Exercise Number: 4.5.7

Proposition. *(Principle of inclusion-exclusion, general case)* Let A_1, A_2, ..., $A_n \in \mathcal{F}$. Then the general principle of inclusion-exclusion holds, i.e.:

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1 \le i < j \le n} \mathbb{P}(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots \pm \mathbb{P}(A_1 \cap \cdots \cap A_n)$$

Proof. To prove the general inclusion-exclusion formula efficiently, we exploit a convenient trick. We write $P(A_1 \cup \cdots \cup A_n)$ in a convenient form (expectation combined with indicator variables), and then expand it into two different forms, applying the expectation differently. A trivial result of expectation/indicator vairables is the following: $\mathbb{E}(1_A) = \mathbb{P}(A)$. Thus, we have

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = \mathbb{E}(1_{A_1 \cup \cdots \cup A_n}) = \mathbb{E}(1 - 1_{(A_1 \cup \cdots \cup A_n)^c}) = \mathbb{E}(1 - 1_{A_1^c \cap \cdots \cap A_n^c}) = \mathbb{E}\big(1 - \Pi_{i=1}^{n}(1 - 1_{A_i})\big).$$

Now focusing on $1 - \Pi_{i=1}^{n}(1 - 1_{A_i})$, we may rewrite it as:

$$1 - \Pi_{i=1}^{n}(1 - 1_{A_i}) = (1 - 1_{A_1})(1 - 1_{A_2})(1 - 1_{A_3}) \cdots (1 - 1_{A_n}).$$

But notice that all the terms of the sum (when all the components of the product are multiplied out) is essentially just

$$\left(\sum_{\forall S \in \mathcal{P}(\{1, \cdots, n\})} (-1)^{|S|} \Big(\Pi_{\forall s \in S} 1_{A_s} \Big) \right).$$

By linearity of expectation, we may write

$$\mathbb{E}\left(\sum_{\forall S \in \mathcal{P}(\{1, \cdots, n\})} (-1)^{|S|} \Big(\Pi_{\forall s \in S} 1_{A_s} \Big) \right) = \sum_{\forall S \in \mathcal{P}(\{1, \cdots, n\})} \mathbb{E}\Big((-1)^{|S|} \Pi_{\forall s \in S} 1_{A_s} \Big)$$

$$= \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1 \le i < j \le n} \mathbb{P}(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots \pm \mathbb{P}(A_1 \cap \cdots \cap A_n)$$