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Reinforcement Learning

Lecture 3 - Dynamic Programming

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Department of Information Technology

Repetition

Important concepts

- **States, actions and rewards:** $s \in \mathcal{S}$, $a \in \mathcal{A}$, $r \in \mathcal{R}$.

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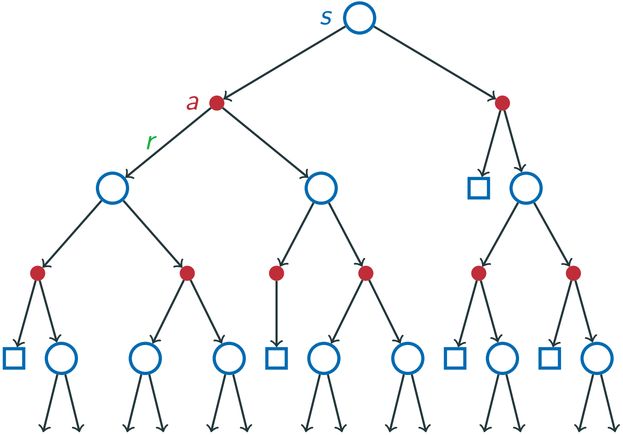
  Expected return when starting in s, taking action a and *then* follow π,

$$q_\pi(s, a) = \mathbb{E}_\pi \left[G_t | S_t = s, A_t = a\right]$$

Markov Decision Process Tree



- **Relations:**

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Relations and Bellman equations

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- **Bellman equation for state-values:**

$$\begin{aligned} v_\pi(s) &= \sum_a \pi(a|s) \sum_{r,s'} p(s',r|s,a)[r + \gamma v_\pi(s')] \\ &= \mathbb{E}_\pi \left[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s\right] \end{aligned}$$

Optimal value function and policy

- **Optimal value functions:**

$$v_*(s) = \max_\pi v_\pi(s), \quad \text{for all } s \in \mathcal{S}$$
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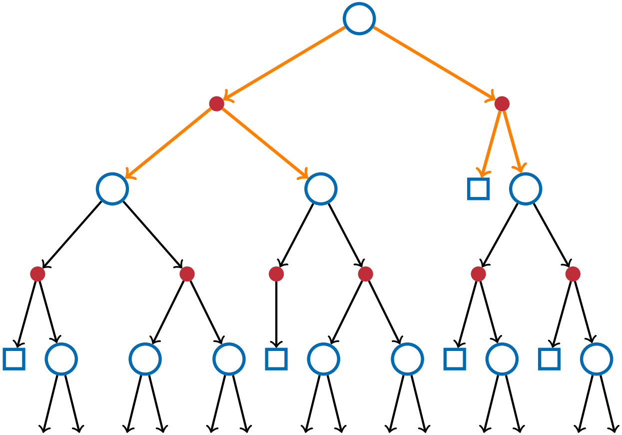
- **Optimal policy:** We get an optimal policy if we act greedily w.r.t v_*, i.e.,

$$\pi_*(s) = \arg\max_a q_*(s, a).$$

Dynamic Programming?

- Dynamic: Sequential or temporal component to the problem.
- Programming: optimizing a “program” (c.f. linear programming)
- Solving complex problems by breaking them down into subproblems.

Bellman equation: $v_\pi(s) = \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s]$



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Control:

- Given an MDP, find an optimal policy π_*.
- If we first compute v_*, then we can use

$$q_*(s, a) = \sum_{r, s'} p(s', r|s, a)[r + \gamma v_*(s')]$$

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- For large state and/or action space, more efficient with iterative solutions.

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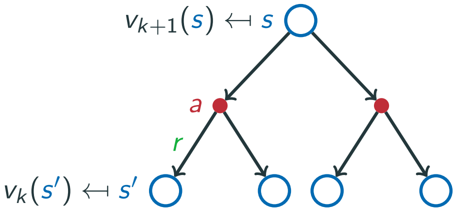
  - With $u_k = v_\pi$:

$$\|v_\pi - v_{k+1}\|_\infty \leq \gamma \|v_\pi - v_k\|_\infty.$$

Iterative policy iteration



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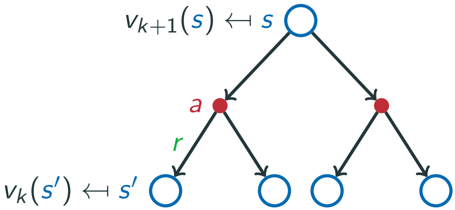

