Linear system theory 2024 Problem set [2]

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1.

Let A be the linear operator in the plane corresponding to the counter-clockwise rotation around the origin by some given angle θ (see Problem 5 in Homework 1). Compute the matrix of A relative to the basis $\left\{\binom{-1}{0}, \binom{2}{1}\right\}$.

Answer:

Suppose the point A is the counter-clockwise rotation around the initial reference frame.
And we want to know the point by frame B. (change basis by $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$) Then, we know that

$$\begin{aligned} B &= P^{-1}AP \\ &= \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) - 2\sin(\theta) & 5\sin(\theta) \\ -\sin(\theta) & \cos(\theta) + 2\sin(\theta) \end{bmatrix} \end{aligned}$$

Or, use linear combination. Let the column vector of basis $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ be v_1 and v_2.

$$A(v_1) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos\theta \\ -\sin\theta \end{bmatrix}$$

$$A(v_2) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cos\theta - \sin\theta \\ 2\sin\theta + \cos\theta \end{bmatrix}$$

Then, we can represented $A(v_1), A(v_2)$ by linear combination.

$$\begin{aligned} A(v_1) &= a_{11}v_1 + a_{21}v_2 \\ A(v_2) &= a_{21}v_1 + a_{22}v_2 \end{aligned}$$

$$\begin{bmatrix} -\cos\theta \\ -\sin\theta \end{bmatrix} = a_{11} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + a_{21} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We can get $a_{11} = \cos\theta - 2\sin\theta$ and $a_{21} = -\sin\theta$.
In second equation,

$$\begin{bmatrix} 2\cos\theta - \sin\theta \\ 2\sin\theta + \cos\theta \end{bmatrix} = a_{12} \begin{bmatrix} -1 \\ 0 \end{bmatrix} + a_{22} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We can get $a_{12} = 5\sin\theta$ and $a_{22} = 2\sin\theta + \cos\theta$.
Therefore, the matrix of A relative to the basis is

$$\begin{bmatrix} \cos\theta - 2\sin\theta & 5\sin\theta \\ -\sin\theta & 2\sin\theta + \cos\theta \end{bmatrix}$$

2.

We revisit Problem 6 in Homework 1. Alice and Bob have reason to believe that Cheng's performance scores and bonus amount that he gave them were incorrect. They still want to figure out their unknown weights, but now they have to use only their own scores and bonus amounts, without Cheng's.

a)

Set up this problem as solving a linear equation of the form $Ax = b$ for the unknown vector x. The new matrix A will be 2×3.

Answer:

$$\text{Bonus} = a \cdot \text{Leadership} + b \cdot \text{Communication} + c \cdot \text{Work Quality}$$

Without Cheng, we can write the equation,

$$Ax = b$$

$$A = \begin{bmatrix} 4 & 4 & 5 \\ 3 & 5 & 4 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, b = \begin{bmatrix} 18,000 \\ 16,000 \end{bmatrix}$$

b)

Let $A^\dagger$ be a *right inverse* of A, which by definition is a matrix such that $AA^\dagger = I_{2\times 2}$. Express all solutions x of the equation in a) in terms of such a right inverse $A^\dagger$, the vector b, and the null space of A. Carefully state and justify your answer.

Answer:

The matrix A is 2×3, non-invertible matrix.
The solution set of non-invertible matrix is "no solution" or "infinitely many solution".
In this case, A has infinitely many solution.
Suppose the solution is X. Then, we can write,

$$X = X_n + X_p$$

X_n is null space and X_p is particular solution.
To provide a brief proof of this,

$$AX = A(X_n + X_p) = AX_n + AX_p = b \rightarrow AX_p = b$$

So, express all solutions x is

$$x = A^{\dagger}b + X_n$$

c)

Find a particular solution x^* of the equation in a). (Hint: use your solution from Homework 1.)
Verify numerically that x^* indeed belongs to the solution space you described in b). To do this, you can use MATLAB commands **pinv** to compute a right inverse and **null** to compute a vector in the null space.

Answer:

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A = [4, 4, 5; 3, 5, 4];
b = [18000; 16000];
x_star = pinv(A) * b;
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Then, x_star = $\begin{bmatrix} 1.0e+03* \\ 1.3699 \\ 1.0411 \\ 1.6712 \end{bmatrix}$
And to verifying that x^* indeed belongs to the solution space,

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z = null(A);
is_solution = A * (x_star + z) == b
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The answer is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so x^* is satisfying $Ax = b$, and adding any vector from the null space to x^* still results in a valid solution for the system.

3.

Let A be a real-valued $n \times n$ matrix. Suppose that $\lambda + i\mu$ is a complex eigenvalue of A and $x + iy$ is a corresponding complex eigenvector. (Here, $\lambda, \mu \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$.)

a)

Show that $x - iy$ is also an eigenvector, with eigenvalue $\lambda - iu$.

Answer:

In the class, we learned $AS = SD$ (S is eigenvector and D is eigenvalue).
Therefore, by substituting the eigenvector and eigenvalue into the above equation and expanding,

$$A(x + iy) = (x + iy)(\lambda + i\mu)$$
$$\Leftrightarrow Ax + Aiy = \lambda x + i\mu x + \lambda iy - \mu y$$

And by separating the equation into real number and complex number,

$$Ax = \lambda x - \mu y$$
$$iAy = i\mu x + i\lambda y$$

Then, substituting the values to second equation.

$$\begin{aligned} A(x - iy) &= Ax - Aiy \\ &= (\lambda x - \mu y) - (i\mu x + i\lambda y) \\ &= (x - iy)(\lambda - i\mu) \end{aligned}$$

Therefore, $(x - iy)$ is also an eigenvector, with eigenvalue $\lambda - i\mu$.

b)

Let V be the 2-dimensional subspace over $\mathbb{R}$ spanned by x and y. (In other words, V is the set of linear combinations, with real coefficients, of the real-valued vectors x and y.) Show that V is an *invariant subspace* for A, in the sense that for every $z \in V$ we have $Az \in V$.

