Analytical Estimation of Joint Two-level Markov–Gibbs Model Parameters

Let $\mathbf{Q} = \{0, \ldots, Q-1\}$ and $\mathbf{K} = \{0, \ldots, K-1\}$ denote sets of gray levels q and region labels k, respectively. Here, Q is the number of gray levels and K is the number of image modes, i.e. peaks in the gray level frequency distribution, e.g., for a bimodal image, $K = 2$. We assume that each dominant image mode corresponds to a particular class of objects to be found in the image.

Let $\mathbf{R} = \{(x, y) : 0 \le x \le X-1, 0 \le y \le Y-1\}$ be a 2-D (x, y)-arithmetic grid supporting gray level images $\mathbf{g} : \mathbf{R} \to \mathbf{Q}$ and their region maps $\mathbf{m} : \mathbf{R} \to \mathbf{K}$. A two-level probability model of original images to segment and their desired region maps is given by a joint distribution $P(\mathbf{g}, \mathbf{m}) = P(\mathbf{m})P(\mathbf{g}|\mathbf{m})$ where $P(\mathbf{m})$ is an unconditional probability distribution of maps (the higher level) and $P(\mathbf{g}|\mathbf{m})$ is a conditional distribution of images, given the map (the lower level of the model).

The Bayesian Maximum *a Posteriori* (MAP) estimate of the map $\mathbf{m}$, given the image $\mathbf{g}$:

$$\mathbf{m}^* = \arg\max_{\mathbf{m}\in\mathcal{X}} L(\mathbf{g}, \mathbf{m})$$

where $\mathcal{X}$ is the set of all region maps with labels $k \in \mathbf{K}$ on $\mathbf{R}$, maximizes the log-likelihood function:

$$L(\mathbf{g}, \mathbf{m}) = \frac{1}{|\mathbf{R}|}\left(\log P(\mathbf{g}|\mathbf{m}) + \log P(\mathbf{m})\right) \tag{1}$$

To find this estimate, we need to select the low- and high-level models and identify their parameters.

I. Unconditional Region Map Model

The simplest model of interdependent region labels is the MGRF with the nearest 8-neighborhood of each pixel involving the closest horizontal-vertical and left-right-diagonal pixel pairs. By symmetry considerations, we assume the Gibbs potentials are independent of relative orientation of pixel pairs, are the same for all classes, and depend only on whether the pair of labels are equal or not. Under these assumptions, it is the simplest auto-binomial model, the Potts one, being for a long time a popular region map model [1]–[4]. But unlike the conventional counterparts, its Gibbs potential is obtained analytically using the maximum likelihood estimator for a generic MGRF derived in [5]. The 8-neighborhood results in a family $\mathbf{C_N} = [\mathbf{c}_{x,y,\xi,\eta} = ((x, y), (x+\xi, y+\eta)) : (x, y) \in \mathbf{R}; (x+\xi, y+\eta) \in \mathbf{R}; (\xi, \eta) \in \mathbf{N} = \{(1, 0), (-1, 1), (0, 1), (1, 1)\}]$ of the neighboring pixel pairs supporting the Gibbs potentials. The potentials are bi-valued because only the coincidence of the labels is taken into account: $V(k, k') = \Gamma$ if $k = k'$ and $-\Gamma$ if $k \ne k'$. Then the MGRF model of region maps is as follows:

$$\begin{aligned} P(\mathbf{m}) &= \frac{1}{Z_\mathbf{N}} \exp \sum_{(x,y)\in\mathbf{R}} \sum_{(\xi,\eta)\in\mathbf{N}} V(m_{x,y}, m_{x+\xi,y+\eta}) \\ &= \frac{1}{Z_\mathbf{N}} \exp\left(|\mathbf{C_N}|\Gamma\left(2f_{\mathrm{eq}}(\mathbf{m}) - 1\right)\right) \end{aligned} \tag{2}$$

where $|\mathbf{C_N}|$ is the cardinality of the family $\mathbf{C_N}$ and $f_{\mathrm{eq}}(\mathbf{m})$ denotes the relative frequency of the equal labels in the pixel pairs of this family:

$$f_{\mathrm{eq}}(\mathbf{m}) = \frac{1}{|\mathbf{C_N}|} \sum_{\mathbf{c}_{x,y,\xi,\eta}\in\mathbf{C_N}} \delta(m_{x,y} - m_{x+\xi,y+\eta}) \tag{3}$$

Here, $\delta()$ denotes the Kronecker delta function: $\delta(0) = 1$ and 0 otherwise. To identify this high-level model, we have to estimate only the potential value Γ.

To compute the second term, $\frac{1}{|\mathbf{R}|}\log P(\mathbf{m})$, in Eq. (1) for a region map $\mathbf{m}$, we use the approximate partition function $Z_\mathbf{N}$ in [6] (see also [1], p.156) reduced in our case to:

$$\begin{aligned} Z_\mathbf{N} &\approx \exp\left(\sum_{x,y\in\mathbf{R}} \sum_{\xi,\eta\in\mathbf{N}} \sum_{k\in\mathbf{K}} V(l, m_{x+\xi,y+\eta})\right) \\ &= \exp\left(|\mathbf{C_N}| \sum_{k\in\mathbf{K}} \left(\Gamma f_k(\mathbf{m}) - \Gamma(1 - f_k(\mathbf{m}))\right)\right) \\ &= \exp\left(\Gamma|\mathbf{C_N}|(2 - K)\right) \end{aligned}$$

where $f_k(\mathbf{m})$ is the marginal frequency of the label k in the map $\mathbf{m}$. The above approximate partition function (which becomes too trivial for $K = 2$) results in the following approximation of the second term $\frac{1}{|\mathbf{R}|}\log P(\mathbf{m})$ in Eq. (1):

$$\varrho\Gamma(2f_{\mathrm{eq}}(\mathbf{m}) + K - 3) \approx 4\Gamma(2f_{\mathrm{eq}}(\mathbf{m}) + K - 3) \tag{4}$$

where $\varrho = \frac{|\mathbf{C_N}|}{|\mathbf{R}|} \approx |\mathbf{N}| = 4$.

