Substitution and sameness: Two components of a relational conception of the equals sign

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Abstract

A sophisticated and flexible understanding of the equals sign is important for arithmetic competence and for learning further mathematics, particularly algebra. Research has identified two common conceptions held by children: the equals sign as an operator, and the equals sign as signalling the same value on both sides of the equation. We argue here that as well as these two conceptions, the notion of *substitution* is also an important part of a sophisticated understanding of mathematical equivalence. We provide evidence from a cross-cultural study in which English and Chinese children were asked to rate the "cleverness" of operational, sameness and substitutive definitions of the equals sign. A Principle Components Analysis revealed the substitutive items were distinct from the sameness items. Furthermore, Chinese children rated the substitutive items as 'very clever', whereas the English children rated them as 'not so clever', suggesting that the notion of substitution develops differently across the two countries. Implications for developmental models of children's understanding of equivalence are discussed.

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Introduction

Children need to learn that the equals sign symbolises an equivalence relation between two mathematical objects, such as numbers or expressions. Their understanding of the meaning of the equals sign is correlated with arithmetic competence and success at further mathematics (Kieran, 1981; Knuth, Stephens, McNeil, & Alibali, 2006; Li, Ding, Capraro, & Capraro, 2008; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011). However, many young children, in Western countries at least, view the equals sign not as expressing a *relation* but as an *operation* meaning "work out the answer" (Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1976; Kieran, 1981; Knuth, Stephens, et al., 2006; Rittle-Johnson & Alibali, 1999). Children are rarely explicitly taught the meaning of the equals sign (Li et al., 2008). They typically encounter it in equations such as $3 + 4 = ?$, where an expression is evaluated to produce a result, and thus an operational interpretation is consistent with a correct solution of the problem (Baroody & Ginsburg, 1983; Denmark, Barco, & Voran, 1976; McNeil et al., 2006; Seo & Ginsburg, 2003). The operational view is reinforced by mathematics lessons and textbooks throughout the early years of schooling and so may become resistant to change (McNeil, 2008; McNeil & Alibali, 2005). Such a resistant operational view is problematic because it can hinder the development of arithmetic competence and learning of algebra in later schooling (Knuth et al., 2006; McNeil & Alibali, 2002). The operational and relational interpretations of the equals sign have been demonstrated to have good criterion validity, for example predicting performance on equation solving even after controlling for general mathematical achievement (Knuth et al., 2006). However, the operation-relation dichotomy risks neglecting other important components of mathematical equivalence. In this paper we present evidence that

substitution, i.e. the replacement of one representation with another, is an important and under-researched component of understanding mathematical equivalence relations.

The equals sign is defined and introduced differently in teacher guidebooks around the world. Li et al. (2008) compared teacher guidebooks in China and the U.S. and found that in the U.S. the equals sign is rarely defined, and is often used interchangeably with computational terms such as "makes". The majority of arithmetic equations presented in U.S. guidebooks are canonical (i.e. of the form *expression = result*). In China the equals sign, along with the greater-than and less-than symbols, is defined in the first year of schooling and presented in a variety of symbolic, verbal and pictorial contexts. For example, an exercise might ask students to choose which relational symbol (<, > or =) should go between a presented pair of numerals or pictures of objects. Arithmetic operations, expressions and equations are typically introduced only after students have learnt the meaning of the equals sign and other relational symbols, Li et al. investigated the impact these two approaches have on Chinese and U.S. students' performance with equation solving at the end of primary schooling. Strikingly, 98% of the Chinese children solved equations and justified their answers correctly whereas only 28% of the American children were successful.

Nevertheless, most Western children can and do develop a more relational understanding of the equals sign as they progress through the middle years of schooling. Over the past three and a half decades researchers have presented increasingly detailed models of development from operational to relational conceptions (Baroody & Ginsburg, 1983; Behr et al., 1976; Carpenter, Franke, & Levi, 2003; Rittle-Johnson et al., 2011). The most recent model, by Rittle-Johnson et al., comprises four levels from operational through to relational, as shown in Table 1.

