PROBLEM SET 2

Reading: Class Notes, Sections 2.7, 3.1–3.5.

Problems:

1. Let A be the linear operator in the plane corresponding to the counter-clockwise rotation around the origin by some given angle θ (see Problem 5 in Homework 1). Compute the matrix of A relative to the basis $\left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

2. We revisit Problem 6 in Homework 1. Alice and Bob have reason to believe that Cheng's performance scores and bonus amount that he gave them were incorrect. They still want to figure out their unknown weights, but now they have to use only their own scores and bonus amounts, without Cheng's.

a) Set up this problem as solving a linear equation of the form $Ax = b$ for the unknown vector x. The new matrix A will be 2×3.

b) Let $A^\dagger$ be a *right inverse* of A, which by definition is a matrix such that $AA^\dagger = I_{2\times 2}$. Express all solutions x of the equation in a) in terms of such a right inverse $A^\dagger$, the vector b, and the null space of A. Carefully state and justify your answer.

c) Find a particular solution x^* of the equation in a). (Hint: use your solution from Homework 1.) Verify numerically that x^* indeed belongs to the solution space you described in b). To do this, you can use MATLAB commands `pinv` to compute a right inverse and `null` to compute a vector in the null space.

3. Let A be a real-valued $n \times n$ matrix. Suppose that $\lambda + i\mu$ is a complex eigenvalue of A and $x + iy$ is a corresponding complex eigenvector. (Here $\lambda, \mu \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$.)

a) Show that $x - iy$ is also an eigenvector, with eigenvalue $\lambda - i\mu$.

b) Let V be the 2-dimensional subspace over $\mathbb{R}$ spanned by x and y. (In other words, V is the set of linear combinations, with real coefficients, of the real-valued vectors x and y.) Show that V is an *invariant subspace* for A, in the sense that for every $z \in V$ we have $Az \in V$.

4. We know from the Cayley-Hamilton theorem that the characteristic polynomial p of a square matrix A satisfies $p(A) = 0$. Let us call a polynomial q a *minimal* polynomial for the matrix A if $q(A) = 0$ and q has the smallest possible degree among all polynomials with this property.

a) Give an example of a matrix A whose characteristic polynomial is minimal (justify rigorously).

b) Give an example of a matrix A whose characteristic polynomial is not minimal (explain why not).

5. Consider the following matrix, whose exponential was computed in class: $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $a, b \in \mathbb{R}$.

a) Re-derive the expression for e^{At} by diagonalizing A over $\mathbb{C}$ and using the formula $e^{a \pm bi} := e^a \cos b \pm ie^a \sin b$ (treat this as the definition of complex exponential).

b) Using the formula $e^{A(t+\sigma)} = e^{At}e^{A\sigma}$ and the result of a) for suitably chosen values of a and b, verify the trigonometric identities $\cos(t+\sigma) = \cos t \cos\sigma - \sin t \sin\sigma$ and $\sin(t+\sigma) = \sin t \cos\sigma + \cos t \sin\sigma$.