Linear system theory 2024 Problem set [1]

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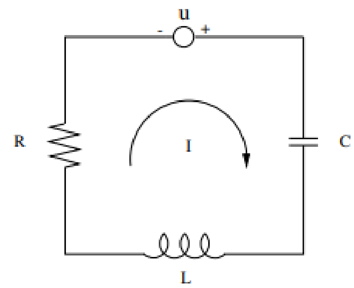
1.

Consider the electrical circuit discussed in class (shown in the figure), but now suppose that its characteristics R, L and C vary with time. Starting with the same non-dynamic physical laws as in class ($q = CV_c$ for the capacitor charge, $\phi = LI$ for the inductor flux), derive a dynamical model of this circuit. It should take the form $\dot{x} = A(t)x + B(t)u$.



Answer:

First, summarize the formulas

$$V_R + V_C + V_L = u$$
$$V_R = IR,\ V_C = \frac{1}{C}\int I(t)dt,\ V_L(t) = L \cdot \frac{dI}{dt}$$
$$q = CV_c,\ \phi = LI$$

The state variables $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} V_c \\ I \end{bmatrix}$

Then, substitute $V_c(t) = \frac{1}{C}\int I(t)dt$ into $q(t)$,

$$q(t) = C(t) \cdot \frac{1}{C(t)}\int I(t)dt = \int I(t)dt$$
$$\frac{d}{dt}q(t) = \dot{C}(t)V_c(t) + C(t)\dot{V}_c(t) = I(t)$$

and not we rearrange the equation for $\dot{V}_c(t)$,

$$\dot{V}_c(t) = \frac{I(t)}{C(t)} - \frac{\dot{C}(t)}{C(t)}V_c(t)$$

Now, for $\dot{I}$

$$V_L(t) = \frac{d\phi}{dt} = \dot{L}I + L\dot{I}$$

$$\dot{I} = \frac{1}{L(t)}(V_L(t) - \dot{L}(t)I(t)) = \frac{1}{L(t)}(V_L(t) - \dot{L}(t)I(t)) = \frac{1}{L(t)}(-V_R(t) - V_C(t) + u(t))$$
$$= \frac{1}{L(t)}(-R(t)I(t) - V_C(t) + u(t))$$

Thus, we can denote the linear form

$\dot{x} = A(t)x + B(t)u$:

$$\begin{bmatrix} \dot{V}_c \\ \dot{I} \end{bmatrix} = \begin{bmatrix} -\frac{\dot{C}(t)}{C(t)} & \frac{1}{C(t)} \\ -\frac{1}{L(t)} & \frac{-R(t)-\dot{L}(t)}{L(t)} \end{bmatrix} \begin{bmatrix} V_c \\ I \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L(t)} \end{bmatrix} u(t)$$

2.

Which of the following are vector spaces over $\mathbb{R}$ (with respect to the standard addition and scalar multiplication)? Justify your answers.

a)

The set of real-valued $n \times n$ matrices with nonnegative entries, where n is a given positive integer.

Answer: No. For random scalar multiplication, $a \in \mathbb{F}, x \in \mathcal{X} \to a \cdot x \in \mathcal{X},\ \mathbb{F} = \mathbb{R}, \mathbb{C}$ Let x be an $n \times n$ matrix with nonnegative entries, if $a = (-1) \to a \cdot x \notin \mathcal{X}$

b)

The set of rational functions of the form $\frac{p(s)}{q(s)}$, where p and q are polynomials in the complex variable s and the degree of q does not exceed a given fixed positive integer k.

Answer: No. For addition, $\frac{p_1(s)}{q_1(s)}, \frac{p_2(s)}{q_2(s)} \in \mathcal{X} \to \frac{p_1(s)q_2(s)+p_2(s)q_1(s)}{q_1(s)q_2(s)} \in \mathcal{X}$

For following example let $k = 4$, $q_1(s) = s^2$ and $q_2(s) = s^3 \to q_1(s)q_2(s) = s^5 \notin \mathcal{X}$

c)

The space $L^2(\mathbb{R}, \mathbb{R})$ of square-integrable functions, i.e., functions $f : \mathbb{R} \to \mathbb{R}$ with the property that $\int_{-\infty}^{\infty} f^2(t)dt < \infty$

Answer: Yes.