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Insert Table 1 about here

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The literature to date usually operationalises the relational understanding of mathematical equivalence in terms of *sameness* (Jones & Pratt, 2012). For example, McNeil and Alibali (2005) described an instrument that included the two relational definitions "something is equal to another thing" and "two amounts are the same" (p. 887). Knuth et al. (2006) coded children's definitions of the equals sign and provided relational examples including "what is to the left and right of the sign mean the same thing" and "the same as" (p. 303). Much research has investigated how students make use of general arithmetic principles to establish the sameness of both sides of the equals sign (e.g. Carpenter, Franke & Levi, 2003; Molina, Castro & Mason, 2008; Pirie & Lyndon, 1997; Sáenz-Ludlow & Walgamuth, 1998). For example, children can use their knowledge of commutativity to see that the equation $4 + 4 = 3 + 4$ has the same value on both sides without having to calculate an answer (Baroody and Ginsburg, 1983).

We propose that a relational understanding of the equals sign involves both sameness and substitutive components. We offer two justifications for the inclusion of substitution, one mathematical and one cognitive.

Mathematically, any relation that is transitive, symmetric and reflexive is said to be an equivalence relation (e.g. Stewart & Tall, 1977). Importantly, the mathematical definition of equivalence makes no claims about the properties of a and b beyond how they are related via the equivalence relation. Formally $a = b$ does not imply that a and b are "the same", only that they are related by the given equivalence relation. However, $a = b$ does imply, because

of transitivity, that given $b = c$ we can deduce $a = c$ (i.e. substitute a for b). To give a simple example, if one defines a relation by stating that a person a is related to a person b if a and b have the same birthdays, then one has an equivalence relationship (the relation is transitive, symmetric and reflexive). But of course a and b are not "the same" beyond their properties in the reduced context of birth dates. However, because of transitivity, a can be substituted for b in the statement "b has the same birthday as c".

The cognitive justification for considering substitution is based on studies into students' difficulties with the notion when working with algebraic equations. Lima and Tall (2007) reported that secondary students, when solving equations containing an unknown on both sides of the equals sign, often used memorised rules (e.g. "change sides, change signs") rather than substitutive solutions (e.g. substituting $m = 0$ to make $2m = 4m$ true). Filloy, Rojana and Solares (2010) found that some secondary students, when solving simultaneous algebraic equations, could often substitute numbers for unknowns (e.g. $y = 5$) but could not substitute variables for an expression containing another variable (e.g. $y = 2x + 3$).

Little research has directly investigated the role of substitution for understanding equivalence within arithmetic contexts, although it has been implied in some studies. For example, McNeil, Fyfe, Petersen, Dunwiddie and Brletic-Shipley (2011) compared how practice with canonical and reversed canonical (i.e. *result = expression*) equations improved understanding of equivalence. The authors' focus was on "non-traditional problem formats" but their approach can be viewed as substitutive because the reversed canonical equations were generated from the canonical equations by exchanging the two sides of the equals sign (see also Weaver, 1973). In the same way, students at the 'flexible operational' level of Rittle-Johnson and colleagues' construct map (see Table 1) accept reversed canonical equations as correct.

We predicted that a *substitutive-relational* conception is a component of understanding the equals sign, distinct from the *sameness-relational*[1] conception. To explore this hypothesis we adapted the Conceptions of the Equals sign (CES) instrument from Rittle-Johnson & Alibali (1999), which has been demonstrated to identify children at different levels of development (Knuth et al., 2006; McNeil & Alibali, 2005; Rittle-Johnson et al., 2011). Our adapted CES included substitutive-relational items along with sameness-relational and operational items. The language used for the items was informed by previous qualitative studies in which elementary students worked with a software program that supports a substitutive view of the equals sign (e.g. Jones & Pratt, 2012).

The CES instrument was administered to two groups of 11 and 12-year-old students likely to have different levels of understanding of the equals sign. One group was drawn from England where primary schooling can be expected to support a predominantly operational view of the equals sign. Mathematics textbooks are used sparingly if at all in English primary classrooms (Mullis et al., 2008) and so cannot be assumed to impact greatly on students' learning. However, an analysis of compulsory high-stakes national tests taken by our participants at the end of primary schooling indicated that their experience of the equals sign was predominantly operational (Jones, Inglis, Gilmore, & Evans, submitted). We therefore expected that the English students would mostly adhere to Rittle-Johnson et al.'s 'flexible operational' and 'basic relational' views of the equals sign, shown in Table 1.

The other group of children was drawn from China where primary schooling can be expected to support a predominantly sameness-relational view of the equals sign (Li et al.,

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[1] In the remainder of the article we use the terms sameness-relational and substitutive-relational to distinguish between the two components that we argue make up a sophisticated relational understanding of the equals sign.

