Exercise Number: 3.6.4

Proposition. Suppose $\{A_n\} \nearrow A$. Let $f\colon \Omega \to \mathbb{R}$ be an arbitrary function. Then $\lim_{n\to\infty} \inf_{w\in A_n} f(\omega) = \inf_{\omega\in A} f(\omega)$.

Proof. Consider any $\epsilon > 0$ sufficiently small (explained later, we needn't concern ourselves with larger epsilon by the nature of limits). We must now find an $N \in \mathbb{N}$ such that for all $k \geq N$,

$$|\inf_{w\in A_k} f(\omega) - \inf_{\omega\in A} f(\omega)| < \epsilon.$$

Consider the set $B = \{\omega \in A \mid f(\omega) \geq \inf_{\omega\in A} f(\omega) + \frac{\epsilon}{2}\}$. We consider ϵ sufficiently small so that this set is non-empty: such an epsilon must exist by the definition of infimum. Furthermore, since $\cup_n A_n = A$ and the A_n are increasing in size, by assumption, there must exist some index N such that $B \subseteq A_N$.

$\square$

Discussion. The point here is that this applies to any function mapping an arbitrary sample space Ω to the real numbers, and so applies to random variables as well. In other words, the limit of the infimum of a random variable over a sequence of increasing sets is just the infimum of that random variable over the limit of the sets.