Complete Intersections in Projective Space

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1 Introduction

A complete intersection is an intersection of hypersurfaces of the expected dimension, or equivalently, a closed subscheme of projective space cut out by a homogeneous ideal with number of generators equal to its codimension. This paper presents a sequence of results describing properties of a complete intersection that depend only of the degrees of the defining equations of the intersecting hypersurfaces. These results generalize the classical form of Bézout's theorem, that the number of intersections of two curves in the projective plane, counted with multiplicity, is the product of the degrees of their defining equations.

In Section 2, I will introduce the Hilbert polynomial of a closed subscheme of projective space, and the associated invariant of degree, which generalizes the degree of the defining equation of a hypersurface, and the cardinality of a set of points. In Section 3, I define the Koszul complex, which under certain conditions gives us a free resolution of the coordinate ring of a close subscheme, and can be used to compute these invariants. In Section 4, I apply this to express the degree of a complete intersection in terms of the degrees of the intersecting hypersurfaces, in Theorem 4.3. This will allow us to obtain a weak form of Bézout's Theorem in the zero dimensional case, Corollary 4.4. As another application of degree, I will show in Section 4.1 that the twisted cubic is not a scheme theoretic complete intersection, Proposition 4.6. Finally, in Section 5 I will introduce intersection multiplicity for irreducible components of complete intersections. Via a relation between these and the expression for degree obtained in Section 4, we can obtain a stronger form of Bézout's theorem for hypersurfaces in $\mathbb{P}^n_k$, Theorem 5.8.

The general argument through Sections 3 and 4 follows [EH00] Section III.3, and the presentation of Hilbert polynomials in Section 2 is adapted from [Har77], Chapter I.7. We will also use some results from Chapter II of [Har77], characterizing closed subschemes of projective space in terms of their defining homogeneous ideals. A few technical results in commutative algebra are deferred to [Eis95].

For an ideal I in a polynomial ring $S = k[x_0, \ldots, x_n]$, $V(I)$ denotes the closed subset of the associated affine space $\mathbb{A}_k^{n+1}$ cut out by I, while $V_+(I)$ denotes the closed subset of the associated projective space $\mathbb{P}_k^n$ cut out by I. When these are treated as closed subschemes, rather than just closed subsets, they have the subscheme structures Spec S/I and Proj S/I respectively. The field k is assumed to be infinite, but not assumed to be algebraically closed unless explicitly stated.

2 Hilbert Polynomials of Closed Subschemes of Projective Space

Let k be a field. Given a homogeneous ideal $I \subset k[x_0, \ldots, x_n] = S$, we can define a closed subscheme $X = \text{Proj } S/I \hookrightarrow \text{Proj } S = \mathbb{P}_k^n$. In this case we say that X is defined by I. As a closed set, X is the set of homogeneous primes $\mathfrak{p} \in \text{Proj } S$ containing I, the projective vanishing set $V_+(I)$. Conversely, Corollary II.5.16 of [Har77] shows that closed subschemes $X \hookrightarrow \mathbb{P}_k^n$ correspond to homogeneous ideals $I \subset k[x_0, \ldots, x_n]$. Such an I can be obtained as $I = \Gamma_*(\mathcal{I})$, for $\mathcal{I}$ the ideal sheaf of X, and in general, two ideals I, J define the same closed subscheme of $\mathbb{P}_k^n$ when $\widetilde{I} = \widetilde{J}$. This occurs exactly when when $I_d = J_d$ for d sufficiently large, as in Exercise II.5.9 of [Har77].

Using this correspondence, we will define a series of invariants of closed subschemes of projective space. We begin with the Hilbert function. For $X \hookrightarrow \mathbb{P}_k^n$ a closed subscheme, let $I = \Gamma_*(\mathcal{I})$ for $\mathcal{I}$ the ideal sheaf of X, and $R = S/I$, which we call the homogeneous coordinate ring of X. Note that R is a graded k-algebra. For $m \geq 0$, R_m denotes the mth graded component of R.

Definition 2.1. The Hilbert function of X, $H(X, -) : \mathbb{N} \to \mathbb{N}$ is the function

$$H(X, m) = \dim_k(R_m) = \binom{n+m}{n} - \dim_k(I_m)$$

The following theorem allows us to translate the Hilbert function into a more useful invariant.

Theorem 2.2. *There is a unique polynomial $P(X, m)$ of degree at most n in m such that for some $m_0 \geq 0$, we have*

$$P(X, m) = H(X, m)$$

for any $m \geq m_0$.

Proof. The uniqueness of P follows from the fact that any two such polynomials P and Q agree on infinitely many inputs and are thus equal. We will prove the existence of such a P in a slightly more general setting, by induction on n. For M any finitely generated graded S-module, we define $H(M, m)$ to be $\dim_k(M_m)$. We will prove that for any M, $H(M, -)$ agrees with a polynomial of degree at most n on sufficiently large input.

