**MEEN 408: Intro to Robotics**
**MEEN 612: Mechanics of Robotic Manipulators**
Texas A&M University
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# Homework 6

1. Consider the 2-link Furuta pendulum as shown: The link 1 has a center of mass at

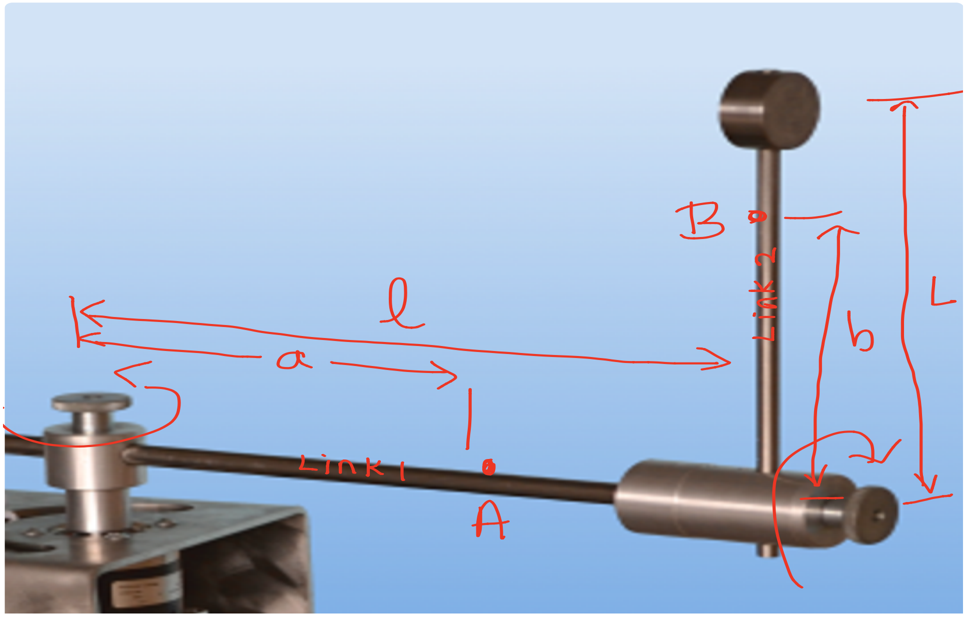



Figure 1: Furuta Pendulum

$A$ as shown and is of mass $m$. The link 2 is vertical and rotates around link 1 as shown. It has a mass $M$ and its center of mass is $b$ units from the center of the link 1. The stator of the motor moving link 2 is rigidly attached at the far right end of link 1. You may assume the moment of inertia of link 1 about its center to be $\bar{I}_1$ and that of link 2 to be $\bar{I}_2$. The motor torques are $\tau_1, \tau_2$. You may assume the angle of rotation of the first link about the vertical axis is $\theta$ and that of the second link about the first to be $\phi$. Attach $\mathcal{F}_1$ to link 1 and $\mathcal{F}_2$ to link 2 as follows: $i_1$ is along the link to the right, $k_1$ is vertically upward, $i_2$ is along the link 2 (and perpendicular to $i_1$) and $k_2$ is aligned along $i_1$.

- In the previous HW, you put the equations in the standard form:

$$M(q)\ddot{q} + C(q, \dot{q}) + G(q) = \tau$$

by identifying $M$, $C$, $G$ and $\tau$ with $\dot{M} - 2C$ is skew symmetric. Derive the control law to track a pre-specified desired trajectory $(\theta_d(t), \phi_d(t))$. Specifically, you may assume that $\theta_d(t) = 2\cos(t) + 3\sin(2t)$ and $\phi(t)_d(t) = 3\cos(t) + 2\sin(2t)$.

- Write a MATLAB/Python program to simulate the open-loop dynamics of this robot.
**Solution:** Governing equations can be specified through the standard equation with the following $M, C$ and $G$ matrices:

$$M(q) = \begin{bmatrix} \bar{I}_1 + \bar{I}_2 \cos^2\phi & -Mlb\sin\phi \\ -Mlb\sin\phi & \bar{I}_2 \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ Mgb\cos\phi \end{bmatrix}$$

$$C = \begin{bmatrix} \bar{I}_2\dot{\phi}\sin\phi\cos\phi & -Mlb\dot{\phi}\cos\phi - \bar{I}_2\dot{\theta}\sin\phi\cos\phi \\ \dot{\theta}\bar{I}_2\sin\phi\cos\phi & 0 \end{bmatrix}$$

For open loop dynamics, $u$ will not be a function of the states. You may simply simulate the system for $u = 0$ and some initial conditions. The state vector will be $q = (\theta, \phi, \dot{\theta}, \dot{\phi})^T$.

