Exercise Number: 1.3.2

Proposition. Suppose $\Omega = \{1, 2, 3\}$ and $\mathcal{F}$ is the collection of all subsets of Ω. The following are the necessary and sufficient conditions on the real numbers x, y, and z such that there exists a countably additive probability measure $\mathbb{P}$ on $\mathcal{F}$, with $x = \mathbb{P}\{1, 2\}$, $y = \mathbb{P}\{2, 3\}$, $z = \mathbb{P}\{1, 3\}$:

1. $x + y + z = 2$
2. Triangle inequality on x, y, z must hold true.
3. x, y, z must all be non-negative.

Proof. We first prove that individually all these conditions are necessary, and then that they are sufficient.

Necessary:

1. Since $\mathbb{P}$ is countably additive, we have that $\mathbb{P}\{1, 2\} = \mathbb{P}\{1\} + \mathbb{P}\{2\} = x$, etc... for each x, y, and z. Summing all of these out yields this fact simply since $\mathbb{P}(\Omega) = 1$.
2. This holds trivially.
3. Obviously since P is a probability measure all of these values must be greater than 0.
4. This holds trivially since all are sums of

Sufficient:
Now we assume the conditions to be true and derive that $\mathbb{P}$ is a countably additive probability measure from them, given the conditions on x, y, and z.