Exercise Number: 4.5.8

Proposition. Let $f(x) = ax^2 + bx + c$ be a second-degree polynomial function (where $a, b, c \in \mathbb{R}$ are constants).

1. $b = c = 0$ are necessary and sufficient conditions such that, for all $\alpha \in \mathbb{R}$ and all random variables X, $\mathbb{E}[f(\alpha X)] = \alpha^2 \mathbb{E}[f(X)]$.
2. $a = b = 0$ are necessary and sufficient conditions such that, for all $\alpha \in \mathbb{R}$ and all random variables X, $\mathbb{E}[f(X - \beta)] = \mathbb{E}[f(X)]$.
3. Not sure what this question is asking...

Proof. These conclusions follow directly from linearity of expecation.

1. $$\mathbb{E}[f(\alpha X)] = \mathbb{E}[a * (\alpha X)^2 + b\alpha X + c]$$

   $$= \alpha^2 \mathbb{E}[aX^2] + \alpha \mathbb{E}[bX] + c.$$

   From the above it is clear that both b and c must be 0 for the desired condition to hold.

2. $$\mathbb{E}[f(X - \beta)] = \mathbb{E}[a * (X - \beta)^2 + b(X - \beta) + c]$$

   $$= a * \mathbb{E}[(X^2 - 2\beta X + \beta^2)] + b * \mathbb{E}[(X - \beta)] + c$$

   Which clearly only equals the original expectation, in general, if $a = b = 0$, as that is the only way to disentangle the X and β terms.

□