Exercise Number: 4.5.2

Proposition. Let X be a random variable with finite mean, and let $a \in \mathbb{R}$ be any real number. Then $\mathbb{E}(\max(X, a)) \geq \max(\mathbb{E}(X), a)$.

Proof. Recall the definition of expected value for a general random variable:

$$\mathbb{E}(X) = \mathbb{E}(X^+) - \mathbb{E}(X^-)$$

where $X^+ = \max(X, 0)$ and $X^- = \max(X, 0)$. The latter expectation function employed, defined for random, non-negative variables (say Y in this case) is

$$\mathbb{E}(Y) = \sup\{\mathbb{E}(Y) \mid \text{all simple random variables } Y \leq X\}.$$

And the expectation function employed in the final case (of simple random variables mapping to n distinct numbers) amounts to:

$$\mathbb{E}(Y) = \sum_{i=1} y_i 1_{A_i}$$

where $A_i \in \mathbb{F}$ is the set such that $Y^{-1}(y_i) = A_i$.

Actually, this problem is simply a simple matter of logic. First consider the case $\mathbb{E}(X) \geq a$. Then evidently $\max(\mathbb{E}(X), a) = \mathbb{E}(X)$. Thus we want to show that

$$\mathbb{E}(\max(X, a)) \geq \mathbb{E}(X) \geq a.$$

Now we exploit the order preservation property of expectation (result of exercise 4.3.2). Define a random variable

$$Z(\omega) = \begin{cases} a, & \text{if } X(\omega) < a \\ X(\omega), & \text{otherwise.} \end{cases}$$

Then note that $Z \geq X$ and therefore $E(\mathbb{Z}) > E(\mathbb{X})$. But $Z = \max(X, a)$, completing the first portion of the proof.

Proving the second case, $\mathbb{E}(X) < a$, follows a similar line of argument

$\square$