For a convenient base case, we consider the zero polynomial to be of degree -1, and S to be k for $n = -1$. In this case, for a finitely generated graded S-module M, $M_m = 0$ for m sufficiently large, which establishes the base case. We now consider $n \geq 0$. To show that $H(M, -)$ agrees with a polynomial of degree at most n on sufficiently large input, it suffices to show that

$$\Delta H(M, m) = H(M, m) - H(M, m-1)$$

agrees with a polynomial of degree at most $n - 1$ for m sufficiently large. Note that this includes the case that $n = 0$, because the zero polynomial is the only polynomial of degree -1. To obtain an expression for $\Delta H(M, m)$, consider the following exact sequence of graded S-modules.

$$0 \longrightarrow K(-1) \longrightarrow M(-1) \xrightarrow{x_n} M \longrightarrow M/x_n M \longrightarrow 0$$

where K is the submodule of M annihilated by x_n, and the map $M(-1) \to M$ is multiplication by x_n. Taking mth graded components in the above sequence, we obtain an exact sequence of k-vector spaces, which implies that

$$\Delta H(M, m) = H(M, m) - H(M(-1), m) = H(K(-1), m) - H(M/x_n M, m)$$

$K(-1)$ and M/x_nM are finitely generated graded modules over $S/(x_n) = k[x_0, \ldots, x_{n-1}]$ so appealing to induction, $H(K(-1), m)$ and $H(M/x_nM, m)$ agree with polynomials of degree at most $n-1$ for m sufficiently large, and consequently $\Delta H(M, m)$ does as well. This completes the proof. $\square$

Definition 2.3. The Hilbert polynomial of X is the polynomial $P(X, m)$ in the above theorem.

Remark 2.4. By definition, $P(X, m)$ depends only on I_d for large d, and so we can compute $P(X, m)$ using any ideal I which defines X as a closed subscheme of $\mathbb{P}^n_k$.

We will now compute the Hilbert polynomial in two simple cases.

Example 2.5. Consider X a hypersurface $V_+(f)$, for f a homogeneous polynomial of degree d. In this case, multiplication by f yields an isomorphism of k-vector spaces $S_{m-d} \to I_m$ for $m \geq d$, and for such m,

$$H(X, m) = \binom{n+m}{n} - \binom{n+m-d}{n} = \frac{(n+m)(n+m-1)\ldots(m) - (n+m-d)\ldots(m-d)}{n!}$$

This is a polynomial in m with leading term $dm^{n-1}/(n-1)!$.

Example 2.6. We will now consider the case that X is a union of d k-rational points, with the reduced induced subscheme structure. We may assume the points all lie in the affine open $U_0 = D_+(x_0)$, and the ith point corresponds to the ideal $\mathfrak{p}_i = (x_1 - a^i_1 x_0, \ldots, x_n - a^i_n x_0)$, for some $a^i_j \in k$. In this case, I_m is the set of homogeneous polynomials $f(x_0, \ldots, x_n)$ of degree m such that $f(1, a^i_1, \ldots, a^i_m) = 0$ for each i, or equivalently, the kernel of the linear map $S_m \to k^d$ defined by

$$f \mapsto \begin{bmatrix} f(1, a^1_1, \ldots, a^1_n) \\ \vdots \\ f(1, a^d_1, \ldots, a^d_n) \end{bmatrix} := \begin{bmatrix} f(P_1) \\ \vdots \\ f(P_d) \end{bmatrix}$$

We will show that this is surjective for m sufficiently large. It suffices to show that for m sufficiently large there is an f_i which vanishes on P_j for $j \neq i$, and does not vanish on P_i. Because k is assumed to be infinite, we can choose linear forms ℓ_i vanishing on P_i and not on P_j for $j \neq i$, and a linear form ℓ not vanishing on any of the P_j and let $f_i = \ell^{m-d+1} \prod_{j \neq i} \ell_j$ for $m \geq d-1$. We conclude that the linear map $S_m \to k^d$ is surjective for all such m, and hence that I_m has codimension d in S_m for all such m. Therefore, $H(X, m) = d$ for m sufficiently large, and so $P(X, m) = d = dm^0/0!$.

2.1 Dimension and Degree

The following theorem generalizes the above two examples.

Theorem 2.7. *The degree of $P(X, m)$ is the dimension of X.*

To prove this, we will need the following lemma.

Lemma 2.8. *If $Y \subset \mathbb{P}^n_k$ is an irreducible closed subset, and $H \subset \mathbb{P}^n_k$ is a hyperplane not containing Y, then* $\dim(Y \cap H) = \dim Y - 1$

Proof. We give Y the reduced induced subscheme structure. Because H does not contain Y, $Y \cap H$ is a proper subset of Y, and so, because Y is irreducible, $\dim(Y \cap H) < \dim Y$. On the other hand, let Z be an irreducible component of $Y \cap H$, and let $U \subset Y$ be an affine open subscheme, such that $Z \cap U$ is nonempty. $H \cap U$ is the vanishing set of some $\ell \in \mathcal{O}_Y(U)$, and $Z \cap U$ corresponds to a minimal prime $\mathfrak{q} \subset \mathcal{O}_Y(U)$ containing ℓ. By the principal ideal theorem, $\mathfrak{q}$ has codimension at most 1, so dimension at least $\dim Y - 1$, because $\mathcal{O}_Y(U)$ is a finitely generated domain over k of dimension $\dim Y$. It remains to show that $Y \cap H$ is nonempty, for $\dim Y \geq 1$. Let $\overline{Y}$ be the affine cone over Y in $\mathbb{A}^{n+1}_k$. Again by the principal ideal theorem, $\overline{Y} \cap \overline{H}$ is of dimension at least 1, and contains the origin, so contains some point other than the origin, corresponding to a point in $Y \cap H$. $\square$

