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Six ungraded HW problems

1) From HW 1: #22

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} f(x) = 3x^2$, $0 \leq x \leq 1$

$\hookrightarrow F(x) = x^3$

We Know: ☆☆☆

$\hookrightarrow 1 \leq k \leq n$

$\hookrightarrow g_{(k)}(y) = \frac{n!}{(k-1)!\cdot(n-k)!} \cdot [F(y)]^{k-1} \cdot [1-F(y)]^{n-k} \cdot f(y)$

a) Find pdf for X_{min}:

$1 \leq 1 \leq n$

$$X_{min} = \frac{n!}{(n-1)!} \cdot [x^3]^0 \cdot [1-x^3]^{n-1} \cdot 3x^2 = \frac{n!}{(n-1)!} \cdot [1-x^3]^{n-1} \cdot 3x^2$$

$$= n \cdot [1-x^3]^{n-1} \cdot 3x^2$$

$$\boxed{X_{min} = n \cdot 3x^2[1-x^3]^{n-1}, \quad 0 \leq x \leq 1.}$$

b) Find pdf for X_{max}:

$1 \leq n \leq n$

$$X_{max} = \frac{n!}{(n-n)!\cdot(n-1)!} \cdot [x^3]^{n-1} \cdot [1-x^3]^{n-n} \cdot 3x^2$$

$0! = 1$

and, $\frac{n!}{(n-1)!} = n$

$$= n \cdot [x^3]^{n-1} \cdot 3x^2$$

$$\boxed{X_{max} = n \cdot 3x^2[x^3]^{n-1}, \quad 0 \leq x \leq 1}$$

c) If $n=10$, find the probability that the largest value, X_{max}, is greater than 0.92?

$$P(X_{max} > 0.92) = 1 - \int_0^{0.92} 10 \cdot 3x^2[x^3]^9 \cdot dx = 1 - \int_0^{0.92} 30x^{29} \cdot dx = 0.91803$$

Can be a beta(30,1) bcz...

$$30 \cdot x^{30-1} \cdot (1-x)^{1-1} \Rightarrow 30x^{29}$$

So, $P(X_{max} > 0.92) = 1 - pbeta(30,1) = 0.918$

$$\boxed{P(X_{max} > 0.92) = 0.918}$$

In conclusion, the probability that the largest value, X_{max}, is greater than 0.92 is 0.918.

2) From Hw 2 : #19

Let $X_1 = x_1,\ X_2 = x_2, \dots, X_n = x_n$, for $f(x|\theta) = \theta \cdot 2^{\theta}/x^{\theta+1}$, $x \geq 2$, $\theta > 1$

a) Use the method of moments to estimate θ.

Theoretical | Sample Moments

$m_1 = \int_2^{\infty} x \cdot f(x|\theta)\,dx = \frac{2\theta}{\theta-1}$ $\qquad$ $\hat{m}_1 = \bar{x}$

$$\theta-1\left(\frac{2\theta}{\cancel{\theta-1}} = \bar{x}\right)\theta-1$$

$$2\theta = \bar{x}(\theta-1)$$

$$2\theta = \bar{x}\theta - \bar{x}$$

$$2\theta - \bar{x}\theta = -\bar{x} \Rightarrow \frac{\theta\cancel{(2-\bar{x})} = -\bar{x}}{2-\bar{x}} \Rightarrow \boxed{\hat{\theta}_{mom} = \frac{-\bar{x}}{2-\bar{x}}}$$

b) Use maximum likelihood to estimate θ.

$$L(\theta|x_1, x_2, \dots, x_n) = \theta \cdot \frac{2^{\theta}}{x_1^{\theta+1}} * \dots * \theta \cdot \frac{2^{\theta}}{x_n^{\theta+1}}$$

$$= \theta^n \cdot \frac{2^{\theta \cdot n}}{(x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{n(\theta+1)}} = \frac{(\theta \cdot 2^{\theta})^n}{(x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{n(\theta+1)}}$$

$$\ln(L) = \ln(\theta^n \cdot 2^{n \cdot \theta}) - \ln\left((x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{n(\theta+1)}\right)$$

$$= n \cdot \ln(\theta) + (n \cdot \theta)\ln(2) - n \cdot (\theta+1)\ln(x_1 \cdot \ldots \cdot x_n)$$