2008). In China, textbooks are central to school teaching (Mullis et al., 2008) and the children in the study were taught using the Beijing Normal University Press series guidebooks which promote a sameness-relational meaning of the equals sign (Li et al., 2008), as well as a regional series (Zhejiang Education Press). We therefore expected that the Chinese sample would predominantly subscribe to Rittle-Johnson et al.'s 'comparative relational' view of the equals sign.

These two groups provided an interesting contrast because the Chinese focus on sameness relations might also foster substitutive views of equivalence through, for example, exposure to reversed canonical equations. We also assumed that the Chinese children would be more competent at arithmetic and more ready to learn algebra than the children English sample. This assumption was based on Li et al.'s (2008) comparison of Chinese and U.S. children, and the similar algebraic and mathematical performance of children in the U.K. and U.S. in contrast to children in China (Mullis et al., 2008; OECD, 2009).

We subjected the students' responses to the CES instrument to two main analyses. First, a Principle Components Analysis (PCA) was undertaken to establish whether the substitutive-relational items loaded separately to sameness-relational and operational items. Second, an Analysis of Variance (ANOVA) was undertaken to establish whether the substitutive-relational items were more strongly endorsed by students in China than those in England.

Method

Participants

A total of 251 children aged 11 and 12 from two urban schools in England ($N = 101$)

and China ($N = 150$) participated in the study. Both groups were in the first year of

secondary school in which students first encounter algebraic equations.

Materials and procedure

The adapted CES instrument presents fictional definitions of the equals sign and

participants rate as "not so clever", "kind of clever" or "very clever". The definitions in the

original instrument correspond to the operational or sameness-relational conceptions, or

were not meaningful in terms of arithmetic equivalence. We added three substitutive-

relational definitions to construct a final instrument with nine primary items and three

distractor items, shown in the Appendix. We developed the adapted instrument in English

and tested it with a small group of children. A Chinese speaker produced a Chinese

language version, and a second Chinese speaker independently translated it back into

English to check for consistency.

The instrument was administered to children in class under test conditions by their

regular mathematics teachers. Children worked individually through the instrument and

were allowed ten minutes to complete all 12 items. The order of the items were randomised

for each participant.

Analysis and results

Preliminary analysis

Children's "cleverness ratings" of the sameness-relational, substitutive-relational and operational items were coded as 0 for "not so clever", 1 for "sort of clever" and 2 for "very clever"[2]. Eight children did not complete all the items or had filled in more than one rating for an item and were removed from the analysis, leaving a total of 243 participants (95 from England, 148 from China).

We checked the performance of the instrument in terms of the internal consistency of the items reflecting each component. We found Cronbach's αs of .713 for the operational items, .644 for the sameness-relational items, and .765 for the substitutive-relational items, suggesting the instrument performed satisfactorily.

Conceptions of the Equals Sign

The suitability of the data for factor analysis was checked by assessing the factorability of the correlation matrix. The Kaiser-Meyer-Oklin value was .751, and Bartlett's Test of Sphericity reached significance, $p < .001$, confirming the suitability of the data. The nine items (excluding the three distracter items) were subjected to a PCA on a matrix of polychoric inter-item correlations, as is advised for ordinal items (Holgado–Tello, Chacón–Moscoso, Barbero–García, & Vila–Abad, 2008).

The analysis revealed the presence of components explaining 34.2%, 25.0%, and 10.7% of the variance respectively. Scrutiny of the screeplot revealed a clear break after the third component, and thus three components were extracted. Oblimin rotation revealed strong loadings for all three components, and the interpretation of the three components was

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[2] Group means for each item are shown in the Appendix.

consistent with our predictions. Operational items loaded strongly onto Component 2, Sameness-relational items loaded strongly onto Component 1, and Substitutive-relational items loaded strongly onto Component 3, as shown in Table 2. We therefore argue that a truly relational conception of the equals sign comprises distinctive substitution and sameness components, as predicted by Jones and Pratt (2012).

