University of Yonsei Mirae, Linear System

HW 3

2020271053, MinJae Kim

October 17, 2024

Problem 1

Consider the following matrices:

$$A_1 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ -3 & 0 & -2 \end{pmatrix}$$

(a) Compute $e^{A_i t}$ for $i = 1, 2, 3$. Note that A_2 is a special case of a matrix whose exponential was computed in class.

(b) For $i = 1, 2, 3$, write down the solution of $\dot{x} = A_i x$ for a general initial condition.

(c) In each case, determine whether the solutions of $\dot{x} = A_i x$ decay to 0, stay bounded, or go to ∞ (for various choices of initial conditions).

(d) Try to state the general rule which can be used to determine, by looking at the eigenstructure of A, whether the solutions of $\dot{x} = Ax$ decay to 0, stay bounded, or go to ∞.

Answer (a)

We can express $A_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ and constant($\frac{1}{2}$) is not necessary to find **nilpotent matirx**, so I'll omit $\frac{1}{2}$.

$$A_1^2 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

And we already know that

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \cdots$$

Therefore,

$$e^{A_1 t} = I + A_1 t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} t & -t \\ t & -t \end{pmatrix}$$

We saw that

$$A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ A_2^2 = -I,\ A_2^3 = -A_2,\ A_2^4 = I$$

Substitute this to exponential matirx's Taylor expansion, we can get

$$e^{At} = I + At - \frac{1}{2!}(It^2) - \frac{1}{3!}(At^3) + \cdots$$

and we can divide to even and odd parts and each part converge to $\cos t$ and $\sin t$. Therefore,

$$\begin{aligned} e^{At} &= I \cdot \cos t + A \cdot \sin t \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos t + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\sin t \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \end{aligned}$$

This is so simple! We can express A_3

$$A_3 = A_{31} + A_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{pmatrix}$$

Then,

$$e^{A_{31}t} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$$

and $A_{32}^2 = 0$, so

$$e^{A_{32}t} = I + A_{32}t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3t & 0 & 1 \end{pmatrix}$$

Finally, we have

$$\begin{aligned} e^{A_3 t} &= e^{A_{31}t} + e^{A_{32}t} = e^{A_{31}tA_{32}t} \\ &= \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ -3te^{-2t} & 0 & e^{-2t} \end{pmatrix} \end{aligned}$$

Answer (b)

We can express the solution of $\dot{x} = A_i x$ as $x(t) = e^{A_i t}x_0$. Therefore,

$$\text{In } A_1, x(t) = e^{A_1 t}x_0 = \begin{pmatrix} 1 + \frac{t}{2} & -\frac{t}{2} \\ \frac{t}{2} & 1 - \frac{t}{2} \end{pmatrix} x_0$$

$$\text{In } A_2, x(t) = e^{A_2 t}x_0 = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} x_0$$

$$\text{In } A_3, x(t) = e^{A_3 t}x_0 = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ -3te^{-2t} & 0 & e^{-2t} \end{pmatrix} x_0$$

Answer (c, d)

In general. te linear system($x\dot{x} = Ax$)'s solution can be expressed:

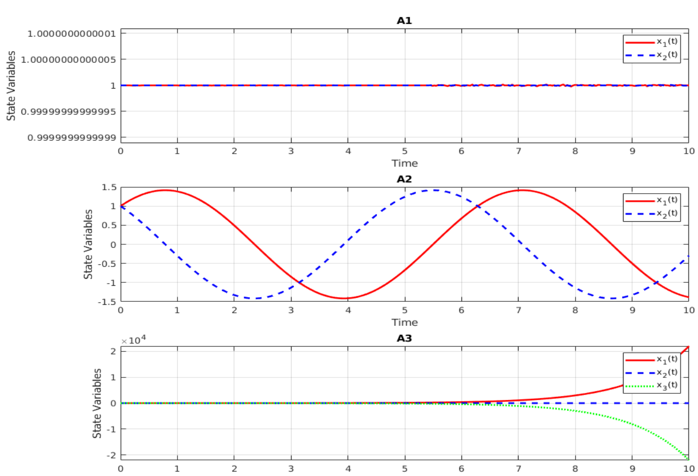
$$x(t) = e^{At}x_0$$

and e^{At} is the exponential matrix of A. Therefore, we can determine the behavior of the solution by looking at the eigenvalues of A.
Then, we can express the solution of the system with the eigenvalues of A:

$$x(t) = e^{\lambda t}v$$

1) If $\lambda < 0$ the solution will decay to 0.
2) If $\lambda = 0$ the solution will stay bounded.
3) If $\lambda > 0$ the solution will go to ∞.
4) If λ is complex, the solution will oscillate.

