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TRANSFER OF POLARIZED RADIATION IN SELF-ABSORBED SYNCHROTRON SOURCES. I. RESULTS FOR A HOMOGENEOUS SOURCE

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ABSTRACT

The solution to the equation of transfer of polarized radiation in a stationary, homogeneous, rarefied medium is applied to self-absorbed synchrotron sources. Relativistic electrons (independent of the presence of any cold plasma) can quite easily produce in such sources significant Faraday rotation and/or conversion of linear to circular polarization. Structural inhomogeneities do not obviate the importance of these phenomena in cosmic, compact nonthermal sources. Contrary to the calculation of Pacholczyk and Swihart, the circular polarization for a homogeneous source changes sign just below the self-absorption turnover as the source becomes opaque, even when polarization conversion dominates; however, for a physically realistic source, structural inhomogeneity may alter this behavior. The observational evidence bearing upon these effects is reviewed.

Subject headings: polarization — radiative transfer — radio sources: variable — synchrotron radiation

I. INTRODUCTION

In previous papers [Jones, O'Dell, and Stein 1974*a* (JOS I); Jones, O'Dell, and Stein 1974*b* (JOS II); Burbidge, Jones, and O'Dell 1974 (BJO)], we have discussed various restrictions on physical parameters (JOS I; JOS II) and energetic requirements (BJO) of compact nonthermal sources derivable from observation and the theory of incoherent electron synchrotron radiation, including effects of possible anisotropies in the electron pitch angle distribution. Here we consider the polarization properties expected for cosmic, self-absorbed synchrotron sources.

In § II we give the appropriate transfer equation and its solution for nearly transverse electromagnetic waves (valid in the limit of a weakly "anisotropic" dielectric tensor) propagating in a stationary, homogeneous medium. Although the necessary coefficients for this transfer equation may be determined directly from the dielectric tensor without prior knowledge of the plasma characteristic waves, the nature of these waves is automatically included so that the effects of Faraday rotation and interconversion of linear and circular polarization as well as polarized absorption are properly accounted for. We summarize in § III the transfer coefficients relevant to cosmic, self-absorbed synchrotron sources, and stress the importance of the two birefringent (Faraday) phenomena (rotation and conversion) near the self-absorption turnover, independent of the presence of any cold plasma. Because of the likely importance of birefringence in compact radio sources, in § IV we examine, for a homogeneous, stationary, rarefied synchrotron source, the strong-rotativity limit, in which Faraday rotation constitutes the primary propagation effect. In § V we consider the strong-rotativity limit, with weak anisotropy of the dielectric tensor, for small absorption depth through an inhomogeneous source, modeled as a matrix of homogeneous domains with statistically independent orientations. We compare in § VI the theoretical predictions with the observed polarization properties of compact nonthermal radio sources, and summarize our conclusions in § VII.

II. TRANSFER OF POLARIZED RADIATION

Sazonov (1969*b*) has given the equation of transfer for a homogeneous medium with a weakly anisotropic dielectric tensor (i.e., for nearly transverse electromagnetic waves)

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$$\begin{Vmatrix} \left(\frac{d}{dl}+\kappa_I\right) & \kappa_Q & \kappa_U & \kappa_V \\ \kappa_Q & \left(\frac{d}{dl}+\kappa_I\right) & \kappa^*_V & -\kappa^*_U \\ \kappa_U & -\kappa^*_V & \left(\frac{d}{dl}+\kappa_I\right) & \kappa^*_Q \\ \kappa_V & \kappa^*_U & -\kappa^*_Q & \left(\frac{d}{dl}+\kappa_I\right) \end{Vmatrix} \begin{Vmatrix} I_\nu \\ Q_\nu \\ U_\nu \\ V_\nu \end{Vmatrix} = \begin{Vmatrix} \eta_\nu{}^I \\ \eta_\nu{}^Q \\ \eta_\nu{}^U \\ \eta_\nu{}^V \end{Vmatrix}, \tag{1}$$

with $d(I_\nu, Q_\nu, U_\nu, V_\nu)/dl$ the convective derivative (in spatial units) of the spectral intensities (see Appendix A). This equation includes the effects of emission $[\eta_\nu{}^{(I,Q,U,V)}]$, absorption $[\kappa_{(I,Q,U,V)}]$, and transformation of the polarization either as rotation $[\kappa^*_V]$ of the polarization ellipse (Faraday rotation) or as conversion $[\kappa^*_{(Q,U)}]$ between linear and circular polarization.[1] The two transformation effects occur because of birefringence, i.e., because of the differing phase velocities between the two plasma characteristic waves when they are partly circularly polarized or partly linearly polarized, respectively.

The above equation is applicable only for a homogeneous, stationary medium. Provided that gradients in the medium are sufficiently small, the same transfer equation—but with variable transfer coefficients—may be used. However, in many cases, as we shall show in Paper II, this restriction proves too severe to be satisfied by cosmic, compact radio sources. Nevertheless, since the homogeneous case may be solved analytically, and since many features of this solution also apply for inhomogeneous sources, we here consider a homogeneous, stationary source, and shall in Paper II investigate some effects of inhomogeneity.

For a homogeneous medium, rotation of coordinates permits the simplification $\eta_\nu{}^U = 0$ and $\kappa_U = \kappa^*_U = 0$; the transfer equation (eq. [1]) then becomes

$$\begin{Vmatrix} \left(\frac{d}{d\tau}+1\right) & \zeta_Q & 0 & \zeta_V \\ \zeta_Q & \left(\frac{d}{d\tau}+1\right) & \zeta^*_V & 0 \\ 0 & -\zeta^*_V & \left(\frac{d}{d\tau}+1\right) & \zeta^*_Q \\ \zeta_V & 0 & -\zeta^*_Q & \left(\frac{d}{d\tau}+1\right) \end{Vmatrix} \left(\begin{Vmatrix} 1 \\ \pi_Q \\ \pi_U \\ \pi_V \end{Vmatrix} I_\nu \right) = \begin{Vmatrix} 1 \\ \epsilon_Q \\ 0 \\ \epsilon_V \end{Vmatrix} J_\nu\,, \tag{2}$$

with the absorption depth[2] $d\tau \equiv \kappa dl$ and the spectral source function $J_\nu \equiv (\eta_\nu/\kappa)$. The remaining symbols denote normalized spectral emissivities $\epsilon_{(Q,V)} \equiv [\eta^{(Q,V)}/\eta_\nu]$, absorptivities $\zeta_{(Q,V)} \equiv [\kappa_{(Q,V)}/\kappa]$, rotativity $\zeta^*_V \equiv [\kappa^*_V/\kappa]$, and convertibility $\zeta^*_Q \equiv [\kappa^*_Q/\kappa]$. The polarization angle (φ_L) and degree of linear (π_L) and circular (π_c) polarization follow from $\pi_{(Q,U,V)} \equiv [(Q_\nu, U_\nu, V_\nu)/I_\nu]$—namely,

$$\varphi_L = \tfrac{1}{2}\tan^{-1}(\pi_U/\pi_Q)\,, \qquad \pi_L = (\pi_Q{}^2 + \pi_U{}^2)^{1/2}\,, \qquad \pi_C = -\pi_V\,, \tag{3}$$

where the adopted sign of π_C corresponds to the conventional definition of helicity.

Zheleznyakov, Suvorov, and Shaposhnikov (1974) and Pacholczyk and Swihart (1975) obtained solutions to this transfer equation (eq. [2]) for a homogeneous, stationary medium. In Appendix B, we give an alternative representation and—since observations generally measure spectral flux rather than spectral intensity—we also present the solution appropriately averaged for a homogeneous sphere. Averaging tends to wash out oscillations in the degree of polarization and intensity. From a practical observational standpoint, the intensity for polarized,

[1] Some of the manifestations of conversion have been termed "repolarization" (Pacholczyk 1973) and "pulsation" (Pacholczyk and Swihart 1970).

[2] As a simplification the subscript I will henceforth be implicit.

cosmic, nonthermal sources behave like that of an unpolarized source; in particular, oscillations in intensity (Pacholczyk and Swihart 1975) do not generally occur at detectable levels.

III. RADIATIVE TRANSFER IN COMPACT NONTHERMAL SOURCES

Self-absorbed, incoherent electron-synchrotron radiation describes most properties of extragalactic, compact, nonthermal radio sources (see, e.g., Pauliny-Toth and Kellermann 1966). In particular, from a study of several such sources (JOS II), we have concluded that thermal absorption and dielectric suppression (Razin-Tsytovich effect) usually play no role. Accordingly, in Appendix C, we summarize the transfer coefficients for a power-law distribution of synchrotron-emitting electrons, in the absence of dielectric suppression. These assumptions, inferred from observations, insure a weakly anisotropic dielectric tensor (see Appendix A). Here, we list the normalized transfer coefficients (for an isotropic electron distribution) in a form which proves particularly useful for self-absorbed synchrotron sources:

$$\epsilon_Q = \epsilon_\alpha{}^Q\,, \qquad \epsilon_V = -\epsilon_\alpha{}^V \frac{1}{\gamma_n}\left(\frac{\nu}{\nu_n}\right)^{-1/2} \cot\theta\,, \tag{4}$$

$$\zeta_Q = \zeta_\alpha{}^Q\,, \qquad \zeta_V = -\zeta_\alpha{}^V \frac{1}{\gamma_n}\left(\frac{\nu}{\nu_n}\right)^{-1/2} \cot\theta\,, \tag{5}$$

$$\zeta^*{}_Q = -\zeta^*{}_\alpha{}^Q\left(\frac{\gamma_n}{\gamma_i}\right)^{2\alpha-1}\left(\frac{\nu}{\nu_n}\right)^{\alpha-1/2}\left[1-\left(\frac{\gamma_i^2}{\gamma_n^2}\frac{\nu_n}{\nu}\right)^{\alpha-1/2}\right]\Big/(\alpha-\tfrac{1}{2})\,, \qquad \alpha > \tfrac{1}{2}\,, \tag{6}$$

$$\zeta^*{}_V = \zeta^*{}_\alpha{}^V \frac{\ln\gamma_i}{\gamma_i}\left(\frac{\gamma_n}{\gamma_i}\right)^{2\alpha+1}\left(\frac{\nu}{\nu_n}\right)^{\alpha+1/2}\cot\theta\,,$$

with α the spectral index, θ the angle between the magnetic field and the wave-vector, and γ_i the Lorentz factor of the low-energy electron cutoff. We define the order-unity dimensionless factors $\epsilon_\alpha{}^Q$, $\epsilon_\alpha{}^V$, $\zeta_\alpha{}^Q$, $\zeta_\alpha{}^V$, $\zeta^*{}_\alpha{}^Q$, and $\zeta^*{}_\alpha{}^V$ in Appendix C and give numerical values in Table 1. The characteristic Lorentz factor of electrons radiating near a fiducial frequency ν_n is $\gamma_n \equiv \nu_n/\nu_{B\perp}$. For self-absorbed synchrotron sources, it proves convenient to choose a fiducial frequency ν_n such that the absorption depth τ and the spectral source function J_ν such that the absorption depth τ and the spectral source function J_ν satisfy

$$\tau = (\nu/\nu_n)^{-(\alpha+5/2)}\,, \tag{7}$$

$$J_\nu = J_n(\nu/\nu_n)^{5/2}\,, \tag{8}$$

where[3]

$$J_n \equiv J_\alpha m v_{B\perp}{}^2\left(\frac{\nu_n}{\nu_{B\perp}}\right)^{5/2} = J_\alpha m \nu_n{}^2\gamma_n \tag{9}$$

[with $J_\alpha \equiv (\eta_\alpha/\kappa_\alpha)$ of order unity, and η_α and κ_α dimensionless quantities defined in Appendix C]. Under these assumptions, $\gamma_n \approx (kT_n/mc^2)$ with T_n the maximum brightness temperature. Observations of spectral shape and angular size thus permit estimation of ν_n and γ_n, respectively (see BJO). Since observations indicate and various theoretical considerations dictate that for compact radio sources 10^{11} K $\leqslant T_n \leqslant 10^{12}$ K (Kellermann and Pauliny-Toth 1969; and BJO), the characteristic Lorentz factor near the self-absorption turnover invariably satisfies $10^2 \leqslant \gamma_n \leqslant 10^3$.