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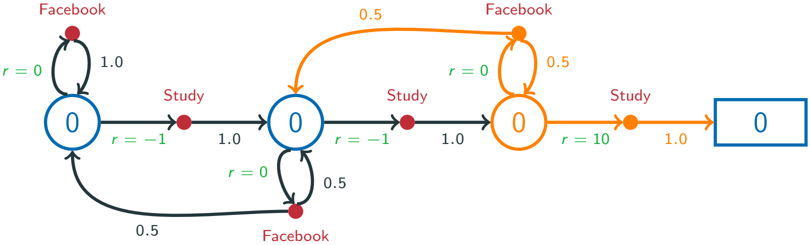
Bootstrapping: We use our old estimate (v_k) to compute a new estimate (v_{k+1}).

Discount: $\gamma = 0.9$.

Policy: $\pi(a|s) = 0.5$ for all a and s.

v_0



$v_1(s) =$

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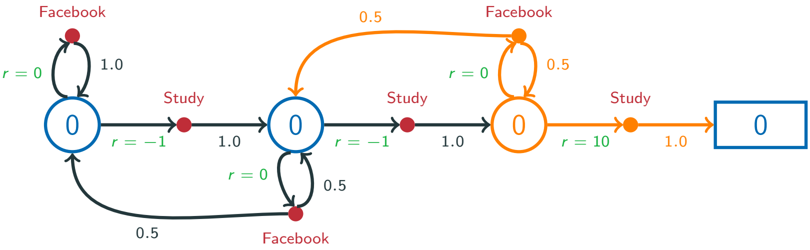

$$v_1(s) = 0.5 \underbrace{[0 + \gamma(0.5 \times 0 + 0.5 \times 0)]}_{\text{facebook}} + 0.5 \underbrace{[10 + 0\gamma]}_{\text{study}} = 5$$

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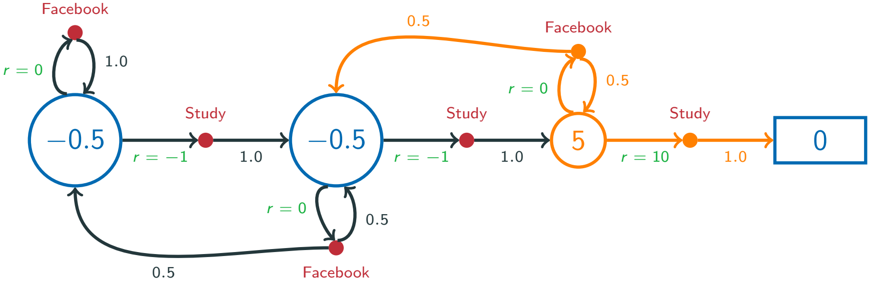
v_1



$$v_2(s) = 0.5\,\underbrace{[0 + \gamma(0.5 \times (-0.5) + 0.5 \times 5)]}_{\text{facebook}} + 0.5\,\underbrace{[10 + 0\gamma]}_{\text{study}} = 6.01$$

Example: Iterative Policy Evaluation

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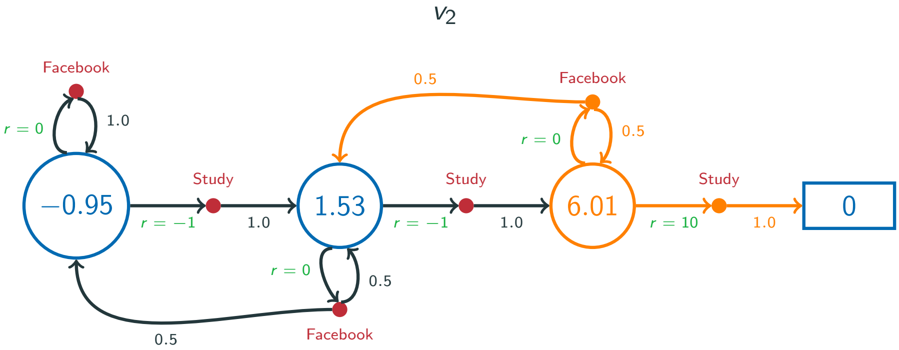