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Group Differences

To investigate differences across the two countries, children's overall cleverness ratings were calculated by summing their responses to the three items associated with each conception. Since each item was scored 0, 1 or 2, cleverness ratings for each conception fell on a 0 to 6 scale. These data are shown in Figure 1. Cleverness ratings were subjected to a 2×3 ANOVA with one between-subjects factor (Country: England, China) and one within-subjects factor (Conception: operational, substitutive-relational, sameness-relational). There was a significant main effect of conception, $F(1.9, 468.3) = 15.9$, $p < .001$ (Greenhouse-Geisser correction), and a significant country × conception interaction effect, $F(1.9, 468.3) = 146.7$, $p < .001$. The main effect of country did not reach significance, $F < 1$. All within-country comparisons of conception were significant ($p < .001$). To compare group differences we conducted three planned comparisons using a Bonferroni correction to adjust for the familywise error rate. The English group had a significantly higher mean operational rating, 4.35, than the Chinese group, 2.09, $t(241) = 11.6$, $p < .001$, and a significantly lower

mean substitutive-relational rating, 1.63 versus 3.61, $t(241) = 9.7$, $p < .001$. The Chinese group had a higher mean sameness-relational rating, 2.77, than the English group, 2.42, but, counter to our expectations, this (after Bonferroni correction) fell some way short of significance, $t(241) = 1.7$, $p = .097$.

In sum, the between-countries difference in operational and substitutive-relational conceptions matched our expectations. We speculate on the implications of these findings for the development of relational understanding in the discussion section.

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Insert Figure 1 about here

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Relationship Between Factors

To explore the relationships of the substitutive-relational conception to the sameness-relational and operational conceptions we calculated correlation coefficients of the children's cleverness ratings summed over the three items associated with each conception, as shown in Table 3. The coefficients revealed a negative significant correlation between the operational and substitutive-relational components, and a positive significant correlation between the sameness- and substitutive-relational components. This suggests the substitutive-relational conception is associated with the higher 'relational' levels but not the lower 'operational' levels of Rittle-Johnson et al.'s construct map (see Table 1). The coefficients also revealed, somewhat unexpectedly, a small but non-significant correlation between the operational and sameness-relational components.

We also investigated whether the sameness- and substitutive-relational components appeared to be contingent on one another, that is whether the emergence of either component precedes the other. We coded children as either "accepting" a conception (having a score of 4, 5 or 6 on the 6-point "cleverness" scale) or "rejecting" a conception (a score of 0, 1 or 2). Of those children who accepted the sameness-relational conception, 22.5% ($n = 16$) rejected the substitutive-relational conception; and of those who accepted the substitutive-relational conception, 31.4% ($n = 27$) rejected the sameness-relational conception. These figures suggest that there may not be a universal order in which children develop the two conceptions.

We further develop our theoretical interpretation of these correlations and contingencies in the discussion section.

Replication

To further investigate the factor structure of our adapted CES, we replicated the study reported above by administering the CES to a further 133 children (England, $N = 81$; China $N = 52$) also aged 11 and 12 years from two further schools. The PCA revealed the presence of components explaining 21%, 20% and 18% of the variance respectively. The screeplot showed a clear break after the third component, and a oblimin rotation resulted in three components showing a number of strong loadings as shown in Table 4. As with the main study, loadings corresponded to operational, sameness- and substitutive-relational items. The pattern of group differences was also similar to that found in the main study.

Discussion

The purpose of the study reported here was to test whether substitution is a distinct way of understanding the equals sign. We administered an instrument to children in England and China comprising operational, sameness-relational and substitutive-relational definitions of the equal sign, and asked them to rate each definition as "not so clever", "kind of clever" or "very clever". A PCA showed that the three different types of items loaded distinctly and strongly, providing evidence that the substitutive-relational conception is a way of understanding the equals sign that is distinct from the sameness-relational conception. Furthermore, we found that the Chinese and English children in our sample endorsed these components to varied extents, suggesting that conceptions of the equals sign are differentially developed across different countries. Whereas the Chinese children rated the substitutive-relational conception as cleverer than did the English children, the reverse was true for the operational conception. The Chinese children endorsed the sameness-relational conception more strongly than the English children although the difference fell short of significance.

We concentrate our discussion on two theoretical positions that have emerged from the literature on children's understanding of the equals sign. First, children develop from an operational conception to a sameness-relational conception in a broadly unidimensional fashion. Second, adopting an operational conception of the equals sign hinders the development of a sameness-relational conception, and so causes difficulty with learning further mathematics, particularly algebra.

Rittle-Johnson et al. (2011) proposed and tested a four-level construct map for mathematical equivalence, as shown in Table 1. The model posits that children develop from a 'rigid operational' conception through to a 'comparative relational' conception, where they solve equations by successfully comparing the expressions on the two sides of

the equals sign. In this paper we have referred to the comparative relational conception as the sameness-relational conception. Our results suggest that there may be more to developing a truly relational conception of the equals sign than the unidimensional model suggests. We found a third distinct conception of the equals sign: one which prioritises the crucial mathematical idea of substitution. The observation of a third component of a relational understanding of equivalence leads to an important question about Rittle-Johnson et al.'s developmental trajectory: does the substitutive-relational conception develop in parallel with or *after* the sameness-relational conception, or do the two conceptions have entirely distinct developmental paths?

