Supplementary Derivations for Gumbel Model

The expressions provided in this document are used in the numerical comparisons in the script `Gumbel_muOnly.R` and `Gumbel_jointEstimation.R`.

1 Setup

1.1 Marginal distribution

$$\begin{aligned}
\theta &= (\mu, \sigma)^\top, \\
F_\theta(x) &= \exp\left\{-e^{-\frac{x-\mu}{\sigma}}\right\}, \\
f(x) &= \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}} F_\theta(x), \\
\ell(\theta) &= \log f(x) = -\log(\sigma) - \frac{x-\mu}{\sigma} - e^{-\frac{x-\mu}{\sigma}}, \\
q(\theta, a) &= \mathbb{P}(Y > a) = 1 - F_\theta(a)
\end{aligned}$$

We utilize a change of variables by expressing the likelihood in terms of the standardized variable $z = \frac{x-\mu}{\sigma}$. Using the identities

$$\frac{\partial z}{\partial \mu} = -\frac{1}{\sigma}, \quad \frac{\partial z}{\partial \sigma} = -\frac{x-\mu}{\sigma^2} = -\frac{1}{\sigma} z, \quad \frac{\partial e^{-z}}{\partial \mu} = \frac{1}{\sigma} e^{-z}, \quad \frac{\partial e^{-z}}{\partial \sigma} = \frac{1}{\sigma} z e^{-z}, \tag{1}$$

we obtain the log-likelihood and its derivatives:

$$\begin{aligned}
\ell(\theta) &= -\log(\sigma) - z - e^{-z}, \\
\frac{\partial \ell}{\partial \theta} &= \begin{pmatrix} \frac{1}{\sigma}(1 - e^{-z}) \\ -\frac{1}{\sigma}(1 - z + z e^{-z}) \end{pmatrix}, \\
\frac{\partial^2 \ell}{\partial \theta^2} &= \begin{pmatrix} -\frac{1}{\sigma^2} e^{-z} & -\frac{1}{\sigma^2}(1 - e^{-z} + z e^{-z}) \\ -\frac{1}{\sigma^2}(1 - e^{-z} + z e^{-z}) & \frac{1}{\sigma^2}(1 - 2z + 2z e^{-z} - z^2 e^{-z}) \end{pmatrix}.
\end{aligned}$$

Using the fact that $Z \sim \text{Gumbel}(0, 1)$, its known moments are: with Euler's constant γ,

$$f_Z(z) = e^{-z - e^{-z}}, \quad \mathbb{E}Z = \gamma, \quad \mathbb{E}e^{-Z} = 1, \quad \mathbb{E}Z e^{-Z} = \gamma - 1, \quad \mathbb{E}Z^2 e^{-Z} = -2\gamma + \gamma^2 + \frac{\pi^2}{6}.$$

Taking the expectation of the observed information yields the Fisher information matrix:

$$\mathbb{E}\left(-\frac{\partial^2 \ell}{\partial \theta^2}\right) = \frac{1}{\sigma^2} \begin{pmatrix} 1 & \gamma - 1 \\ \gamma - 1 & (\gamma - 1)^2 + \frac{\pi^2}{6} \end{pmatrix}. \tag{2}$$

1.2 Joint distribution

$$F(x_1,x_2)=\exp\left\{-(e^{-z_1/r}+e^{-z_2/r})^r\right\}=\exp\{-A^r\},\quad z_i=\frac{x_i-\mu_i}{\sigma_i},$$

where

$$A=e^{-z_1/r}+e^{-z_2/r},\quad \frac{\partial z_i}{\partial x_i}=\frac{1}{\sigma_i},\quad \frac{\partial e^{-z_i}}{\partial x_i}=-\frac{1}{\sigma_i}e^{-z_i},\quad \frac{\partial A}{\partial z_1}=-\frac{1}{r}e^{-z_1/r}.$$

Then, the joint p.d.f. becomes

$$\begin{aligned}
f(z_1,z_2)=\frac{\partial^2 F}{\partial z_1\partial z_2}&=\frac{\partial}{\partial z_2}\left\{\exp\{-A^r\}A^{r-1}e^{-z_1/r}\right\}\\
&=e^{-A^r}A^{2(r-1)}e^{-z_1/r-z_2/r}-\frac{r-1}{r}A^{r-2}e^{-A^r}e^{-z_1/r-z_2/r}\\
&=A^{r-2}e^{-A^r}e^{-(\frac{z_1}{r}+\frac{z_2}{r})}\left\{A^r-\frac{r-1}{r}\right\}.
\end{aligned}$$