Although nonrelativistic ("cold") electrons quite likely contribute negligibly to the emissivity and absorptivity in compact nonthermal sources, they may transform the polarization. The relative importance to the convertibilities and rotativities of the relativistic and nonrelativistic plasmas depends upon the relative number densities and low-energy cutoff of the relativistic electrons as

$$\begin{aligned}\frac{\zeta^*{}_Q{}^{(r)}}{\zeta^*{}_Q{}^{(c)}} &= 2\alpha\,\frac{[1-(\gamma_i^2\gamma_n^{-2}\nu_n/\nu)^{\alpha-1/2}]}{(\alpha-\frac{1}{2})}\,\gamma_i\left(\frac{n_r}{n_c}\right) \qquad (\alpha > \tfrac{1}{2})\,,\\ \frac{\zeta^*{}_V{}^{(r)}}{\zeta^*{}_V{}^{(c)}} &= 2\alpha\,\frac{(\alpha+3/2)}{(\alpha+1)}\,\frac{\ln\gamma_i}{\gamma_i^2}\left(\frac{n_r}{n_c}\right)\,,\end{aligned} \tag{10}$$

where the self-absorption turnover provides an estimate of the number density of relativistic electrons n_r—namely,

$$n_r = \frac{1}{2\alpha\kappa_\alpha}\,\frac{\gamma_n{}^3}{r_e\lambda_n L}\left(\frac{\gamma_n}{\gamma_i}\right)^{2\alpha} \tag{11}$$

with $\lambda_n = (c/\nu_n)$ and L the characteristic longitudinal dimension of the compact source. Provided that dielectric

[3] All these relations apply in the proper-frame of the source.

TABLE 1

DIMENSIONLESS FUNCTIONS OF SPECTRAL INDEX α

α	η_α	κ_α	J_α	ϵ_α^Q	ϵ_α^V	ζ_α^Q	ζ_α^V	ζ^{*Q}_α †	ζ^{*V}_α
-.25	9.3689	1.7264	5.4269	.5294	.4598	.6522	1.3966	.5792	1.9308
-.20	5.7611	1.6900	3.4090	.5455	.6341	.6610	1.4504	.5917	1.9231
-.15	4.1443	1.6665	2.4868	.5604	.7712	.6694	1.5015	.6001	1.9061
-.10	3.2364	1.6540	1.9567	.5745	.8847	.6774	1.5504	.6046	1.8809
-.05	2.6609	1.6511	1.6116	.5876	.9819	.6850	1.5973	.6057	1.8489
.00	2.2672	1.6565	1.3687	.6000	1.0676	.6923	1.6424	.6037	1.8110
.05	1.9838	1.6697	1.1881	.6117	1.1445	.6992	1.6860	.5989	1.7682
.10	1.7720	1.6901	1.0485	.6226	1.2146	.7059	1.7282	.5917	1.7212
.15	1.6095	1.7173	.9372	.6330	1.2794	.7122	1.7691	.5823	1.6709
.20	1.4822	1.7512	.8464	.6429	1.3398	.7183	1.8089	.5710	1.6179
.25	1.3811	1.7916	.7709	.6522	1.3966	.7241	1.8476	.5582	1.5628
.30	1.3000	1.8386	.7071	.6610	1.4504	.7297	1.8853	.5439	1.5062
.35	1.2345	1.8921	.6524	.6694	1.5015	.7351	1.9222	.5285	1.4485
.40	1.1815	1.9524	.6051	.6774	1.5504	.7403	1.9582	.5122	1.3902
.45	1.1387	2.0198	.5638	.6850	1.5973	.7452	1.9935	.4951	1.3317
.50	1.1044	2.0944	.5273	.6923	1.6424	.7500	2.0280	.4775	1.2732
.55	1.0772	2.1767	.4949	.6992	1.6860	.7546	2.0619	.4594	1.2152
.60	1.0563	2.2670	.4660	.7059	1.7282	.7590	2.0952	.4411	1.1579
.65	1.0408	2.3659	.4399	.7122	1.7691	.7633	2.1278	.4227	1.1015
.70	1.0301	2.4739	.4164	.7183	1.8089	.7674	2.1599	.4042	1.0462
.75	1.0238	2.5916	.3950	.7241	1.8476	.7714	2.1914	.3859	.9922
.80	1.0214	2.7197	.3756	.7297	1.8853	.7753	2.2225	.3677	.9396
.85	1.0228	2.8591	.3577	.7351	1.9222	.7790	2.2531	.3498	.8886
.90	1.0276	3.0105	.3413	.7403	1.9582	.7826	2.2832	.3322	.8392
.95	1.0358	3.1749	.3262	.7452	1.9935	.7861	2.3129	.3150	.7915
1.00	1.0472	3.3533	.3123	.7500	2.0280	.7895	2.3422	.2982	.7455
1.05	1.0618	3.5470	.2993	.7546	2.0619	.7927	2.3710	.2819	.7014
1.10	1.0795	3.7572	.2873	.7590	2.0952	.7959	2.3995	.2662	.6591
1.15	1.1004	3.9853	.2761	.7633	2.1278	.7990	2.4277	.2509	.6186
1.20	1.1245	4.2328	.2657	.7674	2.1599	.8020	2.4555	.2363	.5799
1.25	1.1518	4.5015	.2559	.7714	2.1914	.8049	2.4829	.2221	.5430
1.30	1.1825	4.7932	.2467	.7753	2.2225	.8077	2.5100	.2086	.5080
1.35	1.2166	5.1100	.2381	.7790	2.2531	.8104	2.5368	.1957	.4747
1.40	1.2544	5.4542	.2300	.7826	2.2832	.8131	2.5634	.1833	.4431
1.45	1.2959	5.8283	.2223	.7861	2.3129	.8157	2.5896	.1716	.4132
1.50	1.3413	6.2349	.2151	.7895	2.3422	.8182	2.6155	.1604	.3849

† Note that eq. C.15 for $\zeta_Q{}^*$ is strictly applicable only for $\alpha > 0.5$.

suppression is not important, each transfer coefficient for a mixture of relativistic and thermal plasmas is merely the sum of the respective transfer coefficients for all species.

For the range of parameters observed—$\alpha \gtrsim 0.5$ and $10^2 \lesssim \gamma_n \lesssim 10^3$—the normalized transfer coefficients for synchrotron-emitting electrons conform to the hierarchy $\zeta^*{}_Q{}^2 \gtrsim 1 > \zeta_Q{}^2 > \epsilon_Q{}^2 \gg \zeta_V{}^2 > \epsilon_V{}^2$. The normalized rotativity $\zeta^*{}_V$, which determines the importance of Faraday rotation relative to self-absorption, depends upon the low-energy cutoff γ_i (eq. [6]) and upon the density of nonrelativistic electrons (eq. [10]). For the relativistic electrons alone, the strong-rotativity limit ($\zeta^*{}_V{}^2 \gg \zeta^*{}_Q{}^2$) applies near $\nu \approx \nu_n$ and cot $\theta \approx 1$ unless the restriction $(\gamma_i/\gamma_n) > 0.1$ holds. Since this limit has been discussed elsewhere (see, e.g., Pacholczyk and Swihart 1974) and seems applicable over much of the range of parameters expected for cosmic, self-absorbed synchrotron sources, we devote the next section to it as applied to homogeneous sources.

We have, however, calculated the proper frequency dependence of the polarization angle (φ_L), and degrees of linear (π_L) and circular (π_C) polarization, for a uniform sphere, from the full solution (Appendix B), without making the strong-rotativity approximation. We present these results in Figure 1 for the probable range of parameters γ_n and γ_i (neglecting any contribution from a possible cold plasma), and shall refer to them throughout the remainder of this discussion.

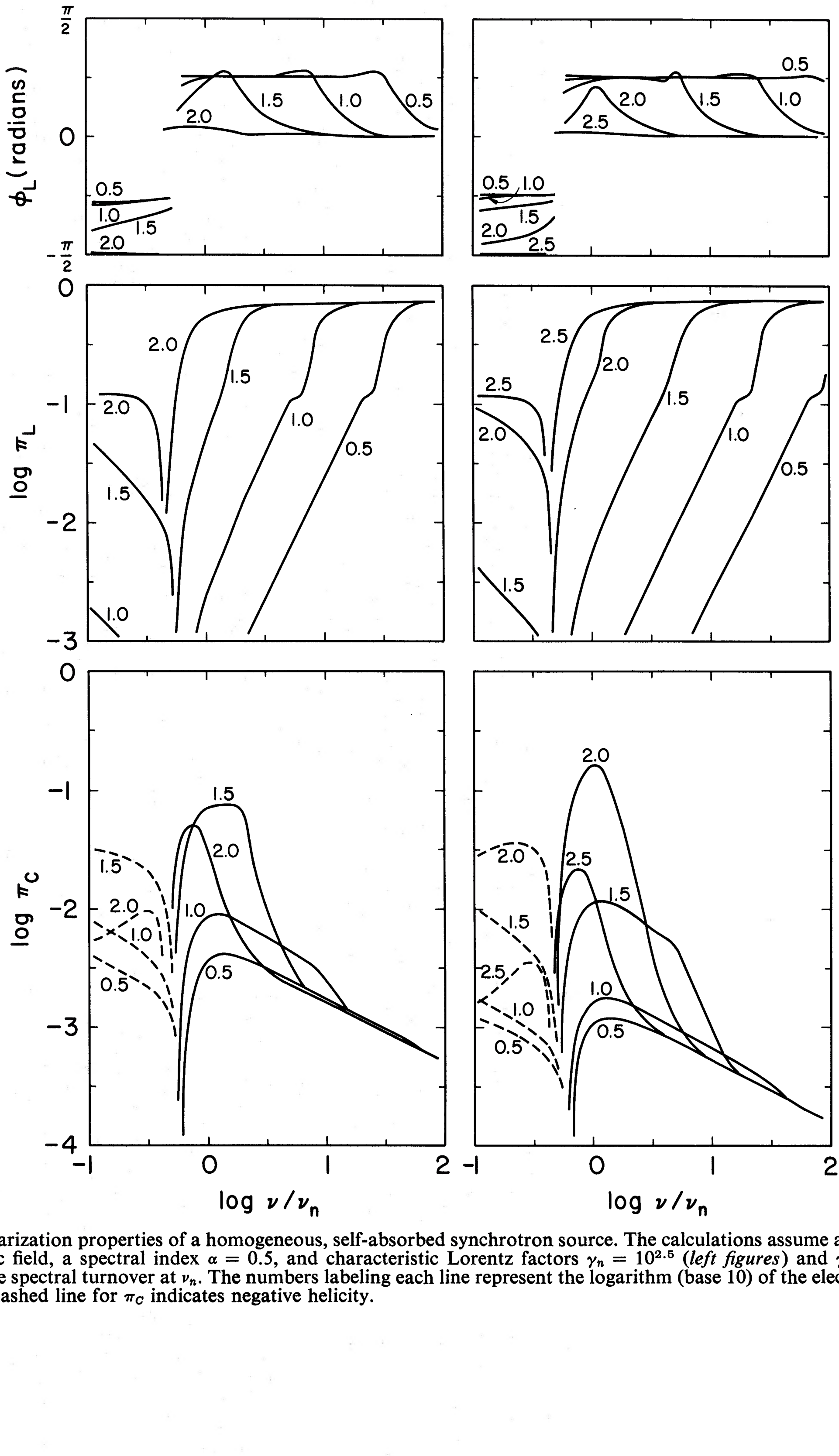

FIG. 1.—Polarization properties of a homogeneous, self-absorbed synchrotron source. The calculations assume an angle $\theta = \pi/4$ to the magnetic field, a spectral index $\alpha = 0.5$, and characteristic Lorentz factors $\gamma_n = 10^{2.5}$ (*left figures*) and $\gamma_n = 10^{3.0}$ (*right figures*) near the spectral turnover at ν_n. The numbers labeling each line represent the logarithm (base 10) of the electron low-energy cutoff γ_l. The dashed line for π_C indicates negative helicity.