In view of the substitutive-relational conception emerging as a separate component in our PCA, one might expect that the substitutive- and sameness-relational conceptions have separate developmental paths. However, as shown in Table 3 children's cleverness ratings of the two conceptions were positively correlated in our data, indicating that it is possible that substitutive-relational and sameness-relational do develop somewhat concurrently, albeit at different rates across individuals. Another possibility is that children develop the substitutive-relational conception only after they have developed a sophisticated understanding of sameness. In other words, perhaps having reached Rittle-Johnson et al.'s fourth level, some children supplement their sophisticated sameness-relational conception with the substitutive-relational conception. If so, we would expect few children to rate the substitutive-relational conception as clever without also rating the sameness-relational conception as clever, but that many children may accept the sameness-relational conception while rejecting the substitutive-relational conception. In fact, our analysis of the contingency of the two conceptions suggests that there is no consistent order in which children develop them. Further work, with a longitudinal component, will be necessary to

disentangle the relationship between the development of sameness-relational and substitutive- relational conceptions.

One of the strengths of Rittle-Johnson et al.'s construct modelling approach is that it provides a rigorous method for testing proposed learning trajectories and leads to a criterion-referenced measure that can be used to identify where individual children are within the trajectory. However, because the method relies upon a hypothesised construct map, it cannot lead to the identification of novel conceptions that may exist but have not already been hypothesised. We believe that our findings in this paper indicate that future studies on the development of children's understanding of the equals sign should consider the substitutive-relational conception independently of the sameness-relational conception.

Our findings also have implications for interpretations of the change resistance account of the development of equality. In an ingenious series of experiments, McNeil and colleagues have demonstrated that adhering to operational patterns of equality harms arithmetic and algebraic performance (e.g. McNeil & Alibali, 2002; McNeil, et al., 2010a; McNeil, et al., 2010b). Operational patterns have traditionally been interpreted as related to students' lack or suppression of a sameness-relational conception of the equals sign. For example, McNeil, Rittle-Johnson et al. (2010) asked undergraduates to solve simultaneous equations of the form "John bought three shirts and two caps for $58. Sue bought two shirts and three caps for $52. What is the cost of one shirt?" Participants who tackled such tasks after having been asked to solve canonical single digit addition problems performed significantly worse than participants who had been asked to tackle non-arithmetic numerical tasks. McNeil et al. argued that being asked to perform arithmetic tasks activated an operational conception of equality, which hindered the necessary sameness-relational conception required to solve such problems.

However, solving simultaneous equations effectively, including those used by McNeil et al. (2010), necessitates an understanding of the mathematical notion of substitution: the solver must express one unknown in terms of another and substitute that expression into the second equation. Perhaps, then, a key reason why activating operational conceptions of equality hinders algebraic performance is that it de-emphasises substitutive-relational conceptions, aside from it de-emphasising sameness-relational conceptions. This would be consistent with our unexpected finding that Chinese children's endorsement of sameness-relational items was not significantly stronger than English children's. It would also be consistent with the strong inverse relationship in our data between cleverness ratings of the operational and substitutive-relational conceptions, shown in Table 3. (It is also consistent with the weak positive relationship between cleverness ratings of operational and sameness-relational conceptions, although this was non-significant and unexpected.) If our interpretation of the findings in Table 3 is correct then we would expect a manipulation which emphasised the substitutive-relational conception of the equals sign to *improve* algebraic performance compared to a control condition, and perhaps even to a condition in which the sameness-relational conception was made salient.

We note that we have only used a single instrument to assess children's understanding of the substitutive component of equivalence relations, based on endorsements of fictitious definitions. Further work is required to establish the extent to which our measure corresponds to others based on definition generation, and equation analysis and solving (Rittle-Johnson et al., 2011). Such evidence would enable us to explore the predictive power and validity of measures of the substitutive-relational conception. Equation based instruments would also enable us to directly test the assumption that Chinese children are more proficient at relational thinking and better prepared to learn algebra than English children. Neither definition nor equation based instruments have previously detected the

substitutive-relational component and may require adaptation as was the case for the instrument used in the present study.

