University of Yonsei Mirae, Linear System

HW 6

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November 18, 2024

Problem 1

Lyapunov's second method (for asymptotic stability) generalizes to time-varying systems $\dot{x} = f(t, x)$. Let $V(t, x)$ as follows. Let $V(t, x)$ be a function such that for positive definite functions $W_1(x), W_2(x), W_3(x)$ we have:

$$W_1(x) \leq V(t, x) \leq W_2(x)$$

and

$$\dot{V}(t, x) = \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) \cdot f(t, x) \leq -W_3(x) \tag{1}$$

for all t and x. Then the system is globally asymptotically stable (globally if W_1 is radially unbounded).

Now, consider an LTV system $\dot{x} = A(t)x$ and let V be of the form $V(t, x) = x^T P(t)x$. Derive the time-varying analogue of the Lyapnov equation, in other words, derive an equation that $P(t)$ needs to satisfy to guarantee asymptotic stability. Carefully specify all required properties of quantities appearing in your equation so that the above stability result, based on (1), is applicable.

Answer Just substitute all properties to $\dot{V}(t,x)$, then we get

$$\begin{aligned}\dot{V}(t,x) &= \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x) \cdot f(t,x) \\ &= x^T \dot{P}(t) x + \dot{x}^T P(t) x + x^T P(t) \dot{x} \\ &= x^T \dot{P}(t) x + x^T A(t)^T P(t) x + x^T P(t) A(t) x \\ &= x^T \left(\dot{P}(t) + P(t)A(t) + A(t)^T P(t) \right) x \le -W_3(x)\end{aligned}$$

Since $\dot{V}(t,x) \le -W_3(x)$ and $W_3(x)$ is positive definite, it should be satisfied that

$$\dot{P}(t) + P(t)A(t) + A(t)^T P(t) \le -Q(t)$$

Problem 2

In class we are focusing on continuous-time systems, but we will occasionally mention discrete-time systems in the exercises. The two cases are usually quite similar but there are some notable differences.

a) Write the formula for the solution $x(k)$ at time k of the discrete-time control system

$$x(k+1) = Ax(k) + Bu(k)$$

starting from some initial state $x(0)$ at time 0. (Your answer will be the discrete counterpart of the variation-of-constants formula.)

b) Under what conditions on the eigenvalues of A is the discrete-time system $x(k+1) = Ax(k)$ (no controls) asymptotically stable? stable? Justify your answer. (Stability definitions are the same as for continuous-time systems, just replace t by k.)

c) Lyapunov's second method for the discrete-time system $x(k+1) = f(x(k))$ involves the difference $\Delta V(x) := V(f(x)) - V(x)$ instead of the derivative $\dot{V}(x)$; with this substitution, the statement is the same as in the continuous-time case. Derive the counterpart of the Lyapunov equation for the LTI discrete-time system $x(k+1) = Ax(k)$.

Answer (a) For discrete-time system, when $k = 0$,

$$x(1) = Ax(0) + Bu(0)$$

and when writing the equation for k,

$$\begin{aligned}
&x(2) = A(Ax(0) + Bu(0)) + Bu(1) = A^2x(0) + ABu(0) + Bu(1) \\
&x(3) = A(A^2x(0) + ABu(0) + Bu(1)) + Bu(2) = A^3x(0) + A^2Bu(0) + ABu(1) + Bu(2) \\
&\quad \vdots \\
&x(k) = A^{k-1}Bu(0) + A^{k-2}Bu(1) + \cdots Bu(k-1)
\end{aligned}$$

Thus, the general formula for $x(k)$ is

$$x(k) = [B \quad AB \quad A^2B \quad \cdots \quad A^{k-1}B] \begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}$$

Answer (b) The AS condition is

$$|\lambda| < 1 \quad \forall \lambda \in \text{eig}(A)$$

Then the system is asymptotically stable. The stability condition is

$$|\lambda| \leq 1 \quad \forall \lambda \in \text{eig}(A)$$

and

$$Real(\lambda) \leq 0$$

Then the system is stable.

Answer (c) Let Lyapunov function $V(x) = x^TPx$. P should be positive definite. Then,

$$\begin{aligned}
\Delta V(x) &= V(f(x)) - V(x) \\
&= V(Ax) - V(x) \\
&= (Ax)^TP(Ax) - x^TPx \\
&= x^TA^TPAx - x^TPx \\
&= x^T(A^TPA - P)x < 0
\end{aligned}$$

Then, we get

$$A^T PA - P < 0$$

I don't know after that... 

Problem 3

Consider the system $\dot{x} = Ax + Bu$ such that $||e^{At}|| \leq ce^{-\mu t}$ for some constants $c, \mu > 0$.

a) Prove that if u is bounded over all time (in the sense that $\sup_{0\leq t<\infty} |u(t)| \leq M$ for some M) then x is also bounded, for arbitrary initial conditions.

b) Now restrict attention to the zero initial condition ($x_0 = 0$). We can view the above system as a linear operator from $\mathcal{U}$ to $\mathcal{X}$, where $\mathcal{U}$ is the space of bounded functions $u : [0, \infty) \to \mathbb{R}^m$ with norm $\|u\| := \sup_{0\leq t<\infty} |u(t)|$ and $\mathcal{X}$ is defined analogously. What can you say about the induced norm of this operator, using the calculations you made in part a)?

c) Repeat parts a)-b) but now assuming that the norm in the $\mathcal{U}$ space is the L_2 norm (so that "bounded u" now means square-integrable u); the norm on $\mathcal{X}$ is still the supremum norm as before. (Hint: use the Cauchy-Schwarz inequality.)

