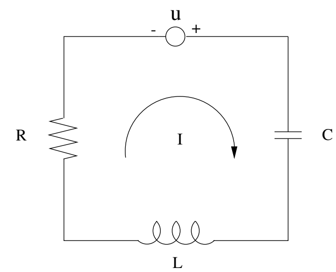
PROBLEM SET 1

Reading: Class Notes, Sections 1.1, 1.4, 1.5, 2.1–2.6.

Problems:



1. Consider the electrical circuit discussed in class (shown in the figure), but now suppose that its characteristics R, L and C vary with time. Starting with the same *non-dynamic* physical laws as in class ($q = CV_c$ for the capacitor charge, $\phi = LI$ for the inductor flux), derive a dynamical model of this circuit. It should take the form $\dot{x} = A(t)x + B(t)u$.

2. Which of the following are vector spaces over $\mathbb{R}$ (with respect to the standard addition and scalar multiplication)? Justify your answers.

a) The set of real-valued $n \times n$ matrices with nonnegative entries, where n is a given positive integer.

b) The set of rational functions of the form $\frac{p(s)}{q(s)}$, where p and q are polynomials in the complex variable s and the degree of q does not exceed a given fixed positive integer k.

c) The space $L^2(\mathbb{R}, \mathbb{R})$ of square-integrable functions, i.e., functions $f : \mathbb{R} \to \mathbb{R}$ with the property that $\int_{-\infty}^{\infty} f^2(t)dt < \infty$.

3. Show that vectors $v_1, v_2, \dots, v_n$ in a vector space X form a basis according to the definition given in class (which says that they are linearly independent and $n = \dim X$) if and only if they are linearly independent and each $x \in X$ can be uniquely written as a linear combination $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$.

4. Let $A : X \to Y$ be a linear operator.

a) Prove that $\dim N(A) + \dim R(A) = \dim X$ (the sum of the dimension of the nullspace of A and the dimension of the range of A equals the dimension of X).

b) Now assume that $X = Y$. It is *not* always true that X is a direct sum of $N(A)$ and $R(A)$. Find a counterexample demonstrating this. Also, describe a class of linear operators (as general as you can think of) for which this statement *is* true.

5. Let A be the linear operator in the plane corresponding to the counter-clockwise rotation around the origin by some given angle θ. Compute the matrix of A relative to the standard basis in $\mathbb{R}^2$.

6. Three employees—let's call them Alice, Bob, and Cheng—received their end-of-the-year bonuses which their boss calculated as a linear combination of three performance scores: leadership, communication, and work quality. The coefficients (weights) in this linear combination are the same for all three employees, but the boss doesn't disclose them. Alice knows that she got the score of 4 for leadership, 4 for communication, and 5 for work quality. Bob's scores for the same categories were 3, 5, and 4, and Cheng's scores were 5, 3, and 3. The bonus amounts are \$18,000 for Alice, \$16,000 for Bob, and \$14,000 for Cheng. The employees are now curious to determine the unknown coefficients (weights).

a) Set up this problem as solving a linear equation of the form $Ax = b$ for the unknown vector x.

b) Calculate the unknown weights. It's up to you whether you use a) for this or do it another way.

c) Are the weights that you computed unique? Explain why or why not.