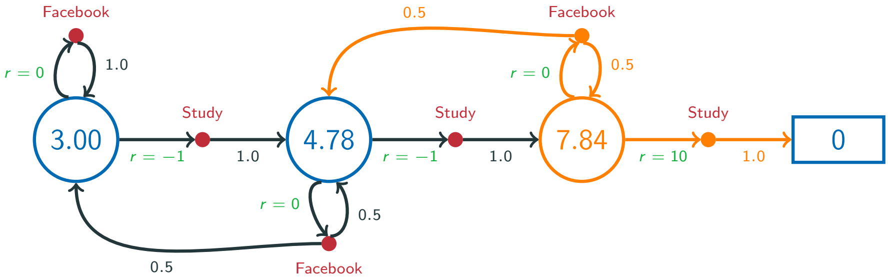
$$v_3(s) = 0.5\underbrace{\left[0 + \gamma(0.5 \times 1.53 + 0.5 \times 6.01)\right]}_{\text{facebook}} + 0.5\underbrace{\left[10 + 0\gamma\right]}_{\text{study}} = 6.70$$

Example: Iterative Policy Evaluation

Discount: $\gamma = 0.9$.

Policy: $\pi(a|s) = 0.5$ for all a and s.

$v_{30} \approx v_\pi$



$$v_{31}(s) = 0.5\underbrace{\left[0 + \gamma(0.5 \times 4.78 + 0.5 \times 7.84)\right]}_{\text{facebook}} + 0.5\underbrace{\left[10 + 0\gamma\right]}_{\text{study}} \approx 7.84$$

Discount: $\gamma = 1$

Policy: Uniform policy (all actions equally likely)

$k = 0$

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Example: Iterative Policy Evaluation

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Policy: Uniform policy (all actions equally likely)

$k = 0$

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

$k = 1$

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	0

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0	0	0	0
0	0	0	0
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$k = 1$

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	0

$k = 2$

0	-1.75	-2	-2
-1.75	-2	-2	-2
-2	-2	-2	-1.75
-2	-2	-1.75	0

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0	-1.75	-2	-2
-1.75	-2	-2	-2
-2	-2	-2	-1.75
-2	-2	-1.75	0

$k = 3$

0	-2.43	-2.94	-3
-2.43	-2.88	-3	-2.94
-2.94	-3	-2.88	-2.43
-3	-2.94	-2.43	0

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0	0	0	0
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-2.43	-2.88	-3	-2.94
-2.94	-3	-2.88	-2.43
-3	-2.94	-2.43	0

$k = \infty$

	-14	-20	-22
-14	-18	-20	-20
-20	-20	-18	-14
-22	-20	-14	

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  1. Initialize v_{old} (e.g. $v_{\text{old}}(s) = 0$ for all s)
  2. For all $s \in \mathcal{S}$:

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  3. If $|v_{\text{old}}(s) - v_{\text{new}}(s)| < \text{tol}$ for all s, output v_{new} and stop.
  4. Otherwise, let $v_{\text{old}} \leftarrow v_{\text{new}}$ and go back to step 2.

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$$v_{\text{new}}(s) = \sum_a \pi(a|s) \sum_{r,s'} p(s', r|s, a)[r + \gamma v_{\text{old}}(s')]$$

  3. If $|v_{\text{old}}(s) - v_{\text{new}}(s)| < \text{tol}$ for all s, output v_{new} and stop.
  4. Otherwise, let $v_{\text{old}} \leftarrow v_{\text{new}}$ and go back to step 2.
- This algorithm do *synchronous* updates.

- For finite $\mathcal{S}$ we can represent $v(s)$ as an array with one element for each sate in $\mathcal{S}$.
- $v_k \rightarrow v_\pi$ as $k \rightarrow \infty$. But in practice we stop when the difference between v_{k+1} and v_k is small enough.
- The algorithm is then
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- This algorithm do *synchronous* updates.
- Asynchronous updates also converge to $v_\pi(s)$, as long as we keep updating all states.

In place updates

1. Start with initial $v(s)$ (e.g. $v(s) = 0$).
2. For all $s \in \mathcal{S}$:

$$v(s) \leftarrow \sum_{a} \pi(a|s) \sum_{r,s'} p(s', r|s, a)[r + \gamma v(s')]$$

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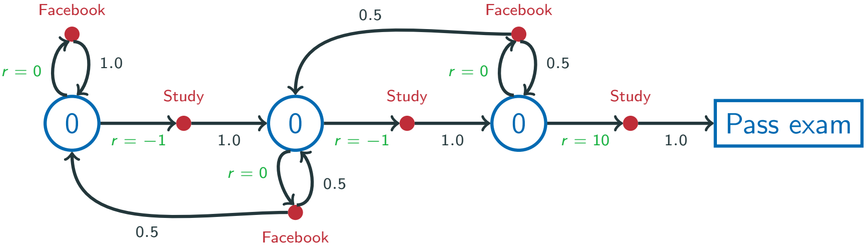
- Easier to implement (only needs one array $v(s)$).
- Now the updates depends on which order we sweep through the states.
- Also converges to $v_\pi(s)$, often faster even.

