Homework 1

1. **(Review of numerical integration of ODEs & MATLAB):** The equations of motion of a two D. O. F mechanical system are given by:

$$\begin{aligned}(\alpha + \beta \sin^2 \theta)\ddot{\phi} + \gamma\ddot{\theta}\cos\theta + 2\beta\dot{\phi}\dot{\theta}\cos\theta\sin\theta - \gamma\dot{\theta}^2\sin\theta &= \tau_\phi, \\ \gamma\ddot{\phi}\cos\theta + \beta\ddot{\theta} - \beta\dot{\phi}^2\sin\theta\cos\theta - \delta\sin\theta &= \tau_\theta.\end{aligned}$$

The terms $\alpha, \beta, \gamma, \delta$ in the above equations are constants. The control inputs are τ_θ, τ_ϕ. Let the state vector

$$q(t) = \begin{bmatrix} \phi(t) \\ \theta(t) \end{bmatrix}.$$

These equations can be put in the "standard" form of robotic systems as:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau,$$

where

$$M = \begin{pmatrix} \alpha + \beta\sin^2\theta & \gamma\cos\theta \\ \gamma\cos\theta & \beta \end{pmatrix}, \quad C = \begin{pmatrix} \beta\dot{\theta}\sin\theta\cos\theta & -\gamma\dot{\theta}\sin\theta + \beta\dot{\phi}\sin\theta \\ -\beta\dot{\phi}\sin\theta & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ -\delta\sin\theta \end{pmatrix}.$$

Pick $\alpha = 10, \beta = 3, \gamma = 2, \delta = 5$.

- Simulate, using MATLAB or Python, the response of this mechanical system when the control inputs are identically zero. You may choose the following initial conditions:

$$q(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \dot{q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- Simulate for the same initial conditions the closed loop response with a PD controller:

$$\begin{pmatrix} \tau_\phi \\ \tau_\theta \end{pmatrix} = -K_p q - K_D \dot{q}.$$

  You may pick $K_p = 10, K_d = 10$ or any suitable value that ensures that $q(t) \to 0$ as $t \to \infty$.

- **(Review of Interpolation for Robot Trajectory Planning):** Find a desired trajectory, i.e., a polynomial $\theta_d(t), \phi_d(t)$ that satisfy the following conditions:

$$\theta_d(0) = 0, \theta_d(1) = \frac{\pi}{2}, \theta_d(2) = \frac{3\pi}{2}, \theta_d(3) = 2\pi.$$

$$\phi_d(0) = 0, \phi_d(1) = \frac{\pi}{3}, \phi_d(2) = \frac{2\pi}{3}, \phi_d(3) = \pi.$$

- Suppose $\theta(t)$ must satisfy additional specifications below:
$$\dot{\theta}_d(0) = 0, \dot{\theta}_d(1) = 1, \dot{\theta}_d(2) = 1, \dot{\theta}_d(3) = 0.$$
Similarly,
$$\dot{\phi}_d(0) = 0, \dot{\phi}_d(1) = 1, \dot{\phi}_d(2) = 2, \dot{\phi}_d(3) = 0.$$
Will the earlier polynomial you constructed satisfy this requirement? If not, how would you construct a polynomial satisfying both requirements?
- Simulate the following closed loop response with the following feedback control input:
$$\begin{pmatrix} \tau_\phi \\ \tau_\theta \end{pmatrix} = M \begin{pmatrix} \ddot{\phi}_d(t) \\ \ddot{\theta}_d(t) \end{pmatrix} + C \begin{pmatrix} \dot{\phi}_d(t) \\ \dot{\theta}_d(t) \end{pmatrix} - K_d \begin{pmatrix} \dot{\phi}(t) - \dot{\phi}_d(t) \\ \dot{\theta}(t) - \dot{\theta}_d(t) \end{pmatrix} - \begin{pmatrix} \phi(t) - \phi_d(t) \\ \theta(t) - \theta_d(t) \end{pmatrix}.$$

A note on its application: Suppose you want the end effector to track a desired trajectory. The idea is to discretize the trajectory, say at k points and get the corresponding homogeneous transformation at each and every one of the k discrete points. Then, solve the inverse kinematics at each discrete point to get the corresponding joint angles. In the first problem, you can think of the end effector's trajectory being discretized at four points in such a way that one of the joint angles (given here by θ here) satisfies the constraints given in the problem). One can use this polynomial function $\theta(t), t \in [0, 3]$ to construct a periodic trajectory of period 3 sec as $\theta(0) = \theta(3)$. If there are specifications on the end effector's velocity, they are correspondingly transferred to joint velocities in the second part).

2. **(Review of Linear Algebra):**

- Let
$$A = \begin{pmatrix} 10 & -5 & -3 \\ -5 & 11 & -4 \\ -3 & -4 & 9 \end{pmatrix}$$
In the context of this matrix, what is the definition of an eigenvalue (natural frequency) and eigenvector (mode shape)? Can you think of the corresponding mechanical vibratory system with a stiffness matrix specified by A?
- Find the eigenvalues and eigenvectors of A.
- Suppose
$$P(x_1, x_2, x_3) = 10x_1^2 - 10x_1x_2 - 6x_1x_3 + 11x_2^2 - 8x_2x_3 + 9x_3^3.$$
Show that it can be written as
$$P(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$
In general, verify that a polynomial
$$P(x_1, x_2, x_3) = a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2$$
can be expressed as
$$P(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

- Show that the eigenvalues of a real, symmetric matrix are real; moreover, its eigenvectors are mutually orthogonal.
- Suppose

$$A\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad A\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

  Without computiong the explicit inverse of A, find A^{-1} in terms of $x_1, x_2, x_3, y_1, y_2, y_3$ and z_1, z_2, z_3.
- Suppose the eigenvalues of A are $\lambda_1 > \lambda_2 > \lambda_3 > 0$ and the corresponding eigenvectors are $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$. What is A^{-1} in terms of $\lambda_1, \lambda_2, \lambda_3$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$?
- For any vector $\mathbf{v}$, show that

$$\lambda_3 \mathbf{v}^T \mathbf{v} \leq \mathbf{v}^T A \mathbf{v} \leq \lambda_1 \mathbf{v}^T \mathbf{v}$$