Determination of regularization parameter in discrete linear ill-posed problem using DNN

- Examples: V1 -

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In this note, a couple of ill-posed problems are illustrated with numerical simulations. The goal of our work is to develop algorithms to estimate the optimal regularization parameters for each problem.

1 Cauchy problem for the backward heat equation

Consider the one-dimensional heat equation.

$$u_t = u_{xx}, \quad 0 < x < \pi, \quad t > 0 \tag{1a}$$

$$u(t, 0) = u(t, \pi) = 0, \quad t > 0 \tag{1b}$$

To solve this problem additional condition is required. In general, the initial condition is given,

$$u(0, x) = q(x), \quad 0 < x < \pi \tag{2}$$

Then u is uniquely solvable under suitable condition on q. Thus, we know the temperature at any time $t = T$, say,

$$u(T, x) = f(x), \quad 0 < x < \pi \tag{3}$$

The backward heat equation is an inverse problem to reconstruct the initial temperature q from the measurement of f. It is well-known that it is an ill-posed problem.

Numerical solutions

From the method of separation of variables, the general solution to (1) is given by

$$u(t,x) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin nx$$

We assume that $q, f \in L_0^2(0,\pi)$, which is a set of square integrable functions vanishing at the boundary. As

$$\mathcal{S} := \{\sin nx : 0 \le x \le \pi\}_{n=1}^{\infty} \tag{4}$$

is dense in L_0^2, we have

$$q = \sum_{n=1}^{\infty} q_n \sin nx,$$
$$f = \sum_{n=1}^{\infty} f_n \sin nx,$$

where

$$q_n = \frac{2}{\pi}\int_0^\pi q(x)\sin nx dx,$$
$$f_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx.$$

The initial condition (2) gives that

$$u(0,x) = \sum_{n=1}^{\infty} c_n \sin nx = \sum_{n=1}^{\infty} q_n \sin nx,$$

or

$$c_n = q_n, \quad n = 1,2,3,\cdots.$$

Together with (3), we have

$$q_n e^{-n^2 T} = f_n, \quad n = 1,2,3,\ldots \tag{5}$$

or

$$q_n = e^{n^2 T} f_n, \quad n = 1,2,3,\ldots.$$

Algorithm 1. *Backward heat equation*

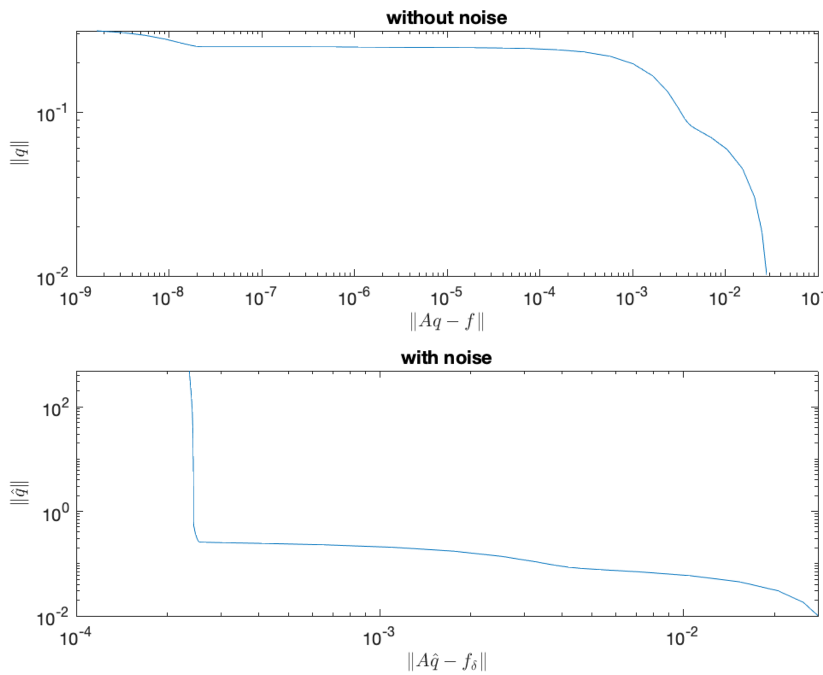

Figure 1: L-curves

1. *Set T. Take N.*
2. *For a given $f \in L^2_0(0,\pi)$ compute $\{f_n\}_{n=1}^N$.*
3. *Solve*
$$F = AQ \tag{6}$$
where $F = (f_1, \cdots, f_N)^T$, $Q = (q_1, \cdots, q_N)^T$, and $A = diag(e^{-T}, e^{-2^2T}, \cdots, e^{-N^2T})$. Note that we need a regularization scheme to solve (6) when F has a noise. We use Tikhonov regularization method with the parameter α. Then approximation solution $\hat{Q}$ is given by
$$\hat{Q} = (A^*A + \alpha I)^{-1}A^*F$$
or
$$\hat{q}_n = \frac{f_n}{\alpha e^{n^2T} + e^{-n^2T}}, \quad n = 1, 2, \cdots, N.$$

Figure 1 shows that log-log plots for $\|Aq - f\|$ and $\|q\|$. Let $q = \sum_{n=1}^5 q_n \sin nx$. $\{q_n\}_{n=1}^5$ is randomly generated satisfying $\|q\| \le \tau = 1$. (see ‘*generate_Q_rej.m’*).

Solve the direct problem to obtain F at the terminal time $T = 1$ by 'sol_act.m'. Run 'noise_data.m' to get noisy data with noise level $\delta = 0.01$, denoted by nF or f_δ. Now we have one data set (Q, F, nF) from the direct problem solver. Conversely, we solve the inverse problem with the Tikhonov method by 'sol_Tik.m' with the various regularization parameter α. All procedures can be done by 'result_Lcurve.m'. See the comments on it. We set min_al $= -20$ for Figure 1.

To obtain data set for training, see 'result_genData.m'.

Task

- *Understand the theory and algorithm*
- *Run (and modify) the appended codes with various parameters*
- *Generate data set*
- *Train a model, say $\mathcal{M}_1$, with $\{(Q, F)\}$. What is the accuracy? Can we use $\mathcal{M}_1$ to solve $AQ = nF$, i.e., for the noisy data f_δ?*
- *Train a model, say $\mathcal{M}_2$ for $\{nF\}$.*
- *Train a model to find the best regularization parameter.*

2 Integral equation of the first kind

The standard form of the integral equation of the first kind with kernel K is given by

$$f(x) = \int_a^b K(x, y)u(y)dy. \tag{7}$$

We seek a function u satisfying (7) for a given f. In general, it is ill-posed problem. We consider the following problem.

$$f(x) = \int_0^\pi K(x, y)u(y)dy \tag{8}$$

where

$$K(x, y) := \begin{cases} \frac{1}{\pi}(\pi - x)y, & 0 \le y \le x \le \pi \\ \frac{1}{\pi}(\pi - y)x, & 0 \le x \le y \le \pi \end{cases}$$

We are interested in solving (8) *for any* $f \in L^2_0(0,\pi) := \{f \in L^2 : f(0) = f(\pi) = 0\}$. *Note that* (8) *can be converted to an the second order differential equation;*

$$f''(x) = -u(x), \quad 0 \le x \le \pi \tag{9a}$$
$$f(0) = f(\pi) = 0 \tag{9b}$$

We see that if f *is not twice differentiable,* u *satisfying* (9) *does not exist in the classical sense. This may give the ill-posedness for* (8). *Nevertheless, we want to solve (or find approximation)* (8) *for a given* $f \in L^2_0(0,\pi)$ *numerically.*

Numerical solutions

Will be discussed.

3 Some scattering problems

Will be discussed.