IV. STRONG-ROTATIVITY LIMIT

Because over a wide range of parameters characteristic of cosmic, self-absorbed synchrotron sources, Faraday rotation (due to relativistic and/or nonrelativistic electrons) constitutes the dominant propagation effect near the self-absorption turnover, we consider here the solution to the transfer equation (for weakly anisotropic dielectric tensor), in the corresponding limit

$$\zeta^*{}_V{}^2 \gg \zeta^*{}_Q{}^2 \gtrsim 1 > \zeta_Q{}^2 > \epsilon_Q{}^2 \gg \zeta_V{}^2 > \epsilon_V{}^2 \,. \tag{12}$$

In terms of characteristic-wave (normal-mode) propagation, the limit $(\zeta^*{}_V/\zeta^*{}_Q)^2 \gg 1$ is equivalent to the quasi-longitudinal (QL) approximation (see Jones and O'Dell 1977 [Paper II]). For a homogeneous sphere with central absorption depth τ_s, we summarize the more detailed results, given in Appendix D: For large absorption depth $(\tau_s \gg 1)$

$$\begin{aligned}
\bar{\varphi}_L &\rightarrow \frac{\pi}{4}\,\mathrm{sgn}\,(\zeta^*{}_V)\,\mathrm{sgn}\,(\epsilon_Q - \zeta_Q)\,, \\
\bar{\pi}_L &\rightarrow |\bar{\pi}_U| \approx \left|\frac{\epsilon_Q - \zeta_Q}{\zeta^*{}_V}\right| \qquad (\tau_s \gg 1)\,, \\
\bar{\pi}_C &\rightarrow -\left[(\epsilon_V - \zeta_V) + \left(\frac{\zeta^*{}_Q}{\zeta^*{}_V}\right)(\epsilon_Q - \zeta_Q)\right];
\end{aligned} \tag{13}$$

for small absorption depth $(\tau_s \ll 1)$ but large rotation depth $(|\zeta^*{}_V|\tau_s \gg 1)$

$$\begin{aligned}
\bar{\varphi}_L &\rightarrow \frac{\pi}{4}\,\mathrm{sgn}\,(\zeta^*{}_V)\,, \\
\bar{\pi}_L &\rightarrow |\bar{\pi}_U| \approx \frac{3}{2}\frac{\epsilon_Q}{|\zeta^*{}_V|\tau_s} \qquad (\tau \ll 1,\, |\zeta^*{}_V|\tau_s \gg 1)\,, \\
\bar{\pi}_C &\rightarrow -\left[\epsilon_V + \left(\frac{\zeta^*{}_Q}{\zeta^*{}_V}\right)\epsilon_Q\right];
\end{aligned} \tag{14}$$

while for both small absorption $(\tau_s \ll 1)$ and rotation $(|\zeta^*{}_V|\tau_s \ll 1)$ depths

$$\begin{aligned}
\bar{\varphi}_L &\rightarrow \tfrac{3}{16}(\zeta^*{}_V\tau_s) \rightarrow 0\,, \\
\bar{\pi}_L &\rightarrow |\bar{\pi}_Q| \approx \epsilon_Q \qquad (\tau_s \ll 1,\, |\zeta^*{}_V|\tau_s \ll 1)\,, \\
\bar{\pi}_C &\rightarrow -\epsilon_V\,.
\end{aligned} \tag{15}$$

Note (eqs. [6] and [10]) that there always exists a frequency above which the strong-rotativity limit applies $(\zeta^*{}_V{}^2 \gg 1)$ and below which it does not. However, detectable Faraday rotation occurs only when $|\zeta^*{}_V|\tau \gtrsim 1$ (eq. [14]).

Contrary to earlier results (Pacholczyk and Swihart 1974), equations (13) and (14) demonstrate that, for a homogeneous source, the circular polarization changes sign just below the self-absorption turnover in the presence of Faraday rotation and polarization conversion (also see Fig. 1), as it does in the absence of such propagation effects (Pacholczyk and Swihart 1971). This result follows from the fact that both $(\epsilon_Q/\zeta_Q) < 1$ and $(\epsilon_V/\zeta_V) < 1$ (cf. Table 1). The earlier erroneous claim of no change in sign resulted from neglect of ζ_Q (the Q-absorptivity) in the adopted transfer equations, an invalid procedure for physical parameters characteristic of cosmic, self-absorbed synchrotron sources. Melrose (1971), who also considered a strong-rotativity limit, did obtain the necessary change in sign, but he neglected the conversion of linear into circular polarization as well as the Q-absorptivity by assuming purely circular orthogonal normal-wave polarizations.[4] Existing observations do not demonstrate such a reversal in sign; however, this may reflect structural inhomogeneity, as no source with measured circular polarization has really simple spectral and angular appearance (see § VI), and every source contains a finite transition region at its boundary (see Paper II).

[4] Purely circular characteristic waves occur only when $\zeta^*{}_Q \equiv \zeta_Q \equiv 0$, in which case the transfer equations decouple into two separate relations (Zheleznyakov 1968; Melrose 1971; cf. eq. [2]). However, this limit does not generally apply in cosmic sources to sufficient accuracy to allow determination of small quantities such as circular polarization. Furthermore, it is inconsistent to retain ϵ_Q while dropping ζ_Q since each gives a comparable contribution for large absorption depths (see Appendix A for further remarks).

Equations (13) and (14) illustrate that for large (small) absorption depth, the polarization angle $\varphi_L = \mp\pi/4$ ($\pm\pi/4$)—depending upon polarity of the magnetic field—when $|\zeta^*_V|\tau_s \gg 1$; while for $|\zeta^*_V|\tau_s \to 0$, φ_L approaches the intrinsic emission value 0 (eq. [15]). This compares with the values $\varphi_L = \pi/2$ (0) for large (small) absorption depths in the absence of Faraday rotation (Kellermann and Pauliny-Toth 1968; Aller 1970). Thus, in a homogeneous source, the polarization angle φ_L must rotate by $\sim\pi/2$ across the self-absorption turnover both in the presence and in the absence of a large Faraday depth; however, the presence of a transition region at the boundary or complex structure may alter this behavior as well (Paper II).

In the strong-rotativity limit, the degree of circular polarization may significantly exceed the intrinsic value, so that, except when $\nu \approx \nu_n$, $\pi_C \propto \nu^{-1}$ rather than $\propto \nu^{-1/2}$ (Pacholczyk 1973; Pacholczyk and Swihart 1974), provided that polarization conversion (Sazonov 1969*b*) operates effectively. As equations (13) and (14) indicate, detectable circular polarization from conversion demands, in a uniform source, not only a large rotation depth ($|\zeta^*_V|\tau_s \gtrsim 1$), but also a negligible intrinsic contribution $|\epsilon_V| < \epsilon_Q|\zeta^*_Q/\zeta^*_V|$, so that

$$\tau_s \gtrsim \frac{1}{|\zeta^*_V|} > \left|\frac{\epsilon_V}{\zeta^*_Q\epsilon_Q}\right| = \frac{\epsilon_\alpha{}^V}{\zeta^*{}_\alpha{}^Q\epsilon_\alpha{}^Q}\frac{1}{\gamma_n}\left(\frac{\nu}{\nu_n}\right)^{-1/2}|\cot\theta|\left\{\left(\frac{\gamma_n}{\gamma_i}\right)^{2\alpha-1}\left(\frac{\nu}{\nu_n}\right)^{\alpha-1/2}\frac{[1-(\gamma_i{}^2\nu_n/\gamma_n{}^2\nu)^{\alpha-1/2}]}{(\alpha-\frac{1}{2})}\right\}^{-1} \tag{16}$$

or

$$\left(\frac{\nu}{\nu_n}\right) < \left(\frac{\zeta^*{}_\alpha{}^Q\epsilon_\alpha{}^Q}{\epsilon_\alpha{}^V}\right)\frac{\gamma_n}{|\cot\theta|}\left\{\left(\frac{\gamma_n}{\gamma_i}\right)^{2\alpha-1}\frac{1-(\gamma_i{}^2\nu_n/\gamma_n{}^2\nu)^{\alpha-1/2}}{(\alpha-\frac{1}{2})}\right\}^{2/5}.$$

Thus for compact nonthermal sources with $\gamma_n \lesssim 10^3$, the circular polarization may significantly exceed the intrinsic value only for $\nu \lesssim 10\nu_n$ (assuming $\cot\theta \approx 1$). However, as a practical matter, the dominant compact component contributing at a given radio frequency generally satisfies this condition. Note that this constraint applies independent of the cause of the Faraday rotation. Because of the restricted range of frequencies over which conversion may dominate, the circular polarization in general obeys neither a ν^{-1} nor a $\nu^{-1/2}$ dependence (see Fig. 1).

A large rotation depth also effects depolarization. In the strong rotativity limit ($|\zeta^*_V| \gg 1$) for a uniform source, $\pi_L \approx |\zeta^*_V|^{-1} \propto \nu^{-(\alpha+1/2)}$ for $\tau_s \gg 1$ (eq. [13]), whereas for $\tau_s \ll 1$ but $|\zeta^*_V|\tau_s \gg 1$, one recovers the familiar result $\pi_L \approx (|\zeta^*_V|\tau_s)^{-1} \propto \nu^2$ (eq. [14]). In the limit of small rotation ($|\zeta^*_V|\tau_s \ll 1$), the intrinsic polarization $\pi_L = \epsilon_Q \approx 1$ (eq. [15]), of course, returns. The full solution for a uniform self-absorbed synchrotron source appears in Figure 1.

V. POLARIZATION OF AN INHOMOGENEOUS SOURCE

Inhomogeneity in physically-realistic cosmic sources must, to some degree, modify the polarization behavior from that obtained for a homogeneous source; consequently, we investigate a tractable, though necessarily simple, model for structural inhomogeneity. This extends the earlier investigations of Burn (1966) and Sazonov (1973) who considered depolarization due to cold electrons in inhomogeneous, optically thin sources. We consider a synchrotron source, composed of N uniform domains arranged with random orientations in a three-dimensional array. We further restrict our considerations to the range of frequencies consistent with small absorption depth ($\tau \ll 1$), and the range of low-energy electron cutoffs compatible with the strong-rotativity limit ($|\zeta^*_V| \gg 1$). Under these conditions, the effects of boundaries between adjacent domains do not appreciably alter the transfer of polarized radiation (Paper II), provided that the boundaries are sufficiently abrupt. Furthermore, for cosmic synchrotron sources, the proximity of the dielectric coefficient to unity permits neglect of reflection and refraction.

On the basis of these considerations, we have calculated (Appendix E) the expected statistical standard deviations of the linear (π_L) and circular (π_C) polarizations for the ensemble of sources comprised of N randomly oriented uniform domains, described by a low-energy cutoff (γ_i) and maximum brightness temperature ($\gamma_1 \approx kT_1/mc^2$). Further, we have neglected any possible contribution of a thermal plasma to the transfer coefficients. As a first approximation, the presence of nonrelativistic electrons merely lowers the effective value for the low-energy cutoff γ_i (see eqs. [6] and [10]).

Figure 2 exhibits the results of these calculations. (We should stress that the frequency dependence shown in this figure is statistical in character and does not in general predict the behavior of an individual source.) As Figure 2 indicates, and as the crudest statistical arguments suggest, the degrees of both linear and circular polarization decrease with increasing N. Aside from providing numerical estimates of $\langle\pi_L{}^2\rangle^{1/2}$ and $\langle\pi^2\rangle^{1/2}$ (which we shall, in the next section, compare with observations), the study of this simple model of structural inhomogeneity demonstrates that, for a wide range of source parameters, conversion produces the primary contribution to the circular polarization at frequencies near the self-absorption turnover, and that conversion becomes unimportant only at frequencies $\nu \gtrsim 10\nu_1$.