Addition: $f, g \in L^2(\mathbb{R}, \mathbb{R}) \to \int_{\infty}^{\infty} (f(t) + g(t))^2 dt < \infty$

Scalar multiplication: $a \in \mathcal{F}, x \in \mathcal{X} \longrightarrow a \cdot x = c^2 \int_{-\infty}^{\infty} f^2(t)dt < \infty \in \mathcal{X}$

$\exists 0$ and $\forall x, \exists(-x) \to f \in L^2(\mathbb{R},\mathbb{R}),\ -f \in L^2(\mathbb{R},\mathbb{R}) \to f + (-f) = 0$

$a(bf) = (ab)f$ and $a(f+g) = af + ag$ and $(a+b)f = af + bf$

3.

Show that vectors $v_1, v_2, \cdots, v_n$ in a vector space X form a basis according of the definition given in class (which says that they are linearly independent and $n = \dim X$) if and only if they are linearly independent and each $x \in X$ can be uniquely written as a linear combination $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$.

Answer:

Suppose vector $v_1, \cdots, v_n$ is linearly independent, $n = \dim X$
The vector V is linearly independent, and each $x \in X$ can be uniquely written as a linear combination $x = a_1v_1 + \cdots + a_nv_n$
suppose there are two different representations,

$$x = a_1v_1 + \cdots + a_nv_n = b_1v_1 + \cdots + b_nv_n$$

and substracting two equations gives:

$$(a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n = 0$$

Since the vectors are linearly independent, the only solution to this equation is $a_i - b_i = 0$ so it's unique. and in the class, a set of n linearly independent vectors in an n dim, vector space is basis.

4.

Let $A : X \to Y$ be a linear operator.

a)

Prove that $\dim N(A) + \dim R(A) = \dim X$ ((the sum of the dimension of the nullspace of A and the dimension of the range of A equals the dimension of X).

Answer:

1. Let $N(A)$ denote the nullspace of A. Since $N(A)$ is a subspace of X, we can find a basis for $N(A)$. Suppose that the basis of $N(A)$ consists of m vectors. Then, $\dim N(A) = m$.

2. Now, extend this basis to span the whole space X. By adding k additional vectors to the basis of $N(A)$, we obtain a basis for the entire space X. Therefore, the dimension of X is $\dim X = m + k$.

3. So $\dim X = m + k$, and $m = \dim N(A), k = \dim R(A)$ thus,

$$\dim N(A) + \dim R(A) = \dim X$$

b)

Now assume that $X = Y$. It is not always true that X is a direct sum of $N(A)$ and $R(A)$. Find a counterexample demonstrating this. Also, describe a class of linear operators (as general as you can think of) for which this statement is true.

Answer: The conditions for a direct sum are:

$$u + w = X$$

$$w \cap w = \{0\}$$

This mean $X = w \oplus u$, u and w us not 0 vector, and unique.

1. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ The matrix A defines a linear transformation on $\mathbb{R}^2$.

2. The null space $N(A)$ consists of all vectors $\mathbf{x} \in \mathbb{R}^2$ such that $A\mathbf{x} = 0$

$$A \begin{bmatrix} x_1 \\ x2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This implies $x_1 + x_2 = 0$. So the null space $N(A)$ is given by:

$$N(A) = \left\{ \begin{bmatrix} x \\ -x \end{bmatrix} : x \in \mathbb{R} \right\} = span \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

3. $R(A)$ consists of all possible outputs of the form $A\mathbf{x}$:

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix}$$

Thus, the $R(A)$ is given by:

$$R(A) = \left\{ \begin{bmatrix} y \\ 0 \end{bmatrix} : y \in \mathbb{R} \right\} = span \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

4. Why $N(A) \oplus R(A) \neq X$

   Consider the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

   $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in R(A)$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \notin N(A)$

