Exercise Number: 2.3.16

Proposition. The extension constructed in the proof of Theorem 2.3.1 must be complete, meaning that if $A \in \mathcal{M}$, $P^*(A) = 0$, then any $B \subset A \in \mathcal{M}$.

Proof. Consider any such A and B. We have by monotonicity that

$$P^*(E) \leq P^*(B \cap E) + P(B^c \cap E)$$

we just need to prove the reverse direction. Since we are given $A \in \mathcal{M}$ we have

$$P^*(E) = P^*(A \cap E) + P(A^c \cap E) = P^*(A^c \cap E) \leq P^*(B^c \cap E) \leq P^*(E).$$

But because $P^*(A \cap E) = P^*(B \cap E) = 0$ (by monotonicity 0 is the min value of this function), the proof is complete. $\square$