HW 5

2020271053, MinJae Kim

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Problem 1

Let x_1, x_2, x_3 be positions of 3 moving points on the real line (let's call them "agents"). For each agent i, let N_i denote the set of all other agents that agent i can "see" (meaning it can measure their positions), and consider the consensus equations

$$\dot{x}_i = -\sum_{j \in N_i} (x_i - x_j), \quad i = 1, 2, 3$$

Assume initially that all agents see each other, so that $N_1 = \{2, 3\}$, $N_2 = \{1, 3\}$, $N_3 = \{1, 2\}$.

a) Write down the consensus equations as an LTI system $\dot{x} = Ax$.

Answer

$$\begin{aligned}
\dot{x}_1 &= -[(x_1 - x_2) + (x_1 - x_3)] = -2x_1 + x_2 + x_3 \\
\dot{x}_2 &= -[(x_2 - x_1) + (x_2 - x_3)] = x_1 + -2x_2 + x_3 \\
\dot{x}_3 &= -[(x_3 - x_1) + (x_3 - x_2)] = x_1 + x_2 - 2x_3
\end{aligned}$$

Then,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

b) Define the following 3×3 matrices:

The degree matrix D is the diagonal matrix with diagonal elements D_{ii} equal to the number of agents in N_i. The adjacency matrix J has elements $J_{ij} = 1$ if agents i and j see each other, and 0 if they don't (by assumption, J is symmetric and its diagonal elements are 0). The Laplacian matrix $L := D - J$. Verify that the matrix A in part a) equals $-L$.

Answer

The diagonal matrix D's diagonal elements are 2.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The adjacency matrix J is

$$J = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Then, the Laplacian matrix L is

$$L = D - J = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

And $-L$ is the matrix A in part a).

c) Is it true that system equilibria are exactly points in $\mathbb{R}^3$ with $x_1 = x_2 = x_3$ (agents' positions coincide)?

Answer

System becomes to equilibria, when $\dot{x} = 0$. (because equilibria means that the system doesn't change.)
Then, we have to solve the equation $Ax = 0$.
We already know that $A = -L$. Then, we have to solve the equation $-Lx = 0$.
Laplacian matrix L's null space is the composed of all constant vectors(when the graph is connected).
Thus,

$$Ax = -Lx = 0 \Rightarrow x = [c, c, c]^T$$

Therefore, the system equilibria are exactly points in $\mathbb{R}^3$ with $x_1 = x_2 = x_3$.

d) Are the system equilibria stable? Asymptotically stable?

Answer

We can check stability by the eigenvalues of the matrix A.
The $Real(\lambda) < 0$ means that the system is AS.
Plus, the $Real(\lambda) \leq 0$ means that the system is **stable** (we saw in class).

```
A =
[-2, 1, 1;
 1, -2, 1;
 1, 1, -2]

eig(A) =
-3.0000
-3.0000
 0.0000
```

The eigenvalues of the matrix A are $-3, -3, 0$.
More detail, $\dot{x}_1 = Ax$ is stable $\Leftrightarrow$ all solutions are bounded!
$\Leftrightarrow$ $Real(\lambda) \leq 0$ for all eigenvalues of A and $Real(\lambda) < 0$ for only λ with multiplicity > 1.
Therefore, this system is stable.

e) Now assume that agents 2 and 3 don't see each other. Answer questions a)–d) for this case.

$$N_1 = \{2, 3\}, N_2 = \{1\}, N_3 = \{1\}$$

$$\dot{x}_1 = -[(x_1 - x_2) + (x_1 - x_3)] = -2x_1 + x_2 + x_3$$
$$\dot{x}_2 = -[(x_2 - x_1)] = x_1 - x_2$$
$$\dot{x}_3 = -[(x_3 - x_1)] = x_1 - x_3$$

Then,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

```
new_A =
[-2, 1, 1;
 1, -1, 0;
 1, 0, -1]

eig(new_A) =
-3.0000
-1.0000
-0.0000
```

Therefore, the system is stable.

f) Now assume that agent 3 doesn't see agents 1 and 2 and is not seen by them. Answer a)–d) again.

Answer

$$
\begin{aligned}
N_1 &= \{2\}, N_2 = \{1\}, N_3 = \{\} \\
\dot{x}_1 &= -[(x_1 - x_2)] = -x_1 + x_2 \\
\dot{x}_2 &= -[(x_2 - x_1)] = x_1 - x_2 \\
\dot{x}_3 &= 0
\end{aligned}
$$

Then,

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

```
B =
[-1, 1, 0;
 1, -1, 0;
 0, 0, 0]

eig(B) =
-2
0
0
```

Therefore, the system is stable.

Problem 2

Consider the system

$$\dot{x_1} = x_2, \quad \dot{x_2} = -x_2 - \sin x_1$$

and the three candidate Lyapunov functions

$$V_1(x) = \frac{1}{2}x_2^2 - \cos x_1, \quad V_2(x) = 1 + \frac{1}{2}x_2^2 - \cos x_1, \quad V_3(x) = 1 + \frac{1}{4}x_1^2 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_2^2 - \cos x_1$$

a) For $i = 1, 2, 3$, compute $\dot{V}_i(x)$.

Answer

$\dot{V} = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial V}{\partial x} \cdot \dot{x}$

$$\begin{aligned}
\dot{V}_1(x) &= \frac{\partial V_1}{\partial x_1} \cdot \dot{x}_1 + \frac{\partial V_1}{\partial x_2} \cdot \dot{x}_2 = -x_2^2 \\
\dot{V}_2(x) &= \frac{\partial V_2}{\partial x_1} \cdot \dot{x}_1 + \frac{\partial V_2}{\partial x_2} \cdot \dot{x}_2 = -x_2^2 \\
\dot{V}_3(x) &= \frac{\partial V_3}{\partial x_1} \cdot \dot{x}_1 + \frac{\partial V_3}{\partial x_2} \cdot \dot{x}_2 = -\frac{1}{2}x_2^2 - -\frac{1}{2}x_1 \sin x_1
\end{aligned}$$

b) In each case, explain what conclusions (if any) you can reach using Lyapunov's second method.

