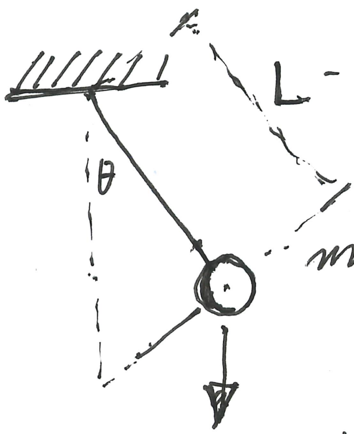
second order ordinary differential equations

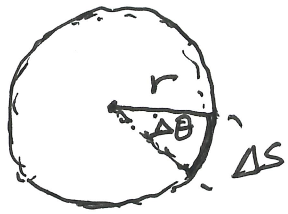
2024-11-11

Pendulum



key parameters
m, g, L.

radians



perimeter of a circle is $2\pi r$.

$\Delta\theta$ = angle
Δs = arc length.

$$\Delta s = r\,\Delta\theta$$

radius of the circle.

$$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

speed (velocity in the θ-direction)

angular velocity.

$$v_\theta = L\frac{d\theta}{dt}$$

For the pendulum $r = L$.

θ-direction momentum balance.
consider a time interval from t to $t+\Delta t$:

acc = in - out + gen.

$$m v_\theta(t+\Delta t) - m v_\theta(t) = -mg\sin\theta\cdot\Delta t$$

$$m\frac{dv_\theta}{dt} = -mg\sin\theta.$$

$$m\frac{d}{dt}\left(L\frac{d\theta}{dt}\right) = -mg\sin\theta$$

$$\frac{L}{g}\frac{d^2\theta}{dt^2} = -\sin\theta$$

Let $y=\theta$ & $x = t/\tau$,
where $\tau = \sqrt{L/g}$

$$\frac{d^2y}{dx^2} = -\sin y \quad \Longleftarrow \quad \frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt} = \frac{1}{\tau}\frac{d\theta}{dx}$$

$$\frac{d^2\theta}{dt^2} = \frac{1}{\tau}\left(\frac{d}{dx}\frac{d\theta}{dx}\right)\frac{dx}{dt} = \frac{1}{\tau^2}\frac{d^2\theta}{dx^2}$$

Dimensional analysis

pendulum: Let τ be the period of the pendulum.

$$\tau = f(m, L, g) \leftarrow$$ dimensional relation

$$\frac{\tau}{T} = F\left(\frac{m}{M}, \frac{L}{\ell}, \frac{gT^2}{\ell}\right) \leftarrow$$ dimensionless relation

~~the~~ We are free to choose the units we use. Let

$$M = m$$
$$\ell = L$$
$$\frac{gT^2}{\ell} = 1$$
$$T = \left(\frac{\ell}{g}\right)^{1/2} = \left(\frac{L}{g}\right)^{1/2}$$

ℓ = unit of length

T = unit of time

M = unit of mass.

$$\frac{\tau}{T} = F(1, 1, 1) \leftarrow$$ This is just a constant

$$\frac{\tau}{\sqrt{L/g}} = \text{const}$$

$$\tau = \text{const} \cdot \sqrt{\frac{L}{g}}$$

The period of a pendulum is independent of mass and proportional to $\sqrt{L/g}$.

Dimensional analysis: Guitar string

Key parameters: L = length

T = string tension (has units of force)

λ = linear mass density (has units of mass per length).

frequency of vibration

$$\omega = f(L, T, \lambda)$$

$$\omega\tau = F\left(\frac{L}{\ell}, \frac{\tau^2 T}{m\ell}, \frac{\ell\lambda}{m}\right)$$

$$\omega\tau = F(1, 1, 1) \leftarrow \text{constant.}$$

$$\omega L\sqrt{\frac{\lambda}{T}} = \text{const}$$

$$\omega \propto \frac{1}{L}\sqrt{\frac{T}{\lambda}}$$

ℓ = unit of length

m = unit of mass

τ = unit of time.

