Exercise Number: 2.7.10

Proposition. Letting $\Omega = \mathbb{R}$, define

$$\mathcal{J} = \Big\{(-\infty, x] \mid x \in \mathbb{R}\Big\} \cup \Big\{(y, \infty) \mid y \in \mathbb{R}\Big\} \cup \Big\{(x, y] \mid x, y \in \mathbb{R}\Big\} \cup \Big\{\emptyset, \mathbb{R}\Big\}$$

where the union is taken over "collections" of sets (not a union in the strict set-theoretic sense, I believe, though I need to learn more rigorous set theory to determine further).

Then this may be a

Proof. Recall the definition of a semi-algebra. $\mathcal{J}$ is a semi-algebra over an event space Ω if:

1. Ω, $\emptyset \in \mathcal{J}$
2. $A, B \in \mathcal{J} \Rightarrow A \cap B \in \mathcal{J}$ (i.e. $\mathcal{J}$ is closed under finite intersection
3. For any $A \in \mathcal{J}$, A^C is a union of disjoint sets in $\mathcal{J}$

As a minor digression, note what this implies about unions in $\mathcal{J}$: consider $A, B \in \mathcal{J}$. Then by DeMorgan's Laws:

$$A \cap B = (A^C \cap B^C)^C.$$

Since A^C, B^C are disjoint unions of sets in $\mathcal{J}$, and $\mathcal{J}$ is closed under intersection, we have that $A^C \cap B^C$ is a disjoint union of sets of $\mathcal{J}$. Then taking the complement of this entire hodgepodge of sets must again yield a disjoint union in the gaps of the sets (although further investigation may be necessary...).

Anyways, we verify the necessary conditions in the order presented:

1. Follows directly by the definition.
2. Order the sets in the order presented in the union. If the sets in question are disjoint, then they trivially yield a set of type 4. So assume not. Taking any intersection of types 1 and 2 yields a set of type 3. Taking any intersection of type 1 and 3 yields a set of type 3 again. Taking the intersection of sets 2 and 3 yields again a set of type 3. Finally if we take the intersection of any set of type 4 with any other set, we either again get the empty set or the other set stays the same (if $\mathbb{R}$ is used).
3. Complement of set of type 1 is type 2, and vice versa. Complement of a set of type 3 is just a disjoint union of a set of type 1 and a set of type 2. Collections 4 are closed under complements.

$\square$

Discussion.

Not much here, fairly intuitive. In fact this very semi-algebra, the set of intervals in $\mathbb{R}$, was the intuition originally provided to motivate the creation of a σ-algebra via the extension theorem. Just a lot of book-keeping required.

Note exactly what the collection $\mathcal{J}$ is defined to be here exactly: it is an amalgamation of infinite intervals bounded from above by a hard number (x, note this type of interval is key for the definition of cumulative distribution and the development of more applied theory later on), the complement of these types of intervals, *half open intervals* that are open below and closed above, and then Ω and the empty set. This is basically the simplest type of semi-algebra that can contain the sets useful for cumulative distribution computation. Not by concidence, this semi-algebra is therefore used to derive the equivalence of measures on *all* borel sets when they are equivalent on all intervals of the type 1 (i.e. when their cumulative distributions for any random variable defined as a continous function $\mathbb{R} \to \mathbb{R}$ are the same, their distributions must be the same).