Exercise Number: 4.5.3

Proposition. Let X and Y be two general random variables with finite means, and let $Z = X + Y$.

1. $Z^+ - Z^- = X^+ - X^- + Y^+ - Y^-$.
2. $\mathbb{E}(Z) = \mathbb{E}(X) + \mathbb{E}(Y)$
3. The general definition of expectation is finitely linear, for general random variables with finite means.

Proof. Proven in the order presented.

1. Follows by definition of expectation.
2. We may re-arrange the first relation to write
$$Z^+ + X^- + Y^- = X^+ + Y^+ + Z^+.$$
Since these are positive random variables with positive coefficients, we may apply the result of equation 4.2.6 of the chapter to write
$$\mathbb{E}(Z^+) + \mathbb{E}(X^-) + \mathbb{E}(Y^-) = \mathbb{E}(X^+) + \mathbb{E}(Y^+) + \mathbb{E}(Z^+)$$
which through trivial re-arranging gives the desired result.
3. From (2) it is clear as to why this general definition of expectation is finitely linear. Consider any arbitrary linear combination of random variables X_i with scalars a_i: $Z = \sum_{i=1}^{n} a_i X_i$. If any of the scalars are negative, we may simply switch them to the other side of the equation and apply the same logic as in (2): all that is needed is positive scalar multiples.

$\square$