We now can prove Theorem 2.7

Proof. We will prove the slightly more general statement that for M a finitely generated graded S-module with annihilator $\mathfrak{a}$, the degree of $P(M,m)$ is the dimension of the vanishing set of $\mathfrak{a}$. By Proposition I.7.4 of [Har77], M has a finite filtration by modules of the form $S/\mathfrak{p}(d)$ for $\mathfrak{p}$ a homogeneous prime. $P(M,m)$ is then the sum of the polynomials $P(S/\mathfrak{p}, m+d)$, and $V_+(\mathfrak{a})$ is the union of $V_+(\mathfrak{p})$, so it suffices to consider the case that $M = S/\mathfrak{p}$. If $\mathfrak{p}$ contains the irrelevant ideal $S_+ = (x_0, \ldots, x_n)$, then $V_+(\mathfrak{p})$ is empty, of dimension -1, and $P(S/\mathfrak{p}, m) = 0$, of degree -1. Otherwise, we may assume $x_n \notin \mathfrak{p}$. Let H be the hyperplane $V_+(x_n)$. Consider the following exact sequence of graded S-modules.

$$0 \longrightarrow S/\mathfrak{p}(-1) \xrightarrow{x_n} S/\mathfrak{p} \longrightarrow S/(\mathfrak{p} + (x_n)) \longrightarrow 0$$

Taking the dimension of the mth graded component, this implies that

$$\Delta P(S/\mathfrak{p}, m) = P(S/\mathfrak{p}, m) - P(S/\mathfrak{p}, m-1) = P(S/(\mathfrak{p} + (x_n)), m)$$

The vanishing set of $\mathfrak{p} + (x_n)$ is $V_+(\mathfrak{p}) \cap H$, of dimension $\dim V_+(\mathfrak{a}) - 1$ by the preceding lemma. Appealing to induction on dimension, this shows that the degree of $P(X,m)$. is at most $\dim X$. To show that the coefficient on $m^{\dim X}$ is nonzero for X nonempty, we again appeal to induction on dimension, using as a base case the fact that $P(X,m)$ is nonzero for X a nonempty set of points. $\square$

Motivated by this theorem and the two examples, we make the following definition.

Definition 2.9. For $X \subset \mathbb{P}^n_k$ of dimension r, the degree of X is $r!$ multiplied by the leading coefficient of $P(X,m)$.

Remark 2.10. For X a union of d closed points of $\mathbb{P}^n_k$ with any closed subscheme structure, R is the product of the homogeneous coordinate rings of each point. Consequently, $P(X,m)$ is the sum of $P(\{x\}, M)$ over $x \in X$, each of which is a constant which is a positive integer, as it agrees with the dimension of some nonzero vector space. This implies that the degree of X is at least d.

3 The Koszul Complex

3.1 The Intersection of Two Hypersurfaces

To compute the Hilbert polynomial of a hypersurface $V_+(f)$, we used a resolution

$$0 \to S(-d) \to S$$

of $R = S/(f)$, where the map $S(-d) \to S$ is multiplication by f. Under certain conditions, we can use a generalization of this to compute the Hilbert polynomial of a variety $V_+(f_1, \ldots, f_r)$. As a motivating case, we will compute the Hilbert polynomial of a complete intersection of two hypersurfaces, $V_+(f) \cap V_+(g)$. This case will be enough to establish the classical Bézout's theorem for plane curves as a special case of Theorem 5.8, and for the analysis of the twisted cubic in Section 4.1.

In this case, that $V_+(f) \cap V_+(g)$ is a complete intersection just means that f and g have no common factors in $k[x_0, \ldots, x_n]$. Let d, e be the degrees of f, g, respectively. Consider the following complex.

$$0 \longrightarrow S(-d-e) \xrightarrow{(g,-f)} S(-d) \oplus S(-e) \xrightarrow{(f,g)} S \longrightarrow S/(f,g) \longrightarrow 0$$

I claim that this is an exact sequence and therefore a resolution of $S/(f,g)$, the coordinate ring of the intersection. The only place where exactness is not clear is $S(-d) \oplus S(-e)$. An element of the kernel of the map labelled (f,g) is a pair (s,r), $s, r \in S$, such that $s \cdot f + r \cdot g = 0$. Because S is an unique factorization domain and f and g have no common factors, we must have $g|s$ and $f|g$, so we can write $s = s' \cdot g$ and

$r = r' \cdot f$. Then, the fact that $s' \cdot g \cdot f + r' \cdot f \cdot g = 0$ implies that $r' = -s'$, so (s, r) is the image of s' under the map labelled $(g, -f)$.

We can now apply this resolution of $S/(f, g)$ to compute the Hilbert polynomial of the intersection. The complex yields

$$P(S/(f,g), m) = P(S, m) - P(S(-d), m) - P(S(-e), m) + P(S(-d-e), m) =$$

$$\binom{n+m}{n} - \binom{n+m-d}{n} - \binom{n+m-e}{n} + \binom{n+m-e-d}{n}$$

The leading term of this polynomial comes from the terms

$$\frac{m^n}{n!} - \frac{(m-d)^n}{n!} - \frac{(m-e)^n}{n!} + \frac{(m-d-e)^n}{n!} = \frac{m^n - (m-d)^n}{n!} - \frac{(m-e)^n - (m-e-d)^n}{n!}$$

These in turn contribute the following to the leading term

$$\frac{d \cdot m^{n-1}}{(n-1)!} - \frac{d \cdot (m-e)^{n-1}}{(n-1)!}$$

This, finally, has leading term

$$\frac{de \cdot m^{n-2}}{(n-2)!}$$

We conclude that the degree of $V_+(f) \cap V_+(g)$ is de.

