University of Yonsei Mirae, Linear System

HW 4

2020271053, MinJae Kim

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Problem 1

Prove that the Euclidean norm $|x| := \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + \cdots + x_n^2}$ satisfies the triangle inequality. (Hint: use the Cauchy-Schwarz inequality $|\langle x, y\rangle|^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle$.)

Answer

Triangle inequality is $||x + y|| \leq ||x|| + ||y||$.
Then, apply to Euclidean norm,

$$\begin{aligned} |x+y|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\ &= \langle x, x\rangle + 2|\langle x, y\rangle| + \langle y, y\rangle \end{aligned}$$

By Cauchy-Schwarz inequality, $|\langle x, y\rangle| \leq |x| \cdot |y|$, then

$$\begin{aligned} |x+y|^2 &\leq \langle x, x\rangle + 2|x| \cdot |y| + \langle y, y\rangle \\ &= |x|^2 + 2|x| \cdot |y| + |y|^2 \\ &= (|x| + |y|)^2 \end{aligned}$$

Therefore, $|x + y| \leq |x| + |y|$. $\Rightarrow$ Triangle inequality holds.

Problem 2

Let A be a symmetric real-valued square matrix.

a) Show that if $\lambda+i\mu$ is an eigenvalue of A and $z = x+iy$ is a corresponding eigenvector, then $\mu = 0$ and x is an eigenvector. In other words, eigenvalues of symmetric matrices are always real and eigenvectors can always be chosen to be real. (Hint: show that $\bar{z}^T Az$ is real.)

Answer

$$\begin{aligned} &Az = (\lambda + i\mu)z \\ \Leftrightarrow &A(x+iy) = (\lambda+i\mu)(x+iy) \\ \Leftrightarrow &Ax + iAy = \lambda x - \mu y + i(\mu x + \lambda y) \end{aligned}$$

Devide the equation into real and imaginary parts,
Real parts : $Ax = \lambda x - \mu y$
imaginary parts : $Ay = \mu x + \lambda y$

Now, use the $\bar{z}^T Az$

$$\begin{aligned} \bar{z}^T Az &= (x^T - iy^T)A(x+iy) \\ &= x^T Ax + ix^T Ay - ixAy^T + y^T Ay \end{aligned}$$

Since A is symmetric, $A = A^T$, so $x^T Ay = y^T Ax$
Then,

$$\begin{aligned} \bar{z}^T Az &= x^T Ax + ix^T Ay - ix^T Ay + y^T Ay \\ &= x^T Ax + y^T Ay \end{aligned}$$

Use this to the origin equation,

$$\begin{aligned} Az = (\lambda+i\mu)z \Rightarrow \bar{z}^T Az &= \bar{z}^T(\lambda+i\mu)z = (\lambda+i\mu)\bar{z}^T z \\ \text{and, } \bar{z}^T z &= x^T x + y^T y \geq 0 \end{aligned}$$

Thus, $\bar{z}^T Az$ is real.
Therefore, $\mu = 0$ and x is an eigenvector.

b) Show that eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

Answer

Suppose that exist two distinct eigenvalues and corresponding eigenvectors.

$$\lambda_1 \to v_1$$

$$\lambda_2 \to v_2$$

Then, $Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$
Now,

$$v_2^T A v_1 = v_2^T(\lambda_1 v_1) = \lambda_1(v_2^T v_1)$$

And A is symmetric, so $(Av_2)^T v_1 = v_2^T A v_1$
Thus,

$$\begin{aligned} v_2^T A v_1 = (Av_2)^T v_1 &= \lambda_1(v_2^T v_1) \\ \Leftrightarrow \lambda_2(v_2^T v_1) &= \lambda_1(v_2^T v_1) \end{aligned}$$

Finally, we can get

$$(v_2^T v_1)(\lambda_2 - \lambda_1) = 0$$

Since λ_1 and λ_2 are distinct, $v_2^T v_1 = 0$
Therefore, eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

Problem 3

Let M be a symmetric real-valued $n \times n$ matrix. Show that the following three statements are equivalent:
1. M is positive definite.
2. All eigenvalues of M are positive.
3. $M = N^T N$ for some nonsingular $n \times n$ matrix N.
(Hint: show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.)

Answer $1 \Rightarrow 2$

Suppose that M is positive definite. Then, $\forall x \neq 0$, $x^T M x > 0$
M is symmetric, so all eigenvalues are real$(\lambda_1, \lambda_2, \cdots, \lambda_n)$ and corresponding real eigenvectors $(v_1, v_2, \cdots, v_i)$

Then, $Mv_i\lambda_i v_i$, and set $v_i = x$

$$\lambda_i = v_i^T M v_i \rightarrow x^T M x > 0$$

Therefore, all eigenvalues λ_i are **real number**.

Answer $2 \Rightarrow 3$

Now, A's all eigenvalues are real number and all eigenvectors are orthogonal(we proofed in problem 2).
Then, we can use eigenvalue decomposition(EVD) by Spectral Theorem.
Then,

$$M = Q\Sigma Q^T$$

where Q is $n \times n$ matrix composed by M's eigenvalues and Σ is diagonal matrix composed by M's eigenvalues.