Discount: $\gamma = 0.9$.

Policy: $\pi(a|s) = 0.5$ for all a and s.



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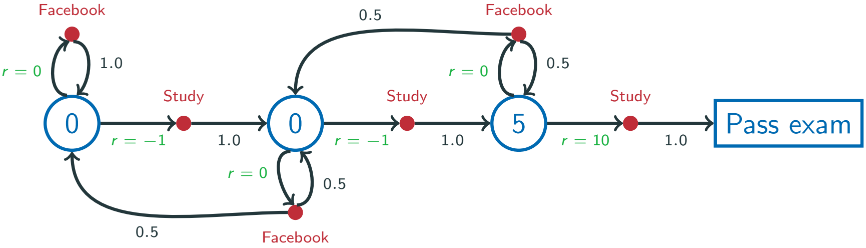

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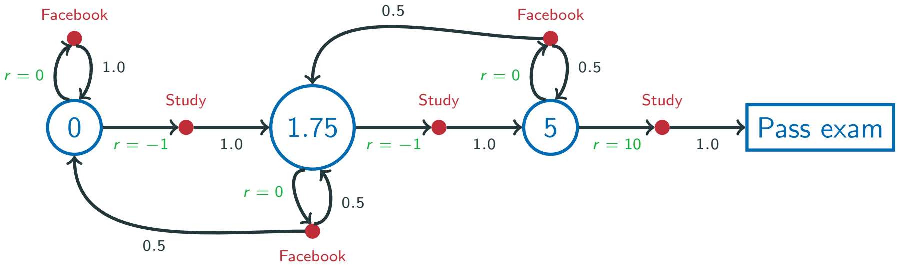

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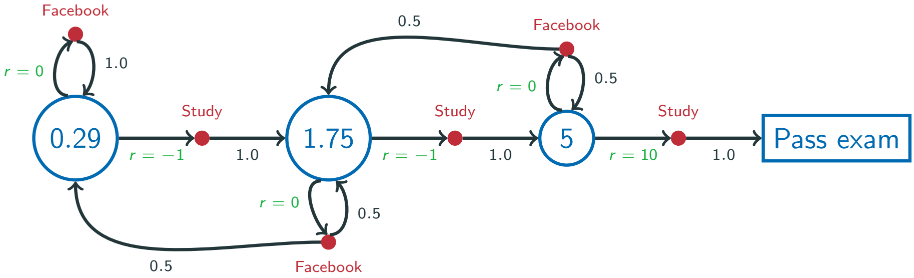

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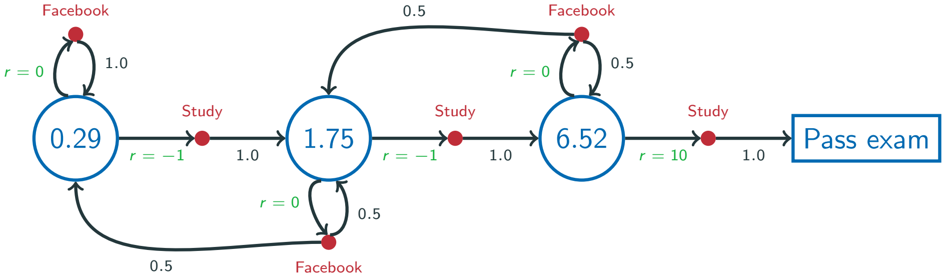

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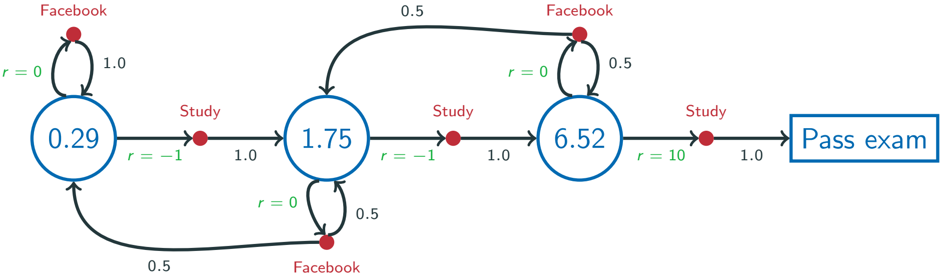

If we continue like this, we will converge to $v_\pi(s)$

Example: In-place updates

Discount: $\gamma = 1$

Policy: Uniform.