VI. COMPARISON WITH OBSERVATION

Several factors—including the structural complexity and variability of sources, and the paucity of circular and linear polarization data—make a detailed quantitative comparison of observation with theory impossible at present, particularly since polarization characteristics depend upon structure more strongly than does, say, brightness temperature. However, some features, such as the order of magnitude for the linear (π_L) and circular (π_C) polarization,

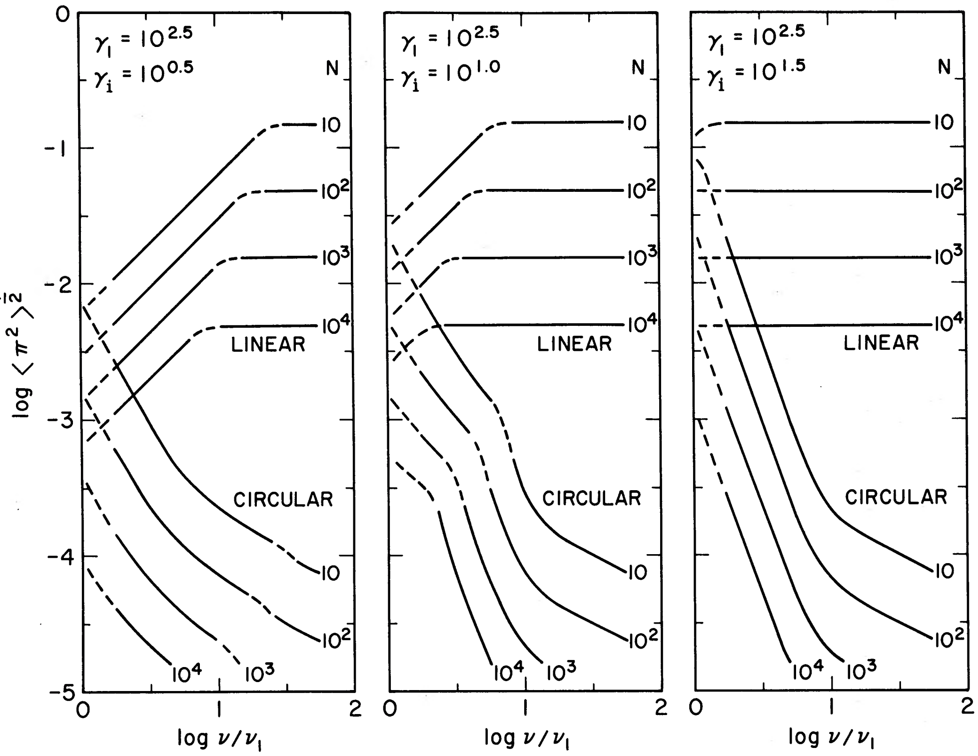

FIG. 2.—Expected standard deviations of polarizations for an inhomogeneous, self-absorbed synchrotron source consisting of N uniform domains. The calculations assume a spectral index $\alpha = 0.5$, a characteristic Lorentz factor $\gamma_1 = 10^{2.5}$ near the spectral turnover at ν_1, and a range of low-energy cutoff γ_i consistent with the strong-rotativity limit. Note that the expected values $\langle \pi_C{}^2 \rangle^{1/2}$ and $\langle \pi_L{}^2 \rangle^{1/2}$ are statistical in nature and do not necessarily predict the frequency dependence of a specific inhomogeneous source.

do permit comparison. As Figure 1 illustrates, homogeneous, self-absorbed sources have considerable difficulty in producing the combination of π_L and π_C typically observed in compact radio sources—namely, $\pi_L \sim 10^{-1.5}$ and $\pi_C \sim 10^{-3}$ for $\nu \lesssim 3\nu_n$.

Since both the spectral shape and VLBI angular structure of compact nonthermal radio sources usually indicate the presence of two or more distinct components, the observed polarization values could represent averages of a small number of more highly polarized components as appears the case for some extended sources (see, e.g., Price and Stull 1973; Miley 1973).

The simple statistical model of N independent, homogeneous cells (§ V, Appendix E, and Fig. 2) produces at high frequencies $\langle \pi_L{}^2 \rangle^{1/2} \approx \epsilon_Q/\sqrt{N} \approx 10^{-1.5}$ for $N \approx 200$, i.e., for a component-to-cell size ratio of $N^{1/3} \approx 6$. Such a degree of structural depolarization reduces the intrinsic circular polarization to approximately $|\epsilon_V|/\sqrt{N} \lesssim 10^{-4}$; however, over most of the allowed range of low-energy cutoffs ($1 < \gamma_i < \gamma_1 \approx 10^3$) polarization conversion provides the dominant contribution to the circular polarization, resulting in $10^{-3} \lesssim \langle \pi_C{}^2 \rangle^{1/2} < 10^{-2}$ in the immediate vicinity of the self-absorption turnover. Note that because of polarization conversion, a variable source of sufficient homogeneity could exhibit rather short-lived increases in circular polarization, possibly to values in excess of a percent when $\nu \sim \nu_1$, a condition under which the spectral flux should change relatively slowly. Some data suggestive of this phenomenon in some sources do exist (see, e.g., Seaquist 1973; Seielstad and Berge 1975), but require further verification.

In the event that a source suffers no significant Faraday (i.e., frequency-dependent) depolarization above the self-absorption turnover, the typical rotation depth through each cell ($\zeta^*{}_V{}^0{}_K\tau^0{}_K$, see Appendix E) at such frequencies must not exceed unity by a large factor. This restricts the range of low-energy electron cut-offs to $(\gamma_i/\gamma_1) \gtrsim 10^{-1}(N)^{-1/6(\alpha+1)}$ (eq. [6]) and the density of nonrelativistic electrons to $(n_c/n_r) \lesssim 10^{-2.5}(\gamma_i/\gamma_1)^{2\alpha}N^{1/3}$ (eqs. [6] and [10]), where the brightness temperature and spectral index determine n_r (eq. [11]).

Pacholczyk (1973) cites three kinds of observational data to support the hypothesis that polarization conversion provides the dominant contribution to circular polarization in compact sources: (i) a frequency dependence for π_C generally steeper than the intrinsic $\nu^{-1/2}$, (ii) an apparent excess circular polarization over intrinsic values calculated using magnetic field strengths derived from self-absorption considerations and (iii) no change in sign of π_C across the self-absorption turnover. While we would argue that, except under special conditions, polarization conversion and/or Faraday rotation must play a significant role near the self-absorption turnover, for the following reasons we question, as do Roberts *et al.* (1975), the strength of the observational support for occurrence of the phenomenon.

Although some data do suggest a π_C frequency dependence steeper than $\nu^{-1/2}$ (e.g., Roberts *et al.* 1975), they could in part reflect structural complexity. Clarification of this point will depend upon simultaneous measurements at several closely spaced frequencies, possibly using VLBI techniques to isolate structural effects.

Pacholczyk and Swihart (1974) have argued that the magnetic fields derived assuming intrinsic circular polarization, B(CP), frequently exceed those derived from self-absorption, B(SA), by one or more orders of magnitude. However, given that $B(\mathrm{CP}) \propto \pi_C^2$ and $B(\mathrm{SA}) \propto \nu_n^5\theta_s^4$, with θ_s the source angular radius, the uncertainties in π_C, θ_s, and ν_n demand extreme caution in interpreting any difference between B(CP) and B(SA), especially since π_C lies on the threshold of detectability, tending to bias the sample toward atypically large values of π_C (Roberts *et al.* 1975). Furthermore, variability data frequently suggest the presence of at least mildly relativistic bulk motions toward the observer (see JOS II and BJO); correction of the resulting enhancement in apparent surface brightness would increase B(SA) (although one must also make the related but opposing correction for redshift). Using the circular polarization data of Roberts *et al.* (1975), including only $\geq 3\ \sigma$ measurements for which a best-fit decomposition of the spectrum into homogeneous components indicates that one component contributes at least 50% of the spectral flux, we find in fact (by the methods developed in JOS I and II for unresolved sources), for $\cot\theta \approx 1$, a mean value $\log\,[B(\mathrm{CP})/B(\mathrm{SA, SC})] = 0.0$, with a logarithmic standard deviation 0.6, for 14 measurements. We regard the mean agreement, based on a simple homogeneous model without the inclusion of any radiative transfer effects, as fortuitous, since we expect that structural inhomogeneity and conversion frequently do play a role.

The presence or absence of a circular-polarization sign change across the self-absorption turnover does not distinguish between converted and intrinsic circular polarization, since for a homogeneous source the sign should change in either case (§ IV). However, the sign change occurs at $\nu \sim 0.5\nu_n$, or well below the spectral peak of a homogeneous component, whereas below the turnover the spectral flux of such a component decreases rapidly. Therefore, in a multicomponent source, less opaque components may dominate the integrated circular polarization. Even in the two best cases for an absence of the expected sign change (0237−233 and 2134+004), the spectral shapes and for one (2134+004) VLBI observations (Kellermann *et al.* 1971; Schilizzi *et al.* 1975) indicate inhomogeneous structure. In addition, calculations summarized in Paper II show that the presence of a finite boundary for a single component can quite easily eliminate the sign reversal of the circular polarization and the 90° rotation in position angle of the linear polarization.

Thus, present observational data do not conflict with an important role for polarization conversion, but they do not provide strong support either.

VII. CONCLUSIONS

In this paper we have reviewed the equation of radiative transfer for nearly transverse propagation and its solution for polarized radiation in a homogeneous, rarefied, stationary medium, including the effects of emission, absorption, and transformation of polarization, either as Faraday rotation of the polarization axis or as interconversion of linear and circular polarization. We have also summarized the transfer coefficients relevant to cosmic, compact nonthermal sources—namely, those of a power-law distribution of synchrotron-emitting electrons and the two Faraday coefficients of a nonrelativistic plasma. Comparison of the relative importance of the transfer coefficients indicates that, except for a restricted range of parameters, the two Faraday phenomena (rotation and conversion) must affect appreciably the propagation of polarized radiation in cosmic, self-absorbed synchrotron sources. While a cold plasma may significantly enhance Faraday rotation, the relativistic electrons alone generally suffice to make rotation and conversion important phenomena in the vicinity of the self-absorption turnover.

Because Faraday rotation dominates the other propagation effects over a sizable range of parameters characteristic of cosmic, nonthermal radio sources, we have considered the strong-rotativity limit of the full solution for nearly transverse waves in a homogeneous medium. In this limit the polarization angle flips from $\pm\pi/4$ to $\mp\pi/4$ across the self-absorption turnover, and then goes to 0 as the rotation depth becomes small at high frequencies. This compares with a flip from $\pi/2$ to 0 across the self-absorption turnover in the limit of negligible Faraday effects. However, for a homogeneous source, the circular polarization must change sign across the self-absorption turnover whether or not Faraday rotation dominates. Independent of the cause of Faraday rotation, conversion serves as an effective generator of circular polarization only at frequencies ν less than about 10 times the turnover frequency ν_n.

We have examined the full weakly anisotropic solution for a homogeneous (unresolved) sphere, over a range of parameters characteristic of cosmic, compact, nonthermal radio sources (Fig. 1), assuming negligible cold plasma, an isotropic distribution of relativistic electrons, and reasonable observation angles to the magnetic field ($\cot\theta \approx 1$). For the homogeneous model, observations of circular and linear polarization thus restrict the allowed range of values for the low-energy cutoff γ_i, in terms of the maximum brightness temperature $\gamma_n \approx (kT_n/mc^2)$. Also, such observations provide restrictions on the number density of any cold plasma.

The degrees of circular ($\pi_C \approx 0.1\%$) and linear ($\pi_L \approx 3\%$) polarization typically observed in compact nonthermal sources do not approach the maximum values predicted for a homogeneous source (Fig. 1); instead, structural inhomogeneities act to depolarize the source, particularly for polarization measurements which do not resolve the angular structure. Consequently, we have investigated the effects of structural inhomogeneity in terms of a simple model source composed of N homogeneous domains, arranged in a three-dimensional array, with random orientations. We have examined this model in the strong-rotativity limit for small absorption depths—a regime relevant over an extensive range of parameters characteristic of cosmic, self-absorbed synchrotron sources. From a study of this model (Fig. 2), we find the following: inhomogeneities do not appreciably reduce the importance, near the self-absorption turnover (ν_1), of circular polarization converted from linear relative to intrinsic circular polarization; conversion constitutes the primary contribution to circular polarization up to $\nu \approx 10\nu_1$; above the self-absorption turnover, the intrinsic circular polarization seems undetectable at $\lesssim 0.01\%$; and the effective number of regions along a given line-of-sight $K \approx N^{1/3} \approx 6$. Although, due to the simplicity of its assumptions and the statistical character of its predictions, this model for structural inhomogeneities does not correspond quantitatively to a given cosmic source, it does serve to suggest strongly that measurements of circular polarization in cosmic, self-absorbed synchrotron sources usually reflect the conversion process rather than the intrinsic circular polarization, and thus provide no direct measure of the magnetic field.

In Paper II we investigate the transfer of polarized radiation in an inhomogeneous medium, by means of a different theoretical formulation—one more generally appropriate to such problems. There we show that transition regions and other structural inhomogeneities can appreciably alter the detailed behavior of the emergent linear and circular polarization—particularly for frequencies at which the source is opaque. However, it is important to emphasize that Faraday rotation and conversion of linear into circular polarization, due to the synchrotron-emitting electrons alone, remain important phenomena which cannot be neglected.

We have pointed out that, in terms of observed parameters, relativistic bulk motions—small pitch angles, ejection, or expansion—may allow more intrinsic circular polarization than predicted if such motions are neglected. While several arguments suggest the presence of relativistic bulk motions in many compact nonthermal sources, in most such sources these effects do not in general suffice to alter appreciably the conclusions concerning polarization reached for a stationary source.