Let $N = \Sigma^{1/2}Q$, then

$$N^T N = Q\Sigma Q^T = M$$

Answer $3 \Rightarrow 1$

Suppose that $M = N^T N$ for some nonsingular $n \times n$ matrix N.
Then, $\forall x \neq 0$, $x^T M x = x^T N^T N x = (Nx)^T (Nx) = ||Nx||^2 > 0$
Therefore, M is positive definite. Q.E.D

Problem 4

Let X and Y be linear vector spaces over $\mathbb{R}$ equipped with inner products $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$, respectively. Let $L : X \to Y$ be a linear operator. We define the adjoint of L to be a linear operator $L^* : Y \to X$ with the property that

$$\langle y, Lx \rangle_Y = \langle L^* y, x \rangle_X \quad \forall x \in X, y \in Y$$

Assume that the map $LL^* : Y \to Y$ is invertible. Then the equation $Lx = y_0$ has a solution

$$x_0 = L^*(LL^*)^{-1} y_0$$

for each $y_0 \in Y$. Prove that if x_1 is any other solution of $Lx = y_0$, then $\langle x_1, x_1 \rangle \geq \langle x_0, x_0 \rangle$. (Hint: Let $y_1 := (LL^*)^{-1} y_0$. Using the definition of adjoint, show that $\langle y_1, Lx_0 \rangle = \langle x_0, x_0 \rangle$ and also that $\langle x_0, x_1 \rangle = \langle y_1, Lx_0 \rangle$. Complete the proof by using the fact that $\langle x_1 - x_0, x_1 - x_0 \rangle \geq 0$.)

Answer

Let $y_1 = (LL^*)^{-1} y_0$.
Then, $y_0 = LL^* y_1$ and $x_0 = L^* y_1$
To show that $\langle y_1, Lx_0 \rangle = \langle x_0, x_0 \rangle$,

$$\langle y_1, Lx_0 \rangle_Y = \langle y_1, L(L^* y_1) \rangle_Y = \langle y_1, (LL^*) y_1 \rangle_Y$$

by adjoint of L, $\langle y_1, LL^* y_1 \rangle_Y = \langle L^* y_1, L^* y_1 \rangle_X = \langle x_0, x_0 \rangle_X$
Also, $\langle x_0, x_1 \rangle_X = \langle y_1, Lx_1 \rangle_Y$

$$\langle x_0, x_1 \rangle_X * = \langle L^* y_1, x_1 \rangle_X = \langle y_1, Lx_1 \rangle_Y$$

Now,

$$\begin{aligned} \langle x_1, x_1 \rangle &= \langle x_0 + (x_1 - x_0), x_0 + (x_1 - x_0) \rangle \\ &= \langle x_0, x_0 \rangle + 2(\langle x_0, x_1 - x_0 \rangle) + \langle x_1 - x_0, x_1 - x_0 \rangle \end{aligned}$$

Problem 5

Using the stability definitions given in class, determine if the systems below are stable, asymptotically stable, globally asymptotically stable, or neither. The first two systems are in $\mathbb{R}^2$, the last is in $\mathbb{R}$.

a)

$$\dot{x}_1 = 0, \quad \dot{x}_2 = -x_2$$

b)

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = 0$$

c)

$$\dot{x} = 0 \text{ if } |x| > 1, \quad \dot{x} = -x \text{ if } |x| \leq 1$$

Justify your answers using only the definitions of stability (not eigenvalues or Lyapunov's method).

Answer

a) Have to AS, $x_1 = 0$ and $x_2 = 0$. But, $\dot{x}_1 = 0$, so x_1 is maintain the initial value. So this system is stable.

b) $\dot{x}_2 = 0$ is maintain the initial value, but $\dot{x}_1 = -x_2$ is linearly change the value. So this system is not stable, not AS, not GAS.

c) If $|x| > 1$, then the system is stable. Also, $|x| \leq 1$ is go to the equlibrium point, so when $|x| \leq 1$, the system is AS.

Problem 6

Consider the LTI system $\dot{x} = Ax$ where

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

Identify the stable and unstable invariant subspaces by giving a real basis for each of them.

Answer

Stability of LTI system is determined by the eigenvalues of the matrix A.
The general solution of the system is $x(t) = e^{At}x(0)$, so negative eigenvalues are stable and positive eigenvalues are unstable.

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>> A = [-1, 0, 0; 0, 2, 1; 0, -1, 2]
A =
    -1     0     0
     0     2     1
     0    -1     2

>> eig(A)
e =
    2.0000 + 1.0000i
    2.0000 - 1.0000i
   -1.0000 + 0.0000i
```

The eigenvalues of the matrix A are $2 \pm i$ and -1.
Therefore,

$$\text{Stable invariant subspace } = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\text{Unstable invariant subspace } = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$