Therefore, we can determine the behavior of the solution by looking at the eigenvalues of A_i.
1) In A_1, the eigenvalues are $\lambda = 0, 0$, so the solution will stay bounded.
2) In A_2, the eigenvalues are $\lambda = \pm i$, so the solution will oscillate.
3) In A_3, the eigenvalues are $\lambda = 1, -2, -2$, may be the solution will go to saddle point.



Problem 2

The pictures show possible trajectories of a linear system $\dot{x} = Ax$ in the plane, when the eigenvalues λ_1 and λ_2 of A are in one of the following configurations:

$$(a)\lambda_1 < \lambda_2 < 0, \quad (b)\lambda_1 < 0 < \lambda_2, \quad (c)0 < \lambda_1 < \lambda_2,$$

$$(d)\lambda_1, \lambda_2 = a \pm bi, a < 0, \quad (e)\lambda_1, \lambda_2 = a \pm bi, a = 0, \quad (f)\lambda_1, \lambda_2 = a \pm bi, a > 0$$

Match each picture with the corresponding eigenvalue distribution. Justify your answers (plotting in Matlab cannot be used as a justification).

Answer

We saw that the behavior of the solution can be determined by the eigenvalues of A. (check in Problem 1.)

a) If $\lambda_1 < \lambda_2 < 0$, $e^{\lambda_1 t}, e^{\lambda_2 t}$ will decay to 0. Therefore, the trajectories will converge to the origin. This is Stable node.

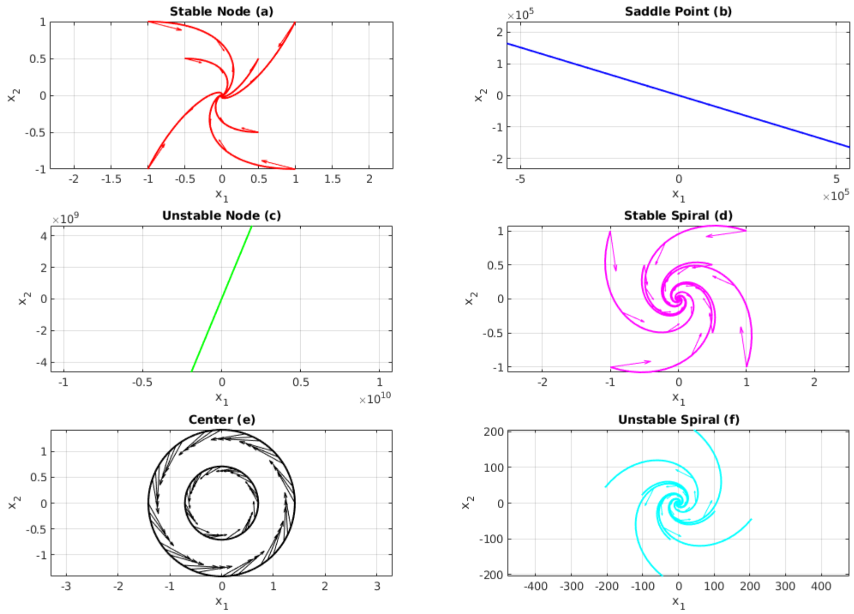
b) If $\lambda_1 < 0 < \lambda_2$, $e^{\lambda_1 t}$ will decay to 0 and $e^{\lambda_2 t}$ will go to ∞. Therefore, the trajectories will make the saddle point.

c) If $0 < \lambda_1 < \lambda_2$, $e^{\lambda_1 t}, e^{\lambda_2 t}$ will go to ∞. Therefore, the trajectories will go to the infinity. This is Unstable node.