Finally, we note that the presence of linear polarization in a compact radio source can place severe limits on the Faraday rotation within the source. Such limits restrict the distributions of relativistic electrons and of any thermal plasma. These restrictions prove important in assessing the energetics and dynamics of compact sources. In particular, the simplest interpretation of the data does not allow sufficient internal thermal plasma to retard relativistic expansion of the source and to provide an effective *in situ* acceleration mechanism. Elsewhere, we shall discuss in more detail the physical inferences to be drawn from measurements of linear and circular polarization.

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APPENDIX A

THE APPROPRIATE EQUATION OF TRANSFER FOR HOMOGENEOUS, COMPACT NONTHERMAL SOURCES

Considerable confusion persists over the transfer of polarized radiation in a homogeneous medium: However, the work of Zheleznyakov, Suvorov, and Shaposhnikov (1974) completely clarifies this issue. These authors have presented the general transfer equation for a homogeneous medium, valid for arbitrary anisotropy of the dielectric tensor and arbitrary nonorthogonality of the characteristic-wave (normal-mode) polarizations. As they note, the equation of Sazonov and Tsytovich (1968; also Sazonov 1969*a*, *b*) follows from their general equation under the single assumption of a weakly anisotropic dielectric tensor, while that of Zheleznyakov (1968) follows in the limit of orthogonal characteristic-wave polarizations.

For nearly transverse characteristic waves, the anisotropic part of the dielectric tensor becomes (Sazonov 1969*a*)

$$e'_{jk} = \frac{2i}{\nu}\sigma_{jk} = \frac{c\kappa}{2\pi\nu}\left\{\begin{bmatrix} \zeta^*_Q & i\zeta^*_V \\ -i\zeta^*_V & -\zeta^*_Q \end{bmatrix} + i\begin{bmatrix} 1+\zeta_Q & i\zeta_V \\ -i\zeta_V & 1-\zeta_Q \end{bmatrix}\right\}, \tag{A1}$$

where σ_{jk} is the transverse part of the conductivity tensor and the other symbols are defined in the text. We emphasize that one may calculate these quantities from the linearized kinetic and Maxwell's equations without prior knowledge of the normal modes: Indeed, the dielectric tensor determines the characteristic-wave polarizations (Sazonov 1969*a*).

We may set the absorption coefficient $\kappa = \tau/L$, with τ the absorption depth of the source and L its characteristic longitudinal dimension, and the frequency $\nu = c/\lambda$, with λ the vacuum wavelength of the radiation. Doing so,

$$e'_{jk} = \frac{\tau}{2\pi}\frac{\lambda}{L}\{\cdots\}. \tag{A2}$$

The coefficient in front of the bracket $(\tau/2\pi)(\lambda/L) \lesssim 10^{-17}$ for observable cosmic sources; consequently, generally speaking, no reasonable value for the normalized transfer coefficients within the bracket (for synchrotron sources, see text eqs. [4], [5], [6], and [10]) is sufficiently large to obviate the assumption of a weakly anisotropic dielectric tensor $|e'_{jk}| \ll 1$. Therefore, the transfer equation of Sazonov (1969*b*) nearly invariably applies for homogeneous cosmic radio sources, independent of the nonorthogonality of the characteristic waves.

On the other hand, the applicability of the equation of Zheleznyakov (1968) rests upon negligible nonorthogonality of the normal-wave polarization, although it does hold for arbitrary anisotropy of the dielectric tensor. While small nonorthogonality may pertain over an extensive range of frequencies and parameters characteristic of cosmic synchrotron sources, there frequently exists a range over which it does not. More important, however—even when small, the nonorthogonality is not in general negligible in comparison with other small quantities, such as the degree of intrinsic circular polarization, typically of order 10^{-3}.

Hence, for cosmic synchrotron sources, which invariably satisfy the condition of a weakly anisotropic dielectric tensor, the appropriate limit of the general transfer equation for a homogeneous medium is that of Sazonov (1969*b*), not that of Zheleznyakov (1968). This agrees with the conclusion of Zheleznyakov, Suvorov, and Shaposhnikov (1974) that, for a sufficiently rarefied medium, "the transfer of polarization is more fully determined by the equation" of Sazonov and Tsytovich (1968, also Sazonov 1969*a*, *b*). Our numerical calculations in Paper II, which use the dielectric tensor directly, further confirm the applicability of this transfer equation. Consequently, in the present paper, we have used this equation and its solution with the dielectric tensor appropriate to cosmic synchrotron sources (Sazonov 1969*b*).

APPENDIX B

TRANSFER OF POLARIZED RADIATION IN A HOMOGENEOUS MEDIUM

For a homogeneous, stationary medium, solution of the transfer equation for weakly anisotropic dielectric tensor (eq. 2) yields (cf. Zheleznyakov, Suvorov, and Shaposhnikov 1974; Pacholczyk and Swihart 1975)

$$\begin{vmatrix} I_\nu \\ Q_\nu \\ U_\nu \\ V_\nu \end{vmatrix} = \begin{vmatrix} 1 \\ \pi_Q \\ \pi_U \\ \pi_V \end{vmatrix} I_\nu = \begin{vmatrix} I_\nu^\infty \\ Q_\nu^\infty \\ U_\nu^\infty \\ V_\nu^\infty \end{vmatrix} + e^{-\tau}\left\{ \begin{vmatrix} \frac{1}{2}(1+q^2+v^2) & 0 & q\times v & 0 \\ 0 & \frac{1}{2}(1+q^2-v^2) & 0 & q\cdot v \\ -q\times v & 0 & \frac{1}{2}(1-q^2-v^2) & 0 \\ 0 & q\cdot v & 0 & \frac{1}{2}(1-q^2-v^2) \end{vmatrix} \cosh(\chi\tau) \right.$$

$$- \begin{vmatrix} 0 & q\cdot k & 0 & v\cdot k \\ q\cdot k & 0 & -v\times k & 0 \\ 0 & v+k & 0 & -q\times k \\ v\cdot k & 0 & q\times k & 0 \end{vmatrix} \sinh(\chi\tau)$$

$$+ \begin{vmatrix} \frac{1}{2}(1-q^2-v^2) & 0 & -q\times v & 0 \\ 0 & \frac{1}{2}(1-q^2+v^2) & 0 & -q\cdot v \\ q\times v & 0 & \frac{1}{2}(1+q^2+v^2) & 0 \\ 0 & -q\cdot v & 0 & \frac{1}{2}(1+q^2-v^2) \end{vmatrix} \cos(\chi_*\tau)$$

$$\left. - \begin{vmatrix} 0 & q\times k & 0 & v\times k \\ q\times k & 0 & v\cdot k & 0 \\ 0 & -v\cdot k & 0 & q\cdot k \\ v\times k & 0 & -q\cdot k & 0 \end{vmatrix} \sin(\chi_*\tau) \right\} \begin{vmatrix} I_\nu^0 - I_\nu^\infty \\ Q_\nu^0 - Q_\nu^\infty \\ U_\nu^0 - U_\nu^\infty \\ V_\nu^0 - V_\nu^\infty \end{vmatrix} . \tag{B1}$$

Here the superscript 0 indicates initial values (appropriate for $\tau = 0$); the superscript ∞ denotes particular solutions (dominant as $\tau \to \infty$) given by

$$\begin{vmatrix} I_\nu^\infty \\ Q_\nu^\infty \\ U_\nu^\infty \\ V_\nu^\infty \end{vmatrix} = \frac{J_\nu}{(1-\chi^2)(1+\chi_*^2)} \begin{vmatrix} [1+\zeta_*^2] - [\zeta_Q + (\zeta:\zeta_*)\zeta^*_Q]\epsilon_Q - [\zeta_V + (\zeta:\zeta_*)\zeta^*_V]\epsilon_V \\ -[\zeta_Q + (\zeta:\zeta_*)\zeta^*_Q] + [1 + (\zeta^{*2}_Q - \zeta_V^2)]\epsilon_Q + [\zeta_Q\zeta_V + \zeta^*_Q\zeta^*_V]\epsilon_V \\ -[\zeta_Q\zeta^*_V - \zeta^*_Q\zeta_V] + [\zeta^*_V - (\zeta:\zeta_*)\zeta_V]\epsilon_Q - [\zeta^*_Q - (\zeta:\zeta_*)\zeta_Q]\epsilon_V \\ -[\zeta_V + (\zeta:\zeta_*)\zeta^*_V] + [\zeta_Q\zeta_V + \zeta^*_Q\zeta^*_V]\epsilon_Q + [1 + (\zeta^{*2}_V - \zeta_Q^2)]\epsilon_V \end{vmatrix} , \tag{B2}$$

where

$$\chi = [\![\tfrac{1}{2}\{[(\zeta^2 - \zeta_*^2)^2 + 4(\zeta:\zeta_*)^2]^{1/2} + (\zeta^2 - \zeta_*^2)]\!]^{1/2} ,$$

$$\chi_* = [\![\tfrac{1}{2}\{[(\zeta_*^2 - \zeta^2)^2 + 4(\zeta_*:\zeta)^2]^{1/2} + (\zeta_*^2 - \zeta^2)\}]\!]^{1/2} , \tag{B3}$$

so that

$$(1 - \chi^2)(1 + \chi_*^2) = \{[1 + \zeta_*^2] - [\zeta_Q + (\zeta : \zeta_*)\zeta^*_Q]\zeta_Q - [\zeta_V + (\zeta : \zeta_*)\zeta^*_V]\zeta_V\} . \tag{B4}$$

The symbols ζ, ζ_*, q, v, and k represent two-vectors defined as

$$\zeta \equiv (\zeta_Q; \zeta_V) , \qquad \zeta_* \equiv (\zeta^*_Q; \zeta^*_V) ,$$

$$q \equiv (\zeta_Q, \zeta^*_Q)[\chi^2 + \chi_*^2]^{-1/2} , \qquad v \equiv (\zeta_V, \zeta^*_V)[\chi^2 + \chi_*^2]^{-1/2} , \qquad k \equiv (\chi, \chi_*)[\chi^2 + \chi_*^2]^{-1/2} . \tag{B5}$$

The interior $(\cdot)$ and exterior $(\times)$ products operate on q, v, and k—for example,

$$q \cdot v = \frac{\zeta_Q\zeta_V + \zeta^*_Q\zeta^*_V}{[\chi^2 + \chi_*^2]} \quad \text{and} \quad q \times v = \frac{\zeta_Q\zeta^*_V - \zeta^*_Q\zeta_V}{[\chi^2 + \chi_*^2]}$$

—whereas the other interior product (:) operates on ζ and ζ_*:

$$\zeta : \zeta_* = \zeta_Q\zeta^*_Q + \zeta_V\zeta^*_V .$$

For an unresolved source, the antenna measures only the spectral flux (F_ν, F_ν^Q, F_ν^U, F_ν^F), although one may construct an average spectral intensity $\langle(I_\nu, Q_\nu, U_\nu, V_\nu)\rangle \equiv (F_\nu, F_\nu^Q, F_\nu^U, F_\nu^F)/\Omega_s$, with Ω_s the solid angle subtended by the source. Integration over a realistic source shape tends to wash out oscillations exhibited in the solution for the spectral intensities (eq. [B1]). As an example, for a homogeneous sphere of central absorption depth τ_s,

$$\begin{aligned}
\langle e^{-\tau}\cosh(\chi\tau)\rangle_{\Omega_s} &= 2[\![(1 + \chi^2) - e^{-\tau_s}\{[(1 + \chi^2) + (1 - \chi^2)\tau_s]\cosh(\chi\tau_s) \\
&\quad + [2 + (1 - \chi^2)\tau_s]\chi\sinh(\chi\tau_s)\}]\!][(1 - \chi^2)\tau_s]^{-2} , \\
\langle e^{-\tau}\sinh(\chi\tau)\rangle_{\Omega_s} &= 2[\![2\chi - e^{-\tau_s}\{[(1 + \chi^2) + (1 - \chi^2)\tau_s]\sinh(\chi\tau_s) \\
&\quad + [2 + (1 - \chi^2)\tau_s]\chi\cosh(\chi\tau_s)\}]\!][(1 - \chi^2)\tau_s]^{-2} , \\
\langle e^{-\tau}\cos(\chi_*\tau)\rangle &= 2[\![(1 - \chi_*^2) - e^{-\tau_s}\{[(1 - \chi_*^2) + (1 + \chi_*^2)\tau_s]\cos(\chi_*\tau_s) \\
&\quad - [2 + (1 + \chi_*^2)\tau_s]\chi_*\sin(\chi_*\tau_s)\}]\!][1 + \chi_*^2)\tau_s]^{-2} , \\
\langle e^{-\tau}\sin(\chi_*\tau)\rangle &= 2[\![2\chi_* - e^{-\tau_s}\{[(1 - \chi_*^2) + (1 + \chi_*^2)\tau_s]\sin(\chi_*\tau_s) \\
&\quad + [2 + (1 + \chi_*^2)\tau_s]\chi_*\cos(\chi_*\tau_s)\}]\!][(1 + \chi_*^2)\tau_s]^{-2} .
\end{aligned} \tag{B6}$$

APPENDIX C

TRANSFER COEFFICIENTS FOR THE SYNCHROTRON PROCESS

We summarize here the transfer coefficients appropriate for cosmic synchrotron sources. The rarefaction of the medium and weakness of the magnetic field in such sources produces a nearly isotropic dielectric tensor (to better than $\sim 10^{-12}$; see Appendix A). Consequently, the transfer coefficients readily follow from the dielectric tensor, and may be computed directly from the linearized kinetic equation and without prior specification of the characteristic-wave polarizations (Sazonov 1969*b*).

