Exercise Number: 4.5.1

Proposition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the Lebesgue measure on $[0, 1]$ and set

$$X(\omega) = \begin{cases} 1, & 0 \leq \omega \leq 1/4 \\ 2\omega^2, & \frac{1}{4} \leq \omega \leq 3/4 \\ \omega^2, & \frac{3}{4} \leq \omega \leq 1 \end{cases} .$$

Then

1. $\mathbb{P}(X \in A) = \frac{1}{4} + \sqrt{\frac{1}{2}}$ if $A = [0, 1]$.
2. $\mathbb{P}(X \in A) = \frac{3}{4}$ if $A = [\frac{1}{2}, 1]$.

Proof. This is just a matter of bookkeeping. Since the Lebesgue measure of intervals is just their length, our job is simple in this case. We calculate the cases in the order they were presented.

1. $\mathbb{P}(X \in A) = X^{-1}([0, 1]) = 1 - X^{-1}(\mathbb{R} \setminus [0, 1])$. Looking at the definition of X quickly revels that the only portion that maps to outside 1, maps above it: for $2\omega^2$. Since it is monotonically increasing just set

   $$2\omega^2 = 1 \quad \Rightarrow \quad \omega = \sqrt{\frac{1}{2}}.$$

   Thus

   $$\mathbb{P}(X \in A) = 1 - (\frac{3}{4} - \sqrt{\frac{1}{2}}) = \frac{1}{4} + \sqrt{\frac{1}{2}}.$$

2. In the second case we choose to calculate directly rather than via complement for convenience.

   $$\mathbb{P}(X \in A) = X^{-1}([\frac{1}{2}, 1]) = (\frac{1}{4} - 0) + (\frac{3}{4} - \frac{1}{2}) + (1 - \frac{3}{4}) = 3 * \frac{1}{4} = \frac{3}{4}.$$

□