d) If $\lambda_1, \lambda_2 = a \pm bi, a < 0$, $e^{\lambda_1 t}, e^{\lambda_2 t}$ will oscillate, but the real part is negative, so the trajectories will converge to the origin. This is Stable spiral.

e) If $\lambda_1, \lambda_2 = a \pm bi, a = 0$, $e^{\lambda_1 t}, e^{\lambda_2 t}$ will oscillate, but the real part is 0, so the trajectories will stay bounded. This is Center.

f) If $\lambda_1, \lambda_2 = a \pm bi, a > 0$, $e^{\lambda_1 t}, e^{\lambda_2 t}$ will oscillate, but the real part is positive, so the trajectories will go to the infinity. This is Unstable spiral.



Problem 3

This exercise illustrates the phenomenon known as resonance. Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad y = x_2$$

where $\omega > 0$, u is the control input, and y is the output. Let the input be $u(t) = \cos \nu t$, $\nu > 0$. Using the variation-of-constants formula, compute the response $y(t)$ to this input from the zero initial condition $x(0) = 0$, considering separately two cases: $\nu \neq \omega$ and $\nu = \omega$. In each case, determine whether this response is decaying to 0, bounded, or unbounded.

Answer

In general, the solution of the linear system can be expressed as $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$.

First, calculate e^{At}, it's known matirx:

$$e^{At} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}$$

and Substituting the given values to the formula,

$$\begin{aligned}
x(t) &= \int_0^t e^{A(t-\tau)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(\nu t) d\tau \\
&= \int_0^t \begin{pmatrix} \cos\omega(t-\tau) & \sin\omega(t-\tau) \\ -\sin\omega(t-\tau) & \cos\omega(t-\tau) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos\nu\tau d\tau \\
&= \int_0^t \begin{pmatrix} \sin\omega(t-\tau) \\ \cos\omega(t-\tau) \end{pmatrix} \cos\nu\tau d\tau
\end{aligned}$$

We can divide this integral to two parts, $x_1 = \int_0^t \sin\omega(t-\tau)\cos\nu\tau d\tau$ and $x_2 = \int_0^t \cos\omega(t-\tau)\cos\nu\tau d\tau$.

Then, use triangle formula, $\sin(a-b) = \sin a\cos b - \cos a\sin b$ and $\cos(a-b) = \cos a\cos b + \sin a\sin b$.

$$\begin{aligned}
x_1(t) &= \int_0^t \sin\omega t\cos\omega\tau\cos\nu\tau d\tau - \int_0^t \cos\omega t\sin\omega\tau\cos\nu\tau d\tau \\
&= \frac{\sin(\omega t)}{2}\int_0^t \cos(\omega-\nu)\tau + \cos(\omega+\nu)\tau d\tau - \frac{\cos(\omega t)}{2}\int_0^t \sin(\omega-\nu)\tau + \sin(\omega+\nu)\tau d\tau \\
&= \frac{\sin(\omega t)}{2}\left[\frac{\sin(\omega-\nu)t}{\omega-\nu} + \frac{\sin(\omega+\nu)t}{\omega+\nu}\right] - \frac{\cos(\omega t)}{2}\left[\frac{-\cos(\omega-\nu)t}{\omega-\nu} + \frac{-\cos(\omega+\nu)t}{\omega+\nu}\right]
\end{aligned}$$

calculate $x_2(t)$ in the same way, we can get

$$x_2(t) = \frac{\cos(\omega t)}{2}\left[\frac{\sin(\omega-\nu)t}{\omega-\nu} + \frac{\sin(\omega+\nu)t}{\omega+\nu}\right] + \frac{\sin(\omega t)}{2}\left[\frac{\cos(\omega-\nu)t}{\omega-\nu} + \frac{\cos(\omega+\nu)t}{\omega+\nu}\right]$$

Now, consider the case of $\nu \neq \omega$ and $\nu = \omega$ separately.
1) If $\nu \neq \omega$, the response will be bounded.
2) If $\nu = \omega$, the response will be unbounded.