For a power-law distribution of relativistic electrons,

$$dn = \frac{\partial^2 n}{d\gamma d\Omega_\psi}\, d\gamma d\Omega_\psi = [n_\gamma\gamma^s]\gamma^{-s}\Theta(\gamma - \gamma_l)g(\Psi)d\gamma d\Omega_\Psi , \tag{C1}$$

with Θ the step distribution and $g(\Psi)$ the pitch-angle distribution (supposed separable) normalized to

$$\int g(\Psi)d\Omega_\psi = 1 , \tag{C2}$$

the transfer coefficients at an angle θ to the magnetic field B comprise (Ginzburg and Syrovatskii 1965; Legg and Westfold 1968; Sazonov and Tsytovich 1968; Sazonov 1969*a*, *b*)

$$\eta_\nu = \eta_\alpha\eta_\perp(\nu_{B\perp}/\nu)^\alpha , \qquad \eta_\nu^Q = \eta_\alpha^Q\eta_\perp(\nu_{B\perp}/\nu)^\alpha ,$$

$$\eta_\nu^V = -\eta_\alpha^V\eta_\perp(\nu_{B\perp}/\nu)^{\alpha+1/2}\cot\vartheta\left[1 + \frac{1}{2\alpha + 3}\frac{d\ln g(\vartheta)}{d\ln(\sin\vartheta)}\right] , \tag{C3}$$

$$\kappa = \kappa_\alpha \kappa_\perp (\nu_{B\perp}/\nu)^{\alpha+5/2}, \qquad \kappa_Q = \kappa_\alpha{}^Q \kappa_\perp (\nu_{B\perp}/\nu)^{\alpha+5/2},$$

$$\kappa_V = \kappa_\alpha{}^V \kappa_\perp (\nu_{B\perp}/\nu)^{\alpha+3} \cot\vartheta \left[1 + \frac{1}{2\alpha+3}\frac{d\ln g(\vartheta)}{d\ln(\sin\vartheta)}\right], \tag{C4}$$

and

$$\kappa^*_Q = -\kappa^*_\alpha{}^Q \kappa_\perp (\nu_{B\perp}/\nu)^3 \gamma_i{}^{-(2\alpha-1)}\{[1 - (\nu_i/\nu)^{\alpha-1/2}](\alpha - \tfrac{1}{2})^{-1}\}, \quad (\alpha > \tfrac{1}{2})$$

$$\kappa^*_V = \kappa^*_\alpha{}^V \kappa_\perp (\nu_{B\perp}/\nu)^2 (\ln\gamma_i)\gamma_i{}^{-2(\alpha+1)} \cot\vartheta \left[1 + \frac{\alpha+2}{2\alpha+3}\frac{d\ln g(\vartheta)}{d\ln(\sin\vartheta)}\right]. \tag{C5}$$

In these expressions

$$\eta_\perp \equiv (mc^2)(r_e/c)\nu_{B\perp}[4\pi g(\vartheta)][n_\gamma \gamma^s], \qquad \kappa_\perp \equiv (r_e c)\nu_{B\perp}{}^{-1}[4\pi g(\vartheta)][n_\gamma \gamma^s], \tag{C6}$$

with

$$\nu_{B\perp} \equiv \frac{eB\sin\vartheta}{2\pi mc}, \qquad \nu_i \equiv \gamma_i{}^2 \nu_{B\perp}. \tag{C7}$$

The dimensionless functions of spectral index $\alpha[\alpha = (s-1)/2]$ satisfy

$$\eta_\alpha = \frac{3^{\alpha+1/2}}{4(\alpha+1)}\Gamma\left(\frac{\alpha}{2}+\frac{11}{6}\right)\Gamma\left(\frac{\alpha}{2}+\frac{1}{6}\right),$$

$$\eta_\alpha{}^Q = \frac{3^{\alpha+1/2}}{4(\alpha+\frac{5}{3})}\Gamma\left(\frac{\alpha}{2}+\frac{11}{6}\right)\Gamma\left(\frac{\alpha}{2}+\frac{1}{6}\right),$$

$$\eta_\alpha{}^V = \frac{3^\alpha}{2}\frac{(\alpha+\frac{3}{2})}{(\alpha+\frac{1}{2})}\Gamma\left(\frac{\alpha}{2}+\frac{11}{12}\right)\Gamma\left(\frac{\alpha}{2}+\frac{7}{12}\right), \tag{C8}$$

$$\kappa_\alpha = \frac{3^{\alpha+1}}{4}\Gamma\left(\frac{\alpha}{2}+\frac{25}{12}\right)\Gamma\left(\frac{\alpha}{2}+\frac{5}{12}\right),$$

$$\kappa_\alpha{}^Q = \frac{3^{\alpha+1}}{4}\frac{(\alpha+\frac{3}{2})}{(\alpha+\frac{13}{6})}\Gamma\left(\frac{\alpha}{2}+\frac{25}{12}\right)\Gamma\left(\frac{\alpha}{2}+\frac{5}{12}\right),$$

$$\kappa_\alpha{}^V = \frac{3^{\alpha+1/2}}{2}\frac{(\alpha+2)}{(\alpha+1)}\left(\alpha+\frac{3}{2}\right)\Gamma\left(\frac{\alpha}{2}+\frac{7}{6}\right)\Gamma\left(\frac{\alpha}{2}+\frac{5}{6}\right), \tag{C9}$$

and

$$\kappa^*_\alpha{}^Q = 1, \qquad \kappa^*_\alpha{}^V = 2\frac{(\alpha+\frac{3}{2})}{(\alpha+1)}. \tag{C10}$$

Note that our expression for $\kappa^*_\alpha{}^V$ exceeds that of Sazonov (1969*a*) by a factor $(2\alpha + 3)$. This difference results because we omit any contribution to κ^*_V due to electrons with Lorentz factors less than or equal to the low-energy cutoff γ_i, whereas Sazonov includes a negative contribution due to a formal discontinuity at the low-energy cutoff. However, unless the relativistic-electron population is inverted (i.e., unless $s < -2$) below the nominal cutoff γ_i, our value constitutes a minimum to κ^*_V. Consequently, we adopt this value as a more realistic estimate for κ^*_V.

Validity of the above expressions for the transfer coefficients requires that ν exceed ν_i, and that both ν and ν_i exceed ν_g (the Doppler-shifted gyrofrequency), so that (O'Dell and Sartori 1970)

$$(\nu/\nu_{B\perp}) > (\nu_i/\nu_{B\perp}) = \gamma_i{}^2 > (\cot\vartheta)^2. \tag{C11}$$

The neglect of dielectric suppression (Razin-Tsytovich effect) seems justified in typical extragalactic nonthermal sources (JOS II).

Although emission and absorption appear dominated by the synchrotron process in typical compact extragalactic nonthermal sources (JOS II), nonrelativistic ("cold") electrons may contribute to the transformation of polarization. For a nonrelativistic plasma of number density n_c, at $(\nu/\nu_B) \gg 1$ (see, e.g., Ginzburg 1961),

$$\kappa^*_Q{}^{(c)} = -(r_e c)\nu_{B\perp}{}^{-1} n_c (v_{B\perp}/\nu)^3, \qquad \kappa^*_V{}^{(c)} = 2(r_e c)\nu_{B\perp}{}^{-1} n_c (\nu_{B\perp}/\nu)^2 \cot\vartheta. \tag{C12}$$

The magnitude of the contribution due to protons is smaller by $(m/m_p)^3$ and by $(m/m_p)^2$, respectively.

In order to assess the relative importance of the various transfer coefficients, it proves convenient to normalize them in the form

$$\epsilon_{(Q,V)} \equiv \eta_\nu{}^{(Q,V)}/\eta_\nu, \qquad \zeta_{(Q,V)} \equiv \kappa_{(Q,V)}/\kappa, \quad \text{and} \quad \zeta^*_{(Q,V)} \equiv \kappa^*_{(Q,V)}/\kappa,$$

so that

$$\epsilon_Q = \epsilon_\alpha{}^Q\,, \qquad \epsilon_V = -\epsilon_\alpha{}^V(\nu_{B\perp}/\nu)^{1/2}\cot\vartheta\left[1 + \frac{1}{2\alpha+3}\frac{d\ln g(\vartheta)}{d\ln(\sin\vartheta)}\right], \tag{C13}$$

$$\zeta_Q = \zeta_\alpha{}^Q\,, \qquad \zeta_V = -\zeta_\alpha{}^V(\nu_{B\perp}/\nu)^{1/2}\cot\vartheta\left[1 + \frac{1}{2\alpha+3}\frac{d\ln g(\vartheta)}{d\ln(\sin\vartheta)}\right], \tag{C14}$$

and

$$\zeta^*{}_Q = -\zeta^*{}_\alpha{}^Q(\nu/\nu_i)^{\alpha-1/2}\{[1-(\nu_i/\nu)^{\alpha-1/2}](\alpha-\tfrac{1}{2})^{-1}\}\,, \quad \alpha > \tfrac{1}{2}$$

$$\zeta^*{}_V = \zeta^*{}_\alpha{}^V(\nu/\nu_i)^{\alpha+1/2}\frac{\ln\gamma_i}{\gamma_i}\cot\vartheta\left[1 + \frac{\alpha+2}{2\alpha+3}\frac{d\ln g(\vartheta)}{d\ln(\sin\vartheta)}\right]. \tag{C15}$$

Table 1 summarizes the dimensionless functions

$$\epsilon_\alpha{}^{(Q,V)} \equiv \eta_\alpha{}^{(Q,V)}/\eta_\alpha\,, \qquad \zeta_\alpha{}^{(Q,V)} \equiv \kappa_\alpha{}^{(Q,V)}/\kappa\,, \quad \text{and} \quad \zeta^*{}_\alpha{}^{(Q,V)} \equiv \kappa^*{}_\alpha{}^{(Q,V)}/\kappa\,.$$

The relative contributions of nonrelativistic and relativistic electrons to the convertibility $\kappa^*{}_Q$ and rotativity $\kappa^*{}_V$ follow from equations (C5) and (C12):

$$\frac{\kappa^*{}_Q{}^{(r)}}{\kappa^*{}_Q{}^{(c)}} = 2\alpha\frac{[1-(\nu_i/\nu)^{\alpha-1/2}]}{(\alpha-\frac{1}{2})}\gamma_i\left(\frac{n_r}{n_c}\right) \quad (\alpha > \tfrac{1}{2})\,,$$

$$\frac{\kappa^*{}_V{}^{(r)}}{\kappa^*{}_V{}^{(c)}} = 2\alpha\frac{(\alpha+\frac{3}{2})}{(\alpha+1)}\frac{\ln\gamma_i}{\gamma_i{}^2}\left(\frac{n_r}{n_c}\right) \qquad (\alpha > 0)\,, \tag{C16}$$

for isotropic distributions. Here n_r represents the number density of relativistic electrons given by

$$n_r = \int_{\gamma_i}^{\infty}[n_\gamma\gamma^s]\gamma^{-s}d\gamma = [n_\gamma\gamma^s]\frac{\gamma_i{}^{-(s-1)}}{(s-1)} \quad (s > 1)\,. \tag{C17}$$

APPENDIX D

STRONG-ROTATIVITY LIMIT FOR A HOMOGENEOUS SOURCE

Since it appears likely that Faraday rotation plays a significant role in compact nonthermal radio sources, even in the absence of cold plasma, we here consider the corresponding strong-rotativity limit ($\zeta^*{}_V{}^2 \gg \zeta^*{}_Q{}^2 \gtrsim 1$) of the full solution for weakly anisotropic dielectric tensor (Appendix B). This limit is equivalent to the well-known quasi-longitudinal (QL) limit (see Paper II). For the hierarchy of transfer coefficients to cosmic self-absorbed synchrotron sources—namely,

$$\zeta^*{}_V{}^2 \gg \zeta^*{}_Q{}^2 \gtrsim 1 > \zeta_Q{}^2 > \epsilon_Q{}^2 \gg \zeta_V{}^2 > \epsilon_V{}^2 \tag{D1}$$

—the parameters χ and χ_* (eq. [B3]) become

$$\chi \approx \zeta_V + \left(\frac{\zeta^*{}_Q}{\zeta^*{}_V}\right)\zeta_Q\,,$$

$$\chi_* \approx \zeta^*{}_V\,, \tag{D2}$$

so that $\chi_*{}^2 \gg 1 \gg \chi^2$.