$$\frac{d}{d\theta}\ln(L) = \frac{n}{\theta} + n \cdot \ln(2) - n \cdot \ln(\Pi x_i)$$

$$0 = \frac{n}{\theta} + n \cdot \ln(2) - n \cdot \ln(\Pi x_i)$$

$$\cancel{n}\left(\frac{\theta}{\cancel{n}}\right) = \frac{1}{\cancel{n} \cdot \ln(2) - \cancel{n} \cdot \ln(\Pi x_i)}\cancel{-n} \Rightarrow \boxed{\hat{\theta}_{MLE} = \frac{1}{-\ln(2) + \ln(\Pi x_i)}}$$

4) from Hw 7: #16

3)
from Hw 5: # 34

Let $X \sim \text{Gamma}(2, \lambda)$ w/ $(2 \cdot \lambda \cdot X) \sim \chi^2_4$

Find 95% CI for λ...

$$0.95 = P\left(\chi^2_{0.025} < 2 \cdot \lambda \cdot X < \chi^2_{0.975} \right)$$

$$= P\left(\frac{\chi^2_{0.025}}{2 \cdot X} < \lambda < \frac{\chi^2_{0.975}}{2 \cdot X} \right)$$

In StatKey:
- ↳ using chi. squ. dist
  - ↳ two-tail w/ 0.95 in center
  - ↳ df = 4

$\chi^2_{.025} = 0.484$; $\chi^2_{.975} = 11.143$

$$0.95 = P\left(\frac{0.484}{2 \cdot X} < \lambda < \frac{11.143}{2 \cdot X} \right) \Rightarrow \boxed{\left(\frac{0.484}{2 \cdot X}, \frac{11.143}{2 \cdot X} \right)}$$

4) From Hw 7: #16

Let $x_1, x_2, \ldots, x_n$ be a random sample for Poisson

pdf: $f(x) = \dfrac{\theta^x \cdot e^{-\theta}}{x!}$, w/ $x = 0, 1, 2, 3, \ldots$

a) Write down likelihood function $f(\theta)$.

$$L(\theta \mid x_1, x_2, \ldots, x_n) = \frac{e^{-\theta} \cdot \theta^{x_1}}{x_1!} * \ldots * \frac{e^{-\theta} \cdot \theta^{x_n}}{x_n!}$$

$$\boxed{= \frac{e^{-n \cdot \theta} \cdot \theta^{\Sigma x_i}}{\prod x_i!}}$$

b) Suppose the prior for θ is the gamma dist. w/ parameters r, λ.
Find the posterior dist./density...

$$\underbrace{g(\theta \mid x_{obs})}_{} \propto \overbrace{g(\theta)}^{\text{prior dist}} * \overbrace{f(x_{obs} \mid \theta)}^{L(\theta)}$$

$$g(\theta \mid x) \propto \left(\theta^{r-1} \cdot e^{-\lambda\theta}\right) \cdot \left(e^{-\theta n} \cdot \theta^{\Sigma x_i}\right)$$

$$\boxed{g(\theta \mid x) \propto \theta^{r + \Sigma x_i - 1} \cdot e^{-\lambda\theta - \theta n}} \rightarrow -\lambda\theta - \theta n = -\theta(\lambda + n)$$

c) Identify distribution of posterior...
where $\alpha = r$

$$\boxed{\theta \mid x \sim \text{Gamma}(r + \Sigma x_i, \lambda + n)}$$

d) Suppose we have the values, [6, 7, 9, 9, 16]
Find posterior given that... $r = 15$ and $\lambda = 3$

$\alpha = r + \Sigma x_i \rightarrow = 47$	$\lambda = \lambda + n \rightarrow = 5$
$= 15 + (47)$	$= 3 + 5$
$\alpha = 62$	$\lambda = 8$

$$\boxed{\theta \mid x \sim \text{Gamma}(62, 8)}$$

e) Find 95% credible interval...

In R: qgamma(c(0.025, 0.975), 62, 8)

(5.942, 9.795)

We are 95% confident that w/ the posterior distribution, theta (θ) is between 5.942 and 9.795.

5) From HW 9: #16

Assume lengths of snakes are normally distributed w/ an unknown mean, but sd of $\sigma = 4$

$H_0: \mu = 25$ vs. $H_a: \mu > 25$ at $\alpha = 0.05$ and $n = 30$

What's power of test, if $\mu = 27$?