3.2 The Koszul Complex in General

For a graded ring S and homogeneous elements $s_1, \ldots, s_r$, we can define a complex $K(s_1, \ldots, s_r)$, as follows, generalizing the case $r = 2$ above. Let d_i be the degree of s_i. For $I \subset \{1, \ldots, r\}$, let $|I|$ denote the cardinality of I, and let d_I denote $\sum\limits_{i \in I} d_i$. For $0 \leq m \leq r$, let

$$F_m = \bigoplus_{|I|=m} S(-d_I)$$

For $j \notin I$, let $\operatorname{place}(j, I)$ be the number of elements of I less than j. We can define a degree preserving map $\partial_m : F_{m+1} \to F_m$ as follows. A homogeneous element of F_{m+1} of degree d is of the form $(h_J)_J$ for h_J homogeneous of degree $d - d_J$. We then let $\partial_m : (h_J)_J \longmapsto (h'_I)_I$, where for $|I| = m$ we define

$$h'_I = \sum_{j \notin I} (-1)^{\operatorname{place}(j,I)} s_j \cdot h_{I \cup \{j\}}$$

Each h'_I is then homogeneous of degree $d - d_I$, so ∂_m is a degree preserving map. We will now show that this defines a complex

$$0 \longrightarrow F_r \xrightarrow{\partial_{r-1}} F_{r-1} \xrightarrow{\partial_{r-2}} \ldots \xrightarrow{\partial_1} F_1 \xrightarrow{\partial_0} F_0 = S$$

by showing that $\partial_m \circ \partial_{m+1} = 0$ for each m. For this computation, we let I range over cardinality m subsets, J over cardinality $m+1$ subsets, and K range over cardinality $m+1$ subsets. Consider an element $(h_K)_K$ of F_{m+2}. This is mapped to a $(h'_J)_J \in F_{m+1}$, which is then mapped to a $(h''_I)_I \in F_m$. We can expand h''_I as follows

$$h''_I = \sum_{j \notin I} (-1)^{\operatorname{place}(j,I)} s_j \cdot h_{I \cup \{j\}} = \sum_{j \notin I} \sum_{k \notin I \cup \{j\}} (-1)^{\operatorname{place}(j,I)} (-1)^{\operatorname{place}(k, I \cup \{j\})} s_k \cdot s_j \cdot h_{I \cup \{j,k\}}$$

For a pair $j \neq k$, a term $s_k \cdot s_j \cdot h_{I \cup \{j,k\}}$ appears exactly twice in the sum, once for adding j followed by k, and again for adding k followed by j. Supposing $j < k$, we have that $\mathrm{place}(k, I \cup \{j\}) = \mathrm{place}(k, I) + 1$, whereas $\mathrm{place}(j, I \cup \{k\}) = \mathrm{place}(j, I)$. Therefore, the two appearances of the term have opposite sign, and cancel out. We conclude that each $h''_I = 0$, so $\partial_m \circ \partial_{m+1} = 0$.

The image of ∂_0 in S is exactly the ideal $(s_1, \ldots, s_r)$. We will see that under certain conditions, $K(s_1, \ldots, s_r)$ is a free resolution of $S/(s_1, \ldots, s_r)$.

Definition 3.1. For A a ring, a sequence of elements $a_1, \ldots, a_r$ is a regular sequence if $(a_1, \ldots, a_r) \neq A$ and for each i a_i is a nonzerodivisor in $A/(a_1, \ldots, a_{i-1})$.

This can be thought of as a generalization of the condition that f and g have no common factors in the motivating case.

Proposition 3.2. *If $s_1, \ldots, s_r$ is a regular sequence of homogeneous elements of S, then $K(s_1, \ldots, s_r)$ is an exact sequence.*

Proof. We will prove this by induction on r. For $r = 1$, this is the statement that for s_1 a nonzero divisor,

$$0 \longrightarrow S \xrightarrow{s_1} S$$

is exact. Suppose now that $s_1, \ldots, s_r, s_{r+1}$ is a regular sequence. Then, $s_1, \ldots, s_r$ is a regular sequence, so appealing to induction, $K(s_1, \ldots, s_r)$ is exact. Let F_m be the mth module of $K(s_1, \ldots, s_r)$ and G_m be the mth module of $K(s_1, \ldots, s_r, s_{r+1})$. Dividing subsets I with $|I| = m$ into those that contain $r+1$ and those that do not, we have $G_m \simeq F_{m-1}(-d_{r+1}) \oplus F_m$, and we can express the map $G_{m+1} \to G_m$ as the sum of $\partial : F_m(-d_{r+1}) \to F_{m-1}(-d_{r+1})$, $\partial : F_{m+1} \to F_m$, and $\pm s_{r+1} : F_m(-d_{r+1}) \to F_m$, as shown in the following diagram, where the sign on s_{r+1} is $(-1)^{m-1}$.

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr] & F_m(-d_{r+1}) \arrow[r, "\partial"] \arrow[dr, "\pm s_{r+1}"] \arrow[d, phantom, "\oplus"] & F_{m-1}(-d_{r+1}) \arrow[r] \arrow[dr] \arrow[d, phantom, "\oplus"] & \cdots \\
\cdots \arrow[r] & F_{m+1} \arrow[r, "\partial"] & F_m \arrow[r] & \cdots
\end{tikzcd}$$