- Write a MATLAB/Python program closed-loop dynamics of this robot.
**Solution:** You may implement a variety of control inputs, like PD, computed torque and modified computed torque. The following is for PD control:

$$\tau = K_P(q - q_d) + K_D(\dot{q} - \dot{q}_d)$$

where

$$K_P = \begin{bmatrix} k_{p\theta} & 0 \\ 0 & k_{p\phi} \end{bmatrix} \qquad K_D = \begin{bmatrix} k_{d\theta} & 0 \\ 0 & k_{d\phi} \end{bmatrix}$$

$$q_d = \begin{bmatrix} \theta_d(t) \\ \phi_d(t) \end{bmatrix} = \begin{bmatrix} 2\cos(t) + 3\sin(2t) \\ 3\cos(t) + 2\sin(2t) \end{bmatrix} \qquad \dot{q}_d = \begin{bmatrix} \dot{\theta}_d(t) \\ \dot{\phi}_d(t) \end{bmatrix} = \begin{bmatrix} -2\sin(t) + 6\cos(2t) \\ -3\sin(t) + 4\cos(2t) \end{bmatrix}$$

You must pick and tune $K_P$ and $K_D$ for good tracking.

- Assume that the masses $M$ and $m$ are not exactly known. Derive an adaptive control law to track the desired trajectories. Show the associated guarantees.
**Solution:**

$$S = \dot{q} - \dot{q}_r = \dot{q} - \dot{q}_d + \lambda(q - q_d)$$
$$\tau = \hat{M}(q)\ddot{q}_r + \hat{C}(q,\dot{q})\dot{q}_r + \hat{G}(q) - KS$$

You must pick $K$, which is a gain matrix such that $K = K^T > 0$. The following parameterizes the system. Note $Y_0 = 0$ since all terms either depend on $M$ or $m$.

$$\tau = \begin{bmatrix} \hat{I}_1 + \hat{m}a^2 + \hat{M}l^2 + (\hat{I}_2 + \hat{M}b^2)\cos^2\phi & -\hat{M}lb\sin\phi \\ -\hat{M}lb\sin\phi & (\hat{I}_2 + \hat{M}b^2) \end{bmatrix}\ddot{q}_r +$$
$$\begin{bmatrix} (\hat{I}_2 + \hat{M}b^2)\dot{\phi}\sin\phi\cos\phi & -\hat{M}lb\dot{\phi}\cos\phi - (\hat{I}_2 + \hat{M}b^2)\dot{\theta}\sin\phi\cos\phi \\ \dot{\theta}(\hat{I}_2 + \hat{M}b^2)\sin\phi\cos\phi & 0 \end{bmatrix}\dot{q}_r + \begin{bmatrix} 0 \\ \hat{M}gb\cos\phi \end{bmatrix} - KS$$

$$Y_1 = \begin{bmatrix} I_1(1) + l^2 + b^2\cos^2\phi & -lb\sin\phi \\ -lb\sin\phi & b^2 \end{bmatrix}\ddot{q}_r +$$
$$\begin{bmatrix} b^2\dot{\phi}\sin\phi\cos\phi & -lb\dot{\phi}\cos\phi - b^2\dot{\theta}\sin\phi\cos\phi \\ \dot{\theta}b^2\sin\phi\cos\phi & 0 \end{bmatrix}\dot{q}_r + \begin{bmatrix} 0 \\ gb\cos\phi \end{bmatrix}$$

$$Y_2 = \begin{bmatrix} I_2(1)\cos^2\phi + a^2\phi & 0 \\ 0 & I_2(1) \end{bmatrix}\ddot{q}_r + \begin{bmatrix} I_2(1)\dot{\phi}\sin\phi\cos\phi & I_2(1)\dot{\theta}\sin\phi\cos\phi + \\ \dot{\theta}I_2(1)\sin\phi\cos\phi & 0 \end{bmatrix}\dot{q}_r$$

$$\tau = Y_1\hat{M} + Y_2\hat{m} - KS$$

Adaptation laws:

$$\dot{\hat{M}} = -\frac{1}{\gamma_1} S^T Y_1 \qquad \dot{\hat{m}} = -\frac{1}{\gamma_2} S^T Y_2$$

Pick $\gamma_1 > 0$ and $\gamma_2 > 0$. Consider the Lypunov function:

$$V = \frac{1}{2} S^T M(q) S + \frac{1}{2}\gamma_1 \tilde{M}^2 + \frac{1}{2}\gamma_2 \tilde{m}^2$$

Note: $V >= 0$ for every $S, \tilde{M}$, and $\tilde{m}$. $V = 0$ if and only if $S = \tilde{M} = \tilde{m} = 0$. Thus, $V$ is Positive definite.

$$\begin{aligned}\dot{V} &= S^T M(q)\dot{S} + \frac{1}{2} S^T \dot{M}(q) S + \gamma_1 \tilde{M}\dot{\hat{M}} + \gamma_2 \tilde{m}\dot{\hat{m}} \\ &= S^T (M(q)\dot{S} + C(q,\dot{q})S) + \frac{1}{2} S^T (\dot{M}(q) - 2C(q,\dot{q}))S + \gamma_1 \tilde{M}\dot{\hat{M}} + \gamma_2 \tilde{m}\dot{\hat{m}}\end{aligned}$$

The above was acquired by adding and subtracting $S^T C(q,\dot{q})S$ and using the fact that $\tilde{M} = \hat{M} - M \implies \dot{\tilde{M}} = \dot{\hat{M}}$.

