공업수학

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#1-1 다음 행렬의 eigenvalues와 eigenvectors를 구하시오.

- $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, A\mathbf{v} = \lambda\mathbf{v}, \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
- $\det(A - \lambda I_2) = \begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2 = 0$
- Eigenvalues : $\boxed{\lambda = a + bi,\ a - bi}$

#1-1 다음 행렬의 eigenvalues와 eigenvectors를 구하시오.

- i) $\lambda_1 = a + bi$

$$A - \lambda_1 I_2 = \begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix}$$

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \rightarrow \begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

$$\left[\begin{array}{cc|c} -bi & b & 0 \\ -b & -bi & 0 \end{array}\right] \xrightarrow{iR_1 + R_2 \rightarrow R_2} \left[\begin{array}{cc|c} -bi & b & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow -bix_1 + bx_2 = 0 \rightarrow x_2 = ix_1$$

$$\therefore \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ ix_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\boxed{\therefore\ eigen\ vector\ \mathbf{v_1} = \begin{bmatrix} 1 \\ i \end{bmatrix}}$$

#1-1 다음 행렬의 eigenvalues와 eigenvectors를 구하시오.

- ii) $\lambda_2 = a - bi$

$$A - \lambda_2 I_2 = \begin{bmatrix} bi & b \\ -b & bi \end{bmatrix}$$

$$(A - \lambda I_2)\mathbf{v} = \mathbf{0} \rightarrow \begin{bmatrix} bi & b \\ -b & bi \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} bi & b & | & 0 \\ -b & bi & | & 0 \end{bmatrix} \xrightarrow{iR_1 - R_2 \rightarrow R_2} \begin{bmatrix} bi & b & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\rightarrow bix_1 + bx_2 = 0 \rightarrow x_2 = -ix_1$$

$$\therefore \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -ix_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\boxed{\therefore eigen\ vector\ \mathbf{v_2} = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$$

#1-2 다음 행렬의 eigenvalues와 eigenvectors를 구하시오.

- $B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $B\mathbf{v} = \lambda\mathbf{v}, \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\det(B - \lambda I_2) = 0 \;\rightarrow\; \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = (\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta = \lambda^2 - 2\lambda\cos\theta + 1 = 0$$

$$\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm \sqrt{-\sin^2\theta} = cos\theta \pm isin\theta$$

$$\therefore eigenvalues = \boxed{\cos\theta + i\sin\theta\,, \cos\theta - i\sin\theta}$$

#1-2 다음 행렬의 eigenvalues와 eigenvectors를 구하시오.

- i)$\lambda_1 = \cos\theta + i\sin\theta$

$$B - \lambda_1 I_2 = \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}$$

$$(B - \lambda_1 I_2)\mathbf{v} = \mathbf{0} \quad \rightarrow \quad \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -i\sin\theta & -\sin\theta & 0 \\ \sin\theta & -i\sin\theta & 0 \end{array}\right] \xrightarrow{iR_1 - R_2 \to R_2} \left[\begin{array}{cc|c} -i\sin\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-i\sin\theta\, x_1 - \sin\theta\, x_2 = 0 \ \rightarrow \ x_2 = -ix_1$$

$$\therefore \mathbf{v} = \begin{bmatrix} x_1 \\ -ix_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\boxed{\therefore eigenvector\ \mathbf{v_1} = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$$

#1-2 다음 행렬의 eigenvalues와 eigenvectors를 구하시오.

- ii) $\lambda_2 = \cos\theta - i\sin\theta$

$$B - \lambda_2 I_2 = \begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix}$$

$$(B - \lambda_2 I_2)\mathbf{v} = \mathbf{0} \quad \rightarrow \quad \begin{bmatrix} i\sin\theta & -\sin\theta \\ \sin\theta & i\sin\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i\sin\theta & -\sin\theta & | & 0 \\ \sin\theta & i\sin\theta & | & 0 \end{bmatrix} \xrightarrow{iR_1 + R_2 \to R_2} \begin{bmatrix} i\sin\theta & -\sin\theta & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$i\sin\theta\, x_1 - \sin\theta\, x_2 = 0 \quad \rightarrow \quad x_2 = ix_1$$

$$\therefore \mathbf{v} = \begin{bmatrix} x_1 \\ ix_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\boxed{\therefore eigenvector\ \mathbf{v_2} = \begin{bmatrix} 1 \\ i \end{bmatrix}}$$

#1-3 다음 행렬의 eigenvalues와 eigenvectors를 구하시오.

- $C = \begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}, C\mathbf{v} = \lambda\mathbf{v}, \quad \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$C - \lambda I_3 = \begin{bmatrix} 3-\lambda & 5 & 3 \\ 0 & 4-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$\det(C - \lambda I_3) = (3-\lambda)(4-\lambda)(1-\lambda) = 0$ (triangular matrix)

→ Eigenvalues = 1, 3, 4

#1-3 다음 행렬의 eigenvalues와 eigenvectors를 구하시오.

- i) $\lambda = 1$

$C - I_3 = \begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad (C - I_3)\mathbf{v} = \begin{bmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$

$0x_3 = 0 \rightarrow x_3 = t,\ t \in \mathbb{C}$

$3x_2 + 6x_3 = 3x_2 + 6t = 0 \rightarrow x_2 = -2t$

$2x_1 + 5x_2 + 3x_3 = 2x_1 - 10t + 3t = 0 \rightarrow x_1 = \frac{7}{2}t$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{7}{2}t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{7}{2} \\ -2 \\ 1 \end{bmatrix} \rightarrow \boxed{\text{eigenvector } \mathbf{v_1} = \begin{bmatrix} \frac{7}{2} \\ -2 \\ 1 \end{bmatrix}}$

#1-3 다음 행렬의 eigenvalues와 eigenvectors를 구하시오.

