Exercise Number: 3.6.6

Proposition. Let X, Y, and Z be three independent random variables, and set $W = X + Y$. Let

$$B_{k,n} = \{(n-1)2^{-k} \leq X < n2^{-k}\}$$

$$C_{k,m} = \{(m-1)2^{-k} \leq Y < m2^{-k}\}$$

and let

$$A_k = \bigcup_{\substack{n,m\in\mathbb{Z} \\ (n+m)2^{-k}<x}} (B_{k,n} \cap C_{k,m}).$$

Finally, let $A = \{X + Y < x\}$ and $D = \{Z < z\}$. Then the following hold:

(a) $\{A_k\} \nearrow A$

(b) A_k and D are independent

(c) A and D are independent

(d) W and Z are independent.

Proof.

(a) First we demonstrate that A_k is increasing. Take any arbitrary $k \in \mathbb{Z}$. Then we must show $A_k \subseteq A_{k+1}$. Consider any $\omega \in A_k$. Then for some $n, m \in \mathbb{Z}$, $\omega \in B_{k,n} \cap C_{k,m}$, where $(n+m)2^{-k} < x.$, and so

$$\{(n-1)2^{-k} \leq X(\omega) < n2^{-k}\} \quad \text{and} \quad \{(m-1)2^{-k} \leq Y(\omega) < m2^{-k}\}.$$

Writing these in terms of $2^{-(k+1)}$ yields

$$\{2(n-1)2^{-k-1} \leq X(\omega) < (2n)2^{-k-1}\} \quad \text{and} \quad \{2(m-1)2^{-k-1} \leq Y(\omega) < (2m)2^{-k-1}\}$$

which immediately implies ω lies in some intersection of $B_{k+1,2n-1}$ or $B_{k+1,2n}$, and then $C_{k+1,2m-1}$ or $C_{k+1,2m}$ (choosing one of the B's and one of the C's). We need only demonstrate now that $(2n + 2m)2^{-k-1} < x$ (the worst case scenario), which follows immediately since $(n+m)2^{-k}$ is true by assumption.

Secondly we demonstrate that $\cup A_k = A$. The fact that $\cup A_k \subseteq A$ follows immediately, so we focus on the reverse. Consider any $\gamma \in \{\omega \mid (X+Y)(\omega) < x\}$. Let $a = X(\gamma)$ and $b = Y(\gamma)$ and $a + b = c < x$. Since $2^{-k} \to 0$ as $k \to \infty$, there must exist a p such that $2^{-p} < x - c$, which implies there exists a $z \in \mathbb{Z}$ such that $c < z2^{-p} < x$. Thus

$$A_z = \bigcup_{\substack{n,m\in\mathbb{Z} \\ (n+m)2^{-z}<x}} (B_{z,n} \cap C_{z,m})$$

and, since the $B_{z,n}$ and $C_{z,m}$ each separately partition the set $[0, 2^z)$, there is some n and m satisfying these conditions such that $(n-1)2^{-z} \leq a < n2^{-z}$ and $(m-1)2^{-z} \leq b < m2^{-z}$, and so we finally have $\gamma \in A_z$.

(b) Consider $\mathbb{P}(A_k \cap D)$. By definition this equals

$$\mathbb{P}\Bigg[D \cap \Bigg(\bigcup_{\substack{n,m\in\mathbb{Z} \\ (n+m)2^{-k}<x}} (B_{k,n} \cap C_{k,m})\Bigg)\Bigg].$$

By DeMorgan's Laws and/or basic set theory, we may rewrite as

$$\mathbb{P}\Bigg[\bigcup_{\substack{n,m\in\mathbb{Z} \\ (n+m)2^{-k}<x}} (D \cap B_{k,n} \cap C_{k,m})\Bigg].$$

Now note that each of the $(D \cap B_{k,n} \cap C_{k,m})$ are disjoint from another, because they each are subsets of respective partitions of Ω (the partitions being the B's and the C's, partitioned by how X and Y respectively map to the reals) and so we may apply countable subadditivity of your measure to yield

$$\mathbb{P}\Bigg[\bigcup_{\substack{n,m\in\mathbb{Z} \\ (n+m)2^{-k}<x}} (D \cap B_{k,n} \cap C_{k,m})\Bigg] = \sum_{\substack{n,m\in\mathbb{Z} \\ (n+m)2^{-k}<x}} \mathbb{P}(D \cap B_{k,n} \cap C_{k,m})$$

which, since D, $B_{k,n}$, $C_{k,m}$ are all Borel and therefore independent (by virtue of being defined by X, Y, and Z, respectively and exclusively), is equivalent to

$$\sum_{\substack{n,m\in\mathbb{Z}\\(n+m)2^{-k}<x}} \mathbb{P}(D)\mathbb{P}(B_{k,n}\cap C_{k,m}) = \mathbb{P}(D)\mathbb{P}\Big[\bigcup_{\substack{n,m\in\mathbb{Z}\\(n+m)2^{-k}<x}} B_{k,n}\cap C_{k,m}\Big] = \mathbb{P}(D)\mathbb{P}(A_k).$$

(c) By (a), (b), and continuity of probabilities (monotonically increasing/decreasing sequences of events),

$$\mathbb{P}(A\cap D) = \mathbb{P}(\lim_{k\to\infty} A_k \cap D) = \lim_{k\to\infty} \mathbb{P}(A_k)\mathbb{D} = \mathbb{P}(A)\mathbb{P}(D).$$

(d) By definition, W and Z are independent if for all borel sets S_1, S_2, $W^{-1}(S_1)$ and $Z^{-1}(S_2)$ are independent events. By Chap 2 we know the sets $(-\infty, x]$ generate all Borel sets, so setting $A_x = \{X + Y < x\}$ and $D_z = \{Z < z\}$ and remembering that inverse images preserve all set operations under which σ-algebras are closed, there must exist countable collections of A_x and D_z generating $W^{-1}(S_1)$ and $Z^{-1}(S_2)$, respectively. By Lemma 3.5.2 these events are still independent since they are composed on independent events, completing the proof.

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