NOTES FOR BALDWIN-LACHLAN PRESENTATION

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Contents

1. Summary

The Balwin-Lachlan proof of the categoricity theorem can be summarized as follows. We find two obstructions to categoricity in any given uncountable cardinal κ, then show that in the absence of these obstructions, we have categoricity in every uncountable cardinality. We have met the first obstruction, having too many types. The other obstruction is the existence of **Vaughtian pairs**, intuitively two pieces of structure which can vary in size independently.

In the absence of these obstructions, we have a good picture of the structure of models of an uncountably categorical theory. A "typical" example of what this looks like is the group $G = Z_4^{\omega}$, an infinite product of cyclic groups. Here the 2-torsion subgroup $H = \{g \in G \mid 2 \cdot g = 0\}$ is an F_2-vector space, and we build isomorphisms between models of $\mathrm{Th}(G)$ by finding bijections between bases of these definable vector spaces. In general there will be a defianble strongly minimal set with a basis whose size controls the isomorphism type of the model.

2. Vaughtian Pairs

Definition. *A **Vaughtian pair** is a proper elementary extension $M \prec N$, $M \neq N$, such that for some formula $\varphi \in \mathcal{L}(M)$, $\varphi(M)$ is infinite and $\varphi(M) = \varphi(N)$.*

A closely related phenomenon which directly obstructs categoricity is the following.

Definition. *For infinite cardinals $\kappa > \lambda$, a (κ, λ)-model is a model M such that $|M| = \kappa$ but for some formula $\varphi \in \mathcal{L}(M)$ we have $|\varphi(M)| = \lambda$.*[1]

As an example, consider the theory of an equivalence relation with two infinite equivalence classes. This has a (κ, λ)-model for any infinite $\kappa > \lambda$, taking one class to be of size κ and the other to be of size λ.

Lemma. *If T has a (κ, λ)-model, T is not categorical in κ.*

Proof. If M is a (κ, λ)-model, we can by compactness produce an $N \models T$ such that for any $\overline{a} \in N$ with $\varphi(N, \overline{a})$ infinite, we have $|\varphi(N, \overline{a})| = \kappa$. Such an N can be taken so that $|N| = \kappa$, but $N \not\cong M$. □

So now our goal is to prove that the existence of a Vaughtian pair implies the existence of a (κ, ω)-model, in order to show that κ-categorical theories have no Vaughtian pairs. In this argument we will also assume that T is ω-stable, as we already know that κ-categorical theories are ω-stable.

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[1] Results about the existence or nonexistence of (κ, λ)-models of general theories go by the names "two cardinal theorems" and "one cardinal theorems" respectively. There is a large literature on these things, some of it involving set theory beyond ZFC. We will only need a small part of this theory.

2.1. **Stretching Vaughtian Pairs.**

Lemma. *If T has a Vaughtian pair, then T has a countable Vaughtian pair.*

Proof. Let $M \prec N$ be a Vaughtian pair. We can view the pair (N, M) as a structure for an expanded language $\mathcal{L}^+$ with a new predicate naming the elementary substructure M. Now apply the downward Lowenheim-Skolem theorem to (N, M). $\square$

Theorem. *Suppose that T is ω-stable and has a countable Vaughtian pair. Then T has a (ω_1, ω)-model.*[2]

Proof sketch. The idea is to get a Vaughtian pair whose constituent structures are isomorphic over the parameters of φ, so we can stack up copies of it. To do this, we construct a countable ω-saturated Vaughtian pair (M, N), i.e. one such that M and N are both ω-saturated countable models of T. This is possible as T is ω-stable. Then as ω-saturated countable models are isomorphic (over a finite set of parameters) by a back and forth argument, we have $M \simeq N$. Now we construct an elementary chain

$$M_0 \prec M_1 \prec M_2 \prec \dots$$

such that $M_0 = M$, $M_1 = N$, and for all $\alpha < \omega_1$, $(M_{\alpha+1}, M_\alpha) \simeq (N, M)$. We take unions at limit stages, which still gives something isomorphic to M. Then taking $M_* = \bigcup_{\alpha<\omega_1} M_\alpha$, we have $|M_*| = \omega_1$. On the other hand, we have by induction on α that $\varphi(M_\alpha) = \varphi(M_0)$ for all α, so $\varphi(M_*)$ is countable. $\square$

Theorem. *If T is ω-stable and has a (ω_1, ω)-model, then it has a (κ, ω)-model for any uncountable κ.*

This is the hardest part of our analysis of Vaughtian pairs. To prove this we need the following lemmas.