2 Gumbel distribution: mu unknown

2.1 JML

Continuing with the results in Section 1.2,

$$\begin{aligned}
f(x_1,x_2)&=f(z_1,z_2)\frac{\partial z_1}{\partial x_1}\frac{\partial z_2}{\partial x_2}=e^{-(\frac{z_1}{r}+\frac{z_2}{r})}A^{r-2}e^{-A^r}\left\{A^r-\frac{r-1}{r}\right\}\frac{1}{\sigma_1\sigma_2},\\
\ell(x_1,x_2)&=-\frac{1}{r}(z_1+z_2)+(r-2)\log A-A^r+\log\{A^r-\frac{r-1}{r}\}-\log(\sigma_1\sigma_2).
\end{aligned}$$

Define

$$B:=\frac{\partial A}{\partial \mu_1}=\frac{1}{r\sigma_1}e^{-z_1/r},\ \text{then}\ \frac{\partial B}{\partial \mu_1}=\frac{1}{r^2\sigma_1^2}e^{-z_1/r}=\frac{B}{r\sigma_1}.$$

Using (1), we have:

$$\begin{aligned}
\frac{\partial \ell}{\partial \mu_1}&=\frac{1}{r\sigma_1}+\frac{r-2}{A}B-rA^{r-1}B\left\{1-\frac{1}{A^r-\frac{r-1}{r}}\right\},\\
\frac{\partial^2 \ell}{\partial \mu_1^2}&=(r-2)\left\{\frac{B}{r\sigma_1 A}-\frac{B^2}{A^2}\right\}-\{r(r-1)A^{r-2}B^2+\frac{1}{\sigma_1}A^{r-1}B\}\left\{1-\frac{1}{A^r-\frac{r-1}{r}}\right\}-\frac{(rA^{r-1}B)^2}{(A^r-\frac{r-1}{r})^2}.
\end{aligned}$$

We then use the second derivative of the log-likelihood derived above to compute the Fisher information for μ_1 under the joint model. Specifically, we have

$$I(\mu_1)=-\mathbb{E}\left[\frac{\partial^2\ell(x_1,x_2)}{\partial\mu_1^2}\right]=-\iint\frac{\partial^2\ell(z_1,z_2)}{\partial\mu_1^2}\cdot f(z_1,z_2)\,dz_1\,dz_2,$$

and the estimated variance of the JML estimator for μ_1 is given by the inverse of $I(\mu_1)$.

2.2 MoM

See the manuscript for details.

2.3 MML

The necessary details are provided by the results derived in Section 1.1 above and those already presented in the manuscript.

3 Gumbel distribution: mu, sigma unknown

3.1 Baseline

Baseline MLE uses (2).

3.2 MML

Continuing with the results and notation in Section 1.1, the log-likelihood and score functions for the marginal distribution are given as

$$\begin{aligned}
\ell(\theta) &= -\log(\sigma) - z - e^{-z}, \\
\frac{\partial \ell}{\partial \theta} &= \begin{pmatrix} \frac{1}{\sigma}(1-e^{-z}) \\ -\frac{1}{\sigma}(1-z+ze^{-z}) \end{pmatrix}, \\
\frac{\partial^2 \ell}{\partial \theta^2} &= \begin{pmatrix} -\frac{1}{\sigma^2}e^{-z} & -\frac{1}{\sigma^2}(1-e^{-z}+ze^{-z}) \\ -\frac{1}{\sigma^2}(1-e^{-z}+ze^{-z}) & \frac{1}{\sigma^2}(1-2z+2ze^{-z}-z^2e^{-z}) \end{pmatrix}.
\end{aligned}$$

Again, using the fact that $Z \sim \text{Gumbel}(0,1)$, its known moments are:

$$f_Z(z) = \exp\{-z-e^{-z}\}, \quad \mathbb{E}Z = \gamma, \quad \mathbb{E}Z^2 = \gamma^2 + \frac{\pi^2}{6}, \quad \mathbb{E}e^{-Z} = 1, \quad \mathbb{E}e^{-2Z} = 2,$$

$$\mathbb{E}Ze^{-Z} = \gamma - 1, \quad \mathbb{E}Ze^{-2Z} = 2\gamma - 3, \quad \mathbb{E}Z^2e^{-Z} = -2\gamma + \gamma^2 + \frac{\pi^2}{6}, \quad \mathbb{E}Z^2e^{-2Z} = 2 - 6\gamma + 2\gamma^2 + \frac{\pi^2}{3}.$$