Suppose now that (g, h) is in the kernel of the map

$$F_{m-1}(-d_{r+1}) \oplus F_m \longrightarrow F_{m-2}(-d_{r+1}) \oplus F_{m-1}$$

where we assume $2 \leq m$. Then $\partial(g) = 0$, and so $g = \partial(\overline{g})$ for some $\overline{g} \in F_m(-d_{r+1})$. Moreover,

$$\partial(h) \pm s_{r+1} g = 0$$

Thus, $\partial(h \pm s_{r+1}\overline{g}) = 0$, so $h \pm s_{r+1}\overline{g} = \partial(\overline{h})$ for some $\overline{h} \in F_{m+1}$. This implies that (g, h) is the image of $(\overline{g}, \overline{h})$ under the map

$$F_m(-d_{r+1}) \oplus F_{m+1} \longrightarrow F_{m-1}(-d_{r+1}) \oplus F_m$$

We will now treat the case that $m = 1$. In this case, we have the following diagram:

$$\begin{tikzcd}
F_1(-d_{r+1}) \arrow[r, "\partial"] \arrow[dr, "-s_{r+1}"] \arrow[d, phantom, "\oplus"] & F_0(-d_{r+1}) \arrow[r, "\pi"] \arrow[dr, "s_{r+1}"] \arrow[d, phantom, "\oplus"] & S/(s_1, \ldots, s_r) \arrow[r] \arrow[dr, "s_{r+1}"] \arrow[d, phantom, "\circlearrowleft"] & 0 & \\
F_2 \arrow[r, "\partial"] & F_1 \arrow[r, "\partial"] & F_0 \arrow[r, "\pi"] & S/(s_1, \ldots, s_r) \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact and the diamond on the right commutes. Here, $G_0 = F_0$. Suppose (g, h) is in the kernel of

$$F_0(-d_{r+1}) \oplus F_1 \to F_0$$

Then, $\partial(h) + s_{r+1}g = 0$, which implies that

$$s_{r+1} \cdot \pi(g) = \pi(s_{r+1}g) = \pi(-\partial(h)) = 0$$

Using the fact that s_{r+1} is a nonzerodivisor in $S/(s_1, \dots, s_r)$, this implies that $\pi(g) = 0$, so $g = \partial(\overline{g})$ for some $\overline{g} \in F_1(-d_{r+1})$. Then, $\partial(h + s_{r+1}\overline{g}) = 0$, so $h + s_{r+1}\overline{g} = \partial(\overline{h})$ for some $h \in F_2$. Then, (g, h) is the image of $(\overline{g}, \overline{h})$ under the map

$$F_1(-d_{r+1}) \oplus F_2 \to F_0(-d_{r+1}) \oplus F_1$$

We conclude that the Koszul complex $K(s_1, \dots, s_r, s_{r+1})$ is exact. □

4 Degrees of Complete Intersections

When the Koszul complex is exact, it is a free resolution of $S/(s_1, \dots, s_r)$, which we can use to compute the Hilbert function. Suppose that Z is the scheme theoretic intersection $\bigcap_{i \leq r} Y_i$, with $Y_i = V_+(s_i)$ for homogeneous s_i. Then, Z is defined by the ideal $(s_1, \dots, s_r) \subset S$.

Definition 4.1. A complete intersection is an intersection $Z = \bigcap_{i \leq r} Y_i$ of hypersurfaces $Y_i \subset \mathbb{P}^n_k$ of dimension $n - r$.

Lemma 4.2. *If $\bigcap_{i \leq r} V_+(s_i)$ is a nonempty complete intersection, $s_1, \dots, s_r$ is a regular sequence.*

This follows from the fact that $S = k[x_0, \dots, x_n]$ is a Cohen-Macaulay ring by Proposition 18.9 of [Eis95]. We sketch a proof below, using the Unmixedness Theorem for Cohen-Macaulay rings.

Proof Sketch. We will prove this by induction on r. In general, the fact that $Z = V_+(s_1, \dots, s_r)$ is nonempty implies that $(s_1, \dots, s_r)$ is a proper ideal. For $r = 1$, the fact that $Z = V_+(s_1)$ has codimension 1 implies that $s_1 \neq 0$, so s_1 is a nonzerodivisor.

Suppose now that $Z = \bigcap_{i \leq r+1} Y_i$ is a complete intersection, for $Y_i = V_+(s_i)$. By the principal ideal theorem, $\bigcap_{i \leq r} Y_i$ has codimension at most r, and so because $\bigcap_{i \leq r} Y_i \cap V(s_{r+1})$ has codimension $r + 1$, it must have codimension exactly r. That is, $\bigcap_{i \leq r} Y_i$ is a complete intersection. Appealing to induction, $s_1, \dots, s_r$ is a regular sequence. It remains to show that s_{r+1} is a nonzerodivisor in $S/(s_1, \dots, s_r)$. It suffices to show that s_{r+1} does not lie in any associated prime of $S/(s_1, \dots, s_r)$. For this, we use the following.

Theorem (Unmixedness, [Eis95] Corollary 18.14)**.** *If $s_1, \dots, s_r$ generate a codimension r ideal of S, every associated prime of $S/(s_1, \dots, s_r)$ is of codimension r.*

Then, the fact that $(s_1, \dots, s_{r+1})$ is of codimension $r + 1$ implies that s_{r+1} is not contained in any of the associated primes. □

We can now apply the Koszul complex to compute the degree of a complete intersection.

Theorem 4.3. *If $Z = \bigcap_{i \leq r} Y_i$ is a complete intersection of hypersurfaces $Y_i = V_+(s_i)$ of degree d_i, then the degree of Z is $d_1 \dots d_r$.*

Proof. By the preceding lemma, $s_1, \dots, s_r$ is a regular sequence, so the Koszul complex gives us a resolution of $S/(s_1, \dots, s_r)$. We can read the Hilbert polynomial of Z off of this resolution, as follows. First, note that

$$P(S(-d), m) = P(S, m - d) = \binom{n + m - d}{n}$$

so summing over the modules F_i in the Koszul complex

$$P(Z,m) = \sum_i (-1)^i P(F_i, m) = \sum_I (-1)^{|I|} \binom{n+m-d_I}{n}$$

We will prove by induction on r that this has leading coefficient $\frac{d_1 \dots d_r}{(n-r)!}$, which will imply that Z has degree $d_1, \dots, d_r$, as required. For $r = 1$, this is the case of a hypersurface, done earlier.