Lemma. *Suppose M is an uncountable model of T an ω-stable theory. Then there is a formula $\theta(x)$ of $\mathcal{L}$ such that $\theta(M)$ is uncountable, and for any formula ψ, exactly one of $(\theta \wedge \psi)(M)$ or $(\theta \wedge \neg\psi)(M)$ is countable.*

Proof. If not we could build a perfect tree of formulas so as to contradict ω-stability. $\square$

Lemma. *Suppose M is a (λ, ω)-model of an ω-stable theory T, $\lambda \geq \omega_1$, witnessed by a formula φ. Then M has a proper elementary extension $N \succ M$, $N \neq M$, with $\varphi(M) = \varphi(N)$.*

Proof. Take θ as in the previous lemma and define

$$p(x) = \{\psi(x) \mid |(\theta \wedge \psi)(M)| \geq \omega_1\} \in S(M)$$

Now one can check that p is a type, and contains $x \neq a$ for all $a \in M$, so in particular is not realized in M. However, one can check that any countable subset of this type is realized in M. Let g be a realization of p in some elementary extension and let N be a prime model over $M \cup \{g\}$. One could perhaps compare this N to a forcing extension, as it is generated over M by a single highly generic element. This element looks exactly like the elements of M up to countable amounts of information, so will not disturb the countable set $\varphi(M)$.

To make this precise, suppose $b \in \varphi(N) \setminus \varphi(M)$, towards a contradiction. Thus b is a realization of the type

$$q(y) = \{\varphi(y)\} \cup \{y \neq a \mid a \in M\}$$

On the other hand, the type of b is isolated by a formula $\eta(y, g, \overline{m})$ using parameters from $M \cup \{g\}$. Thus we have $\eta \vdash q$. We conclude that the following sentences are all in p.

$$\exists y \eta(y, x, \overline{m})$$

$$\forall y\, (\eta(y, x, \overline{m}) \to \varphi(y))$$

$$\forall y\, (\eta(y, x, \overline{m}) \to y \neq a) \text{ for each } a \in \varphi(M)$$

This is a countable set of sentences, so is realized in M, which is a contradiction, as it implies the existence of a tuple which satisfying φ but not equal to any of the elements of M which satisfy φ.

$\square$

[2] One can get away with not assuming ω-stability here, but as our interest is in categoricity and not two cardinal theorems, I will assume it.

Now to construct a (κ, ω)-model, we build an elementary chain, using this lemma at each stage to extend M while keeping $\varphi(M)$ the same.

Conclusion. *If T is κ-categorical, it has no Vaughtian pairs.*

Having no Vaughtian pairs is a quite strong condition on T. We will now explore some consequences of it.

Theorem. *If M is a model of a theory with no Vaughtian pairs, and $\varphi(M)$ is an infinite definable subset, M is prime over $\varphi(M)$.*

Proof. We can construct an elementary substructure $M_* \subset M$ which is prime over $\varphi(M)$. Because there are not Vaughtian pairs, $M_* = M$. □

Thus any infinite definable set "generates" the model. This will allow us to reduce our classification of models to a classification of relatively simple definable sets that live inside them. What follows is a nice definability property which will be useful later.

Theorem. *If T has no Vaughtian pairs, T eliminates the quantifier $\exists_\infty$. This is to say, if $M \models T$, $\varphi(x, y)$ is a formula with parameters from M, them there is an n_φ such that for $a \in M$*

$$|\varphi(M, a)| > n_\varphi \Rightarrow |\varphi(M, a)| \geq \omega$$

Proof. Supposing otherwise, let a_n be such that $\varphi(M, a_n)$ is finite, but has at least n elements. We will use this to build a Vaughtian pair. Add a predicate U and constant a_* to the language, and consider the set of sentences saying

(1) U defines a proper elementary substructure.
(2) $\varphi(U, a_*)$ has at least n elements, for each $n < \omega$.
(3) $\forall x(\varphi(x, a_*) \to x \in U)$

Using the a_n, this is consistent, but a model of it yields a Vaughtian pair. Intuitively, finite sets do not change size in elementary extensions, so T must enforce a strict boundary between finite and infinite sets. □