$$\begin{aligned}\dot{V} &= S^T (M(q)(\ddot{q} - \ddot{q}_r) + C(q,\dot{q})(\dot{q} - \dot{q}_r)) + \gamma_1 \tilde{M}\dot{\hat{M}} + \gamma_2 \tilde{m}\dot{\hat{m}} \\ &= S^T (u - C(q,\dot{q})\dot{q} - G(q) - M(q)\ddot{q}_r + C(q,\dot{q})(\dot{q} - \dot{q}_r)) + \gamma_1 \tilde{M}\dot{\hat{M}} + \gamma_2 \tilde{m}\dot{\hat{m}} \\ &= S^T (\hat{M}(q)\ddot{q}_r + \hat{C}(q,\dot{q})\dot{q}_r + \hat{G}(q) - KS - G(q) - M(q)\ddot{q}_r - C(q,\dot{q})\dot{q}_r) + \gamma_1 \tilde{M}\dot{\hat{M}} + \gamma_2 \tilde{m}\dot{\hat{m}} \\ &= S^T (\tilde{M}(q)\ddot{q}_r + \tilde{C}(q,\dot{q})\dot{q}_r + \tilde{G}(q) - KS) + \gamma_1 \tilde{M}\dot{\hat{M}} + \gamma_2 \tilde{m}\dot{\hat{m}} \\ &= S^T (Y_1 \tilde{M} + Y_2 \tilde{m} - KS) + \gamma_1 \tilde{M}\dot{\hat{M}} + \gamma_2 \tilde{m}\dot{\hat{m}} \\ &= (S^T Y_1 + \gamma_1 \dot{\hat{M}})\tilde{M} + (S^T Y_2 + \gamma_2 \dot{\hat{m}})\tilde{m} - S^T K S\end{aligned}$$

Using the prior discussed adaptation law.

$$\begin{aligned}&\dot{V} = -S^T K S \leq 0 \text{ for every } S \text{ and } \dot{V} = 0 \iff S = 0 \\ &\dot{V} = -S^T K S \leq -\lambda_{min}(K)||S||^2 \\ \implies &\int_0^t \dot{V}(\tau)d\tau \leq \int_0^t \lambda_{min}(K)||S||^2 d\tau \implies \int_0^t ||S||^2 d\tau \leq \frac{V(0) - V(t)}{\lambda_{min}(K)} \leq \frac{V(0)}{\lambda_{min}(K)} \\ \implies &S \text{ is square integrable.}\end{aligned}$$

$$\begin{aligned}&V(t) \leq V(0) \\ \implies &\frac{1}{2} S^T M(q) S \leq V(0) \implies \frac{1}{2}\lambda_{min}(M)||S||^2 \leq V(0) \implies ||S|| \leq \sqrt{\frac{2V(0)}{\lambda_{min}(M)}} \\ \implies &S \text{ is bounded.} \implies e \text{ and } \dot{e} \text{ are bounded.} \\ \implies &\frac{1}{2}\gamma_1 \tilde{M}^2 \leq V(0) \implies |\tilde{M}| \leq \sqrt{\frac{2V(0)}{\gamma_1}} \\ \implies &\text{Similarly, } |\tilde{m}| \leq \sqrt{\frac{2V(0)}{\gamma_2}}\end{aligned}$$

Thus, $Y_1$ and $Y_2$ are bounded. Assuming, $q_d, \dot{q}_d$, and $\ddot{q}_d$ are bounded and using the fact that $e$ and $\dot{e}$ are bounded, it follows that $||\ddot{q}_r|| = ||\ddot{q}_d + \lambda e|| \leq ||\ddot{q}_d|| + \lambda||e||$ is bounded. Hence $\dot{S}$ is bounded. Thus, per Barbalat's lemma, $S$ assymptotically goes to 0, implying $e$ and $\dot{e}$ also got to 0 as $t \to \infty$.

- Corroborate the performance of the adaptive controller using numerical simulations in MATLAB/Python.
  **Solution:** Modify the code from closed loop dynamics. Firstly, augment the state vector to include $\hat{M}$ and $\hat{m}$.

```
x = [theta, phi, thetaDot, phiDot, Mhat, mhat]
```

  To compute the control input:

```
e = x[1:2]-[theta_des;phi_des]
eDot = x[3:4]-[thetaDot_des;phiDot_des]
S = eDot + lambda*e
x_r_dot = [theta_des_dot;phi_des_dot] - lambda*(x[1:2]
        - [theta_des;phi_des])
x_r_ddot = [theta_des_ddot;phi_des_ddot] - lambda*(x[3:4]
        - [theta_des_dot;phi_des_dot])
u = M_hat*x_r_ddot + C_hat*x_r_dot + G_hat - K*S
```

  The initial conditions for `x[5]` and `x[6]` are the initial estimates for $\hat{M}$ and $\hat{m}$ respectively. `M_hat`, `C_hat`, and `G_hat` are the $M$, $C$, and $G$ matrices with the estimates $\hat{M}$ and $\hat{m}$. The adaptation laws for $\hat{M}$ and $\hat{m}$ should be coded in the ODE function as follows.

```
dx[1:2] = x[3:4];
dx[3:4] = inv(M)*(u-C*x[3:4]-G);
dx[5] = -(transpose(S)*Y1)/gamma1;
dx[6] = -(transpose(S)*Y2)/gamma2;
```

