Exercise Number: 4.5.3

Proposition. Let X be a random variable with finite mean, and let $a \in \mathbb{R}$ be any real number. Then $\mathbb{E}(\max(X, a)) \geq \max(\mathbb{E}(X), a)$.

Proof. First consider the case $\mathbb{E}(X) \geq a$. Then evidently $\max(\mathbb{E}(X), a) = \mathbb{E}(X)$. Thus we want to show that

$$\mathbb{E}(\max(X, a)) \geq \mathbb{E}(X) \geq a.$$

Now we exploit the order preservation property of expectation (result of exercise 4.3.2). Define a random variable

$$Z(\omega) = \begin{cases} a, & \text{if } X(\omega) < a \\ X(\omega), & \text{otherwise.} \end{cases}$$

Then note that $Z \geq X$ and therefore $\mathbb{E}(Z) > \mathbb{E}(X)$. But $Z = \max(X, a)$, completing the first portion of the proof.

Proving the second case, $\mathbb{E}(X) < a$, follows a similar but reversed line of argument.

$\square$