Exercise Number: 3.2.2

Proposition. Consider a possibly infinite collection $\{A_\alpha\}_{\alpha \in I}$ of events. Suppose they are independent: i.e. for each $j \in \mathbb{N}$ and each distinct finite choice $\alpha_1, ..., \alpha_j$ the following holds:

$$\mathbb{P}(A_{\alpha_1} \cap A_{\alpha_2} \cap \cdots \cap A_{\alpha_j}) = \mathbb{P}(A_{\alpha_1})\mathbb{P}(A_{\alpha_2}) \cdots \mathbb{P}(A_{\alpha_j}).$$

Then if any arbitrary A_{α_i} is replaced by $A_{\alpha_i}^C$, the independence property still holds. Logically this implies any arbitrary number of the events may be replaced by their complements, and independence will still hold.

Proof. WLOG let $i = 1$ in the proposition, and let $B = A_{\alpha_2} \cap \cdots \cap A_{\alpha_j}$. Then by countable additivity of our probability measure the following must hold:

$$\mathbb{P}(A_{\alpha_1} \cap B) + \mathbb{P}(A_{\alpha_1}^C \cap B) = \mathbb{P}(B).$$

By the proposition statement this then implies

$$\mathbb{P}(A_{\alpha_1}^C \cap B) = \mathbb{P}(B) - \mathbb{P}(A_{\alpha_1} \cap B) = \mathbb{P}(A_{\alpha_2}) \cdots \mathbb{P}(A_{\alpha_j}) - \mathbb{P}(A_{\alpha_1})\mathbb{P}(A_{\alpha_2}) \cdots \mathbb{P}(A_{\alpha_j}) =$$

$$\mathbb{P}(A_{\alpha_2}) \cdots \mathbb{P}(A_{\alpha_j}) * (1 - \mathbb{P}(A_{\alpha_1})) = \mathbb{P}(A_{\alpha_1}^C)\mathbb{P}(A_{\alpha_2}) \cdots \mathbb{P}(A_{\alpha_j})$$

as desired.

$\square$