Problem 4

Compute the state transition matrix $\Phi(t, t_0)$ of $\dot{x} = A(t)x$ with

$$A(t) = \begin{pmatrix} -1 + \cos t & 0 \\ 0 & -2 + \cos t \end{pmatrix}.$$

Answer

By, **Peano-Baker series**, we can express the solution of the linear system as

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A(\tau_1) d\tau_1 + \int_{t_0}^{t} A(\tau_1) \int_{t_0}^{\tau_1} A(\tau_2) d\tau_2 d\tau_1 + \cdots$$

Therefore, we can calculate the state transition matrix.
First term of the series is I and the second term is

$$\int_{t_0}^{t} A(\tau_1) d\tau_1 = \begin{pmatrix} -(t - t_0) + \sin t - \sin t_0 & 0 \\ 0 & -2(t - t_0) + \sin t - \sin t_0 \end{pmatrix}$$

and from the third term onwards, it disappears because its periodicity.
Therefore,

$$\Phi(t, t_0) = \begin{pmatrix} e^{-(t-t_0)+\sin t - \sin t_0} & 0 \\ 0 & e^{-2(t-t_0)+\sin t - \sin t_0} \end{pmatrix}$$

Problem 5

Consider the LTV system $\dot{x} = A(t)x$, where $A(t)$ is periodic with period T, i.e., $A(t + T) = A(t)$. Let $\Phi(t, t_0)$ denote the corresponding state transition matrix. The goal of this exercise is to show that we can simplify the system and characterize its transition matrix with the help of a suitable (time-varying) coordinate transformation. Being a nonsingular matrix, $\Phi(T, 0)$ can be written as an exponential: $\Phi(T, 0) = e^{RT}$, where R is some matrix. Define also $P(t) := \Phi(t, 0)e^{-Rt}$.

(a) Using the properties of state transition matrices from class, show that $\Phi(t, t_0) = P(t)e^{R(t-t_0)}P^{-1}(t_0)$.

(b) Deduce from (a) that $\bar{x}(t) := P^{-1}(t)x(t)$ satisfies the time-invariant differential equation $\dot{\bar{x}} = R\bar{x}$.

(c) Prove that $P(t)$ is periodic with period T.

(d) Consider the LTV system from Problem 4. Find new coordinates $\bar{x}(t)$ in which, according to part (b), this system should become time-invariant. Confirm this fact by differentiating $\bar{x}(t)$.

Answer (a)

We know that $P(t) = \Phi(t, 0)e^{-Rt}$ in other words, $\Phi(t, 0) = P(t)e^{Rt}$. And applying the property of the state transition matirx,

$$\Phi(t, t_0) = \Phi(t, 0)\Phi^{-1}(t_0, 0)$$

,

$$\Phi(t, 0) = P(t)e^{Rt}, \text{ and } \Phi(t_0, 0) = e^{-Rt_0}P^{-1}(t_0)$$

Therefore,

$$\Phi(t, t_0) = P(t)e^{R(t-t_0)}P^{-1}(t_0)$$

Answer (b)

Answer (c)

For the periodicity of $P(t)$, we can express $P(t + T)$ as

$$\Phi(t + T, 0) = \Phi(t, 0)\Phi(T, 0)$$

And we know that $\Phi(T, 0) = e^{RT}$, so

$$\Phi(t + T, 0) = \Phi(t, 0)e^{RT}$$

Now calculate $P(t + T)$,

$$\Phi(t + T, 0)e^{-R(t+T)}$$

$$P(t + T) = \Phi(t, 0)e^{RT}e^{-R(t+T)} = \Phi(t, 0)e^{RT}e^{-Rt}e^{-RT}$$

and $e^{RT}e^{-RT} = I$,

$$P(t + T) = \Phi(t, 0)e^{-Rt} = P(t)$$

Answer (d)

Problem 6

Prove the variation-of-constants formula for linear time-varying control systems stated in class (see also Class Notes, end of Section 3.7).

Answer

In class, we saw that the solution of the LTV system,

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

To prove this, if we differentiate both sides with respect to time,

$$\begin{aligned}\dot{x}(t) &= \frac{d}{dt}\left(\Phi(t, t_0)x_0\right) + \frac{d}{dt}\left(\int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau\right) \\ &= A(t)\Phi(t, t_0)x_0 + \Phi(t, t)B(t)u(t) \\ &= A(t)\Phi(t, t_0)x_0 + B(t)u(t)\end{aligned}$$