Define

$$I(\theta) = \mathbb{E}\left(-\frac{\partial^2 \ell}{\partial \theta^2}\right) = \frac{1}{\sigma^2}\begin{pmatrix} 1 & \gamma-1 \\ \gamma-1 & (\gamma-1)^2 + \frac{\pi^2}{6} \end{pmatrix} =: C/\sigma^2$$

We have

$$h^{(1)}(x) = C^{-1}\sigma_1^2\dot{\ell}_1(x) = \frac{6}{\pi^2}\begin{pmatrix} (\gamma-1)^2 + \frac{\pi^2}{6} & 1-\gamma \\ 1-\gamma & 1 \end{pmatrix}\begin{pmatrix} \sigma_1(1-e^{-z}) \\ \sigma_1(-1+z-ze^{-z}) \end{pmatrix}.$$

In other words,

$$
\begin{aligned}
h^{(1)}(Y^{(1)})_1 &= \sigma_1(a_1 - b_1 - a_1 e^{-Z_1} + b_1 Z_1 - b_1 Z_1 e^{-Z_1}),\\
h^{(1)}(Y^{(1)})_2 &= \sigma_1(a_2 - b_2 - a_2 e^{-Z_1} + b_2 Z_1 - b_2 Z_1 e^{-Z_1}),
\end{aligned}
$$

where $a_1 = \frac{6}{\pi^2}(\gamma-1)^2 + 1$ and $b_1 = \frac{6}{\pi^2}(1-\gamma)$, $a_2 = \frac{6}{\pi^2}(1-\gamma), b_2 = \frac{6}{\pi^2}$. Observe that

$$
\mathbb{E}(a - b - ae^{-Z_1} + bZ_1 - bZ_1 e^{-Z_1}) = a - b - a + b\gamma - b(\gamma - 1) = 0.
$$

We have

$$
\beta_{l,\text{opt}} = \frac{\operatorname{Cov}\left(h_l^{(1)}(Y^{(1)}), h_l^{(2)}(Y^{(2)})\right)}{\operatorname{Var}\left(h_l^{(2)}(Y^{(2)})\right)},
$$

where

$$
\begin{aligned}
\operatorname{Cov}\left(h_1^{(1)}(Y^{(1)}), h_1^{(2)}(Y^{(2)})\right) &= \sigma_1\sigma_2\mathbb{E}((-ae^{-Z_1} - bZ_1 + bZ_1e^{-Z_1})(-ae^{-Z_2} - bZ_2 + bZ_2e^{-Z_2})) - \sigma_1\sigma_2(a+b)^2,\\
\operatorname{Var}\left(h_1^{(2)}(Y^{(2)})\right) &= \sigma_2^2((a+b)^2 + b^2(\gamma^2 - 2\gamma + \frac{\pi^2}{6}) - 2ab\gamma).
\end{aligned}
$$

Finally, one computes

$$
\begin{aligned}
\widehat{\theta}_{\text{mml}} &= \overline{h^{(1)}(Y^{(1)})}_n + \theta_1^* + \beta_{\text{opt}} \odot \left(\overline{h^{(2)}(Y^{(2)})}_{n+m} - \overline{h^{(2)}(Y^{(2)})}_n\right),\\
\lim n \operatorname{Var}((\widehat{\theta}_{1,\text{mml}})_l) &= \operatorname{Var}\left(h_l^{(1)}(Y^{(1)})\right)\left(1 - \operatorname{Corr}\left(h_l^{(1)}(Y^{(1)}), h_l^{(2)}(Y^{(2)})\right)^2\right).
\end{aligned}
$$

3.3 JML

We continue with the results in Section 2.1.

