing (what was) a procedure as an object to be manipulated without the need to perform the procedure.

In this paper we situate our investigation into process-to-object theories within the context of early arithmetic (e.g. Davis, 1984; Gray et al., 1999; Kamii, 1985). Specifically we interrogate children's ability to use the relationship between addition and subtraction to solve simple missing number arithmetic problems. A similar approach has been used by several earlier researchers interested in arithmetic development (e.g. Bryant, Christie, & Rendu, 1999; Rasmussen, Ho, & Bisanz, 2003).

Given the missing number problem 14+□-11=14, at least two solution methods are possible. The process-based thinking route would involve explicit calculation of 14 - 11, followed by 14 - 3. An alternative method would be to use the relationship between addition and subtraction to construct the inverse of "-11", determining that the answer must be "+11". Based on Davis's (1984) and Tall et al.'s (2000) characterisation, this method - which relies upon knowledge of the relationship between addition and subtraction, and the construction of the inverse of a subtraction - would appear to use object-based thinking. To perform such an operation the child must treat "-11" as a noun, not a verb: they must not perform the process of -11, but instead must perform a process on -11 by constructing its inverse. Although this higher level process may be straightforward, it does nevertheless involve object-based thinking: "-11" must be treated as a object to be manipulated, not as a process to be performed.

These observations suggest a reasonable way of operationalising the constructs of process- and object-based thinking within the restricted domain of missing number arithmetic problems. If a child is able to quickly and successfully solve missing number problems using an inversion strategy, we can characterise them as exhibiting object-based thinking within this domain. If, however, they are unable to do so, and instead calculate the answer directly, we can characterise them as exhibiting process-based thinking within this domain.

The main goal of the study reported in this paper was to use this operationalisation of process-based and object-based thinking to directly test the predictions of the process-to-object theories: that process-based thinking is a prerequisite to object-based thinking.

METHOD

Fifty-nine children participated in the study, which took place in an English state school. There were 26 children from a Year 4 class (mean age 8 years 11 months). The remaining 33 children were from a Year 5 class (mean age 9 years 10 months. Six children declined to attempt a large number of the problems and their data were discarded, thus the data from 53 participants were included in the analysis.

Each child participated individually in two 20-minute sessions. The children completed 48 arithmetic missing number problems, each with four numbers (i.e. *a*+*b*-*c*=*d*). Half of the questions were inverse problems (where *b*=*c* and *a*=*d*; e.g. 15+12-12=□), and these were matched with a control problem that had the same missing number (e.g. 11+11-7=□). Half of the problems (both inverse and control) had the operator order plus-first (i.e. a+b-b=a and a+b-c=d) and half had the operator order minus-first (i.e. a-b+b=a and a-b+c=d). In each problem, one of the numbers was missing and the child was asked to supply it.). Both class teachers reported that children had no been explicitly taught the short-cut method for solving inversion problems.

Numbers were chosen so that the problems were at the limit of, or just beyond, what could be solved by this age group when using computation. For all of the problems, the first and fourth numbers were between 10 and 30, and the second and third numbers were between 5 and 20. Examples of problems used in the study are given in Table 1. The problems were presented to children on the screen of a laptop. The plus-first problems were presented in one session and the minus-first problems in the other. The order in which the sessions were given was counterbalanced across participants. The problems were presented in a random order for each participant.

Missing number	Inverse		Control	
	Plus-first	Minus-first	Plus-first	Minus-first
Position 1	□+7-7=13	□-9+9=12	□+14-9=18	□-8+12=16
Position 2	13+□-9=13	15-□+13=15	15+□-5=19	18-□+8=13
Position 3	16+14-□=16	16-12+□=16	18+9-□=13	12-8+□=16
Position 4	15+12-12=□	14-5+5=□	11+11-7=□	10-6+10=□

Table 1. Examples of problems used in the study

The task was introduced as a numbers game in which the participants had to work out the missing number. At the beginning of each session there were four familiarisation/practice trials (all control problems). In each trial the problem was presented on the screen and the experimenter read it aloud twice. When the child responded the experimenter pressed a key and recorded the response. Along with accuracy, the child's response time was recorded (the time between the presentation of the problem and the response key being pressed). The children were given positive encouragement without any specific feedback throughout.