$v(s)$

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Example: In-place updates

Discount: $\gamma = 1$

Policy: Uniform.

$v(s)$

0	-1	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Example: In-place updates

Discount: $\gamma = 1$

Policy: Uniform.

$v(s)$

0	-1	-1.25	0
0	0	0	0
0	0	0	0
0	0	0	0

Example: In-place updates

Discount: $\gamma = 1$

Policy: Uniform.

$v(s)$

0	-1	-1.25	-1.31
0	0	0	0
0	0	0	0
0	0	0	0

Example: In-place updates

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0	-1	-1.25	-1.31
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-1	-1.5	0	0
0	0	0	0
0	0	0	0

Example: In-place updates

Discount: $\gamma = 1$

Policy: Uniform.

One sweep over all states

0	-1	-1.25	-1.31
-1	-1.5	-1.69	-1.75
-1.25	-1.69	-1.84	-1.90
-1.31	-1.75	-1.90	0

Policy improvement

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- The value of taking action a in state s and then following policy π afterwards is

$$q_\pi(s, a) = \sum_{r, s'} p(s', r | s, a)[r + \gamma v_\pi(s')].$$

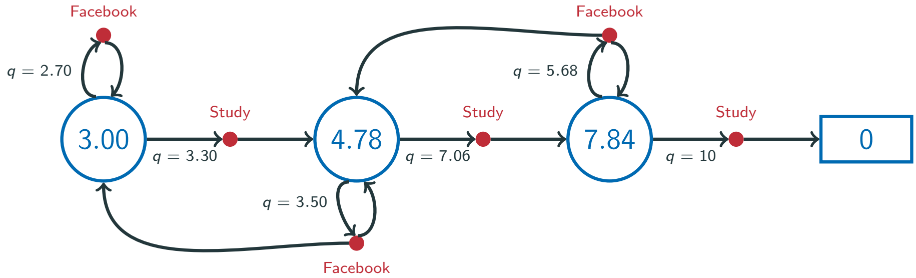
- Let us try to act **greedily** with respect to the action values, i.e.

$$\pi'(s) = \arg\max_a q_\pi(s, a).$$

Example: Greedy

Discount: $\gamma = 0.9$.

Policy: $\pi(a|s) = 0.5$ for all a and s.



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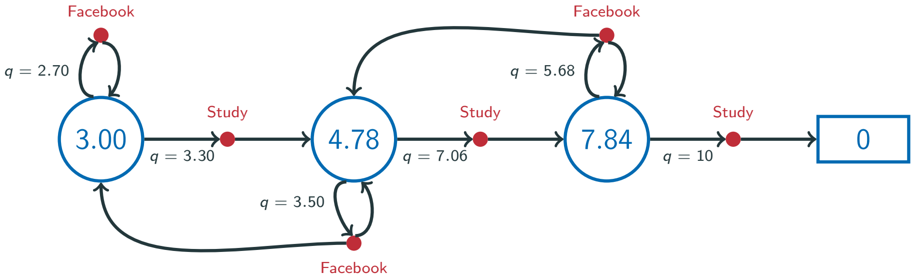

The greedy policy with respect to v_π is

$$\pi'(s) = \arg\max_a q_\pi(s, a) = \text{study}.$$

The Policy Improvement Theorem

- Consider deterministic policy $a = \pi(s)$ (but result holds for stochastic $\pi(a|s)$ too)

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If $q_\pi(s, \pi'(s)) \geq v_\pi(s)$ for all $s \in \mathcal{S}$, then

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- So $\pi'(s) = \arg\max_a q_\pi(s, a)$ is as good as, or better, than $\pi(s)$.

Proof of the policy improvement theorem

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Is it really an improvement?

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$$\pi'(s) = \arg\max_a q_\pi(s, a) = \arg\max_a \sum_{r,s'} p(s', r|s, a)[r + \gamma v_\pi(s')]$$

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$$\pi'(s) = \arg\max_a q_\pi(s, a) = \arg\max_a \sum_{r,s'} p(s', r|s, a)[r + \gamma v_\pi(s')]$$

  will be *at least* as good as π in all states.

- **No improvement?** What if $v_\pi(s) = v_{\pi'}(s)$ for all s?

  Then from previous slide we see that

$$q_\pi(s, \pi'(s)) = \max_a q_\pi(s, a) = q_\pi(s, \pi(s)) = v_\pi(s)$$

  so

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- This is the *Bellman optimality equation*! So π and π' are optimal policies!