- i) $\lambda = 3$

$$C - 3I_3 = \begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \qquad (C - 3I_3)\mathbf{v} = \begin{bmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

$-2x_3 = 0 \rightarrow x_3 = 0$

$x_2 + 6x_3 = x_2 + 0 = 0 \rightarrow x_2 = 0$

$x_1 = t,\ t \in \mathbb{C}$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \boxed{\text{eigenvector } \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

#1-3 다음 행렬의 eigenvalues와 eigenvectors를 구하시오.

- i) λ =4

$$C - 4I_3 = \begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix} \qquad (C - 4I_3)\mathbf{v} = \begin{bmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

$$-3x_3 = 0 \rightarrow x_3 = 0$$

$$-x_1 + 5x_2 + 3x_3 = -x_1 + 5x_2 = 0 \rightarrow x_1 = 5x_2$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} \rightarrow \boxed{\text{eigenvector} \quad \mathbf{v_3} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}}$$

#1-4 다음 행렬의 eigenvalues와 eigenvectors를 구하시오.

- $D = \begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}, D\mathbf{v} = \lambda\mathbf{v}, \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$D - \lambda I_3 = \begin{bmatrix} 13-\lambda & 5 & 2 \\ 2 & 7-\lambda & -8 \\ 5 & 4 & 7-\lambda \end{bmatrix}$$

$$\det(D - \lambda I_3) = (13-\lambda)\begin{vmatrix} 7-\lambda & -8 \\ 4 & 7-\lambda \end{vmatrix} - 5\begin{vmatrix} 2 & -8 \\ 5 & 7-\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & 7-\lambda \\ 5 & 4 \end{vmatrix}$$

$$= (13-\lambda)\left((7-\lambda)^2 + 32\right) - 5(14 - 2\lambda + 40) + 2(8 - 35 + 5\lambda)$$

$$= -\lambda^3 + 27\lambda^2 - 243\lambda + 729 = -\lambda^3 + 3^3\lambda^2 - 3^5\lambda + 3^6 = -(\lambda - 9)^3$$

$$\boxed{\therefore\ eigenvalue\ \lambda = 9}$$

#1-4 다음 행렬의 eigenvalues와 eigenvectors를 구하시오.

- $\lambda = 9$

$$D - 9I_3 = \begin{bmatrix} 4 & 5 & 2 \\ 2 & -2 & -8 \\ 5 & 4 & -2 \end{bmatrix}$$

$$(D - 9I_3)\mathbf{v} = 0 \rightarrow \left[\begin{array}{ccc|c} 4 & 5 & 2 & 0 \\ 2 & -2 & -8 & 0 \\ 5 & 4 & -2 & 0 \end{array}\right] \xrightarrow[-\frac{5}{4}R_1 + R_3 \to R_3]{-\frac{1}{2}R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 4 & 5 & 2 & 0 \\ 0 & -\frac{9}{2} & -9 & 0 \\ 0 & -\frac{9}{4} & -\frac{9}{2} & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 4 & 5 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t, t \in \mathbb{C}$

$x_2 + 2x_3 = 0 \rightarrow x_2 = -2t$

$4x_1 + 5x_2 + 2x_3 = 0 \rightarrow 4x_1 - 10t + 2t = 4(x_1 - 2t) = 0 \rightarrow x_1 = 2t$

$$\therefore v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ -2t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\boxed{\therefore eigenvector\ v = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}$$

MATLAB

```
>> % #1-3
>> A = [3 5 3; 0 4 6; 0 0 1];
>> eig(A)

ans =

     3
     4
     1

>> [V D] = eig(A)

V =

    1.0000    0.9806    0.8427
         0    0.1961   -0.4815
         0         0    0.2408


D =

     3     0     0
     0     4     0
     0     0     1
```

$\lambda = 3 \rightarrow eigenvector\colon \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 4 \rightarrow eigenvector\colon \begin{bmatrix} 0.9806 \\ 0.1961 \\ 0 \end{bmatrix} = k \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 1 \rightarrow eigenvector\colon \begin{bmatrix} 0.8427 \\ -0.4815 \\ 0.2408 \end{bmatrix} = k \begin{bmatrix} \frac{7}{2} \\ -2 \\ 1 \end{bmatrix}$

Eigenvalues : 3, 4, 1

MATLAB

```
>> A = [13 5 2; 2 7 -8; 5 4 7]

A =

    13     5     2
     2     7    -8
     5     4     7

>> eig(A)

ans =

   8.9999 + 0.0000i
   9.0000 + 0.0000i
   9.0000 - 0.0000i

>> [V D] = eig(A)

V =

   0.6667 + 0.0000i   0.6667 + 0.0000i   0.6667 + 0.0000i
  -0.6667 + 0.0000i  -0.6667 + 0.0000i  -0.6667 - 0.0000i
   0.3333 + 0.0000i   0.3333 + 0.0000i   0.3333 - 0.0000i


D =

   8.9999 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   9.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   9.0000 - 0.0000i
```

$$\lambda = 9 \rightarrow eigenvectors : \begin{bmatrix} 0.6667 \\ -0.6667 \\ 0.3333 \end{bmatrix} = k \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Eigenvalues : 9, 9, 9 (3중근)