Lecture 18

Steady-State Markov Chain

Manju M. Johny

STAT 330 - Iowa State University

Steady-State

Steady-State Distribution

$P_h = P_0 P^{(h)}$

Definition

For a Markov chain $\{X_t : t \in \mathcal{T}\}$, if a collection of limiting probabilities

$$\pi_x = \lim_{h\to\infty} P_h(x), \quad x \in \mathcal{X},$$

exists, then π_x is called a ***steady-state distribution*** of the Markov chain. (Note: π is a distribution not the value 3.14...)

- This is the distribution of X_t after many, many transitions! (the long run probability)

many many steps ahead.

About the Steady-State Distribution

1. Obtaining the steady-state distribution: $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ is the solution to the following set of linear equations

$$\boldsymbol{\pi} P = \boldsymbol{\pi}, \qquad \sum_{x\in\mathcal{X}} \pi_x = 1.$$

($\sum_{x\in\mathcal{X}} \pi_x = 1$ since each of the states should be in $\mathcal{X}$)

2. What is meant by the "steady state" of a Markov chain?
   - Suppose the system has reached its steady state, so that the distribution of the states is $P_t = \boldsymbol{\pi}$.
   - The state after one more transitions is: $P_{t+1} = P_t P = \boldsymbol{\pi} P = \boldsymbol{\pi}$
   - Thus, if a chain is in a steady state, the distribution *stays the same* ("steady") after any subsequent transitions.

Steady-State Distribution Cont.

3. The limit for P^h (the h-step transition matrix) is

capital $\downarrow$ Π

$$\Pi = \lim_{h\to\infty} P^h = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \\ \pi_1 & \pi_2 & \cdots & \pi_n \\ \vdots & \vdots & \ddots & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}.$$

All the rows are equal and consist of the steady state probabilities π_x.

4. The steady-state distribution is not guaranteed to exist.
   - Steady-state distribution may or may not exist.
   - If a Markov chain is *regular*, then it has a steady state distribution. (This is what we will check).

Example

Example

Example 1: Ames weather problem: Suppose the state space is ("*sunny*", "*rainy*") = (1, 2), with initial probability $P_0 = (p, 1-p)$

- Can approximate the *steady state distribution* (π), by $P \cdot P \cdots P$ until convergence

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}, \underset{P \cdot P}{\overset{\parallel}{P^{(2)}}} = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}, \underset{P \cdot P \cdot P}{\overset{\parallel}{P^{(3)}}} = \begin{pmatrix} 0.583 & 0.417 \\ 0.556 & 0.444 \end{pmatrix}$$

$$P^{(15)} \approx \ldots \approx P^{(30)} \approx \begin{pmatrix} 0.5714 & 0.4286 \\ 0.5714 & 0.4286 \end{pmatrix} = \begin{pmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 \\ \pi_1 & \pi_2 \end{pmatrix}$$

- For any given starting state distribution $P_0 = (p,\ 1-p)$,

$$P_0\pi = (p,\ 1-p) \begin{pmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{pmatrix} = (4/7, 3/7)$$

Example

- Alternatively, we can obtain *steady state distribution* (π) by solving the system of equations:
  1. $\pi P = \pi$ $\longrightarrow$ $(\pi_1\ \pi_2)\begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = (\pi_1\ \pi_2)$
  2. $\sum_{x \in \mathcal{X}} \pi_x = 1$

$$\begin{cases} 0.7\pi_1 + 0.4\pi_2 = \pi_1 \\ 0.3\pi_1 + 0.6\pi_2 = \pi_2 \end{cases}$$

$$\Rightarrow \begin{cases} 0.4\pi_2 = 0.3\pi_1 \\ 0.3\pi_1 = 0.4\pi_2 \end{cases}$$

one row will always reduce to be same as another row. If I had 3 rows, I would end up w/ 2 unique rows.

$$\Rightarrow 0.4\pi_2 = 0.3\pi_1$$

$$\Rightarrow \boxed{\pi_2 = \frac{3}{4}\pi_1}$$

Now use $\sum_x \pi_x = 1$ to obtain values for π_1 & π_2

Example

$\sum_x \pi_x = 1 \quad = \quad ① \ \pi_1 + \pi_2 = 1$

$② \ \pi_2 = \frac{3}{4}\pi_1$

We have system of Equations 2 Eq's & 2 unknowns

plug ② into ①

$\pi_1 + \pi_2 = 1$

$\Rightarrow \pi_1 + \frac{3}{4}\pi_1 = 1$

$\Rightarrow \frac{7}{4}\pi_1 = 1$

$\Rightarrow \boxed{\pi_1 = \frac{4}{7}}$

and

$\pi_1 + \pi_2 = 1$

$\Rightarrow \frac{4}{7} + \pi_2 = 1$

$\Rightarrow \boxed{\pi_2 = \frac{3}{7}}$

So my steady state distribution

$\underline{\pi} = (\pi_1 \ \ \pi_2)$

$= (4/7 \ \ 3/7)$

Main Idea: No matter what initial distribution P_0 we start with, after a large number of steps, the probability distribution converges to approximately (4/7, 3/7). This (4/7, 3/7) is called the *"steady-state distribution" or π*

How to find steady state Distribution $\underline{\pi}$

① Multiply P by itself many many times until $P^{(n)}$ converges (doesn't change). The rows of $P^{(n)} = \underline{\pi}$

② Solve System Eq's

① $\underline{\pi} P = \underline{\pi}$

② $\sum_x \pi_x = 1$

Regular Markov Chain

Regular MC

Definition

A Markov Chain $\{X_t\}$ with transition matrix P is said to be *regular* if, for some n, *all* entries of $P^{(n)}$ are positive (> 0).
Any regular Markov chain has a steady-state distribution.

1. Not every Markov chain has a steady-state distribution. Why? Consider the following transition matrix:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then

$$\overbrace{P^{2k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{\text{Even Transition}}, \quad \overbrace{P^{2k-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}^{\text{odd transition}} \quad \forall k \in \mathbb{N}$$

$P^{(n)}$ will keep oscillating b/w these two matrices forverer.

Checking Regular MC

2. As long as we find *some* n such that *all* entries of $P^{(n)}$ are (>0) positive, then the chain is *regular*. This does not mean that a regular Markov chain has to possess this property for all n. Consider the following transition matrix,

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 2/3 & 0 & 1/3 \\ 1/2 & 1/4 & 1/4 \end{pmatrix},$$

then

$$P^2 = \begin{pmatrix} .500 & .250 & .250 \\ .167 & .083 & .750 \\ .292 & .063 & .646 \end{pmatrix}$$

This Markov chain is regular since $P^{(2)}$ contains all positive elements even though the one-step transition matrix P contain non-positive elements.