For very large absorption depth ($\tau \gg 1$),

$$\begin{vmatrix} 1 \\ \pi_Q \\ \pi_U \\ \pi_V \end{vmatrix} I_\nu \to \begin{vmatrix} 1 \\ \dfrac{1+\zeta^*{}_Q{}^2}{\zeta^*{}_V{}^2}(\epsilon_Q-\zeta_Q) + \dfrac{\zeta^*{}_Q}{\zeta^*{}_V}(\epsilon_V-\zeta_V) \\ \dfrac{1}{\zeta^*{}_V}(\epsilon_Q-\zeta_Q) \\ (\epsilon_V-\zeta_V) + \dfrac{\zeta^*{}_Q}{\zeta^*{}_V}(\epsilon_Q-\zeta_Q) \end{vmatrix} J_\nu \quad (\tau \gg 1,\ |\zeta^*{}_V| \gg 1)\,; \tag{D3}$$

while for small absorption depth ($\tau \ll 1$) and no incident radiation ($I_\nu{}^0 = Q_\nu{}^0 = U_\nu{}^0 = V_\nu{}^0 = 0$),

$$\begin{vmatrix} 1 \\ \pi_Q \\ \pi_U \\ \pi_V \end{vmatrix} I_\nu \to \begin{vmatrix} 1 \\ \dfrac{\sin(\zeta^*{}_V\tau)}{(\zeta^*{}_V\tau)}\epsilon_Q \\ \dfrac{[1-\cos(\zeta^*{}_V\tau)]}{(\zeta^*{}_V\tau)}\epsilon_Q \\ \epsilon_V + \epsilon_Q\dfrac{\zeta^*{}_Q}{\zeta^*{}_V}\left[1-\dfrac{\sin(\zeta^*{}_V\tau)}{(\zeta^*{}_V\tau)}\right] \end{vmatrix} J_\nu\tau \quad (\tau \ll 1, |\zeta^*{}_V| \gg 1)\,. \tag{D4}$$

Although $\zeta^*{}_V$ increases with frequency, $\zeta^*{}_V\tau \propto \nu^{-2}$; thus, at sufficiently high frequencies $|\zeta^*{}_V|\tau \ll 1$, whereupon

$$\begin{vmatrix} 1 \\ \pi_Q \\ \pi_U \\ \pi_V \end{vmatrix} I_\nu \to \begin{vmatrix} 1 \\ \epsilon_Q \\ \frac{1}{2}(\zeta^*{}_V\tau)\epsilon_Q \\ \epsilon_V \end{vmatrix} J_\nu\tau \quad (\tau \ll 1, |\zeta^*{}_V|\tau \ll 1)\,, \tag{D5}$$

which manifests the well-known frequency dependence for homogeneous, transparent sources with Faraday rotation.

When averaged over a uniform sphere with absorption depth τ_s through a diameter (see eq. [B6]), the latter two limits ($\tau_s \ll 1$) become

$$\begin{vmatrix} 1 \\ \bar{\pi}_Q \\ \bar{\pi}_U \\ \bar{\pi}_V \end{vmatrix} \bar{I}_\nu \to \begin{vmatrix} 1 \\ \dfrac{3\epsilon_Q}{(\zeta^*{}_V\tau_s)^2}\left[\dfrac{\sin(\zeta^*{}_V\tau_s)}{(\zeta^*{}_V\tau_s)} - \cos(\zeta^*{}_V\tau_s)\right] \\ \dfrac{3\epsilon_Q}{(\zeta^*{}_V\tau_s)}\left[\dfrac{1}{2} - \dfrac{\sin(\zeta^*{}_V\tau_s)}{(\zeta^*{}_V\tau_s)} + \dfrac{1-\cos(\zeta^*{}_V\tau_s)}{(\zeta^*{}_V\tau_s)^2}\right] \\ \left(\epsilon_V + \dfrac{\zeta^*{}_Q}{\zeta^*{}_V}\epsilon_Q\right) - \dfrac{3\epsilon_Q}{(\zeta^*{}_V\tau_s)}\left[\dfrac{\sin(\zeta^*{}_V\tau_s)}{(\zeta^*{}_V\tau_s)} - \cos(\zeta^*{}_V\tau_s)\right] \end{vmatrix} \tfrac{2}{3}J_\nu\tau_s \quad (\tau_s \ll 1, |\zeta^*{}_V| \gg 1)\,, \tag{D6}$$

which in the limit $|\zeta^*{}_V|\tau_s \gg 1$

$$\to \begin{vmatrix} 1 \\ -3\dfrac{\cos(\zeta^*{}_V\tau_s)}{(\zeta^*{}_V\tau_s)^2}\epsilon_Q \\ \dfrac{3}{2}\dfrac{1}{(\zeta^*{}_V\tau_s)}\epsilon_Q \\ \left(\epsilon_V + \dfrac{\zeta^*{}_Q}{\zeta^*{}_V}\epsilon_Q\right) \end{vmatrix} \tfrac{2}{3}J_\nu\tau_s \quad (\tau_s \ll 1, |\zeta^*{}_V|\tau_s \gg 1)\,, \tag{D7}$$

while, if $|\zeta^*{}_V|\tau_s \ll 1$,

$$\to \begin{vmatrix} 1 \\ \epsilon_Q \\ \frac{3}{8}(\zeta^*{}_V\tau_s)\epsilon_Q \\ \epsilon_V \end{vmatrix} \tfrac{2}{3}J_\nu\tau_s \quad (\tau_s \ll 1, |\zeta^*{}_V|\tau_s \ll 1)\,. \tag{D8}$$

APPENDIX E

STRONG-ROTATIVITY LIMIT FOR A TRANSPARENT INHOMOGENEOUS SOURCE

We consider here a simple model of inhomogeneity for small absorption depth ($\tau \ll 1$), under the assumption of strong rotativity ($\zeta^*{}_V{}^2 \gg 1$): Such an assumption seems appropriate for cosmic, self-absorbed synchrotron sources over a sizable range of parameters. The simple model consists of K uniform domains—each described by a set of intrinsic parameters $\{\xi_K\}$ (magnetic field, density, and low-energy cutoff), an azimuthal orientation φ_K, and a polar angle φ_K—lying on a given line of sight. Making the relevant approximations for

$$\zeta^*{}_{VK}{}^2 \gg \zeta^*{}_{QK}{}^2 \gtrsim 1 > \zeta_{QK}{}^2 > \epsilon_{QK}{}^2 \gg \zeta_{VK}{}^2 > \epsilon_{VK}{}^2 ,$$

$$\tau = \sum_{K'=1}^{K} \tau_{K'} < 1 , \tag{E1}$$

the solution of the transfer equation for nearly transverse propagation (i.e., for negligibly anisotropic dielectric tensor) yields[5]

$$\begin{vmatrix} 1 \\ \pi_Q \\ \pi_U \\ \pi_V \end{vmatrix} I_\nu \to \sum_{K'=1}^{K} \begin{vmatrix} 1 \\ \pi_{QK'} \\ \pi_{UK'} \\ \pi_{VK'} \end{vmatrix} J_{\nu K'}\tau_{K'} , \tag{E2}$$

where

$$\pi_{QK'} \equiv \frac{\epsilon_{QK'}}{(\zeta^*{}_{VK'}\tau_{K'})}\left\{\sin\left[\sum_{K''=K'}^{K+1}(\zeta^*{}_{VK''}\tau_{K''}) + 2\varphi_{K'}\right] - \sin\left[\sum_{K''=K'+1}^{K+1}(\zeta^*{}_{VK''}\tau_{K''}) + 2\varphi_{K'}\right]\right\},$$

$$\pi_{UK'} \equiv -\frac{\epsilon_{QK'}}{(\zeta^*{}_{VK'}\tau_{K'})}\left\{\cos\left[\sum_{K''=K'}^{K+1}(\zeta^*{}_{VK''}\tau_{K'}) + 2\varphi_{K'}\right] - \cos\left[\sum_{K''=K'+1}^{K+1}(\zeta^*{}_{VK''}\tau_{K''}) + 2\varphi_{K'}\right]\right\},$$

$$\pi_{VK'} \equiv -\frac{\epsilon_{QK'}}{(\zeta^*{}_{VK'}\tau_{K'})}\sum_{K'''=K'+1}^{K+1}\frac{\zeta^*{}_{QK'''}}{\zeta^*{}_{VK'''}}$$

$$\times \left[\!\left[\left\{\sin\left[\sum_{K''=K'}^{K'''}(\zeta^*{}_{VK''}\tau_{K''}) + 2(\varphi_{K'} - \varphi_{K'''})\right] - \sin\left[\sum_{K''=K'+1}^{K'''}(\zeta^*{}_{VK''}\tau_{K''}) + 2(\varphi_{K'} - \varphi_{K'''})\right]\right\}\right.\right.$$

$$\left.\left. - \left\{\sin\left[\sum_{K''=K'}^{K'''-1}(\zeta^*{}_{VK''}\tau_{K''}) + 2(\varphi_{K'}-\varphi_{K'''})\right] - \sin\left[\sum_{K''=K'+1}^{K'''-1}(\zeta^*{}_{VK''}\tau_{K''}) + 2(\varphi_{K'} - \varphi_{K'''})\right]\right\}\right]\!\right]$$

$$+ \frac{\zeta^*{}_{QK'}}{\zeta^*{}_{VK'}}\epsilon_{QK'}\left[1 - \frac{\sin(\zeta^*{}_{VK'}\tau_{K'})}{(\zeta^*{}_{VK'}\tau_{K'})}\right] + \epsilon_{VK'} .$$

(E2) (*continued*)

Physically, the three terms constituting the circular polarization ($-\pi_V$) represent conversion to circular polarization of linearly polarized radiation produced outside the region in which conversion occurs, conversion of internally-generated linear polarization, and the intrinsic circular polarization.