II. Identification of the High-Level Model

The approximate log-likelihood term in Eq. (4) is unsuitable for estimating the model parameter Γ that specifies the Gibbs potential. Thus we identify the high-level model using a reasonably close first approximation of the Maximum Likelihood Estimate (MLE) of Γ derived for a given region map $\mathbf{m}^\circ$ in accord with [5] from the unconditional log-likelihood $L_u(\mathbf{m}^\circ|\Gamma) = \frac{1}{|\mathbf{R}|}\log P(\mathbf{m}^\circ)$ of Eq. (2) with the exact partition function $Z_\mathbf{N} = \sum_{\mathbf{m}\in\mathcal{X}} \exp(\Gamma\varrho|\mathbf{R}|(2f_{\mathrm{eq}}(\mathbf{m}) - 1))$ where $\mathcal{X}$ is the parent population of region maps:

$$\begin{aligned} L_u(\mathbf{m}^\circ|\Gamma) &= \Gamma\varrho(2f_{\mathrm{eq}}(\mathbf{m}^\circ) - 1) \\ &- \frac{1}{|\mathbf{R}|}\log\left(\sum_{\mathbf{m}\in\mathcal{X}} \exp(\Gamma\varrho|\mathbf{R}|(2f_{\mathrm{eq}}(\mathbf{m}) - 1)\right) \end{aligned}$$

The approximation is obtained by truncating the Taylor's series expansion of $L(\mathbf{m}^\circ|\Gamma)$ in the close vicinity of zero potential, $\Gamma = 0$, to the first three terms:

$$L_u(\mathbf{m}^\circ|0) + \Gamma \left.\frac{dL(\mathbf{m}^\circ|\Gamma)}{d\Gamma}\right|_{\Gamma=0} + \frac{1}{2}\Gamma^2 \left.\frac{d^2L_u(\mathbf{m}^\circ|\Gamma)}{d\Gamma^2}\right|_{\Gamma=0} \tag{5}$$

Because zero potential produces an independent random field (IRF) equiprobable region labels $k \in \mathbf{K}$, the relative frequency of the equal pairs of labels over $\mathbf{C_N}$ has in this case the mean value $\frac{1}{K}$ and the variance $\frac{K-1}{K^2}$. Then the following relationships hold:

$$\frac{dL_u(\mathbf{m}^\circ|\Gamma)}{d\Gamma}\Big|_{\Gamma=0} = 2\varrho\left(f_{eq}(\mathbf{m}^\circ) - \frac{1}{K}\right)$$

$$\frac{d^2 LE_u(\mathbf{m}^\circ|\Gamma)}{d\Gamma^2}\Big|_{\Gamma=0} = -4\varrho\frac{K-1}{K^2}$$

where $f_{eq}(\mathbf{m})^\circ$ is the relative frequency of the equal label pairs in the region map $\mathbf{m}^\circ$ specified in Eq. (3). The approximate likelihood of Eq. (5) results in the following MLE of Γ for a given map $\mathbf{m}^\circ$:

$$\Gamma = \frac{K^2}{2(K-1)}\left(f_{\text{eq}}(\mathbf{m}^\circ) - \frac{1}{K}\right) \tag{6}$$

This relationship allows for computing the potentials of the Potts model for each current region map obtained by the Bayesian classification based on the estimated low-level image model. For bimodal images ($K = 2$), the value Γ is estimated as:

$$\Gamma = 2f_{\text{eq}}(\mathbf{m}^\circ) - 1 \tag{7}$$

REFERENCES

[1] R. C. Dubes and A. K. Jain, “Random field models in image analysis,” *J. Applied Statistics*, vol. 16, pp. 131–164, 1989.
[2] G. L. Gimel’farb and A. V. Zalesny, “Probabilistic models of digital region maps based on Markov random fields with short- and long-range interaction,” *Pattern Recognition Letters*, vol. 14, pp. 789–797, 1993.
[3] R. W. Picard and I. M. Elfadel, “Structure of aura and cooccurrence matrices for the Gibbs texture model,” *J. Mathematical Imaging and Vision*, vol. 2, pp. 5–25, 1992.
[4] L. Dryden, M. R. Scarr, and C. C. Taylor, “Bayesian texture segmentation of weed and crop images using reversible jump Markov chain Monte Carlo methods,” *J. Royal Statistical Society*, vol. 52C, pt. 1, pp. 31–50, 2003.
[5] G. Gimel’farb, *Image Textures and Gibbs Random Fields*, Kluwer Academic, Dordrecht, 1999.
[6] J. E. Besag, “Spatial interaction and the statistical analysis of lattice systems,” *J. Royal Statistical Society*, vol. B36, pp. 192–236, 1974.