PROBLEM SET 5

Reading: Class Notes, Sections 4.2–4.4, 4.6, 4.7.

Problems:

1. Let x_1, x_2, x_3 be positions of 3 moving points on the real line (let's call them "agents"). For each agent i, let N_i denote the set of all other agents that agent i can "see" (meaning it can measure their positions), and consider the *consensus equations*

$$\dot{x}_i = -\sum\nolimits_{j \in N_i} (x_i - x_j), \qquad i = 1, 2, 3$$

Assume initially that all agents see each other, so that $N_1 = \{2, 3\}$, $N_2 = \{1, 3\}$, $N_3 = \{1, 2\}$.

a) Write down the consensus equations as an LTI system $\dot{x} = Ax$.

b) Define the following 3×3 matrices: the *degree matrix* D is the diagonal matrix with diagonal elements D_{ii} equal to the number of agents in N_i; the *adjacency matrix* J has elements $J_{ij} = 1$ if agents i and j see each other, and 0 if they don't (by assumption, J is symmetric and its diagonal elements are 0); and the *Laplacian matrix* $L := D - J$. Verify that the matrix A in part a) equals $-L$.

c) Is it true that system equilibria are exactly points in $\mathbb{R}^3$ with $x_1 = x_2 = x_3$ (agents' positions coincide)?

d) Are the system equilibria stable? Asymptotically stable?

e) Now assume that agents 2 and 3 don't see each other. Answer questions a)–d) for this case.

f) Now assume that agent 3 doesn't see agents 1 and 2 and is not seen by them. Answer a)–d) again.

2. Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_2 - \sin x_1$$

and the three candidate Lyapunov functions

$$V_1(x) = \tfrac{1}{2}x_2^2 - \cos x_1, \qquad V_2(x) = 1 + \tfrac{1}{2}x_2^2 - \cos x_1, \qquad V_3(x) = 1 + \tfrac{1}{4}x_1^2 + \tfrac{1}{2}x_1 x_2 + \tfrac{1}{2}x_2^2 - \cos x_1$$

a) For $i = 1, 2, 3$, compute $\dot{V}_i(x)$.

b) In each case, explain what conclusions (if any) you can reach using Lyapunov's second method.

3. Consider a system $\dot{x} = f(x)$. Suppose that $\frac{d}{dt}(x^T P x) \leq -x^T Q x$, where P and Q are symmetric positive definite matrices (here the time derivative is taken along solutions of the system, as explained in class). Prove that under this condition the system is exponentially stable, in the sense that its solutions satisfy $|x(t)| \leq ce^{-\mu t}|x(0)|$ for some $c, \mu > 0$. Note that this statement is true whether the system is linear or not.

Hint: use the inequality $\lambda_{\min}(M)|x|^2 \leq x^T M x \leq \lambda_{\max}(M)|x|^2$ discussed in class. You may also use the fact that if a function $v(t)$ satisfies $\dot{v} \leq -av$, then $v(t) \leq e^{-at}v(0)$.

4. Show that if all eigenvalues of a matrix A have real parts *strictly less* than some $-\mu < 0$, then for every $Q = Q^T > 0$ the equation $PA + A^T P + 2\mu P = -Q$ has a unique solution $P = P^T > 0$. Show that in this case we have $|x(t)| \leq ce^{-\mu t}|x(0)|$ for some $c > 0$. (The number μ is called a *stability margin*.)

5. Investigate asymptotic stability of the origin for the system

$$\dot{x}_1 = -x_1^3 + e^{-x_2} x_2$$
$$\dot{x}_2 = x_1 - x_1^2$$