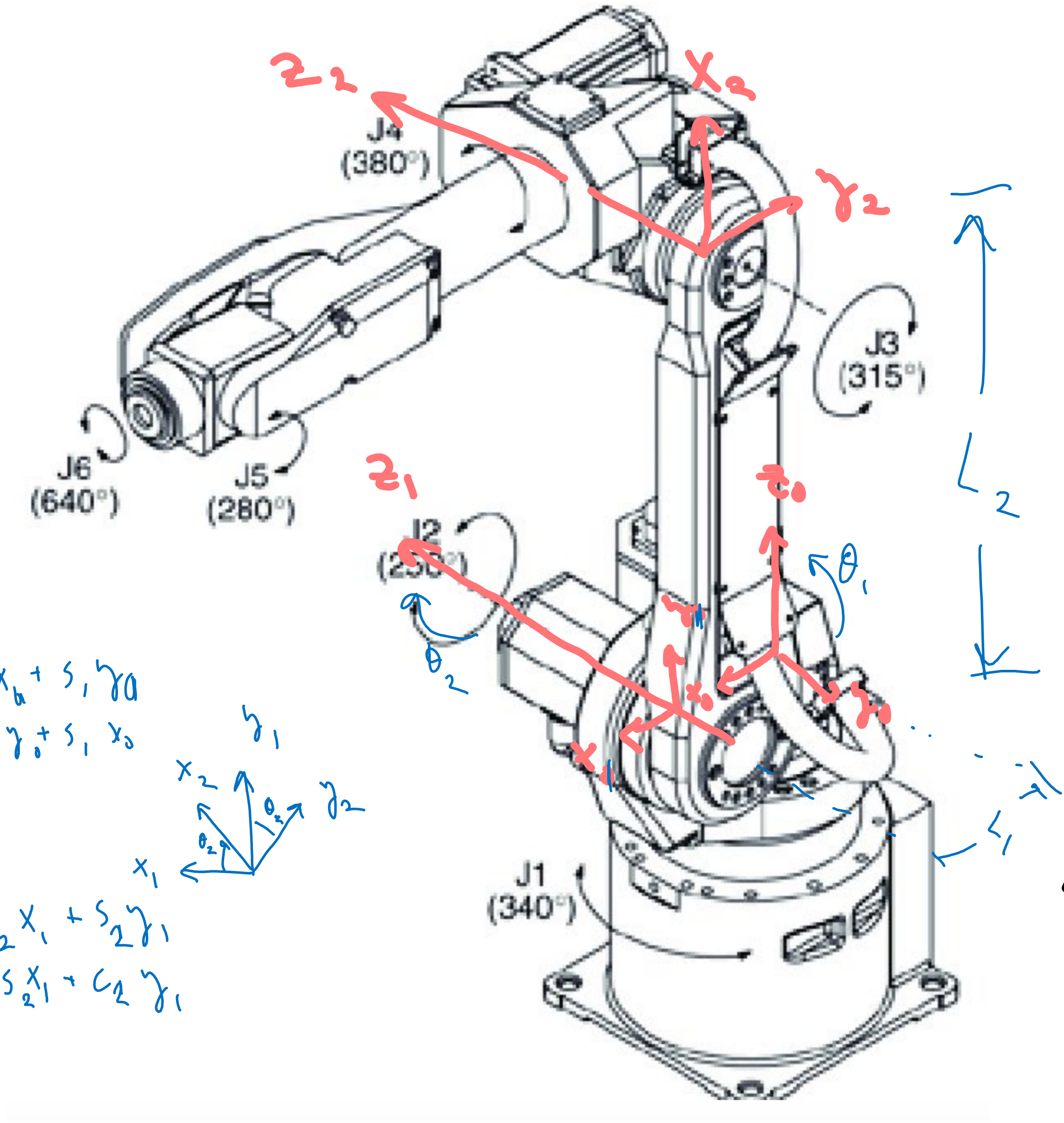
- What can you say about the parameters converging to their true value?
  **Solution:** Plot $|\hat{M}(t) - M|$ and $|\hat{m}(t) - m|$ vs time. The error should reduce. $|\cdot|$ refers to the absolute value. Note that the error may not converge to zero within the simulated time. The convergence rate will vary with your gains.

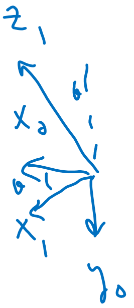
2. Consider a two link robot obtained by removing the last four links of the six-link robot as shown: You may assume that joint axes 1 and 2 (shown by J1 and J2) are not intersecting;

   - Derive the governing equations and put them in the standard form.
   - Pick some reasonable values of inertia and length parameters (from Fanuc's website or other sources or your guess) and simulate the response of the robot to a constant torque input for both joint motors.
   - Suppose the desired joint angle trajectories are:

$$q_{1,des}(t) = 0.5(1 - \cos(\frac{\pi t}{60}))rad, \quad q_{2,des} = 0.5(1 + \cos(\frac{\pi t}{20}))rad.$$

   Derive a computed torque control law to track the desired trajectory and simulate the closed loop response corroborating the effectiveness of the scheme.





$x_1 = c_1 x_0 + s_1 y_0$

$z_1 = -c_1 y_0 + s_1 x_0$

$y_1 = z_0$

$z_2 = z_1$

$x_2 = c_2 x_1 + s_2 y_1$

$y_2 = -s_2 x_1 + c_2 y_1$

You may assume that there is an initial error in position of 0.1 rad, and an error in initial velocity of 0.2 rad/sec.