RESULTS

The main analysis examined individual differences in children's performances using a hierarchical cluster analysis (using Ward's method). Children's accuracy scores on inverse and control problems were entered into a single analysis.

Three clusters were identified which accounted for 81% of the variance in scores. The first cluster identified (N=17, mean age 9 years 11 months) had high scores on both the inverse (91% correct) and the control (79% correct) problems. The second (N=17, mean age 9 years 2 months) had low scores on both problems (25% and 15% correct respectively). Crucially, the third cluster (N=19, mean age 9 years 4 months) showed a high score on the inverse problems (81% correct) but a low score on the control problems (33% correct). Both the first and third clusters were quicker to respond to inverse problems than they were to control problems. No such difference was found for the second cluster. The mean accuracies and response times from the three clusters are summarised in Figure 1.

Two one-way analyses of variance (ANOVA) were conducted to interrogate the differences between the clusters, with cluster membership as a between-groups factor and inverse and control scores as dependent variables. For inverse scores there was a significant effect of group membership, $F(2,50)=151.72$, $p<0.01$. Scheffe post-hoc comparison tests (all at $p < 0.001$), revealed that the children in Clusters 1 and 2 were more accurate than the children in Cluster 3; but that there was no significant difference between those in Clusters 1 and 3. For control scores there was again a significant effect of group membership, $F(2,50)=81.55$, $p<0.001$, with post-hoc tests (all at $p < 0.01$) revealing that the children in Cluster 1 were more accurate than those in Clusters 2 and 3, and that those in Cluster 3 were more accurate than those in Cluster 2. Similar analyses were conducted with respect to response times (as is typical with RT data, the ANOVA homogeneity of variance assumption was violated; thus these analyses were conducted on log-transformed data). A significant effect of group membership on inverse problems was found, $F(2,50)=6.43$, $p<0.01$, with post-hoc tests (all at $p < 0.05$) finding that children in Cluster 2 were slower than those in Clusters 1 and 3, but that there were no significant differences between the transformed response times of children from Clusters 1 and 3. No significant effect was found for control problems, $F(2,50) = 2.83$, NS.

To summarise, the children in Clusters 1 and 3 appeared to be able to successfully use the inverse relationship to solve the problems: their mean accuracy rate on inverse problems was high, and their mean response time on inverse problems was lower than their equivalent figure for control problems. In contrast the children in

Cluster 2 appeared to struggle with both types of problem: their accuracies were low and response times high. Crucially, although they were able to successfully use the inverse relationship, the children in Cluster 3 found the control problems (where no inverse relationship shortcut was available) difficult: they solved these problems correctly just one third of the time.