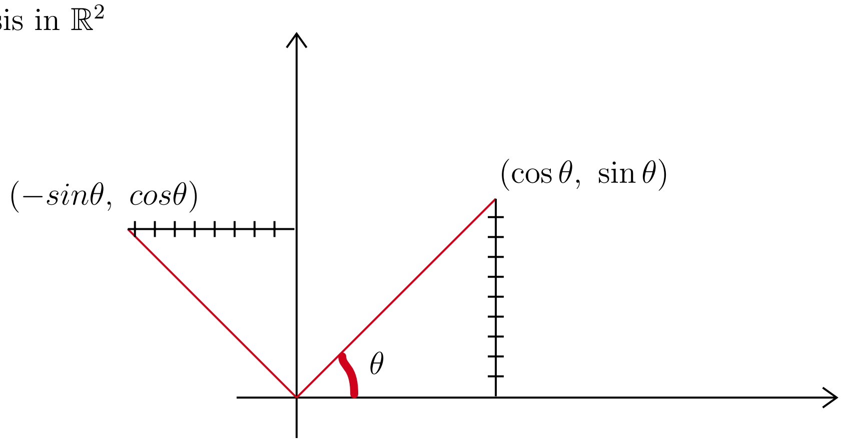
5. The general direct sum example is :

$$\mathbf{w} = span\,\{(1,0,0),(0,1,0)\} \text{ and } \mathbf{u} = span\,\{(0,0,1)\}$$

$$\mathbb{R}^3 = \mathbf{w} \oplus \mathbf{u}$$

5.

Let A be the linear operator in the plane corresponding to the counter-clockwise rotation around the origin by some given angle θ. Compute the matrix of A relative to the standard basis in $\mathbb{R}^2$



Rotate the point $(1,0)$ by angle θ, it becomes $(\cos\theta, \sin\theta)$. Similarly, rotating the point $(0,1)$ by an angle θ results in $(-\sin\theta, \cos\theta)$.

Since the rotation matrix R represents a linear transformation:

$$R\begin{bmatrix} x \\ y \end{bmatrix} = R\begin{bmatrix} x \\ 0 \end{bmatrix} + R\begin{bmatrix} 0 \\ y \end{bmatrix} = R\begin{bmatrix} 1 \\ 0 \end{bmatrix} x + R\begin{bmatrix} 0 \\ 1 \end{bmatrix} y$$

$$R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} x + \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} y$$

So,

$$R\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

6.

Three employees—let's call them Alice, Bob, and Cheng—received their end-of-the-year bonuses which their boss calculated as a linear combination of three performance scores: leadership, communication, and work quality. The coefficients (weights) in this linear combination are the same for all three employees, but the boss doesn't disclose them. Alice knows that she got the score of 4 for leadership, 4 for communication, and 5 for work quality. Bob's scores for the same categories were 3, 5, and 4, and Cheng's scores were 5, 3, and 3. The bonus amounts are \$18,000 for Alice, \$16,000 for Bob, and \$14,000 for Cheng. The employees are now curious to determine the unknown coefficients (weights).

a)

Set up this problem as solving a linear equation of the form $Ax = b$ for the unknown vector x

Answer:

The general formula for the bonus calculation is :

$$\text{Bonus} = a \cdot \text{Leadership} + b \cdot \text{Communication} + c \cdot \text{Work Quality}$$

And we can express this system of linear equation in matrix form as $Ax = b$:

$$A = \begin{bmatrix} 4 & 4 & 5 \\ 3 & 5 & 4 \\ 5 & 3 & 3 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \text{leadership} \\ \text{communication} \\ \text{work quality} \end{bmatrix}, b = \begin{bmatrix} 18000 \\ 16000 \\ 14000 \end{bmatrix}$$

b)

Calculate the unknown weights. It's up to you whether you use a) for this or do it another way.

Answer:

Using matlab, we can solve the system by $x = A^{-1}b$
$\text{inv}(A) * b$

$$x = \begin{bmatrix} 1000 \\ 1000 \\ 2000 \end{bmatrix}$$

More detail in Appendix Figure 1.