- Perform the same task with a modified computed torque method.
- Suppose the second link has an added paylod whose mass is unknown. How do your equations of motion change? How do you derive a parameter adaptive control law?
- Corroborate the effectiveness of your parameter adaptive control law.
- Perform the analysis of guarantees for your control law as in the previous problem.

$\underline{\omega}_1 = \dot\theta_1 z_0 = \dot\theta_1 y_1$

$\underline{\omega}_2 = \dot\theta_2 z_1 + \dot\theta_1 \underbrace{z_0}_{y_1} =$

$\dfrac{x_2 - s_2 y_1}{c_2} = x_1 \qquad y_2 = -s_2\left(\dfrac{x_2 - s_2 y_1}{c_2}\right) + c_2 y_1$

$c_2 y_2 + s_2 x_2 = y_1$

$r_e = L_1 x_1 + L_2 x_2$

$\frac{dr_e}{dt} = L_1 \frac{dx_1}{dt} + L_2 \frac{dx_2}{dt}$

$\frac{dx_1}{dt} = \underline{w}_1 \times x_1 = (\dot\theta_1 y_1) \times x_1 = \underline{-\dot\theta_1 z_1}$

$\frac{dx_2}{dt} = w_2 \times x_2 = (\dot\theta_2 z_1 + \dot\theta_1 y_1) \times x_2 = (\dot\theta_2 z_2 + \dot\theta_1 s_2 x_2 + \dot\theta_1 c_2 y_2) \times x_2$

$= \dot\theta_2 (\underbrace{z_2 \times x_2}_{y_2}) + \dot\theta_1 s_2 (\underbrace{x_2 \times x_2}_{0}) + \dot\theta_1 c_2 (\underbrace{y_2 \times x_2}_{-z_2}) = \underline{\dot\theta_2 y_2 - \dot\theta_1 c_2 z_2}$

$x \times y = z$
$z \times x = y$
$y \times z = x$

$\frac{dr_e}{dt} : V_e = L_1(-\dot\theta_1 z_1) + L_2\left[\dot\theta_2 y_2 - \dot\theta_1 c_2 \overset{z_1}{\cancel{z_2}}\right]$ ; $y_2 = -s_2 x_1 + c_2 y_1$

$= -\dot\theta_1 L_1 z_1 + L_2 \dot\theta_2 c_2 y_1 - L_2 \dot\theta_2 s_2 x_1 - L_2 \dot\theta_1 c_2 z_1$

$V_e = -L_2 \dot\theta_2 s_2 \hat{x}_1 + L_2 \dot\theta_2 c_2 \hat{y}_1 - (\dot\theta_1 L_1 + L_2 \dot\theta_1 c_2)\hat{z}_1$

Assume center of mass [COM] is at A for link 1, or $(L_1 = l_1, L_2 = 0)$ and B is COM for link 2, or $(L_1 = L_1, L_2 = l_2)$

$V_B = -l_2 \dot\theta_2 s_2 \hat{x}_1 + l_2 \dot\theta_2 c_2 \hat{y}_1 - (\dot\theta_1 L_1 + l_2 \dot\theta_1 c_2)\hat{z}_1$

$V_A = -\dot\theta_1 l_1 \hat{z}_1$

Find energies of links

$TKE_1 = \frac{1}{2} m_1 \dot\theta_1^2 l_1^2$

$TKE_2 = \frac{1}{2} m_2 \left[ l_2^2 \dot\theta_2^2 s_2^2 + l_2^2 \dot\theta_2^2 c_2^2 + (L_1 + l_2 c_2)^2 \dot\theta_1^2 \right] = \frac{1}{2} m_2 \left[ l_2^2 \dot\theta_2^2 + (L_1 + l_2 c_2)^2 \dot\theta_1^2 \right]$

$PE = m_2 g L_2 S_2$

$RKE_1 = \frac{1}{2} I_{yy_1} \dot\theta_1^2$

$RKE_2 = \frac{1}{2} I_{zz_2} \dot\theta_2^2 + \frac{1}{2} I_{yy_2} \dot\theta_1^2 c_2^2 + \frac{1}{2} I_{xx_2} \dot\theta_1^2 s_2^2$

Lagrangian

$L = \frac{1}{2} m_1 \dot\theta_1^2 l_1^2 + \frac{1}{2} m_2 \left[ l_2^2 \dot\theta_2^2 + (L_1 + l_2 c_2)^2 \dot\theta_1^2 \right] + \frac{1}{2} I_{yy_1} \dot\theta_1^2 + \frac{1}{2} I_{zz_2} \dot\theta_2^2 + \frac{1}{2} I_{yy} \dot\theta_1^2 c_2^2 + \frac{1}{2} I_{xx_2} \dot\theta_1^2 s_2^2 - m_2 g L_2 S_2$

$\underline{\frac{\partial L}{\partial \theta_1} = 0}$ $\quad \frac{\partial L}{\partial \theta_2} = \underline{-m_2 (L_1 + l_2 c_2) l_2 s_2 \dot\theta_2 \dot\theta^2} - m_2 g L_2 c_2 \dot\theta_2 - I_{yy_2} \dot\theta_1^2 c_2 s_2 \dot\theta_2 + I_{xx_2} \dot\theta_1^2 s_2 c_2 \dot\theta_2$