To conclude, a correct understanding of the equals sign has traditionally been defined in terms of understanding, implicitly or explicitly, that both sides must have the same value. However, the "same value" definition is limited because it does not incorporate the important mathematical idea of substitution, that one side of the equals sign can be used to replace the other. Our study showed that the mathematical idea of substitution has a detectable cognitive analogue, and that children with a sophisticated conception of the equals sign explicitly endorse it. Instruments designed to investigate children's conceptual development should therefore incorporate the substitution component of a truly relational conception.

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Appendix

All 12 items used in the Conceptions of the Equals Sign (CES) instrument along with group means.

Definition = means…	Predicted Conception	Group means	
		English	**Chinese**
…the answer to the problem	Operational	2.52	1.72
…work out the result	Operational	2.24	1.64
…the total	Operational	2.59	1.73
…that two amounts are the same	Sameness	1.64	1.86
…both sides have the same value	Sameness	1.71	1.94
…that something is equal to another thing	Sameness	2.07	1.97
…one side can replace the other	Substitutive	1.46	2.26
…that the right-side can be swapped for the left-side	Substitutive	1.47	2.13
…that two sides can be exchanged	Substitutive	1.69	2.22
…the end of the problem	Distractor	2.19	1.95
…the start of the problem	Distractor	1.35	2.24
…to repeat the numbers	Distractor	1.29	1.71

Table 1: Construct map for mathematical equivalence (adapted from Rittle-Johnson et al., 2011, p.87).

Level	Description
4. Comparative relational	Explicitly view "=" as a relation signalling the same value on each side and able to define it as such. Accept a wide range of arithmetic equation types. Draw on arithmetic principles (commutativity, associativity and inversion) in order to evaluate and solve equations in terms of their structural properties; e.g. recognise that $3 + 5 = 5 + 3$ and $6 + 9 = 7 + 8$ are true by drawing on the commutative and associative properties of addition respectively.
3. Basic relational	Implicitly view "=" as a relation signalling the same value is on each side but unable to define it as such. Accept a wide range of arithmetic equations as properly formed, including those with expressions on both sides.
2. Flexible operational	View "=" as an operator. Accept as properly formed equations that contain a result on at least one side of the equal sign.
1. Rigid operational	View the symbol "=" as an operator. Consider only canonical equations to be properly formed.

Table 2. Results of the Principal Components Analysis, after oblimin rotation. Loadings

>.4 are shown in bold.

Definition = means…	Predicted Conception	Loading on Factor 1	Loading on Factor 2	Loading on Factor 3
…the answer to the problem	Operational	-0.116	**0.896**	0.001
…work out the result	Operational	0.190	**0.733**	-0.071
…the total	Operational	-0.071	**0.832**	-0.034
…that two amounts are the same	Sameness	**0.929**	-0.158	-0.09
…both sides have the same value	Sameness	**0.750**	-0.004	0.213
…that something is equal to another thing	Sameness	**0.596**	0.322	0.090
…one side can replace the other	Substitutive	0.047	-0.08	**0.793**
…that the right-side can be swapped for the left-side	Substitutive	0.040	-0.039	**0.865**
…that two sides can be exchanged	Substitutive	-0.049	0.044	**0.909**

Table 3. Correlations between children's summed cleverness ratings for each component. Fisher's r-to-z Tests showed all correlations to be significantly different to one another at $p < .001$.

	r	p
operational and sameness	.104	.105
operational and substitutive	-.236	< .001
sameness and substitutive	.385	< .001

Table 4. Results of the replicated Principal Components Analysis, after oblimin

rotation, for the replication study. Loadings >.4 are shown in bold.

Definition = means…	Predicted Conception	Loading on Factor 1	Loading on Factor 2	Loading on Factor 3
…the answer to the problem	Operational	0.113	-0.064	**0.826**
…work out the result	Operational	-0.184	0.052	**0.789**
…the total	Operational	0.089	-0.024	**0.698**
…that two amounts are the same	Sameness	**0.773**	0.168	0.187
…both sides have the same value	Sameness	**0.788**	0.085	-0.213
…that something is equal to another thing	Sameness	**0.686**	-0.115	0.042
…one side can replace the other	Substitutive	-0.150	**0.815**	-0.004
…that the right-side can be swapped for the left-side	Substitutive	0.169	**0.725**	0.171
…that two sides can be exchanged	Substitutive	0.079	**0.665**	-0.225

Figure 1. Chinese and British children's conceptions of the equals sign. Error bars give the standard error of the mean.