3. Minimal Formulas

Definition. *Let M be a structure, φ a formula, perhaps with parameters from M. We say φ is* ***minimal*** *if $\varphi(M)$ is infinite any definable subset of $\varphi(M)$ is finite or cofinite. φ is* ***strongly minimal*** *if this remains true in any elementary extension of M.*

Another way to understand strong minimality is via Morley rank. One can define Morley rank for formulas with parameters, which is dual to Morley rank of types in the following sense.

$$R(\varphi) = \sup\{R(p) \mid \varphi \in p\}$$
$$R(p) = \inf\{R(\varphi) \mid \varphi \in p\}$$

Then a strongly minimal formula is one of Morley rank 1. As strongly minimal formulas define infinite subsets, any model is prime over the associated strongly minimal set. Thus a strongly minimal formula is a good step towards an invariant classifying such models up to isomorphism.

To refer back to our running example, the group G itself is not strongly minimal, as H the 2-torsion subgroup is definable, infinite, and has infinite complement. On the other hand, H itself is strongly minimal.

Lemma. *If M is a model of an ω-stable theory, there is a minimal formula in M.*

Proof. If not we could build a perfect tree and contradict ω-stability. □

Lemma. *If M is a model of a theory with no Vaughtian pairs, and φ is a minimal formula, it is strongly minimal.*

Proof. The fact that T eliminates the quantifier $\exists_\infty$ means that minimality can be stated as a set of first order properties, which are preserved by elementary extensions. □

Now as our κ-categorical T is ω-stable, it has a prime model M_0. This model has a minimal, and hence strongly minimal formula φ. As we can embed M_0 in any model of T, this gives us a strongly minimal set in every model of T. Understanding the structure of these simple sets will be the basis of our classification of models of T.

3.1. **Dimension Theory in Strongly Minimal Sets.** Now for our classification of models, we will develop a notion of dimension qualitatively different to Morley rank, but still closely related to transcendence degree. To return to our running example this will be the dimension of our 2-torsion subgroup as an F_2-vector space. Our other motivating example is transcendence degree in algebraically closed fields, which explains the general approach.

Definition. *Let X be a strongly minimal set in a model M. For $Y \subset X$, we define $\mathrm{acl}^X(Y)$ to be $\mathrm{acl}(Y) \cap X$. We say Y is an independent set if for all $y \in Y$, $y \notin \mathrm{acl}^X(Y \setminus \{y\})$. We say Y is a spanning set if $\mathrm{acl}^X(Y) = X$. We say Y is a basis if it is a maximal independent subset.*

One way to think of this is that we are imbuing X with all the definable structure on it from the ambient model, and then viewing it as a structure in its own right with those definable relations. Then $\mathrm{acl}^X(-)$ is just $\mathrm{acl}(-)$ evaluated in this structure.

Lemma. *The operation $\mathrm{acl}^X(-)$ satisfies the following properties:*

(1) ***Monotonicity**: If $A \subset B$, $A \subset \mathrm{acl}^X(A) \subset \mathrm{acl}^X(B)$*
(2) ***Finite Character**: If $a \in \mathrm{acl}^X(A)$, then $a \in \mathrm{acl}^X(F)$ for some finite $F \subset A$.*
(3) ***Transitivity**: If $B \subset \mathrm{acl}^X(A)$, then $\mathrm{acl}^X(B) \subset \mathrm{acl}^X(A)$.*
(4) ***Exchange**: If $b \in \mathrm{acl}^X(A \cup \{a\}) \setminus \mathrm{acl}^X(A)$, then $a \in \mathrm{acl}^X(A \cup \{b\})$.*

Proof. Properties (1), (2), and (3) are general properties of the model theoretic algebraic closure. (4) is a consequence of strong minimality. Let $\varphi(x, a, \bar{c})$ witness that $b \in \mathrm{acl}^X(A \cup \{a\})$, so $(\exists_{=n} x)\varphi(x, a, \bar{c})$ and $\varphi(b, a, \bar{c})$ both hold. Consider the formula

$$\psi(y) = (\exists_{=n} x)\varphi(x, y, \bar{c}) \wedge \varphi(b, y, \bar{c})$$

This is satisfied by a. I claim that it has only finitely many solutions in X. If not, it has cofinitely many, say all but m values of $y \in X$ satisfy it. Now consider

$$\theta(x) = (\text{for all but at most } m \text{ values of } y)((\exists_{=n} z)\varphi(z, y, \bar{c}) \wedge \varphi(x, y, \bar{c}))$$