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- This is the *Bellman optimality equation*! So π and π' are optimal policies!
- **Conclusion:** π' will be strictly better than π, unless π is already optimal!

Policy Iteration

Start with an initial policy π.

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$$\pi_0 \xrightarrow{E} v_{\pi_0} \xrightarrow{I} \pi_1 \xrightarrow{E} v_{\pi_1} \xrightarrow{I} \pi_2 \xrightarrow{E} \cdots \xrightarrow{I} \pi_* \xrightarrow{E} v_*.$$

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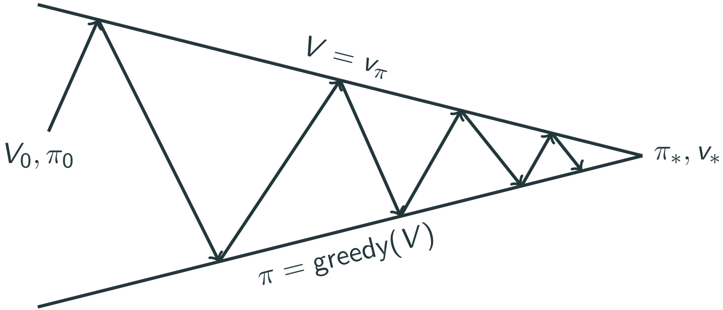
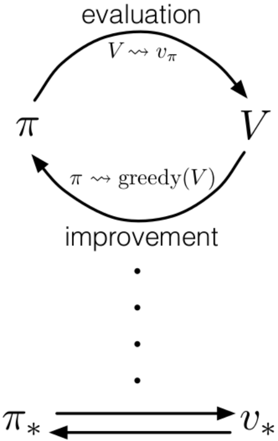
In a finite MDP this will converge in a finite number of iterations.

Some implementation details:

- In E: Start from v for the previous policy to speed up computation.
- In I: If there are several a that maximize $q_\pi(s, a)$, choose arbitrarily (or use a stochastic policy that picks between them with e.g. uniform probability).

Policy Iteration





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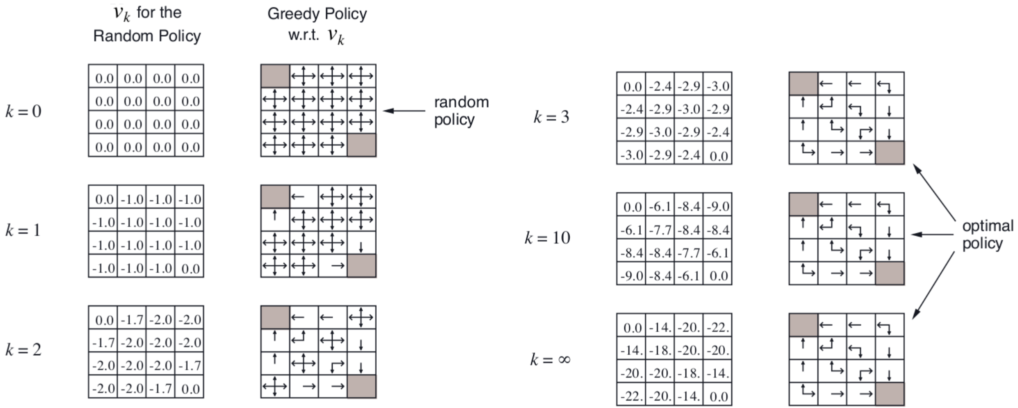
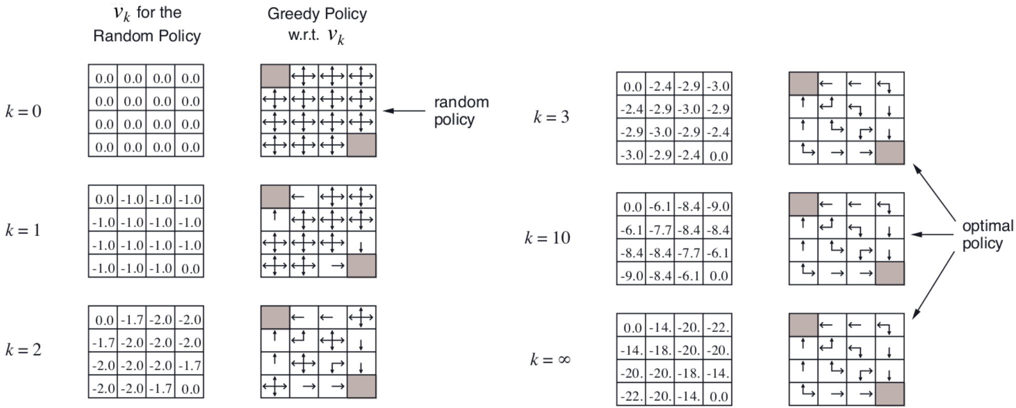
Evaluating the uniform policy.