Answer:

To show that V is an *invariant subspace* for A, let's prove that for $z \in V, Az \in V$.
In question, V is a 2-dimensional subspace over spanned by x and y.
In other words, when $z = ax + by, Az$ must also be an element of V.

$$Az = A(ax + by) = Aax + Aby$$

and we saw that $Ax = \lambda x - \mu y$ and $Ay = \mu x + \lambda y$ in a).
Now, substituting Ax and Ay.

$$\begin{aligned} Az &= a(\lambda x - \mu y) + b(\mu x + \lambda y) \\ &= (a\lambda + b\mu)x + (-a\mu + \lambda b)y \end{aligned}$$

This is also linear combination of x and y.
So V is **invariant subspace** of A.

4.

We know from the Cayley-Hamilton theorem that the characteristic polynomial p of a square matrix A satisfies $p(A) = 0$. Let us call a polynomial q a *minimal* polynomial for the matrix A if $q(A) = 0$ and q has the smallest possible degree among all polynomials with this property.

a)

Give an example of a matrix A whose characteristic polynomial is minimal (justify rigorously).

Answer:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

and this characteristic polynomial p is

$$p(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 3 \end{bmatrix} = (\lambda - 1)(\lambda - 3)$$

For this matrix A, $p(A) = 0$ holds and $q(A) = 0$ is satisfied, the polynomial of smallest degree is the same as the characteristic polynomial.

For q and p to be equal, **Algebraic multiplicity** and **Geometric multiplicity** should be 1.

b)

Give an example of a matrix A whose characteristic polynomial is not minimal (explain why not).

Answer:

Suppose the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.
The characteristic polynomial of A is $p(\lambda) = (\lambda - 1)^2$.
And the Geometric multiplicity is 1, it isn't equal with algebraic multiplicity.
Therefore, it cannot diagonalized.

5.

Consider the following matrix, whose exponential was computed in class:
$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, a, b \in \mathbb{R}$

a)

Re-derive the expression for e^{At} by diagonalizing A over $\mathbb{C}$ and using the formula $e^{a\pm bi} := e^a \cos b \pm ie^a \sin b$ (treat this as the definition of complex exponential).

Answer:

In class we saw $\det(A) = (\lambda - a)^2 + b^2 = 0$

$$\lambda = a \pm bi$$

And we know that

$$A = PDP^{-1}$$

To use this equation,

$$e^{At} = e^{PDP^{-1}t}$$

And we can write

$$e^{PDP^{-1}t} = Pe^{Dt}P^{-1}$$

$$\begin{aligned} e^{Dt} &= \begin{bmatrix} e^{(a+bi)t} & 0 \\ 0 & e^{(a-bi)t} \end{bmatrix} \\ &= e^{at} \begin{bmatrix} \cos b(t) + i \sin b(t) & 0 \\ 0 & \cos b(t) - i \sin b(t) \end{bmatrix} \end{aligned}$$

And now, we know that

$$P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, P^{-1} = -\frac{1}{2i} \begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}$$

Therefore,

$$\begin{aligned} e^{At} &= Pe^{Dt}P^{-1} \\ &= e^{at} \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix} \end{aligned}$$

b)

Using the formula $e^{A(t+\sigma)} = e^{At}e^{A\sigma}$ and the result of a) for suitably chosen values of a and b, verify the trigonometric identities $\cos(t+\sigma) = \cos t \cos \sigma - \sin t \sin \sigma$ and $\sin(t+\sigma) = \sin t \cos \sigma + \cos t \sin \sigma$.

Answer:

We saw $e^{At} = e^{at} \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}$ but, e^{at} is not important for proving the equation, so I'll omit.

Using the formula, $e^{A(t+\sigma)} = e^{At}e^{A\sigma}$,

$$e^{At}e^{A\sigma} = Pe^{Dt}e^{D\sigma}P^{-1}$$

And we can get matrix,

$$e^{A(t+\sigma)} = \begin{bmatrix} \cos(t+\sigma) & -\sin(t+\sigma) \\ \sin(t+\sigma) & \cos(t+\sigma) \end{bmatrix}$$

And $e^{At}e^{A\sigma}$ also represented by

$$\begin{aligned} e^{At}e^{A\sigma} &= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} \cos\sigma & -\sin\sigma \\ \sin\sigma & \cos\sigma \end{bmatrix} \\ &= \begin{bmatrix} \cos t\cos\sigma - \sin t\sin\sigma & -(\cos t\sin\sigma + \sin t\cos\sigma) \\ \sin t\cos\sigma + \cos t\sin\sigma & \cos t\cos\sigma - \sin t\sin\sigma \end{bmatrix} \end{aligned}$$

Therefore,

$$\cos(t+\sigma) = \cos t\cos\sigma - \sin t\sin\sigma$$
$$\sin(t+\sigma) = \sin t\cos\sigma + \cos t\sin\sigma$$