I now claim this has at most n solutions. If not, we can find $b_0, \ldots, b_n$ all satisfying $\theta(x)$. Excluding at most $(n+1) \cdot m$ values of y, we can find an a_* such that for each i,

$$(\exists_{=n} z)\varphi(z, a_*, \bar{c}) \wedge \varphi(b_i, a_*, \bar{c})$$

This is a contradiction. Now we tie everything up. As $\theta(x)$ is satisfied by b, this witnesses that $b \in \mathrm{acl}^X(A)$, contradicting our assumption. Thus, $\psi(y)$ must have finitely many solutions, so $a \in \mathrm{acl}^X(A \cup \{b\})$. □

Lemma. *Any two bases for X have the same cardinality. We call this the dimension of X.*

Proof. Many texts will tell you that this is a standard argument from undergraduate linear algebra, but the infinite dimensional case really requires more thought. One first proves the finite dimensional case, then uses the finite character property and a bit of cardinal arithmetic to reduce the infinite case to the finite one. □

4. Classifying Models

Recall that we have a prime model M_0 of T, and a strongly minimal formula φ over M_0. For any $M \models T$, we have an elementary embedding $M_0 \hookrightarrow M$, so we can interpret φ in M. Moreover, $\varphi(M)$ is a strongly minimal set in M. For simplicity we can add constants to the language naming the parameters of φ, using the elementary embeddings to interpret these in any model. Thus we may assume φ has no parameters, and so interpret it any model without reference to M_0. Using this, we carry out our classification of models. We will build up isomorphims between models starting with maps between bases of the strongly minimal sets. First, we need to know that these bases all look alike.

Lemma. *Let φ be a strongly minimal formula for T. For each n there is a type $p^n_\varphi \in S^n(T)$ depending only on φ such that for $M \models T$, $A \subset \varphi(M)$ an independent set, $a_1, \ldots, a_n \in A$ distinct elements, we have*

$$(a_1, \ldots, a_n) \models p^n_\varphi$$

In particular, an independent set is indiscernible set, and an injective map between independent sets is an elementary map.

Proof. We construct p_φ^n by induction on n. For $n = 1$, we define

$$p_\varphi^1 = \{\psi(x) \mid T \vdash \varphi(x) \wedge \psi(x) \text{ has infinitely many solutions } \}$$

This is a complete type by strong minimality, satisfied by any element which is not algebraic over the empty set. Now supposing we have p_φ^n, we define

$$p_\varphi^{n+1} = \{\psi(\overline{x}, x_{n+1}) \mid p_\varphi^n(\overline{x}) \vdash \varphi(x_{n+1}) \wedge \psi(\overline{x}, x_{n+1}) \text{ has infinitely many solutions } x_{n+1}\}$$

Again, this is a complete type satisfied by any independent sequence. □

Lemma. *Suppose A is a basis of $\varphi(M)$ and B is a basis of $\varphi(N)$. Let $f : A \to B$ be an injective map. Then f extends to an elementary map $M \to N$.*

Proof. From the above, f is an elementary map. As $\varphi(M) \subset \operatorname{acl}(A)$, we can extend f to an elementary map $\varphi(M) \to \varphi(N)$. Then as M is prime over $\varphi(M)$, we can extend f to an elementary embedding $M \to N$. □