Now consider $r+1$, assuming the result for r. Recall that $G_i = F_i \oplus F_{i-1}(-d_{r+1})$, for F_i the ith module in $K(s_1, \dots, s_r)$, and G_i the ith module in $K(s_1, \dots, s_{r+1})$. Let $Z' = \bigcap_{j \leq r} Y_j$. On the level of Hilbert polynomials, this implies that

$$P(Z,m) = P(Z',m) - P(Z', m - d_{r+1})$$

Let $P(Z',m) = \frac{d_1 \dots d_r}{(n-r)!} m^{n-r} + p(m)$, where p is a polynomial of degree at most $n-r-1$. Then, we have that

$$P(Z,m) = \frac{d_1 \dots d_r}{(n-r)!}(m^{n-r} - (m - d_{r+1})^{n-r}) + p(m) - p(m - d_{r+1})$$

The binomial theorem then implies that

$$\frac{d_1 \dots d_r}{(n-r)!}(m^{n-r} - (m - d_{r+1})^{n-r})$$

has leading term

$$\frac{d_1 \dots d_r}{(n-r)!} \cdot (n-r) d_{r+1} m^{n-r-1} = \frac{d_1 \dots d_r d_{r+1}}{(n-r-1)!} m^{n-r-1}$$

whereas $p(m) - p(m - d_{r+1})$ is if degree at most $n-r-2$. This completes the proof. □

As a consequence of this, we obtain a weak form of Bézout's theorem.

Corollary 4.4. *Let $Y_1, \dots, Y_n$ be hypersurfaces $Y_i = V_+(f_i)$, and let d_i be the degree of f_i. If the intersection of the Y_i is zero dimensional, it contains at most $d_1 \dots d_n$ points.*

Proof. If the intersection Z is zero dimensional, it is a finite set of closed points. Then, the fact that the degree of Z is $d_1 \dots d_n$ and Remark 2.10 imply that Z contains no more that $d_1 \dots d_n$ points. □

Moreover, if the intersection Z is reduced, and all the points are k-rational (for instance, if k is algebraically closed), then Z consists of exactly $d_1 \dots d_n$ points, as shown in Example 2.6.

4.1 The Twisted Cubic

As another consequence of Theorem 4.3, we will exhibit a projective variety which is not a complete intersection of hypersurfaces, scheme-theoretically. The Twisted Cubic C in $\mathbb{P}^3_k$ is the closed subscheme corresponding to the degree 3 Veronese embedding $\mathbb{P}^1_k \hookrightarrow \mathbb{P}^3_k$, given by the map $\varphi : k[x_0, x_1, x_2, x_3] \to k[y_0, y_1]$ defined as follows

$$x_0 \mapsto y_0^3$$
$$x_1 \mapsto y_0^2 y_1$$
$$x_2 \mapsto y_0 y_1^2$$
$$x_3 \mapsto y_1^3$$

This map is surjective in high enough degrees, so defines a closed embedding $V : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. To show that C is not a complete intersection, we will need the following.

Proposition 4.5. *On the level of sets, C is not contained in any hyperplane $H \subset \mathbb{P}^3$.*

Proof. Let $\ell(x_0, x_1, x_2, x_3)$ be a nonzero linear form defining H. Then $V^{-1}(H)$ is the closed set defined by $\ell(y_0^3, y_0^2 y_1, y_0 y_1^2, y_1^3)$. This is a nonzero polynomial, so defines a proper closed subset of $\mathbb{P}^1$. Therefore, $C \not\subset H$. $\square$

We will now compute the degree of C. Let $I \subset k[x_0, x_1, x_2, x_3]$ be the kernel of φ as defined above. This is then an ideal defining C. $R = S/I$ is isomorphic to $k[y_0^3, y_0^2 y_1, y_0 y_1^2, y_1^3] \simeq S^{(3)}$, taking $S = k[y_0, y_1]$ and using the notation of Exercise II.5.13 of [Har77]. We then have that $R_m \simeq S_{3m}$, so

$$H(C, m) = \dim_k(S_{3m}) = \binom{1+3m}{1} = 3m + 1$$

Consequently, $P(C, m) = 3m + 1$, so the degree of C is 3.

Proposition 4.6. *C is not a complete intersection of hypersurfaces.*

Proof. Suppose C is a complete intersection. C has dimension 1, so in this case $C = Y_1 \cap Y_2$, for hypersurfaces $Y_1 = V_+(f_1)$ and $Y_2 = V_+(f_2)$. Because C is not contained in any hyperplane, both $\deg f_1$ and $\deg f_2$ are greater than 1. On the other hand, Theorem 4.3 implies that $3 = \deg f_1 \cdot \deg f_2$, which is a contradiction because 3 is prime. $\square$

5 Intersection Multiplicity

We can refine the inequality in Corollary 4.4 to an equation by attributing multiplicities to points of intersection. We will first introduce a more general notion of multiplicity for irreducible components of a complete intersection.

