PROBLEM SET 4

Reading: Class Notes, Sections 2.8–2.10, 4.1, 4.5.

Problems:

1. Prove that the Euclidean norm $|x| := \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + \cdots + x_n^2}$ satisfies the triangle inequality. (Hint: use the Cauchy-Schwarz inequality $|\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle$.)

2. Let A be a symmetric real-valued square matrix.

a) Show that if $\lambda + i\mu$ is an eigenvalue of A and $z = x + iy$ is a corresponding eigenvector, then $\mu = 0$ and x is an eigenvector. In other words, eigenvalues of symmetric matrices are always real and eigenvectors can always be chosen to be real. (Hint: show that $\overline{z}^T A z$ is real.)

b) Show that eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

3. Let M be a symmetric real-valued $n \times n$ matrix. Show that the following three statements are equivalent:

1. M is positive definite.
2. All eigenvalues of M are positive.
3. $M = N^T N$ for some nonsingular $n \times n$ matrix N.

(Hint: show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.)

4. Let X and Y be linear vector spaces over $\mathbb{R}$ equipped with inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$, respectively. Let $L : X \to Y$ be a linear operator. We define the *adjoint* of L to be a linear operator $L^* : Y \to X$ with the property that

$$\langle y, Lx\rangle_Y = \langle L^* y, x\rangle_X \qquad \forall\, x \in X,\ y \in Y$$

Assume that the map $LL^* : Y \to Y$ is invertible. Then the equation $Lx = y_0$ has a solution

$$x_0 = L^*(LL^*)^{-1} y_0$$

for each $y_0 \in Y$. Prove that if x_1 is any other solution of $Lx = y_0$, then $\langle x_1, x_1\rangle \geq \langle x_0, x_0\rangle$.

(Hint: Let $y_1 := (LL^*)^{-1} y_0$. Using the definition of adjoint, show that $\langle y_1, Lx_0\rangle = \langle x_0, x_0\rangle$ and also that $\langle x_0, x_1\rangle = \langle y_1, Lx_0\rangle$. Complete the proof by using the fact that $\langle x_1 - x_0, x_1 - x_0\rangle \geq 0$.)

5. Using the stability definitions given in class, determine if the systems below are stable, asymptotically stable, globally asymptotically stable, or neither. The first two systems are in $\mathbb{R}^2$, the last is in $\mathbb{R}$.

a) $\dot{x}_1 = 0$, $\dot{x}_2 = -x_2$

b) $\dot{x}_1 = -x_2$, $\dot{x}_2 = 0$

c) $\dot{x} = 0$ if $|x| > 1$; $\dot{x} = -x$ if $|x| \leq 1$

Justify your answers using *only* the definitions of stability (not eigenvalues or Lyapunov's method).

6. Consider the LTI system $\dot{x} = Ax$ where

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

Identify the stable and unstable invariant subspaces by giving a *real* basis for each of them.