Exercise Number: 2.4.3

Proposition. Define

$$\mathcal{J} = \{\text{all intervals contained in } [0,1]\}.$$

Let $\mathbb{P}$ be defined as the "length" of I. Then $\mathbb{P}$ as defined here is countably monotonic: that is to say, for any set $A \in \mathcal{J}$ and any countable collection of sets $\{A_i\}$ that are all subsets of $\mathcal{J}$ and $A \subseteq \cup_{\forall i} A_i$, we have

$$P(A) \leq \sum_{\forall i} \mathbb{P}(A_i).$$

Proof. We demonstrate this in several steps, as outlined by the text. Let I have endpoints $a < b$.

1. First we demonstrate that if $I_1, I_2, ..., I_n$ is a finite collection of intervals, and if $\cup_{j=1}^{n} I_j \supseteq I$ for some interval I, then $\sum_{j=1}^{n} \mathbb{P}(I_j) \geq \mathbb{P}(I)$. Since $\cup_{j=1}^{n} I_j \supseteq I$, there must exist a subset of intervals I_{α_j} with $a_{\alpha_j} < b_{\alpha_j}$ being the endpoints, indexed by an index set $\alpha_j \in S$, cardinality k, such that

   $$a_{\alpha_1} \leq a \leq a_{\alpha_2} \leq b_{\alpha_1} < b_{\alpha_2} \cdots < b \leq b_{\alpha_k}.$$

   In other words, this subset of intervals encapsulates I. Then evidently,

   $$\mathbb{P}(I) \leq \sum_{j=1}^{k} \mathbb{P}(I_{\alpha_j}) \leq \sum_{j=1}^{n} \mathbb{P}(I_j).$$

2.

3.