$\frac{\partial L}{\partial \dot\theta_1} = m_1 \dot\theta_1 l_1^2 + m_2 (L_1 + l_2 c_2)^2 \dot\theta_1 + I_{yy_1} \dot\theta_1 + I_{yy_2} \dot\theta_1 c_2^2 + I_{xx_2} \dot\theta_1 s_2^2$

$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot\theta_1} - \frac{\partial L}{\partial \theta_1} = \tau_1$

$\underline{\frac{\partial L}{\partial \dot\theta_2}} = m_2 l_2^2 \dot\theta_2 + I_{zz_2} \dot\theta_2 \rightarrow \frac{\partial}{\partial t} \rightarrow \underline{m_2 l_2^2 \ddot\theta_2 + I_{zz_2} \ddot\theta_2}$

$\frac{\partial}{\partial t} \frac{\partial L}{\partial \dot\theta_1} = m_1 l_1^2 \ddot\theta_1 - 2 m_2 (L_1 + l_2 c_2) \dot\theta_1 l_2 s_2 \dot\theta_2 + m_2 (L_1 + l_2 c_2)^2 \ddot\theta_1 + I_{yy_1} \ddot\theta_1 + I_{yy_2} \ddot\theta_1 c_2^2 - 2 I_{yy_2} \dot\theta_1 c_2 s_2 \dot\theta_2$
$+ I_{xx_2} \ddot\theta_1 s_2^2 + 2 I_{xx_2} \dot\theta_1 s_2 c_2 \dot\theta_2$

EOM

$m_1 l_1^2 \ddot\theta_1 - 2 m_2 (L_1 + l_2 c_2) \dot\theta_1 l_2 s_2 \dot\theta_2 + m_2 (L_1 + l_2 c_2)^2 \ddot\theta_1 + I_{yy_1} \ddot\theta_1 + I_{yy_2} \ddot\theta_1 c_2^2 - 2 I_{yy_2} \dot\theta_1 c_2 s_2 \dot\theta_2 + I_{xx_2} \ddot\theta_1 s_2^2 + 2 I_{xx_2} \dot\theta_1 s_2 c_2 \dot\theta_2$

$m_2 l_2^2 \ddot\theta_2 + I_{zz} \ddot\theta_2 + m_2 (L_1 + l_2 c_2) l_2 s_2 \underset{\dot\theta_1^2}{\dot\theta_2} + m_2 g L_2 c_2 \dot\theta_2 + I_{yy_2} \dot\theta_1^2 c_2 s_2 \dot\theta_2 - I_{xx_2} \dot\theta_1^2 s_2 c_2 \dot\theta_2$

Mass matrix:

$$\begin{bmatrix} m_1 l_1^2 + m_2(L_1+l_2C_2)^2 + I_{yy_1} + I_{yy_2}C_2^2 + I_{xx_2}S_2^2 & 0 \\ 0 & m_2 l_2^2 + I_{zz} \end{bmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix}$$

1) $-2m_2(L_1+l_2C_2)\dot{\theta}_1 l_2 S_2 \dot{\theta}_2 - 2I_{yy_2}\dot{\theta}_1 C_2 S_2 \dot{\theta}_2 + 2I_{xx_2}\dot{\theta}_1 C_2 S_2 \dot{\theta}_2$

$\hookrightarrow -2m_2(L_1+l_2C_2) l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 + 2(I_{xx_2}-I_{yy_2}) C_2 S_2 \dot{\theta}_1 \dot{\theta}_2$

2) $m_2 L_1 l_2 S_2 \dot{\theta}_2 + m_2 l_2^2 S_2 \dot{\theta}_2 + m_2 g L_2 C_2 \dot{\theta}_2 + I_{yy_2}\dot{\theta}_1^2 S_2 C_2 \dot{\theta}_2 - I_{xx_2}\dot{\theta}_1^2 S_2 C_2 \dot{\theta}_2$

$\hookrightarrow m_2 l_2(L_1+cl_2) S_2 \dot{\theta}_2 \dot{\theta}_1^2 + m_2 g L_2 C_2 \dot{\theta}_2 + (I_{yy_2}-I_{xx_2}) S_2 C_2 \dot{\theta}_1^2 \dot{\theta}_2$

$\dot{m}_{11}\dot{\theta}_1 + \dot{m}_{12}\dot{\theta}_2 = 2C_{11}\theta$

$$\dot{M} = \begin{bmatrix} -2m_2(L_1+l_2C_2)l_2S_2\dot{\theta}_2 \mp 2I_{yy_2}C_2S_2\dot{\theta}_2 \pm 2I_{xx_2}S_2C_2\dot{\theta}_2 & 0 \\ 0 & 0 \end{bmatrix} \quad \dot{M} - 2C$$

$$= \begin{bmatrix} -2m_2(L_1+l_2C_2)l_2S_2\dot{\theta}_2 + 2(I_{xx_2}-I_{yy_2})S_2C_2\dot{\theta}_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$2C = 2\begin{bmatrix} -m_2(L_1+l_2C_2)l_2S_2\dot{\theta}_2 + (I_{xx_2}-I_{yy_2})S_2C\dot{\theta}_2 & -m_2(L_1+l_2C_2)\dot{\theta}_1 l_2 S_2 + (I_{xx_2}-I_{yy_2})\dot{\theta}_1 S_2 C_2 \\ -(I_{xx_2}-I_{yy_2})\dot{\theta}_2 S_2 C_2 + m_2(L_1+l_2C_2)\dot{\theta}_1 l_2 S_2 \dot{\theta}_2 + m g L_2 C_2 \dot{\theta}_2 & 0 \end{bmatrix}$$

