Exercise Number: 2.7.5

Proposition. Suppose that $\Omega = \mathbb{N}$ is the set of positive integers and $\mathcal{F}$ is the of all subsets of A such that either A or A^c is finite, and $\mathbb{P}(A) = 0$ if A is finite, and $\mathbb{P}(A) = 1$ if A^C is finite. Then we have the following:

1. $\mathcal{F}$ is an algebra.
2. $\mathcal{F}$ is *not* a σ-algebra.
3. $\mathbb{P}$ is finitely additive.
4. $\mathbb{P}$ is *not* countably additive.

Proof.

1. Closure under complements is trivial. Consider any two sets $A, B \in \mathcal{F}$. If we prove $A \cap B \in \mathbb{F}$ then by induction and DeMorgan's laws the result follows. The only way $A \cap B \notin \mathbb{F}$ would be if both $A \cap B$ *and* $(A \cap B)^C = A^C \cup B^C$ are both infinite (obviously they cannot both be finite given Ω). Assume $A \cap B$ is infinite. Then at least one of A or B must be as well. If both are infinite, then $A^C \cup B^C$ is clearly finite and the result holds. Now assume WLOG that A is infinite, but B is finite. Then $A \cap B$ is clearly finite, and so consider $(A^C \cup B^C)$. Since B^C is infinite this is infinite. Done.

2. Define $A_n = \{2n\}$. Then clearly each $A_n \in \mathbb{F}$. However both $\cup_{n=1}^{\infty} A_n$ *and* $\left(\cup_{n=1}^{\infty} A_n\right)^C$ are clearly infinite.

3. Consider any two disjoint sets events A and B. We first prove a lemma: it is impossible for both A and B to both be infinite. Consider the case where they are. Then since $A^c \supseteq B$ it follows that A^C is infinite as well. Contradiction.

   Now consider all possible combinations of A and B having finite or infinite cardinality. If one has infinite cardinality, the other must be finite and so
   $$\mathbb{P}(A \cup B) = 1 \ \text{ and } \ \mathbb{P}(A) + \mathbb{P}(B) = 0 + 1 = 1.$$
   In the second case, if both are finite, then obviously $\mathbb{P}(A \cup B) = 0$ and $\mathbb{P}(A) + \mathbb{P}(B) = 0 + 0$. Done.

4. This follows immediately by consider the sequence of events $A_i = \{i\}$: the individual probabilities are all 0, but when taken together they are an infinite set

$\square$

Discussion.

The point of this exercise is that it lends a bit of intuition regarding the practical differences between an algebra and a σ-algebra via an example. If we attempt to define a probability triple by the cardinality of sets involving infinity, things tend to break down if we extend from finite unions to countable unions. In general the moral of the story is that it is important to consider behavior on countably infinite unions when extending from finite ones.