For statistically independent domains of random orientation,

$$\langle\pi_Q\rangle_{\xi,\vartheta,\varphi} = \langle\pi_U\rangle_{\xi,\vartheta,\varphi} = \langle\pi_V\rangle_{\xi,\vartheta,\varphi} = 0 ; \tag{E3}$$

however, the expected variances

$$\langle\pi_L{}^2\rangle_{\xi,\vartheta,\varphi} = \langle(\pi_Q{}^2 + \pi_U{}^2)\rangle_{\xi,\vartheta,\varphi} = \left\{2\left\langle\left(\frac{\epsilon_{QK}J_{\nu K}}{\zeta^*{}_{VK}}\right)^2[1 - \cos(\zeta^*{}_{VK}\tau_K)]\right\rangle_{\xi,\vartheta}\right\}(K\langle J_{\nu K}\tau_K\rangle_{\xi,\vartheta}{}^2)^{-1} ,$$

$$\langle\pi_C{}^2\rangle_{\xi,\vartheta,\varphi} = \langle(-\pi_V)^2\rangle_{\xi,\vartheta,\varphi}$$

$$= \left\{2\left\langle\left(\frac{\epsilon_{QK}J_{\nu K}}{\zeta^*{}_{VK}}\right)^2[1 - \cos(\zeta^*{}_{VK}\tau_K)]\right\rangle_{\xi,\vartheta}\frac{K-1}{2}\left\langle\left(\frac{\zeta^*{}_{QK}}{\zeta^*{}_{VK}}\right)^2[1 - \cos(\zeta^*{}_{VK}\tau_K)]\right\rangle_{\xi,\vartheta}\right.$$

$$\left. + \left\langle\left[\left(\frac{\zeta^*{}_{QK}}{\zeta^*{}_{VK}}\epsilon_{QK}\left[1 - \frac{\sin(\zeta^*{}_{VK}\tau_K)}{(\zeta^*{}_{VK}\tau_K)}\right] + \epsilon_{VK}\right)J_{\nu K}\tau_K\right]^2\right\rangle_{\xi,\vartheta}\right\}(K\langle J_{\nu K}\tau_K\rangle_{\xi,\vartheta}{}^2)^{-1} , \tag{E4}$$

[5] We note that for cosmic synchrotron sources, the rarefaction of the medium ensures not only a nearly isotropic dielectric tensor, but also an index of refraction differing from unity by a very small amount ($\sim(\nu_{\mathrm{pa}}/\nu)^2 < \gamma^{-2} < 10^{-6}$ from limits on dielectric suppression). Consequently, we need not consider effects of refraction and reflection due to inhomogeneous structure.

where we have set

$$I_\nu \approx \langle I_\nu \rangle_{\xi,\vartheta} = K \langle J_{\nu K} \tau_K \rangle_{\xi,\vartheta} . \tag{E5}$$

In order to perform the average over polar angle θ, we extract the angularly dependent factors from the normalized transfer coefficients for an isotropic distribution of synchrotron electrons (eqs. [C13]–[C15]), denoting the remaining angularly independent factors by a superscripted o. (For $\zeta^*{}_Q$, we have treated the factor $[1 - (\nu_i/\nu)^{\alpha - 1/2}]$ as independent of polar angle.) The expected variance then becomes

$$\langle \pi_L{}^2 \rangle_{\xi,\vartheta,\varphi} = \left\{ 2 \left\langle \left(\frac{\epsilon_Q{}^o{}_K J_\nu{}^o{}_K}{\zeta^*{}_V{}^o{}_K} \right)^2 \left[\frac{(\sin\vartheta)^{\alpha+1}}{\cos\vartheta} \right]^2 [1 - \cos(\zeta^*{}_V{}^o{}_K \tau^o{}_K \cos\vartheta)] \right\rangle_{\xi,\vartheta} \right\} (K \langle J_{\nu K} \tau_K \rangle_{\xi,\vartheta}{}^2)^{-1} ,$$

$$\begin{aligned} \langle \pi_C{}^2 \rangle_{\xi,\vartheta,\varphi} = \Bigg\{ & 2 \left\langle \left(\frac{\epsilon_Q{}^o{}_K J_\nu{}^o{}_K}{\zeta^*{}_V{}^o{}_K} \right)^2 \left[\frac{(\sin\vartheta)^{\alpha+1}}{\cos\vartheta} \right]^2 [1 - \cos(\zeta^*{}_V{}^o{}_K \tau^o{}_K \cos\vartheta)] \right\rangle_{\xi,\vartheta} \\ & \times \frac{K-1}{2} \left\langle \left(\frac{\zeta^*{}_Q{}^o{}_K}{\zeta^*{}_V{}^o{}_K} \right)^2 \right\rangle \left[\frac{(\sin\vartheta)^2}{\cos\vartheta} \right]^2 [1 - \cos(\zeta^*{}_V{}^o{}_K \tau^o{}_K \cos\vartheta)] \Big\rangle_{\xi,\vartheta} \\ & + \Bigg\langle \Bigg[\left(\frac{\zeta^*{}_Q{}^o{}_K}{\zeta^*{}_V{}^o{}_K} \right) \epsilon_Q{}^o{}_K \left[\frac{(\sin\vartheta)^2}{\cos\vartheta} \right] \left[1 - \frac{\sin(\zeta^*{}_V{}^o{}_K \tau^o{}_K \cos\vartheta)}{(\zeta^*{}_V{}^o{}_K \tau^o{}_K \cos\vartheta)} \right] + \epsilon_V{}^o{}_K (\sin\vartheta)^{\alpha+1/2} (\cos\vartheta) \\ & \times J_\nu{}^o{}_K \tau^o{}_K (\sin\vartheta)^{\alpha+1} \Bigg] \Bigg\rangle_{\xi,\vartheta} \Bigg\} (K \langle J_{\nu K} \tau_K \rangle_{\xi,\vartheta}{}^2)^{-1} , \end{aligned} \tag{E6}$$

with

$$\langle J_{\nu K} \tau_K \rangle_{\xi,\vartheta} = \langle J_\nu{}^o{}_K \tau^o{}_K (\sin\vartheta)^{\alpha+1} \rangle_{\xi,\vartheta} = \langle J_\nu{}^o{}_K \tau^o{}_K \rangle_\xi \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{1}{2}\alpha + \frac{3}{2})}{\Gamma(\frac{1}{2}\alpha + 2)} . \tag{E7}$$

Because not all the indicated averages exist in closed form, we consider the two limits $|\zeta^*{}_V{}^o{}_K| \tau^o{}_K \ll 1$ and $|\zeta^*{}_V{}^o{}_K| \tau^o{}_K \gg 1$. For $|\zeta^*{}_V{}^o{}_K| \tau^o{}_K \ll 1$,

$$\langle \pi_L{}^2 \rangle_{\xi,\vartheta,\varphi} \to \left\{ \frac{\sqrt{\pi}}{2} \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+\frac{5}{2})} \langle (\epsilon_Q{}^o{}_K J_\nu{}^o{}_K \tau^o{}_K)^2 \rangle_\xi \right\} (K \langle J_{\nu K} \tau_K \rangle^2)^{-1} ,$$

$$\begin{aligned} \langle \pi_C{}^2 \rangle_{\xi,\vartheta,\varphi} \to \Bigg\{ & \frac{\sqrt{\pi}}{2} \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+\frac{5}{2})} \langle (\epsilon_Q{}^o{}_K J_\nu{}^o \tau^o{}_K)^2 \rangle_\xi (K-1) \tfrac{2}{15} \langle (\zeta^*{}_Q{}^o{}_K \tau^o{}_K)^2 \rangle_\xi \\ & + \frac{1}{72} \frac{\sqrt{\pi}}{2} \frac{\Gamma(\alpha+4)}{\Gamma(\alpha+\frac{11}{2})} \langle [(\zeta^*{}_Q{}^o{}_K \tau^o{}_K)(\zeta^*{}_V{}^o{}_K \tau^o{}_K)(\epsilon_Q{}^o{}_K J_\nu{}^o{}_K \tau^o{}_K)]^2 \rangle_\xi \\ & + \frac{1}{6} \frac{\sqrt{\pi}}{2} \frac{\Gamma(\alpha+\frac{11}{4})}{\Gamma(\alpha+\frac{17}{4})} \langle (\zeta^*{}_Q{}^o{}_K \tau^o{}_K)(\zeta^*{}_V{}^o{}_K \tau^o{}_K)(\epsilon_Q{}^o{}_K J_\nu{}^o{}_K)(\epsilon_V{}^o{}_K J_\nu{}^o{}_K \tau^o{}_K) \rangle_\xi \\ & + \frac{1}{2} \frac{\sqrt{\pi}}{2} \frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+3)} \langle (\epsilon_V{}^o{}_K J_\nu{}^o{}_K \tau^o{}_K)^2 \rangle_\xi \Bigg\} (K \langle J_{\nu K} \tau_K \rangle_{\xi,\vartheta}{}^2)^{-1} ; \end{aligned} \tag{E8}$$

while for $|\zeta^*{}_V{}^o{}_K| \tau^o{}_K \gg 1$,

$$\langle \pi_L{}^2 \rangle_{\xi,\vartheta,\varphi} \to \left\{ \pi \left\langle \frac{(\epsilon_Q{}^o{}_K J_\nu{}^o{}_K \tau^o{}_K)^2}{(\zeta^*{}_V{}^o{}_K \tau^o{}_K)} \right\rangle_\xi \right\} (K \langle J_{\nu K} \tau_K \rangle_{\xi,\vartheta}{}^2)^{-1} ,$$

$$\begin{aligned} \langle \pi_C{}^2 \rangle_{\xi,\vartheta,\varphi} \to \Bigg\{ & \pi \left\langle \frac{(\epsilon_Q{}^o{}_K J_\nu{}^o{}_K \tau^o{}_K)^2}{(\zeta^*{}_V{}^o{}_K \tau^o{}_K)} \right\rangle_\xi (K-1) \frac{\pi}{4} \left\langle \frac{(\zeta^*{}_Q{}^o{}_K \tau^o{}_K)^2}{(\zeta^*{}_V{}^o{}_K \tau^o{}_K)} \right\rangle_\xi + \frac{\pi}{6} \left\langle \frac{[(\zeta^*{}_Q{}^o{}_K \tau^o{}_K)(\epsilon_Q{}^o{}_K J_\nu{}^o{}_K \tau^o{}_K)]^2}{(\zeta^*{}_V{}^o{}_K \tau^o{}_K)} \right\rangle_\xi \\ & + 2 \frac{\sqrt{\pi}}{2} \frac{\Gamma(\alpha+\frac{11}{4})}{\Gamma(\alpha+\frac{13}{4})} \left\langle \frac{(\zeta^*{}_Q{}^o{}_K \tau^o{}_K)(\epsilon_Q{}^o{}_K J_\nu{}^o{}_K \tau^o{}_K)(\epsilon_V{}^o{}_K J_\nu{}^o{}_K \tau^o{}_K)}{(\zeta^*{}_V{}^o{}_K \tau^o{}_K)} \right\rangle_\xi \\ & + \frac{1}{2} \frac{\sqrt{\pi}}{2} \frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+3)} \langle (\epsilon_V{}^o{}_K J_\nu{}^o{}_K \tau^o{}_K)^2 \rangle_\xi \Bigg\} (K \langle J_{\nu K} \tau_K \rangle_\xi{}^2)^{-1} . \end{aligned} \tag{E9}$$

The total absorption depth through the K regions satisfies

$$\tau = \sum_{K'=1}^{K} \tau_{K'} \approx K \langle \tau_K \rangle_{\xi,\vartheta} = K \langle \tau^o{}_K \rangle_\xi \langle (\sin\vartheta)^{\alpha+1/2} \rangle_\vartheta = K \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{1}{2}\alpha + \frac{7}{4})}{\Gamma(\frac{1}{2}\alpha + \frac{9}{4})} \langle \tau^o{}_K \rangle_\xi , \tag{E10}$$

so that

$$\langle \tau^o{}_K \rangle_\xi = \frac{1}{K}\left[\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1}{2}\alpha+\frac{7}{4})}{\Gamma(\frac{1}{2}\alpha+\frac{9}{4})}\right]^{-1}\tau \equiv \frac{1}{K}\tau_\Sigma(\nu/\nu_1)^{-(\alpha+5/2)}\,, \tag{E11}$$

where $\tau = 1$ defines a reference frequency $\nu = \nu_1$.

Two parameters—γ_i and $\gamma_1 \equiv (\nu_1/\nu_B)^{1/2}$—suffice to specify the dependence of the normalized transfer coefficients (eqs. [C13]–[C15]) upon the dimensionless frequency (ν/ν_1). While observations do not generally determine γ_i, measurements of brightness temperature do frequently permit estimation of γ_1: For large absorption depths, the spectral intensity I_ν must approach the spectral source function $J_\nu = (\eta_\nu/\kappa)$ in the strong-rotativity limit (eq. [D3]), so that

$$\langle I_\nu \rangle \to \langle J_\nu \rangle_{\xi,\vartheta} = J_\alpha m\nu_1{}^2\langle\gamma_1\rangle_\xi\langle(\sin\vartheta)^{-1/2}\rangle_\vartheta(\nu/\nu_1)^{5/2}\,,\quad [(\nu/\nu_1) < 1]$$

$$= \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{5}{4})} J_\alpha m\nu_1{}^2\langle\gamma_1\rangle_\xi(\nu/\nu_1)^{5/2}\,. \tag{E12}$$

We then approximate an unresolved source as a $K \times L \times M$ matrix of randomly oriented, uniform domains. Performing the average over the $L \times M$ independent lines of sight merely results in an additional factor (LM) in the denominators of the expected variances $\langle\pi_L{}^2\rangle$ and $\langle\pi_C{}^2\rangle$ (eqs. [E6], [E8], and [E9]). For the calculations shown in Figure 2, we have taken the same set of parameters ξ for each domain, and have assumed $K \approx L \approx M \approx N^{1/3}$, where N represents the total number of cells comprising the unresolved source.

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