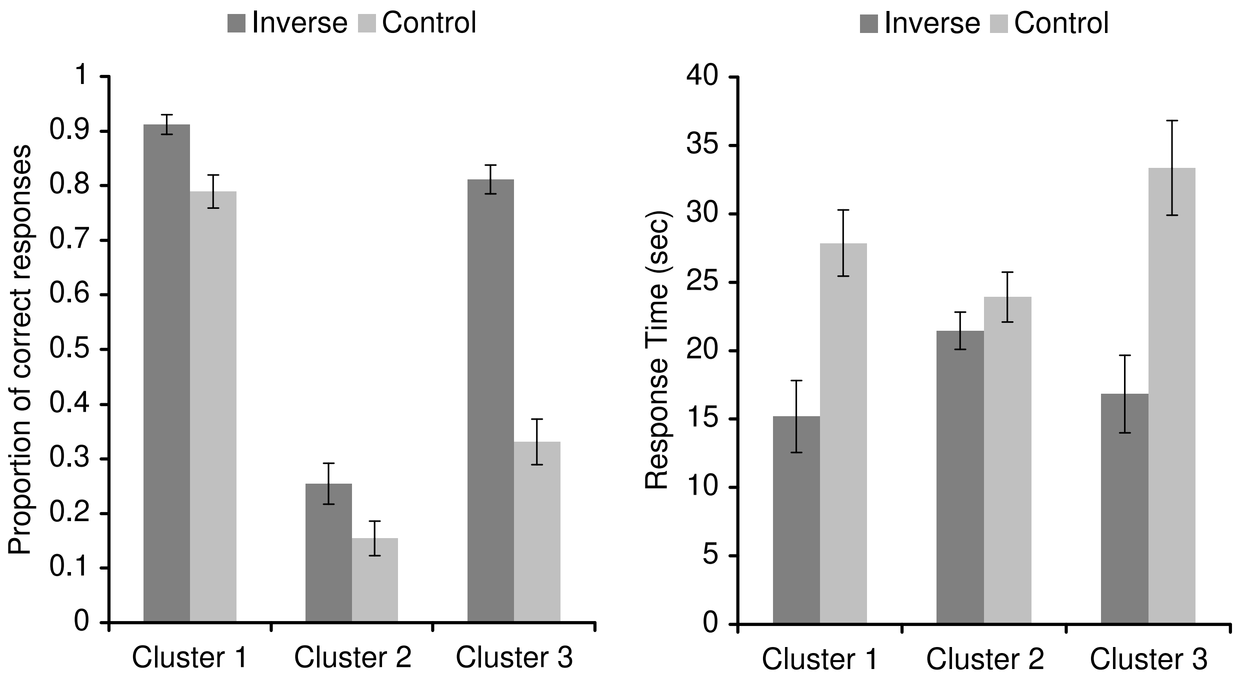

Figure 1. Left: each cluster's mean proportion of correct responses on the two types of problem. Right: each cluster's mean response time. Error bars show SEs of the mean.

DISCUSSION

The behaviour of participants from each of the clusters can be analysed in terms of the process-to-object theories discussed in the introduction to this paper. Cluster 1 appeared to demonstrate both successful object-based thinking and successful process-based thinking. On the inversion problems they showed a high level of accuracy, and were relatively quick. On the control problems - which required an explicit calculation - they were also successful, but were relatively slow (suggesting that they were using the slow process-based method of solving the problems).

Cluster 2, in contrast, demonstrated only unsuccessful process-based thinking. They appeared to be using similar strategies on both the inverse and the control problems, as they had similar (low) accuracy rates and (relatively high) response times.

Cluster 3, however, exhibited a different pattern. They appeared to be using object-based thinking on the inverse problems: their accuracy rate on these problems was high, and their corresponding response times were low, suggesting that they had knowledge of the addition-subtraction inversion relationship and that they could use it appropriately. However, they did not seem to be able to engage in successful process-based thinking. Their accuracy rates were low on the control problems, and their response times were high. In sum, the children from Cluster 3 seemed able to engage in successful object-based thinking, but did not seem able to engage in successful process-based thinking.

It is notable that we did not find a cluster of children who exclusively used process-based thinking to *successfully* tackle the problems. Such a cluster would be expected to have high accuracy rates, and similar (long) response times to the two types of problem. A cluster of successful process-based thinkers would be predicted by the process-to-object theories, which hypothesise that learners on the verge of encapsulation are highly fluent at process-based thinking.

These results seem to suggest that process-to-object theories may not capture every child's developmental route. The children in Cluster 3 *were* apparently aware of the concept of sum, and could think about it in an object-like way (as a noun, not a verb), but *were not* fluent in performing the sum procedure. If a subset of learners do in fact develop an ability to engage in object-based thinking before process-based thinking, then the assumption that deeper understanding can *only* come through the encapsulation (or reification) of processes into objects seems questionable.

CONCLUSION

Process-object theories (Davis, 1984; Dienes, 1960; Gray, & Tall, 1994; Piaget, 1985) posit that learners can construct mathematical objects by becoming highly fluent at process-based thinking. Several highly influential theorists have proposed that this is how a deeper understanding of mathematics comes about (Cotrill et al., 1996; Dubinsky, 1991, Sfard, 1991), especially in the domain of arithemetic (Gray et al., 1999; Gray and Tall, 1994). These theories are still highly influential.

In this paper we reported a study which showed that a subset of children apparently follow a different developmental route to that proposed by the process-to-object theorists. They seemed to have developed an ability to engage in object-based thinking about arithmetic despite being unable to perform calculation procedures. Determining whether this finding reflects an object-to-process developmental route, or some more complex developmental interaction between object- and process-based thinking would be a valuable goal for future research.

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