Technical Note 20.2

To Show that b_1, b_0, and $\hat{Y}_j$ are Unbiased Estimators of β_1, β_1, and $\mu_{Y.X_j}$.

To show that b_1 is an unbiased estimator of β_1, we need to show that $E(b_1) = \beta_1$.

The regression model states that

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

where X_i is a fixed-effect variable. We can re-write this equation as

$$Y_i = \mu_Y + \beta_1(X_i - \overline{X}) + \varepsilon_i$$

(Note that we refer to the mean of X as $\overline{X}$ and the mean of Y as μ_Y because, according to the usual regression model, X is a fixed-effect variable and Y is a random variable.)

From Note 20.1, we have $b_1 = \sum_i f_i Y_i$ where $f_i = \dfrac{X_i - \overline{X}}{SS_X}$. Substituting the expression for the model and taking the expectation, we have

$$\begin{aligned} E(b_1) &= E\left(\sum f_i\left[\mu_Y + \beta_1(X_i - \overline{X}) + \varepsilon_i\right]\right) \\ &= \mu_Y \sum f_i + E\left(\sum f_i \beta_1(X_i - \overline{X})\right) + \sum f_i E(\varepsilon_i) \end{aligned}$$

The first and third terms are equal to zero because $\sum f_i = \sum (X_i - \overline{X}) / SS_X = 0$ and, by assumption, $E(\varepsilon_i) = 0$. The second term is

$$\begin{aligned} E(b_1) &= E\left(\sum \frac{X_i - \overline{X}}{SS_X}\beta_1(X_i - \overline{X})\right) \\ &= E\left(\sum \frac{\beta_1(X_i - \overline{X})^2}{SS_X}\right) \\ &= E(\beta_1) \qquad \text{because } \sum(X_i - \overline{X})^2 = SS_X \\ &= \beta_1 \end{aligned}$$

To show that b_0 is an unbiased estimator of β_0, we need to demonstrate that $E(b_0) = \beta_0$. We begin by noting that

$$b_0 = \overline{Y} - b_1\overline{X}$$

and that from the model

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i$$

Taking the mean of both sides (and noting that β_0 and β_1 are constants), we have

$$\overline{Y} = \frac{1}{N}\sum Y_i = \beta_0 + \beta_1\overline{X} + \overline{\varepsilon}$$

Therefore,

$$\begin{aligned}
E(b_0) &= E(\overline{Y} - b_1\overline{X}) \\
&= E(\beta_0 + \beta_1\overline{X} + \overline{\varepsilon} - b_1\overline{X}) \\
&= \beta_0 + \beta_1\overline{X} + 0 - \beta_1\overline{X} \\
&= \beta_0
\end{aligned}$$