Exercise Number: 3.1.7

Proposition. Consider a sequence of random variables $\{Z_n\}$ over an arbitrary probability triple (standard notation) such that $\lim_{n\to\infty} Z_n(\omega)$ exists. Set this equal to a new function $Z : \Omega \to \mathbb{R}$, i.e.

$$\lim_{n\to\infty} Z_n(\omega) = Z(\omega) \ \ \forall \omega \in \Omega.$$

Then the following holds true:

$$\{Z \leq x\} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{ Z_k \leq x + \frac{1}{m} \right\}$$

Proof. We prove this by demonstrating these two propositions are saying exactly the same thing. In this sense, there really is no proof to be had: just deconstruction of the statements.

$\underline{\{Z \leq x\}}$: Writing this statement out more verbosely, we find it is equivalent to

$$\{\omega \in \Omega \mid \lim_{n\to\infty} Z_n(\omega) \leq x.\}$$

By definition, this is the set of all $\omega \in \Omega$ such that for all $\epsilon > 0$, there exists $N \in \mathbb{Z}$ such that for all $k \geq N$, $Z_k(\omega) \leq x + \epsilon$.

$\underline{\bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{ Z_k \leq x + \frac{1}{m} \right\}}$: We examine this statement from inside out. Consider first of all

$$\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{ Z_k \leq x + \frac{1}{m} \right\}.$$

As discussed later in the chapter, this is equivalent to

$$\liminf_n \left\{ Z_k \leq x + \frac{1}{m} \right\} = \left\{ Z_k \leq x + \frac{1}{m}, \ \ a.a. \right\}.$$

The "almost always" implies that, past a certain finite point in the sequence, the condition must hold true for all subsequent elements in the sequence. In otherwords, we may restate this set as: the set of all $\omega \in \Omega$ such that there exists $N \in \mathbb{Z}$ such that for all $k \geq N$, $Z_k(\omega) \leq x + \frac{1}{m}$. Now we peal back the last layer: the $\frac{1}{m}$. Taking the intersection over all $m \in \mathbb{N}$ means that, if an ω is to be included in the final set, it must satisfy the above condition over all m: that is, no matter how arbitrarily small $\frac{1}{m}$ becomes, the inequality must still be satisfied.

So now it is easy to see why the statements are equivalent: consider any $\omega \in \{Z \leq x\}$. Then since the rationals are dense in the reals, for any $\epsilon > 0$ there exists m s.t. $\epsilon < \frac{1}{m}$. The reverse obviously holds as well, so both sets are subsets of one another, meaning that they are equal.

$\square$