Definition 5.1. Let M be a finitely generated graded module over a Noetherian graded ring S. Let $\mathfrak{p}$ be a prime minimal over ann$M$. Then $M_\mathfrak{p}$ is a zero dimensional $S_\mathfrak{p}$ module, so is of finite length over $S_\mathfrak{p}$. Let $\mu(\mathfrak{p}, M)$, the multiplicity of $\mathfrak{p}$ in M, be the length of $M_\mathfrak{p}$ over $S_\mathfrak{p}$.

Suppose now that $Y_1, \ldots Y_r$ are hypersurfaces $Y_i = V_+(f_i)$ in $\mathbb{P}^n_k$, such that $Z = \bigcap_{i \le r} Y_i$ is a complete intersection. Then, every irreducible component Z_j of Z has codimension r, and corresponds to a homogeneous prime minimal over $(f_1, \ldots, f_r)$. In this case, we can define intersection multiplicity as follows.

Definition 5.2. The intersection multiplicity of $Y_1, \ldots, Y_n$ at Z_j, denoted $\mu(Z_j : Y_1, \ldots, Y_r)$, is

$$\mu(\mathfrak{p}, S/(f_1, \ldots, f_r))$$

for $\mathfrak{p}$ the homogeneous prime corresponding to Z_j.

We will show that this is really a local property at the point $\mathfrak{p}$.

Proposition 5.3. *$\mu(Z_j : Y_1, \ldots, Y_r)$ is the length of $\mathcal{O}_{Z,\mathfrak{p}}$.*

Proof. We have defined $\mu(Z_j : Y_1, \ldots, Y_r)$ to be the length of $(S/(f_1, \ldots, f_r))_\mathfrak{p}$, the degree zero component of which is $\mathcal{O}_{Z,\mathfrak{p}}$. Let $x_i \notin \mathfrak{p}$. Then x_i is a unit in $(S/(f_1, \ldots, f_r))_\mathfrak{p}$ of degree 1. Given a submodule

$$M \subset ((S/(f_1, \ldots, f_r))_\mathfrak{p})_0$$

there is a corresponding submodule

$$\overline{M} = \bigoplus_{z \in \mathbb{Z}} x_i^z M \subset (S/(f_1, \ldots, f_r))_\mathfrak{p}$$

which is the unique submodule whose degree zero component is M. This correspondence between submodules yields a correspondence between maximal filtrations, and hence the rings have the same length. $\square$

This characterization of multiplicity allows us to connect it to another local geometric property of intersections.

Definition 5.4. Let $Y_i = V(f_i) \subset U = \text{Spec } A$ be hypersurfaces in an affine variety. For $p \in \bigcap_{i \leq r} Y_i$, we say that the Y_i intersect transversely at p if the germs $(f_1)_p, \ldots, (f_r)_p$ span the cotangent space $\mathfrak{m}_p/\mathfrak{m}_p^2$ at p. The f_i are determined up to multiplication by a unit in $\mathcal{O}_U(U)$, so their residue classes in the cotangent space are determined up to multiplication by a nonzero scalar. As such, this is independent of the choice of f_i. For Y_i hypersurfaces in projective space, we say that the Y_i intersect transversely at p if $Y_i \cap U$ intersect transversely at p for some affine open U containing p.

Note that in the above definition, we allow $r < n$, and p to be a generic point of a closed subset of positive dimension.

Proposition 5.5. *If $Y_1, \ldots, Y_n \subset \mathbb{P}_k^n$ intersect transversely at p, then $\mu(\overline{\{p\}} : Y_1, \ldots, Y_r) = 1$.*

Proof. We will show that the local ring $\mathcal{O}_{Z,p}$ has length 1. Taking $U = \text{Spec } A$ to be an affine open subset containing p where the Y_i intersect transversely, this isomorphic to the local ring $\mathcal{O}_{Z\cap U,p}$. Let p correspond to a prime $\mathfrak{p} \subset A$. We have that each $Y_i \cap U = V(f_i)$ for some $f_i \in A$. Then

$$\mathcal{O}_{Z\cap U,p} = (A/(f_1, \ldots, f_r))_{\mathfrak{p}} = (A)_{\mathfrak{p}}/(f_1, \ldots, f_n)$$

Because the $f_1, \ldots, f_n$ span $\mathfrak{m}_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}^2$, they generate the ideal $\mathfrak{m}_{\mathfrak{p}}$ by Nakayama's lemma. This implies that the local ring is a field, and hence, has length 1 over itself. □

Remark 5.6. Although we will not use this, the converse of Proposition 5.5 is true as well, because if the local ring of the intersection is a field, any local defining equations of the Y_i must generate the maximal ideal, and hence span the cotangent space. This implies that we can use any affine open U containing p to check if an intersection is transverse.

Having defined multiplicity, we can state a refinement of Theorem 4.3.

Theorem 5.7. *Let $Z = \bigcap_{i \leq r} Y_i$ be a complete intersection of hypersurfaces Y_i of degree d_i. Let $Z_1, \ldots, Z_N$ be the irreducible components of Z, which we endow with the reduced induced subscheme structure. Then*

$$\sum_{j \leq N} \mu(Z_j : Y_1, \ldots, Y_r) \cdot \deg Z_j = d_1 \ldots d_r$$

Proof. The right side of this equation is equal to the degree of Z by Theorem 4.3, so it suffices to show that the left side is also the degree of Z. Let $R = S/(f_1, \ldots, f_r)$. By Proposition I.7.4 of [Har77], R admits a filtration

$$0 = M_0 \subset \cdots \subset M_q = R$$

such that $M_{i+1}/M_i \simeq S/\mathfrak{p}_i(\ell_i)$, for some homogeneous prime $\mathfrak{p}_i$ and integer ℓ_i. Localizing at a minimal prime $\mathfrak{p}$ over $(f_1, \ldots, f_r)$, this becomes a maximal filtration of $R_{\mathfrak{p}}$ whose length is the number of i for which $\mathfrak{p}_i = \mathfrak{p}$. We conclude that this number is $\mu(\mathfrak{p}, R)$.

