**Technical Note 20.3**

**To obtain the Standard Errors of $b_1$, $b_0$, $\hat{Y}_J$, and $e_j$** (Note that we assume here that $X$, the predictor, is a fixed-effect variable that is measured without error – so that if the usual assumptions are satisfied, $Var(Y_i) = \sigma_e^2$.)

(1) From Technical Note 20.1, we know that $b_1$ can be expressed as $b_1 = \sum_i f_i Y_i$ where $f_i = \dfrac{X_i - \overline{X}}{SS_X}$ .

Therefore, $Var(b_1) = Var\left(\sum_i f_i Y_i\right) = Var(f_1 Y_1 + f_2 Y_2 + \ldots + f_N Y_N)$ assuming independence

$$= Var(f_1 Y_1) + Var(f_2 Y_2) + \ldots + Var(f_N Y_N)$$

$$= f_1^2 \sigma_1^2 + f_2^2 \sigma_2^2 + \ldots + f_N^2 \sigma_N^2$$

$$= \sigma_e^2 \left( f_1^2 + f_2^2 + \ldots + f_N^2 \right) \quad \text{assuming homoscedasticity}$$

$$= \sigma_e^2 \sum_i f_i^2$$

Now, because $f_i = \dfrac{X_i - \overline{X}}{SS_X}$ ,

$$Var(b_1) = \sigma_e^2 \sum_i f_i^2 = \sigma_e^2 \sum_i \frac{(X_i - \overline{X})^2}{(SS_X)^2}$$

$$= \frac{\sigma_e^2}{SS_X} \quad \text{because } SS_X = \sum_i (X_i - \overline{X})^2$$

so the estimated standard error is

$$SE(b_1) = \frac{s_{Y.X}}{\sqrt{SS_X}}$$

(2) To find the standard error of $b_0$, we start with $b_0 = \sum g_i Y_i$. BIf we can assume independence and homoscedasticity, $Var(b_0) = \sigma_e^2 \sum_i g_i^2$ . Substituting $g_i = \dfrac{1}{N} - \dfrac{\overline{X}(X_i - \overline{X})}{SS_X}$ and simplifying, we have $Var(b_0) = \sigma_e^2 \left( \dfrac{1}{N} + \dfrac{\overline{X}^2}{SS_X} \right)$, so that $SE(b_0) = s_{Y.X} \sqrt{\dfrac{1}{N} + \dfrac{\overline{X}^2}{SS_X}}$

(3) Similarly, we can show that $SE(\hat{Y}_j) = s_{Y.X}\sqrt{h_{jj}}$ .

From Note 20.1,

$$\hat{Y}_j = \sum_i h_{ij} Y_i \quad \text{where } h_{ij} = \frac{1}{N} + \frac{(X_j - \overline{X})(X_i - \overline{X})}{SS_X}$$

Then

$$Var(\hat{Y}_j) = Var\left[\sum_i h_{ij} Y_i\right] = \sigma_e^2 \sum_i h_{ij}^2$$

But it can be shown that $\sum_i h_{ij}^2 = h_{jj}$ (just expand and simplify to see that this is true). Therefore,

$$Var(\hat{Y}_j) = \sigma_e^2 h_{jj}$$

and so

$$SE(\hat{Y}_j) = s_{Y.X}\sqrt{h_{jj}}$$

(4) Finally, to show that $SE(e_j) = s_{Y.X}\sqrt{1 - h_{jj}}$ , we begin with

$$e_j = Y_j - \hat{Y}_j = Y_j - \sum_i h_{ij} Y$$

so

$$Var(e_j) = Var(Y_j) + Var\left[\sum_i h_{ij} Y_i\right] - 2Cov\left[Y_j, \sum_i h_{ij} Y_i\right]$$

The first term on the right of the equation is equal to $\sigma_e^2$ . The second term is equal to $h_{jj}\sigma_e^2$

from (3) above. The last term is equal to $-2h_{jj}\sigma_e^2$ because $Cov(Y_j, Y_j) = \sigma_e^2$

and $Cov(Y_j, Y_{j'}) = 0$ for $j \neq j'$ . Therefore,

$$Var(e_j) = \sigma_e^2 (1 - h_{jj})$$

and

$$SE(e_j) = s_{Y.X}\sqrt{1 - h_{jj}}$$