Answer

To use Lyapunov's second(direct) method, we have to check two conditions.
1. $V(x)$ is positive definite.
2. $\dot{V}(x)$ is negative semi-definite.
If the two conditions are satisfied, the system is stable.
$V_1(x)$ is not positive definite, so we can't use Lyapunov's second method.
$V_2(x)$ is positive definite, we need to check the second condition.

$$\dot{V}_2(x) = -x_2^2$$

$\dot{V}_2(x)$ is negative semi-definite, so the system is stable.
In $V_3(x)$, $1 - \cos x_1 \geq 0$, we have to check $\frac{1}{4}x_1^2 + \frac{1}{2}x_1 x_2 + \frac{1}{2}x_2^2 \geq 0$.
We can find P by $x^T P x$:

$$P = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$$

```
eig(p) =
0.0955
0.6545
```

So, $V_3(x)$ is positive definite. Now we have to check the second condition.

$$\dot{V}_3(x) = -\frac{1}{2}x_2^2 - \frac{1}{2}x_1 \sin x_1 \leq 0$$

Therefore, the system is stable.

Problem 3

Consider a system $\dot{x} = f(x)$. Suppose that

$$\frac{d}{dt}(x^T P x) \le -x^T Q x$$

where P and Q are symmetric positive definite matrices (here the time derivative is taken along solutions of the system, as explained in class). Prove that under this condition the system is exponentially stable, in the sense that its solutions satisfy

$$|x(t)| \le c e^{-\mu t} |x(0)|$$

for some $c, \mu > 0$. Note that this statement is true whether the system is linear or not.

Hint:

> Use the inequality $\lambda_{\min}(M)|x|^2 \le x^T M x \le \lambda_{\max}(M)|x|^2$ discussed in class. You may also use the fact that if a function $v(t)$ satisfies $\dot{v} \le -av$, then $v(t) \le e^{-at} v(0)$.

Answer

The system $\dot{x} = f(x)$ and given condition is that $\frac{d}{dt}(x^T P x) \le -x^T Q x$ where P and Q are symmetric positive definite matrices. We can set Lyapunov function $V(x) = x^T P x$ plus, use the hint.

$$\lambda_{\min}(P)|x|^2 \le x^T P x \le \lambda_{\max}(P)|x|^2$$
$$\lambda_{\min}(Q)|x|^2 \le x^T Q x \le \lambda_{\max}(Q)|x|^2$$

Then, we can get

$$\frac{d}{dt}(x^T P x) = \dot{V}(x) \le -x^T Q x$$

and

$$\dot{V}(x) \le -x^T Q x \le -\lambda_{\min}(Q)|x|^2$$
$$\Rightarrow \dot{V}(x) \le -\lambda_{\min}(Q)|x|^2$$

This means that $\dot{V}(x) \leq 0$.
Now, we can use the hint.

$$\lambda_{\min}(P)|x|^2 \leq V(x) = x^T P x \leq \lambda_{\max}(P)|x|^2$$

$$|x|^2 \geq \frac{V(x)}{\lambda_{\max}(P)}$$

Then,

$$\dot{V}(x) \leq -\lambda_{\min}(Q) \cdot \frac{V(x)}{\lambda_{\max}(P)} = -\mu V(x)$$

Thus, we can write the equation as

$$\frac{d}{dt} V(x) \leq -\mu V(x)$$

Therefore,

$$V(x) \leq e^{-\mu t} V(0)$$

The system is exponentially stable.

Problem 4

Show that if all eigenvalues of a matrix A have real parts strictly less than some $-\mu < 0$, then for every $Q = Q^T > 0$ the equation

$$PA + A^T P + 2\mu P = -Q$$

has a unique solution $P = P^T > 0$. Show that in this case we have

$$|x(t)| \leq ce^{-\mu t}|x(0)|$$

for some $c > 0$. (The number μ is called a stability margin.)

Answer

We can write the equation as

$$PA + A^T P = -Q - 2\mu P$$

Let system $\dot{x} = Ax$ and $V(x) = x^T Px$.
Then, we can get

$$\begin{aligned}\dot{V}(x) &= x^T P\dot{x} + \dot{x}^T Px \\ &= x^T PAx + x^T A^T Px = x^T(PA + A^T P)x\end{aligned}$$

And we can substitute the equation to $\dot{V}(x)$:

$$\begin{aligned}\dot{V}(x) &= X^T(-2\mu P - Q)x \\ &= -2\mu x^T Px - x^T Qx = -2\mu V(x) - x^T Qx\end{aligned}$$

to be continued........... I'm sorry, I can't solve this problem. :(
I'll try it again later.

Problem 5

Investigate asymptotic stability of the origin for the system

$$\dot{x_1} = -x_1^3 + e^{-x_2}x_2$$

$$\dot{x_2} = x_1 - x_1^2$$

Answer

It cannot be solved by the Lyapunov's second method. So we have to use the Lyapunov's first method.

$$\dot{x} = f(x) = \begin{bmatrix} -x_1^3 + e^{-x_2}x_2 \\ x_1 - x_1^2 \end{bmatrix}, x_e = 0$$

Linearization :

$$A := \frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -3x_1 & -e^{x_2}x_2 + e^{-x_2} \\ 1 - 2x_1 & 0 \end{pmatrix}$$

And substitute $x = 0$ to A:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\det(A - \lambda I) = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$
Therefore, the system is unstable.