$q = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$

$\Rightarrow \underline{M}(q)\ddot{q} + \underline{C}(q,\dot{q})\dot{q} + \underline{G}(q) = \underline{\tau}$

To model it

$q_1 = q \quad \dot{q}_1 = \dot{q}$

$q_2 = \dot{q}_1$

$\dot{q}_2 = \ddot{q}_1$

$M\dot{q}_2 + Cq_2 = \tau$

$q_2 = \dot{q}_1$

find $\dot{q}_2$ for fsolve

$\dot{q}_2 = M^{-1}[\tau - Cq_2] \rightarrow$ solve for constant torque

Derive a control law to track

$$q_{1,d} = \theta_{1,d} = 0.5\left(1-\cos\left(\frac{\pi}{60}t\right)\right) \text{ rad}$$

$$q_{2,d} = \theta_{2,d} = 0.5\left(1+\cos\left(\frac{\pi}{20}t\right)\right) \text{ rad}$$

$$\tilde{q} = q - q_d$$

Torque

$$\tau = M(q)\ddot{q}_d + C(q,\dot{q})\dot{q}_d - K_p\tilde{q} - K_D\dot{\tilde{q}}$$

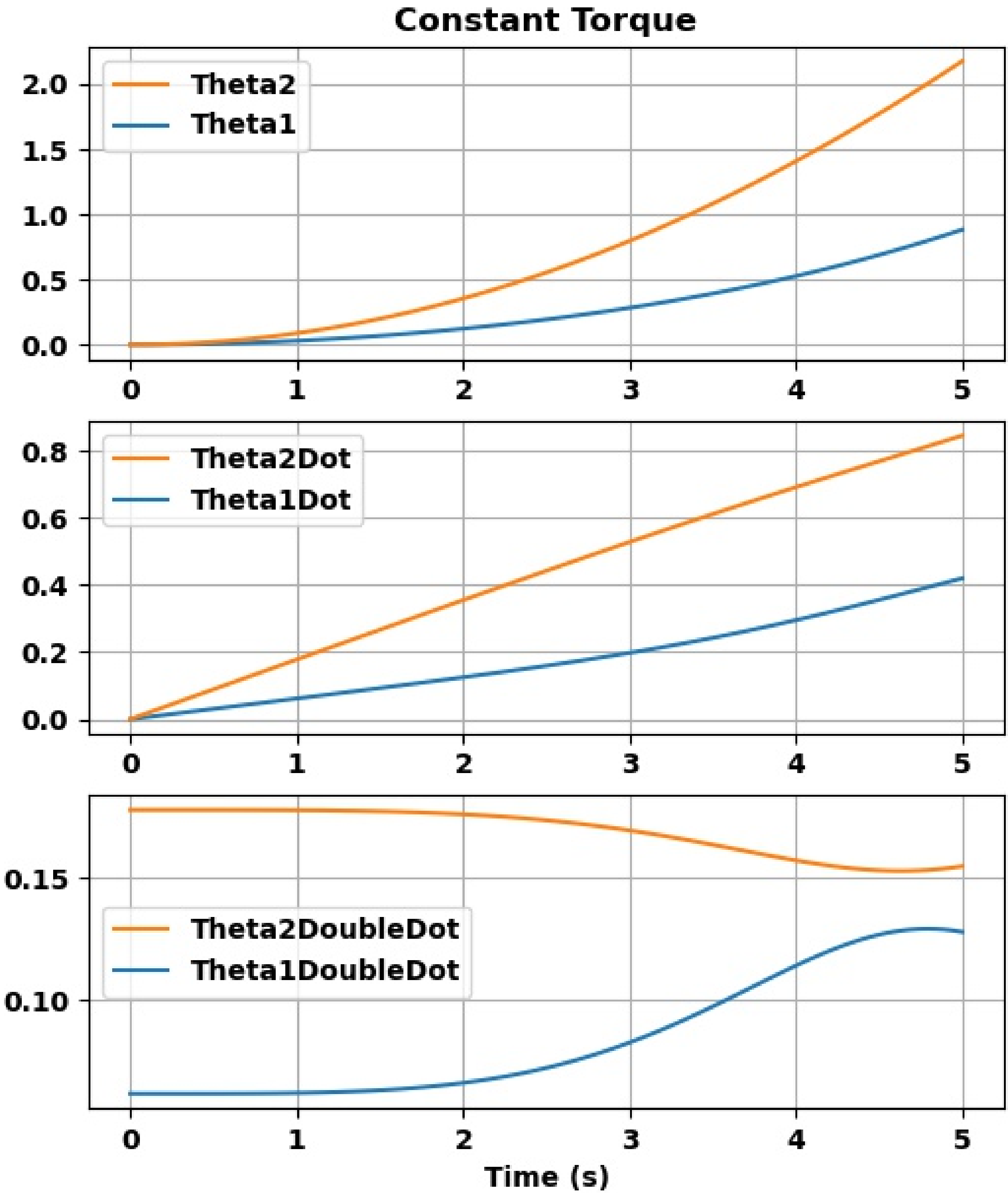
$$\dot{\theta}_{1d} = 0.5\left(\sin\left(\frac{\pi}{60}t\right)\frac{\pi}{60}\right)$$

$$\dot{\theta}_{2d} = -0.5\left(\sin\left(\frac{\pi}{20}t\right)\frac{\pi}{20}\right)$$