Lemma. *Suppose* $\dim(\varphi(M)) = \dim(\varphi(N))$*. Then $M \simeq N$.*

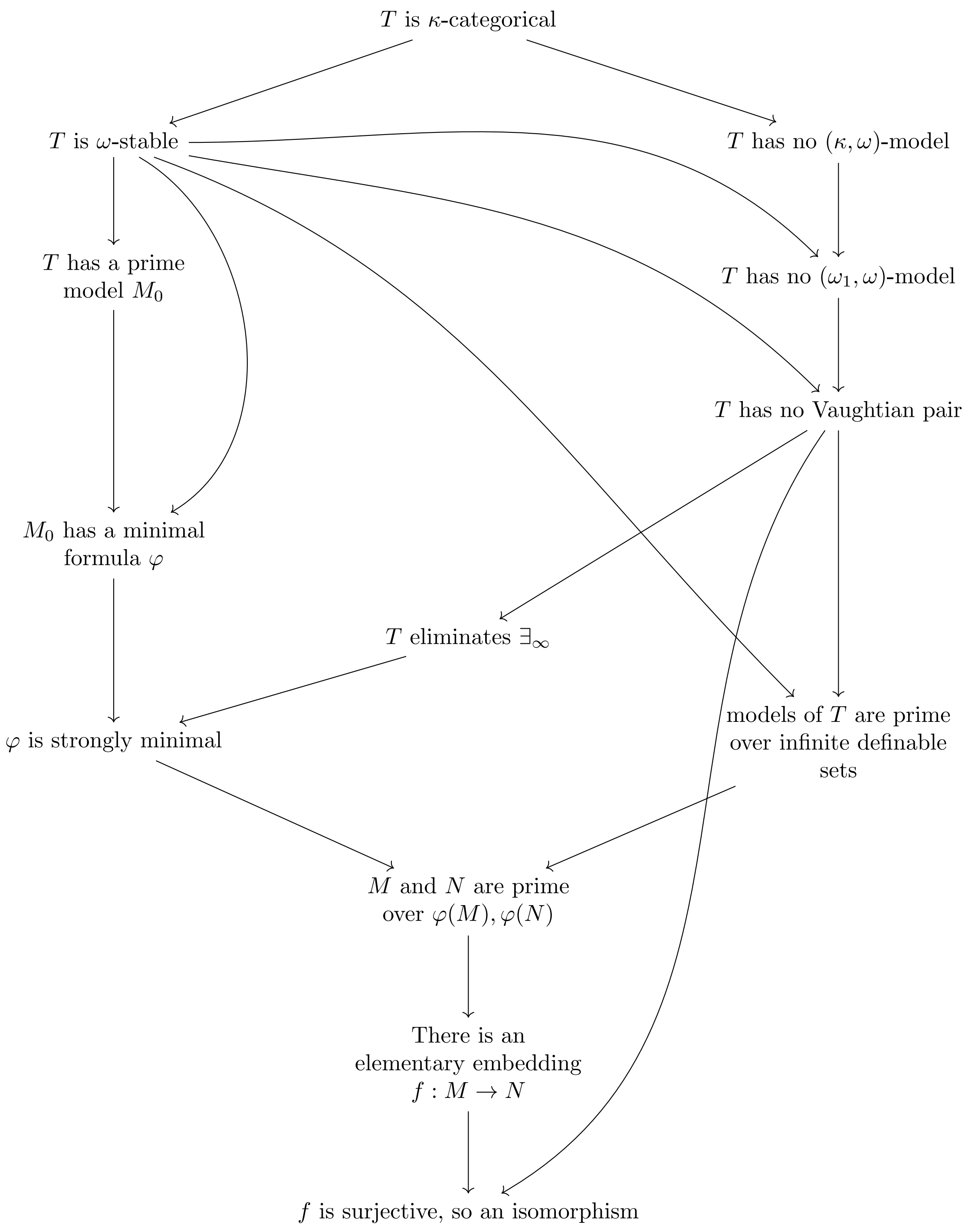
Proof. We can choose a bijection $A \to B$ between bases of $\varphi(M)$ and $\varphi(N)$. This extends to an elementary map $\varphi(M) \to \varphi(N)$, which must be surjective as $\varphi(N) \subset \operatorname{acl}(B)$. This then extends to an elementary embedding $M \to N$ whose image contains $\varphi(N)$. This now must be surjective as T has no Vaughtian pairs. □

Theorem (Categoricity Theorem)**.** *If T is κ-categorical for some $\kappa \geq \omega_1$, T is λ-categorical for every $\lambda \geq \omega_1$.*

Proof. If T is κ categorical, then given two models M, N of size $\lambda \geq \omega_1$, we have $|\varphi(M)| = |\varphi(N)| = \lambda$, and hence $\dim(\varphi(M)) = \dim(\varphi(N)) = \lambda$. Thus, $M \simeq N$. □

Theorem. *If T is ω_1-categorical, then T has at most countably many isomorphism types of countable models.*

Proof. These isomorphism types correspond to possible values of $\dim(\varphi(M))$, which for a countable model is a natural number or ω. □


T is κ-categorical
T is ω-stable
T has no (κ, ω)-model
T has a prime model M_0
T has no (ω_1, ω)-model
T has no Vaughtian pair
M_0 has a minimal formula φ
T eliminates $\exists_\infty$
models of T are prime over infinite definable sets
φ is strongly minimal
M and N are prime over $\varphi(M), \varphi(N)$
There is an elementary embedding $f : M \to N$
f is surjective, so an isomorphism