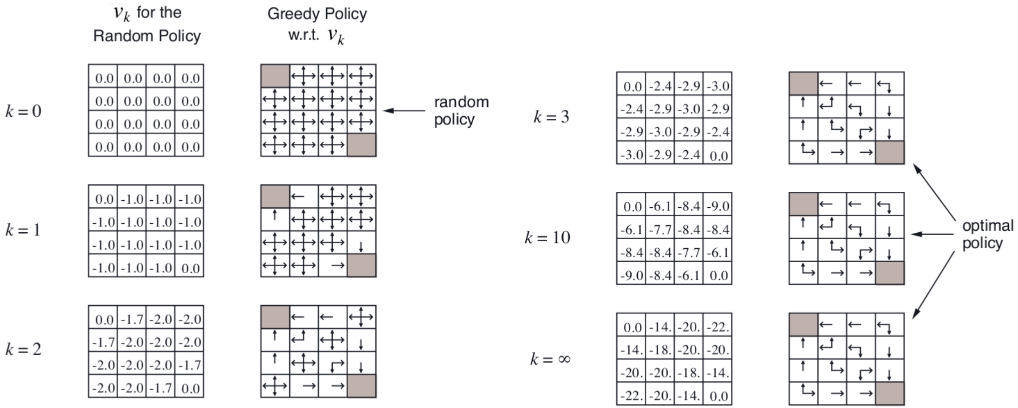
Example 4.1: Policy evaluation

Evaluating the uniform policy.



- Do we really have to wait for the E to converge before we do I?

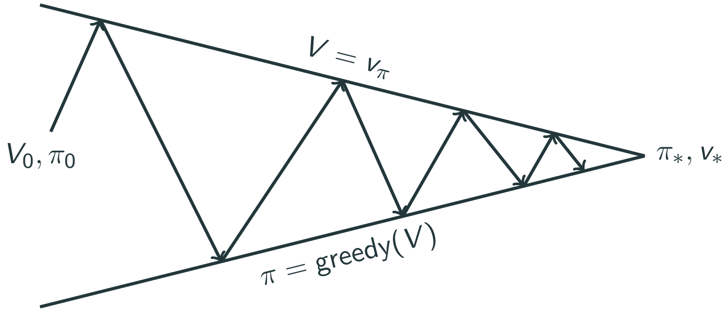
Evaluating the uniform policy.

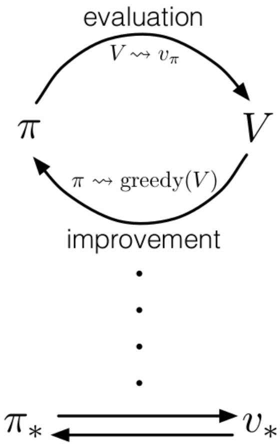


- Do we really have to wait for the E to converge before we do I?
- Definitely not in this case!

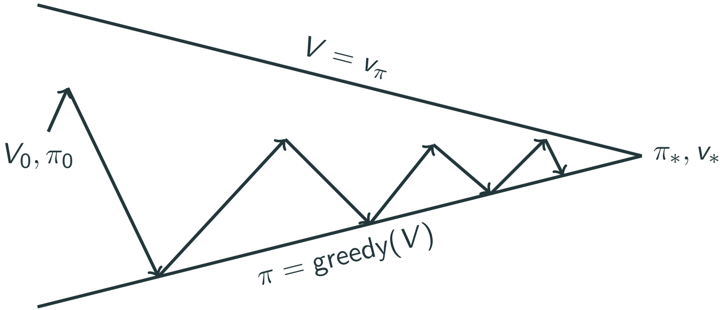
Value iteration

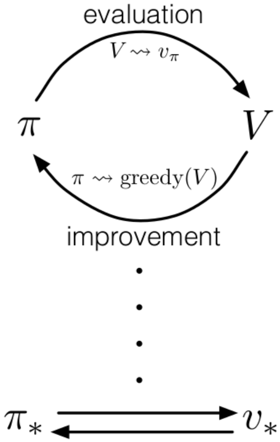
Generalized Policy Iteration





- **Policy iteration:** Evaluation complete before we improve.