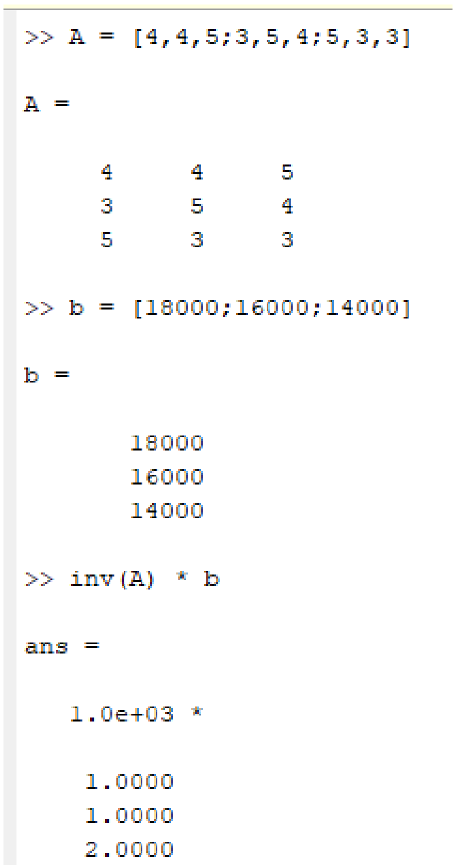
c)

Are the weights that you computed unique? Explain why or why not.

Answer:

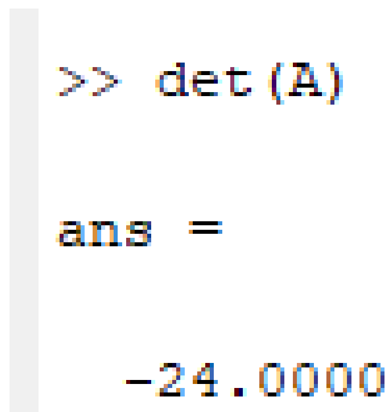
To determine whether the weights from the system are unique, we need to check matrix A is invertible.
If A is invertible, the weights are unique. This means the determinant is not 0.
We can also solve the determinant using matlab $\det(A)$
Thus, $\det(A) = -24$, so weight x is unique. (figure 2.)

Appendix



```
>> A = [4,4,5;3,5,4;5,3,3]

A =

     4     4     5
     3     5     4
     5     3     3

>> b = [18000;16000;14000]

b =

       18000
       16000
       14000

>> inv(A) * b

ans =

   1.0e+03 *

    1.0000
    1.0000
    2.0000
```

Figure 1: Calculate x



```
>> det(A)

ans =

  -24.0000
```

Figure 2: det(A)

Matlab output

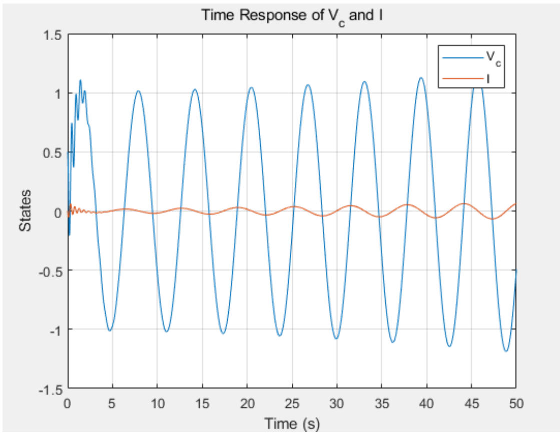


Figure 3: Time response V_c and I

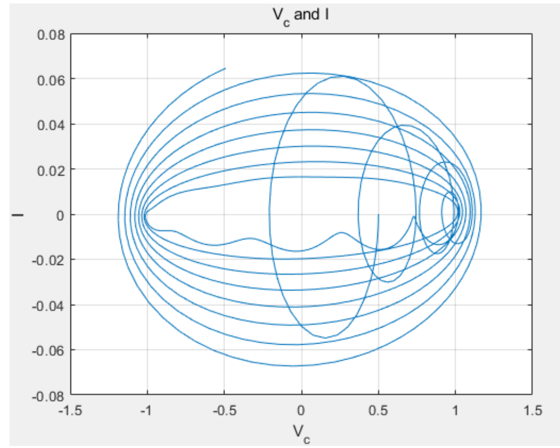


Figure 4: Plane