Answer (a) The solution of the linear system is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

We can use the property of the matrix exponential that $||e^{At}|| \leq ce^{-\mu t}$, plus $\sup_{0\leq t<\infty} |u(t)| \leq M$

$$\begin{aligned}
||x(t)|| &\leq ||e^{At}|| \cdot ||x(0)|| + \int_0^t ||e^{A(t-\tau)}|| \cdot ||B|| \cdot ||u(\tau)||d\tau \\
&\leq ce^{-\mu t}||x(0)|| + \int_0^t ce^{-\mu(t-\tau)}||B||Md\tau
\end{aligned}$$

Now, calculate the integral

$$\begin{aligned}
\int_0^t ce^{-\mu(t-\tau)}||B||M d\tau &= c||B||M \int_0^t e^{-\mu(t-\tau)} d\tau \\
&= c||B||M \int_t^0 e^{-\mu k}(-dk) = c||B||M \int_0^t e^{-\mu k} dk \\
&= c||B||M \left[-\frac{1}{\mu} e^{-\mu k} \right]_0^t \\
&= c||B||M \cdot \frac{1 - e^{-\mu t}}{\mu}
\end{aligned}$$

Then, we get

$$||x(t)|| \leq ce^{-\mu t}||x(0)|| + c||B||M \cdot \frac{1 - e^{-\mu t}}{\mu}$$

Since $e^{-\mu t} \leq 1$,

$$||x(t)|| \leq c||x(0)|| + \frac{c||B||M}{\mu}$$

Answer (b) When $x_0 = 0$, the solution of the linear system is

$$x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

Then, we can find the bound of $x(t)$, using the result of part (a)

$$||x(t)|| \leq \int_0^t ||e^{A(t-\tau)}|| \cdot ||B|| \cdot ||u(\tau)|| d\tau$$

Since $||u(\tau)|| \leq 1$ (because u has exponential, so it's supremum is 1), we get

$$\begin{aligned}
||x(t)|| &\leq \int_0^t ce^{-\mu(t-\tau)}||B|| d\tau = c||B|| \int_0^t e^{-\mu(t-\tau)} d\tau \\
&\leq \frac{c||B||}{\mu}
\end{aligned}$$

Therefore, the induced norm of this operator is

$$||T|| = \sup_{||u||_{\mathcal{U}} \leq 1} ||x||_{\mathcal{X}} \leq \frac{c||B||}{\mu}$$

Answer (c) When $x_0 = 0$, the solution of the linear system is

$$x(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$

Then,

$$\begin{aligned} ||x(t)|| &\leq \int_0^t ||e^{A(t-\tau)}|| \cdot ||B|| \cdot ||u(\tau)|| d\tau \\ &\leq c||B|| \int_0^t e^{-\mu(t-\tau)} ||u(\tau)|| d\tau \end{aligned}$$

Now, we can use the Cauchy-Schwarz inequality

$$\begin{aligned} \int_0^t e^{-\mu(t-\tau)} ||u(\tau)|| d\tau &\leq \left(\int_0^t e^{-\mu(t-\tau)} d\tau \right)^{1/2} \cdot \left(\int_0^t ||u(\tau)||^2 d\tau \right)^{1/2} \\ &= \left(\frac{1 - e^{-2\mu t}}{2\mu} \right)^{1/2} \cdot ||u||_{L_2} \\ &\leq \left(\frac{1}{2\mu} \right)^{1/2} \cdot ||u||_{L_2} \end{aligned}$$

Therefore, we get

$$||x(t)|| \leq c||B|| \left(\frac{1}{2\mu} \right)^{1/2} \cdot ||u||_{L_2}$$

Problem 4

We revisit Problem 1 from Problem Set 5. Assume that agent 1 is the *leader* and knows a desired location $p \in \mathbb{R}$ to which all agents should converge. Agents 2 and 3 do not know p. All agents see each other. Based on this information, write down modified consensus equations for which you can prove that all three agents asymptotically converge to p from arbitrary initial positions.

Answer The agent 1 is the leader and knows the desired location p. And the agents 2 and 3 do not know p. However, all agents can see each other. Then, all agents can reach the desired location if agnets 2 and 3 move to agent 1's location without moving the initial position of agent 1 as p. Then, we can write

$$x_1(t) = p, \quad \dot{x}_1(t) = 0$$

Thus,

$$\begin{aligned}
\dot{x}_1(t) &= 0 \\
\dot{x}_2(t) &= -(x_2 - p) - (x_2 - x_3) = -2x_2 + x_3 + p \\
\dot{x}_3(t) &= -(x_3 - p) - (x_3 - x_2) = -2x_3 + x_2 + p
\end{aligned}$$

And we can write the matrix form as

$$\dot{x}(t) = Ax(t) + Bp$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Now, we can check the stability of the system by checking the eigenvalues of A.

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A = [0 0 0; 0 -2 1; 0 1 -2];
eig(A)

ans =

   -3.0000
   -1.0000
    0
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Therefore, the system is asymptotically stable.

Problem 5

Consider the LTV system $\dot{x} = A(t)x + B(t)u$ and the problem of steering its state from x_0 at time t_0 to x_1 at time t_1. Instead of reducing to a system without A by means of a time-dependent coordinate transformation (as done in class), address this problem directly, using the variation-of-constants formula and applying to the appropriate map the lemma from class about the range of such a map. You should arrive at a slightly different condition from the one obtained in class. Discuss its geometric interpretation (in terms of free motion vs. controlled motion).

Answer I don't know how to approach this problem.