<u>we know</u>

$\mu_a = 27$
$\mu_0 = 25$
$s = 4$
$n = 30$

Power $= 1 - \beta$

☆ In Stat key ☆
↳ use normal dist. w/ $\mu = 25$ and $\sigma = 4/\sqrt{30}$
↳ and we get $\bar{x}$ (cutoff) which is 26.201 at $\alpha = 0.05$
so, $\bar{x} \leq 26.201$

$\beta = P(\text{Type II error})$

$= P\left(\begin{matrix}\text{Don't reject } H_0 \\ \text{not in RR}\end{matrix} \middle| \mu_a = 27\right)$

$= P(\bar{X} \leq 26.201 \mid \mu_a = 27)$

$= \text{Pnorm}(26.201, 27, 4/\text{sqrt}(30))$

$\beta = 0.136961$

Power $= 1 - \beta$
$= 1 - 0.136961$
$= 0.863040$

$\boxed{\text{Power} = 0.863}$

In conclusion, we have a 86.3% chance of rejecting the null hypothesis if we are testing 27 at the μ (mean).

6) From HW 10: #35

Let $X_1, X_2, \ldots, X_n$. Derive most powerful test/LRT for...

$$H_0: p = p_0 \quad \text{vs} \quad H_a: p_0 < p_a$$

pdf: Binomial = Bernoulli

$$f(x) = \binom{n}{x} \cdot p^x (1-p)^{n-x}$$

$$ts = \frac{L(\Omega_0)}{L(\Omega_a)} \qquad L(p \mid x_1, x_2, \ldots, x_n) \propto p^{\Sigma x_i} \cdot (1-p)^{n^2 - \Sigma x_i}$$

Likelihood func

n^2 bcz $(n-x)$ n-times

Find test stat

$$L(p_0) = p_0^{\Sigma x_i} \cdot (1-p_0)^{n^2 - \Sigma x_i}$$

$$L(p_a) = p_a^{\Sigma x_i} \cdot (1-p_a)^{n^2 - \Sigma x_i}$$

$$ts = \frac{p_0^{\Sigma x_i} \cdot (1-p_0)^{n^2 - \Sigma x_i}}{p_a^{\Sigma x_i} \cdot (1-p_a)^{n^2 - \Sigma x_i}} \Rightarrow RR = \{ts < k\}$$

Solve

$$0.05 = P\left(\left(\frac{p_0}{p_a}\right)^{\Sigma x_i} \cdot \left(\frac{(1-p_0)}{(1-p_a)}\right)^{n^2 - \Sigma x_i} < k \;\middle|\; p = p_0\right)$$

$$= P\left(\left(\frac{p_0}{p_a}\right)^{\Sigma x_i} \cdot \left(\frac{(1-p_0)}{(1-p_a)}\right)^{-\Sigma x_i} < \underbrace{k \Big/ \left(\frac{1-p_0}{1-p_a}\right)^{n^2}}_{k_2} \;\middle|\; p = p_0\right)$$

$$= P\left(\left(\frac{p_0}{p_a} \cdot \frac{1-p_0}{1-p_a}\right)^{\Sigma x_i} < k_2 \;\middle|\; p = p_0\right)$$

$$= P\left(\Sigma x_i \cdot \ln\left(\frac{p_0}{p_a} \cdot \frac{1-p_0}{1-p_a}\right) < k_2 \;\middle|\; p = p_0\right)$$

$$= P\left(\Sigma x_i < \underbrace{k_2 \Big/ \ln\left(\frac{p_0}{p_a} \cdot \frac{1-p_0}{1-p_a}\right)}_{k_3} \;\middle|\; p = p_0\right)$$

$$= P(\Sigma x_i < k_3 \mid p = p_0)$$

↳ we know that $m_{\Sigma x_i}(t) = m_y(t) = [[pe^t + (1-p)]^n]^n \rightarrow \Sigma x_i \sim \text{Binom}(n^2, p)$

$$= P(\Sigma x_i > k_3 \mid p_0) \Rightarrow qbinom(0.95, n^2, p_0)$$

In conclusion we will reject the null if $\Sigma x_i > qbinom(0.95, n^2, p_0)$.