Let $\ell = L$

$m = \lambda L \leftarrow$ mass of string.

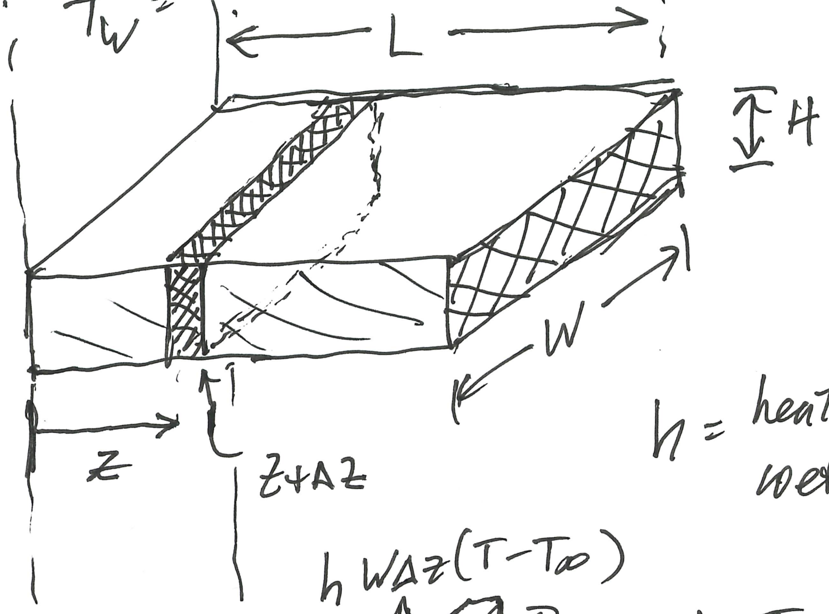
$$\frac{\tau^2 T}{m\ell} = 1$$

$$\tau^2 = \frac{m\ell}{T} = \frac{(\lambda L)L}{T}$$

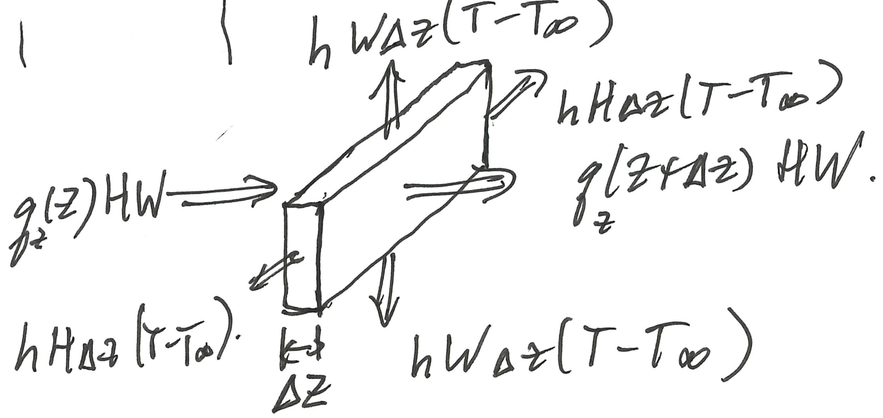
$$\tau = L\sqrt{\frac{\lambda}{T}}$$

Fins

T_w = wall temperature. T_∞ = temperature of surrounding fluid.



h = heat transfer coefficient.



Energy balance: We assume that we are at steady-state & no generation.

$$\underbrace{\text{acc}}_{0} = \text{in} - \text{out} + \underbrace{\text{gen}}_{0}$$

$t \to t + \Delta t$.

conduction

$$0 = q_z(z)\, WH\,\Delta t - q_z(z+\Delta z)\, WH\,\Delta t$$
$$-2h\,\Delta z\, W(T - T_\infty) - 2h\,\Delta z\, H(T - T_\infty)$$

convection.