The filtration of R implies that

$$H(R, m) = \sum_i H(S/\mathfrak{p}_i, m + \ell_i)$$

so, taking m sufficiently large

$$P(Z, m) = \sum_i P(S/\mathfrak{p}_i, m + \ell_i)$$

If $\mathfrak{p}_i$ is not a minimal prime, corresponding to a variety of dimension less than $n-r$, the resulting polynomial is of degree less than $n-r$, and so does not affect the leading term of $P(Z, m)$. Consequently, the leading term of $P(Z,m)$ is the sum over $\mathfrak{p}_i$ such that $\mathfrak{p}_i$ is minimal over $(f_1,\dots,f_r)$ of the leading term of $P(S/\mathfrak{p}_i, m+\ell_i)$. The ℓ_i does not affect the leading term, so we may assume these are all zero. For an irreducible component Z_j of Z corresponding to a minimal prime $\mathfrak{p}$, $P(S/\mathfrak{p}, m) = P(Z,m)$ has leading term $\deg Z_j \cdot \frac{m^{n-r}}{(n-r)!}$. Summing over the i for which $\mathfrak{p} = \mathfrak{p}_i$, and all irreducible components Z_j, the leading term of $P(Z,m)$ is

$$\sum_j \mu(Z_j : Y_1,\dots,Y_r) \cdot \deg Z_j \cdot \frac{m^{n-r}}{(n-r)!}$$

so Z has degree

$$\sum_j \mu(Z_j : Y_1,\dots,Y_r) \cdot \deg Z_j$$

as required. $\square$

Specializing to the case of a dimension zero intersection over an algebraically closed field, each irreducible component Z_j is a k-rational closed point, so $\deg(Z_j) = 1$. We obtain the following version of Bézout's Theorem.

Theorem 5.8. *Suppose k is algebraically closed, and $Z = \bigcap_{i\le n} Y_i$ is a complete intersection of hypersurfaces $Y_i = V_+(f_i) \subset \mathbb{P}^n_k$, where f_i has degree d_i. Then*

$$\sum_{p\in Z} \mu(p : Y_1,\dots,Y_n) = d_1 \dots d_n$$

In particular, if the Y_i intersect transversely at each point in Z, then Z consists of exactly $d_1 \dots d_n$ points.

5.1 Plane Conics

As an example of the above theorem, considering the conic curves $C_1 = V_+(x_2x_1 - x_0^2 + x_2^2)$ and $C_2 = V_+(x_2x_1 + x_0^2 - x_2^2)$ in $\mathbb{P}^2_{\mathbb{C}}$. By Theorem 5.8 these curves intersect with total multiplicity $2\cdot 2 = 4$. Let U_i be the affine open set $D_+(x_i)$. In U_2, $C_1 \cap U_2 = V(x_{1/2} - x_{0/2}^2 + 1) = V(f_1)$ and $C_2 \cap U_2 = V(x_{1/2} + x_{0/2}^2 - 1) = V(f_2)$. These parabolas intersect at the points $p = [-1 : 0 : 1]$ and $q = [1 : 0 : 1]$, in homogeneous coordinates, corresponding to the ideals $(x_{0/2} - 1, x_{1/2})$ and $(x_{0/2} + 1, x_{1/2})$ of $\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(U_2) = k[x_{0/2}, x_{1/2}]$. We will show that these intersections are transverse. For p, we make the change of variables $v = x_{0/2} + 1$, $w = x_{1/2}$. Then

$$\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}},P} = \mathbb{C}[v,w]_{(v,w)}$$

In these coordinates,

$$f_1 = w - v^2 + 2v$$
$$f_2 = w + v^2 - 2v$$

These span $\mathfrak{m}_p/\mathfrak{m}_p^2 = (v,w)/(v,w)^2 = \mathbb{C}\overline{v} \oplus \mathbb{C}\overline{w}$. For q, making the change of variables $v = 1 - x_{0/2}$, $w = x_{1/2}$, the situation is identical. Consequently, $\mu(p : C_1, C_2) = \mu(q : C_1, C_2) = 1$.

We can check directly that $[1 : 0 : 0]$ does not lie in either curve, so all remaining intersections lie in U_1. $C_1 \cap U_1 = V(x_{2/1} - x_{0/1}^2 + x_{2/1}^2)$ and $C_2 \cap U_1 = V(x_{2/1} + x_{0/1}^2 - x_{2/1}^2)$. Solving these equations, we find a third intersection $r = [0 : 1 : 0]$ in homogeneous coordinates. We will compute the multiplicity of this intersection. We let $v = x_{0/1}$, $w = x_{2/1}$.

$$\mathcal{O}_{C_1\cap C_2, r} = \mathbb{C}[v,w]_{(v,w)}/(w - v^2 + w^2, w + v^2 - w^2) = \mathbb{C}[v]_{(v)}/(v^2) = \mathbb{C}[v]/(v^2)$$

which has length 2. Consequently $\mu(r : C_1, C_2) = 2$. We conclude that $C_1 \cap C_2 = \{p, q, r\}$, where p and q have multiplicity 1 and r has multiplicity 2, giving a total multiplicity of 4, as expected.

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