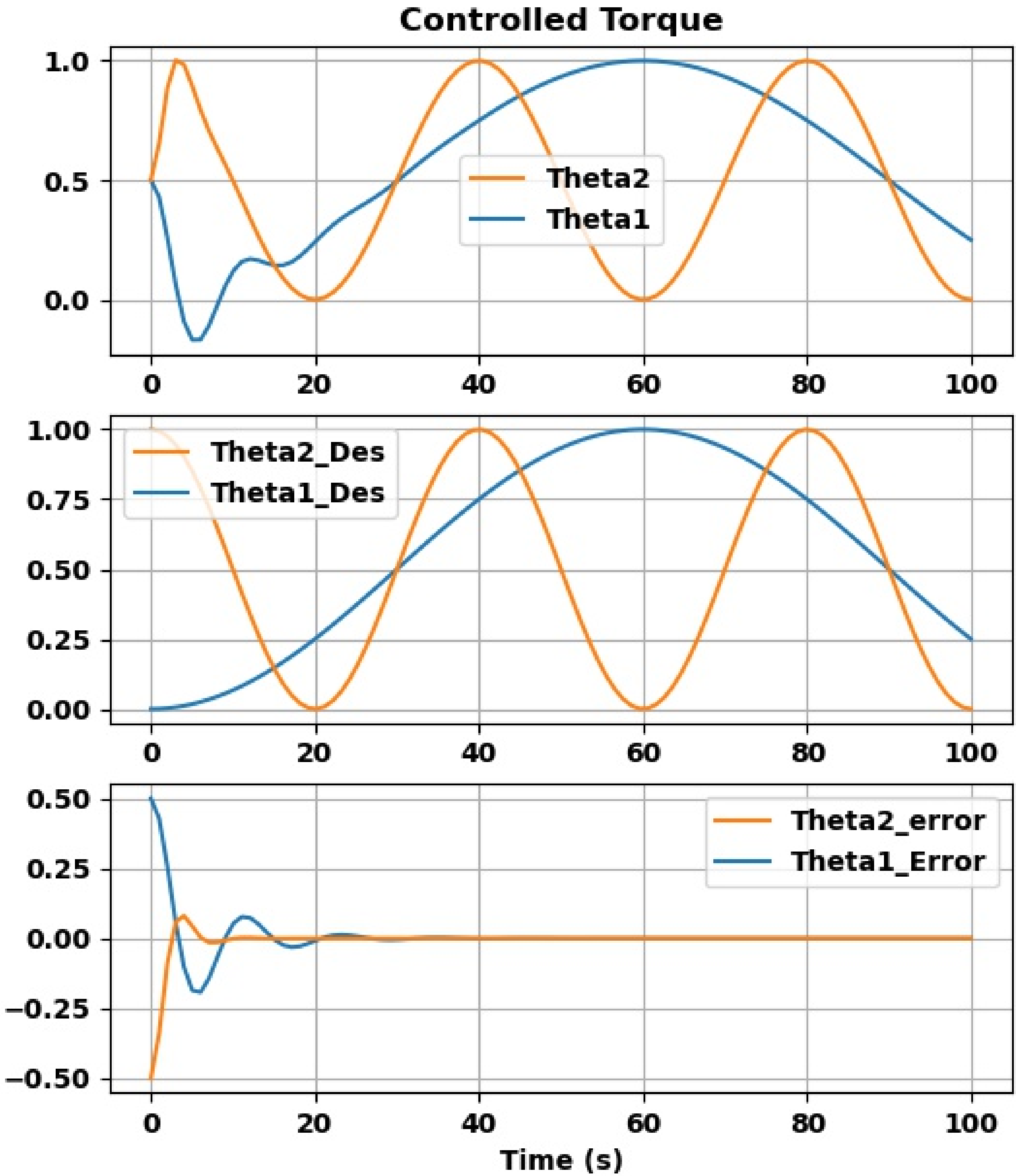
$$\ddot{\theta}_{1d} = 0.5\left(\cos\left(\frac{\pi}{60}t\right)\left(\frac{\pi}{60}\right)^2\right)$$

$$\ddot{\theta}_{2d} = -0.5\left(\cos\left(\frac{\pi}{20}t\right)\left(\frac{\pi}{20}\right)^2\right)$$

Choose $K_p = K_D = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$



Constant Torque
Theta2
Theta1
2.0
1.5
1.0
0.5
0.0
0
1
2
3
4
5
Theta2Dot
Theta1Dot
0.8
0.6
0.4
0.2
0.0
Theta2DoubleDot
Theta1DoubleDot
0.15
0.10
Time (s)


Controlled Torque
Theta2
Theta1
Theta2_Des
Theta1_Des
Theta2_error
Theta1_Error
Time (s)