$$
\begin{aligned}
f(x_1, x_2) &= f(z_1, z_2)\frac{\partial z_1}{\partial x_1}\frac{\partial z_2}{\partial x_2} = e^{-(\frac{z_1}{r} + \frac{z_2}{r})}A^{r-2}e^{-A^r}\left\{A^r - \frac{r-1}{r}\right\}\frac{1}{\sigma_1\sigma_2},\\
\ell(x_1, x_2) &= -\frac{1}{r}(z_1 + z_2) + (r-2)\log A - A^r + \log\{A^r - \frac{r-1}{r}\} - \log(\sigma_1\sigma_2).
\end{aligned}
$$

We again use

$$
\frac{\partial z}{\partial \mu} = -\frac{1}{\sigma}, \quad \frac{\partial z}{\partial \sigma} = -\frac{x-\mu}{\sigma^2} = -\frac{1}{\sigma}z, \quad \frac{\partial e^{-z}}{\partial \mu} = \frac{1}{\sigma}e^{-z}, \quad \frac{\partial e^{-z}}{\partial \sigma} = \frac{1}{\sigma}ze^{-z}.
$$

Let

$$
\begin{aligned}
B &:= \frac{\partial A}{\partial \mu_1} = \frac{1}{r\sigma_1}e^{-z_1/r}, \text{ then } \frac{\partial B}{\partial \mu_1} = \frac{1}{r^2\sigma_1^2}e^{-z_1/r} = \frac{B}{r\sigma_1},\\
D &:= \frac{\partial A}{\partial \sigma_1} = \frac{1}{r\sigma_1}z_1e^{-z_1/r}, \text{ then } \frac{\partial D}{\partial \sigma_1} = \frac{z_1(z_1 - 2r)}{r^2\sigma_1^2}e^{-z_1/r} = \frac{D}{r\sigma_1}(z_1 - 2r),\\
G &:= \frac{\partial B}{\partial \sigma_1} = -\frac{1}{r\sigma_1^2}e^{-z_1/r} + \frac{1}{r^2\sigma_1^2}z_1e^{-z_1/r} = -\frac{B}{\sigma_1} + \frac{D}{r\sigma_1}.
\end{aligned}
$$

We then use the following results to compute the Fisher information:

$$
\begin{aligned}
\frac{\partial \ell}{\partial \mu_1} &= \frac{1}{r\sigma_1} + \frac{r-2}{A} B - rA^{r-1} B \left\{ 1 - \frac{1}{A^r - \frac{r-1}{r}} \right\}, \\
\frac{\partial^2 \ell}{\partial \mu_1^2} &= (r-2) \left\{ \frac{B}{r\sigma_1 A} - \frac{B^2}{A^2} \right\} - \{ r(r-1) A^{r-2} B^2 + \frac{1}{\sigma_1} A^{r-1} B \} \left\{ 1 - \frac{1}{A^r - \frac{r-1}{r}} \right\} - \frac{(rA^{r-1}B)^2}{(A^r - \frac{r-1}{r})^2}, \\
\frac{\partial \ell}{\sigma_1} &= \frac{z_1 - r}{r\sigma_1} + \frac{r-2}{A} D - rA^{r-1} D \left\{ 1 - \frac{1}{A^r - \frac{r-1}{r}} \right\}, \\
\frac{\partial^2 \ell}{\sigma_1^2} &= \frac{r - 2z_1}{r\sigma_1^2} + (r-2) \left\{ \frac{D}{r\sigma_1 A} (z_1 - 2r) - \frac{D^2}{A^2} \right\} \\
&\quad - \{ r(r-1) A^{r-2} D^2 + \frac{z_1 - 2r}{\sigma_1} A^{r-1} D \} \left\{ 1 - \frac{1}{A^r - \frac{r-1}{r}} \right\} - \frac{(rA^{r-1}D)^2}{(A^r - \frac{r-1}{r})^2}, \\
\frac{\partial^2 \ell}{\partial \mu_1 \partial \sigma_1} &= -\frac{1}{r\sigma_1^2} + (r-2) \left\{ \frac{G}{A} - \frac{BD}{A^2} \right\} - \{ r(r-1) A^{r-2} BD + rA^{r-1} G \} \left\{ 1 - \frac{1}{A^r - \frac{r-1}{r}} \right\} - \frac{(rA^{r-1})^2 BD}{(A^r - \frac{r-1}{r})^2}.
\end{aligned}
$$

As in Section 2.1, we numerically integrate the derived second-order derivatives of the log-likelihood over the joint distribution to estimate the variances of the parameters.

3.4 MoM

See the manuscript for details.