- **Policy iteration:** Evaluation complete before we improve.
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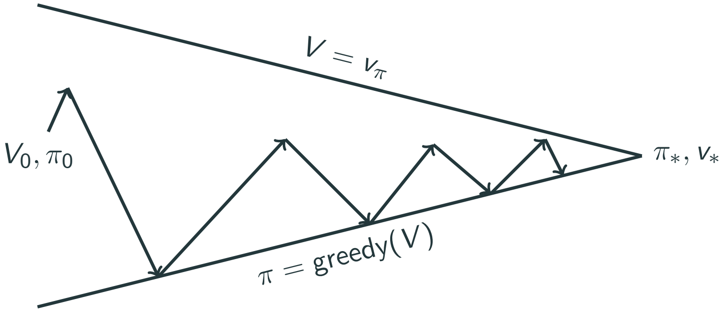

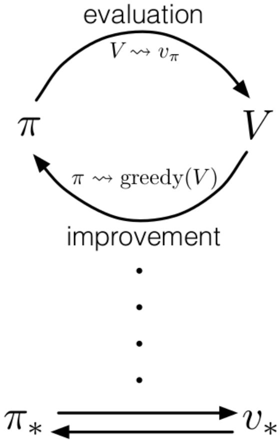

- **Policy iteration:** Evaluation complete before we improve.
- **Generalized policy iteration**.
- **Value iteration:** Stop evaluation after one sweep over all states!

Value iteration

- Lets see what happens if we only do one iteration of evaluation before we improve.

Value iteration

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- Lets see what happens if we only do one iteration of evaluation before we improve.
- That is, for all s

$$q_{k+1}(s, a) = \sum_{r,s'} p(s', r|s, a)[r + \gamma v_k(s')]$$

Value iteration

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$$v_{k+1}(s) = q_{k+1}(s,\pi_{k+1}(s)).$$

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- Or in one equation

$$v_{k+1}(s) =$$

Value iteration

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- With this iteration we will converge to the optimal $v_*(s)$.

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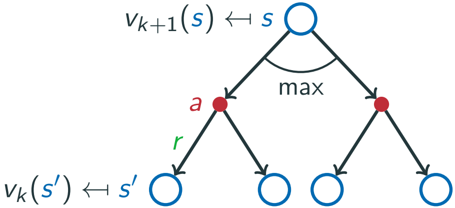
- With this iteration we will converge to the optimal $v_*(s)$.
- Can use e.g. in place updates instead of synchronous updates.

Value Iteration



$$v_{k+1}(s) = \max_a \sum_{r,s'} p(s', r|s, a)[r + \gamma v_k(s')].$$

Value Iteration and the Bellman Optimality Equation

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- **Value iteration:**

$$v_{k+1}(s) = \max_{a} \sum_{r,s'} p(s', r | s, a)[r + \gamma v_k(s')]$$

Value Iteration and the Bellman Optimality Equation

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$$v_{k+1}(s) = \max_a \sum_{r,s'} p(s', r|s, a)[r + \gamma v_k(s')]$$

- **Fixed point:** If $v_k(s) = v_{k+1}(s)$ for all s

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Value Iteration and the Bellman Optimality Equation

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  This is the *Bellman optimality equation*, so $v_k(s)$ is the optimal value function.

- **Optimal policy:** When converged to v_* we can find an optimal policy

$$\pi_*(s) = \arg\max_a q_*(s, a).$$

Example: GridWorld

$$v_{k+1}(s) = \max_a \sum_{r,s'} p(s', r|s, a)[r + \gamma v_k(s')].$$

Discount: $\gamma = 1$, Reward: -1 for each action.

v_0

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Example: GridWorld

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v_0

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

v_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	0

Example: GridWorld

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v_0

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

v_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	0

v_2

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-1
-2	-2	-1	0

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v_0

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

v_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	0

v_2

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-1
-2	-2	-1	0

$v_3 = v_*$

0	-1	-2	-3
-1	-2	-3	-2
-2	-3	-2	-1
-3	-2	-1	0

Summary

Problem	**Based on**	**Algorithm**
Prediction	Bellman Equation for v_π	Iterative Policy Evaluation
Control	Bellman Equation for v_π $+$ Greedy Policy Improvement	Policy Iteration
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Next:

- What to do if $p(s', r|s, a)$ is unknown?
- Before that: Look at Tinkering Notebook 2.
- You are now ready for Assignment 1!