PROBLEM SET 3

Reading: Class Notes, Sections 3.6–3.8.

Problems:

1. Consider the following matrices: $A_1 = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix}$, $\quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\quad A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ -3 & 0 & -2 \end{pmatrix}$.

a) Compute $e^{A_i t}$ for $i = 1, 2, 3$. Note that A_2 is a special case of a matrix whose exponential was computed in class.

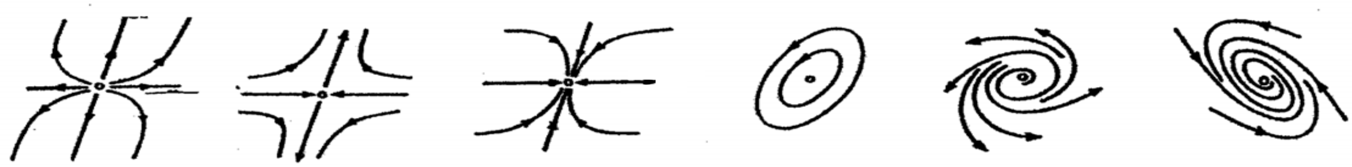
b) For $i = 1, 2, 3$, write down the solution of $\dot{x} = A_i x$ for a general initial condition.

c) In each case, determine whether the solutions of $\dot{x} = A_i x$ decay to 0, stay bounded, or go to ∞ (for various choices of initial conditions).

d) Try to state the general rule which can be used to determine, by looking at the eigenstructure of A, whether the solutions of $\dot{x} = Ax$ decay to 0, stay bounded, or go to ∞.

2. The pictures show possible trajectories of a linear system $\dot{x} = Ax$ in the plane, when the eigenvalues λ_1 and λ_2 of A are in one of the following configurations: a) $\lambda_1 < \lambda_2 < 0$, b) $\lambda_1 < 0 < \lambda_2$, c) $0 < \lambda_1 < \lambda_2$, d) $\lambda_1, \lambda_2 = a \pm bi$, $a < 0$, e) $\lambda_1, \lambda_2 = a \pm bi$, $a = 0$, f) $\lambda_1, \lambda_2 = a \pm bi$, $a > 0$.
Match each picture with the corresponding eigenvalue distribution. Justify your answers (plotting in MATLAB cannot be used as a justification).

Unstable node | Saddle | Stable node | Center | Unstable focus | Stable focus



3. This exercise illustrates the phenomenon known as *resonance*. Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \qquad y = x_2$$

where $\omega > 0$, u is the control input, and y is the output. Let the input be $u(t) = \cos \nu t$, $\nu > 0$. Using the variation-of-constants formula, compute the response $y(t)$ to this input from the zero initial condition $x(0) = 0$, considering separately two cases: $\nu \neq \omega$ and $\nu = \omega$. In each case, determine whether this response is decaying to 0, bounded, or unbounded.

4. Compute the state transition matrix $\Phi(t, t_0)$ of $\dot{x} = A(t)x$ with $A(t) = \begin{pmatrix} -1 + \cos t & 0 \\ 0 & -2 + \cos t \end{pmatrix}$.

5. Consider the LTV system $\dot{x} = A(t)x$, where $A(t)$ is *periodic* with period T, i.e., $A(t + T) = A(t)$. Let $\Phi(t, t_0)$ denote the corresponding state transition matrix. The goal of this exercise is to show that we can simplify the system and characterize its transition matrix with the help of a suitable (time-varying) coordinate transformation. Being a nonsingular matrix, $\Phi(T, 0)$ can be written as an exponential: $\Phi(T, 0) = e^{RT}$, where R is some matrix. Define also $P(t) := \Phi(t, 0)e^{-Rt}$.

a) Using the properties of state transition matrices from class, show that $\Phi(t, t_0) = P(t)e^{R(t-t_0)}P^{-1}(t_0)$.

b) Deduce from a) that $\bar{x}(t) := P^{-1}(t)x(t)$ satisfies the time-invariant differential equation $\dot{\bar{x}} = R\bar{x}$.

c) Prove that $P(t)$ is periodic with period T.

d) Consider the LTV system from Problem 4. Find new coordinates $\bar{x}(t)$ in which, according to part b), this system should become time-invariant. Confirm this fact by differentiating $\bar{x}(t)$.

6. Prove the variation-of-constants formula for linear time-varying control systems stated in class (see also Class Notes, end of Section 3.7).