Assigned: 07Feb2019 Due: 14Feb2019, **4:30 PM**

Homework 04

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1. A CT signal $x(t)$ is converted to a DT signal by sampling with a sampling interval of $T_s = \frac{1}{250}$ secs. This DT signal is passed through a LPF with a cut-off frequency of $\frac{4\pi}{5}$ rad. The output is converted back to a CT signal by passing through an ideal low pass reconstruction filter with cut-off frequency $\omega_c = \frac{\omega_s}{2} = 250\pi$ rad/sec and gain $\frac{1}{250}$. This filter has frequency response

$$H_{\text{r-LP}}(\omega) = \begin{cases} \frac{1}{250} & |\omega| \leq 250\pi \text{ rad/sec} \\ 0 & |\omega| > 250\pi \text{ rad/sec.} \end{cases}$$

Let $y(t)$ denote the output signal.

(a) Characterize $Y(j\omega)$ if $x(t) = \frac{\sin(400\pi t)}{\pi t}$ and thus find $A(\omega) := \frac{Y(j\omega)}{X(j\omega)}$ in the range $|\omega| \leq 200\pi$ rad/sec.

(b) Characterize $Y(j\omega)$ if $x(t) = \frac{\sin(200\pi t)}{\pi t}$ and thus find $B(\omega) := \frac{Y(j\omega)}{X(j\omega)}$ in the range $|\omega| \leq 200\pi$ rad/sec.

(c) Is $A(\omega) = B(\omega)$? What can you conclude on the overall transformation from $x(t)$ to $y(t)$? Is the overall transformation from $x(t)$ to $y(t)$ equivalent to that of an LTI system? Explain your answers.

2. Solve **any two** parts. A signal $x(t)$ is multiplied by an impulse train $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$ to obtain $x_s(t) = x(t)s(t)$. Then $x_s(t)$ is passed through an ideal LPF with cut-off frequency ω_c rad/sec and gain T_s whose frequency response

$$H_{\text{r-LP}}(\omega) = \begin{cases} T_s & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}.$$

Plot the magnitude of the Fourier transform of the output $y(t)$ if

(a) $x(t) = \cos(300\pi t)$, $T_s = \frac{1}{400}$ sec, $\omega_c = \frac{\pi}{T_s} = 400\pi$ rad/sec,

(b) $x(t) = \cos(300\pi t)$, $T_s = \frac{1}{400}$ sec, $\omega_c == 250\pi$ rad/sec,

(c) $x(t) = \cos(300\pi t)$, $T_s = \frac{1}{200}$ sec, $\omega_c = \frac{\pi}{T_s} = 200\pi$ rad/sec,

(d) $x(t) = \cos(300\pi t)$, $T_s = \frac{1}{200}$ sec, $\omega_c = 500\pi$ rad/sec,

(e) $x(t) = \frac{\sin(200\pi t)}{\pi t}$, $T_s = \frac{1}{500}$ sec, $\omega_c = \frac{\pi}{T_s} = 500\pi$ rad/sec,

(f) $x(t) = \frac{\sin(200\pi t)}{\pi t}$, $T_s = \frac{1}{500}$ sec, $\omega_c = 100\pi$ rad/sec,

(g) $x(t) = \frac{\sin(200\pi t)}{\pi t}$, $T_s = \frac{1}{150}$ sec, $\omega_c = 150\pi$ rad/sec,

(h) $x(t) = \frac{\sin(2000\pi t)}{\pi t}\cos(400\pi t)$, $T_s = \frac{1}{1500}$ sec, $\omega_c = 3000\pi$ rad/sec,

3. In this problem, we shall sample the DTFT of an (aperiodic) signal. Essentially, we will derive the Discrete Fourier Transform (DFT) of an aperiodic signal $x_d(n)$.

   (a) We let $X_d(\omega)$ denote the DTFT of $x_d(n)$.[1] Recall the analysis and synthesis equations for the DTFT of a general aperiodic DT signal $x_d(n)$. Write them down and denote them as Eqn. (1) and Eqn. (2) respectively.

   (b) We denote a new sequence $Z(k) : k = 0, \pm 1, \pm 2, \cdots$ as the samples of $X_d(\omega)$. Specifically, let $Z(k) = X_d(\frac{2\pi k}{N})$. Substitute $\omega = \frac{2\pi k}{N}$ in Eqn. (1) and express $Z(k)$ in terms of $x_d(n)$. Denote this as Eqn (3). Using this or otherwise, prove that $Z(k)$ is periodic with period N. Specifically, establish $Z(k) = Z(k+N)$ for any k.

   (c) Recall that the DTFS coefficients of a periodic DT signal is periodic with period N. We shall now characterize the DT signal whose DTFS coefficients are $Z(k)$ as obtained in Eqn. (3). Suppose $z(n)$ denotes the periodic DT signal whose DTFS coefficients are $Z(k)$. Write down the synthesis equation for DTFS and thereby expressing $z(n)$ in terms of $Z(k)$. Now use Eqn. 3 to substitute for $Z(k)$. You should now have a relation relating $z(n)$ and $x_d(n)$. Exchange the summations and denote this as Eqn. (4).

   (d) Show that

$$\sum_{l=-\infty}^{\infty} \delta(n - lN) = \sum_{k=0}^{N-1} \frac{1}{N} e^{j\frac{2\pi kn}{N}}. \tag{1}$$

   To prove this, study Example 5.6 from SAndS. Therein, it is shown that DTFS coefficients of the signal on the LHS of (1) is $\frac{1}{N}$. The RHS of (1) is just the synthesis equation with a_k substituted to be $\frac{1}{N}$. Conclude that for every $r = 0, \pm 1, \pm 2, \pm 3$,

$$\sum_{l=-\infty}^{\infty} \delta(r - lN) = \sum_{k=0}^{N-1} \frac{1}{N} e^{j\frac{2\pi kr}{N}}. \tag{2}$$

   (e) Use Eqn. (4) and (2) to conclude that $z(n) = N \sum_{r=-\infty}^{\infty} x_d(n - rN)$.[2] Interpret this equation and observe the similarity with the CTFT of $x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$.

[1] This is in sync with class, wherein in I did away with the $e^{j*}-$notation and denoted the DTFT of $x_d(n)$ as $X_d(\omega)$.

[2] If you are stuck with this step, refer to the derivation in Lecture 9 or Section 8.4 (*Sampling the Fourier Transform* section) of DTSP textbook. This entire problem is leading you through this entire derivation from Eqn 8.49 to 8.56.