The Bell–CHSH S_1 Statistic in Financial Markets

A Complete, Step-by-Step Derivation
with Mathematical Justifications and Intuitive Explanations

Companion document to:

Bell Inequality Violations in Agriculture Equity Networks

`github.com/mjpuma/StocksBellTest`

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Contents

1 Why Are We Doing This? The Big Picture

★ The Problem in Plain English

Standard tools for measuring how two stocks move together—like Pearson correlation—have a fundamental weakness: they cannot tell the difference between two stocks that are correlated because of a *shared hidden cause* (both react to rising oil prices) versus two stocks whose co-movement is structurally deeper, in a way that no single hidden cause can explain.

During financial crises, markets do not just become *more* correlated; they become correlated in a qualitatively different way. The Bell–CHSH test is a tool borrowed from quantum physics that can detect this qualitative difference. If the S_1 statistic exceeds 2, we have mathematical proof that ordinary "shared hidden-variable" stories are insufficient.

The Bell–CHSH inequality was originally developed by John Bell (1964) and Clauser, Horne, Shimony, and Holt (1969) to probe whether quantum mechanics is fundamentally different from classical physics. The central insight is that *any* classical model with a shared hidden variable must satisfy a specific algebraic bound. Violating that bound is a rigorous signal that the model is wrong.

We apply this logic to pairs of agricultural stocks. The question is not "are these two stocks correlated?" but rather "is the pattern of co-movement consistent with *any* classical hidden-variable story?" If $|S_1| > 2$, the answer is no—and that violation is our crisis indicator.

Road map. We walk through every component in sequence:

Step 1. Raw data: stock returns

Step 2. Converting returns to binary outcomes (signs)

Step 3. Defining measurement regimes via threshold masks

Step 4. Computing four conditional expectation values

Step 5. Combining them into S_1

Step 6. The Bell bound: why $|S_1| > 2$ is special

Step 7. Rolling windows: turning S_1 into a time series

Step 8. Aggregating across pairs: the violation percentage

2 Step 1 — Raw Data: Stock Returns

2.1 What we start with

We work with daily adjusted close prices $P_{i,t}$ for 53 agriculture-related stocks (ticker index i, trading date t), covering 2000-01-01 to the present.

2.2 Computing percentage returns

Definition 2.1 (Daily return)**.**

$$r_{i,t} = \frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}} \tag{1}$$

Σ Why percentage returns, not log returns?

Log returns $\ln(P_t/P_{t-1})$ are popular because they are additive over time. For daily moves, $|r| \ll 1$ in normal conditions, and $\ln(1+r) \approx r$, so both measures are nearly identical. Our pipeline uses only the *sign* and whether the *magnitude* exceeds 5%. Both properties are identical for log and percentage returns at any realistic daily frequency. Percentage returns are used for transparency and interpretability.

★ Returns in plain English

A return of $r = 0.03$ means the stock rose 3% that day. A return of $r = -0.07$ means it fell 7%. We will shortly reduce each return to just two pieces of information: (a) did it go up or down, and (b) was the move large (above 5%) or small?

2.3 Stock pairs

We form all unordered pairs (A, B) with $A \neq B$. For $n = 53$ tickers:

$$\binom{53}{2} = 1{,}378 \text{ pairs}$$

All subsequent steps operate on a single pair (A, B) over a rolling window.

3 Step 2 — Binary Outcomes: The Sign of Returns

3.1 Why binary outcomes?

The Bell–CHSH inequality is stated for **binary** outcomes: each measurement must yield either $+1$ or -1. In physics this corresponds to spin measurements (spin up or spin down). In our financial application, the natural binary is whether a stock rose or fell on a given day.

Definition 3.1 (Binary outcome).

$$a_i = \text{sign}(x_i) = \begin{cases} +1 & x_i > 0 \quad \text{(stock rose)} \\ -1 & x_i < 0 \quad \text{(stock fell)} \\ 0 & x_i = 0 \quad \text{(excluded)} \end{cases} \tag{2}$$

and analogously $b_i = \text{sign}(y_i)$ for stock B, where $x_i = r_{A,i}$ and $y_i = r_{B,i}$ within the rolling window.

★ Signs in plain English

We simply ask: did the stock go up or down today?

Up $\to +1$. Down $\to -1$. Unchanged $\to$ ignored.

The *product* $a_i b_i$ tells us whether both stocks moved the same way ($+1 \cdot +1 = +1$ or $-1 \cdot -1 = +1$) or in opposite directions ($+1 \cdot -1 = -1$).

3.2 Interpreting the product $a_i b_i$

$$a_i b_i = +1 \implies \text{same direction (co-movement)}$$
$$a_i b_i = -1 \implies \text{opposite directions (divergence)}$$

A simple average of $a_i b_i$ over many days gives a correlation-like measure in $[-1, +1]$. Bell–CHSH goes further by conditioning on whether moves were large or small—and this conditioning is what creates the statistical power to detect non-classical behavior.

4 Step 3 — Measurement Settings: The Four Threshold Masks

4.1 Why we need "settings"

In the original Bell experiment, Alice chooses a measurement setting $x \in \{0, 1\}$ and Bob chooses $y \in \{0, 1\}$, creating four setting combinations $(x, y) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. The CHSH inequality uses all four to form S_1.

For stocks, we define two "settings" via the *magnitude* of returns:

- Setting 0 (large / "strong"): $|r| \geq \tau$
- Setting 1 (small / "weak"): $|r| < \tau$

with fixed threshold $\tau = 0.05$ (5%). This creates four regime combinations.

★ **The four regimes in plain English**

Every day, each stock makes either a big move ($\geq 5\%$) or a small move ($< 5\%$). The pair of stocks together fall into one of four regimes:

Stock A	Stock B	Regime	Notation
Big ($\geq 5\%$)	Big ($\geq 5\%$)	both large	ab
Big ($\geq 5\%$)	Small ($< 5\%$)	A large, B small	ab'
Small ($< 5\%$)	Big ($\geq 5\%$)	A small, B large	$a'b$
Small ($< 5\%$)	Small ($< 5\%$)	both small	$a'b'$

Each trading day belongs to exactly one regime (they are mutually exclusive and exhaustive). The Bell test asks: does the direction agreement (up/down) differ systematically across these four regimes in a way that no hidden common factor can explain?

4.2 Formal definition of the four masks

Definition 4.1 (Threshold masks)**.** Fix $\tau = 0.05$. For each day i in the window:

$$m_{ab,i} = \mathbf{1}[|x_i| \geq \tau] \cdot \mathbf{1}[|y_i| \geq \tau] \quad \text{(both large)} \tag{3}$$
$$m_{ab',i} = \mathbf{1}[|x_i| \geq \tau] \cdot \mathbf{1}[|y_i| < \tau] \quad \text{(A large, B small)} \tag{4}$$
$$m_{a'b,i} = \mathbf{1}[|x_i| < \tau] \cdot \mathbf{1}[|y_i| \geq \tau] \quad \text{(A small, B large)} \tag{5}$$
$$m_{a'b',i} = \mathbf{1}[|x_i| < \tau] \cdot \mathbf{1}[|y_i| < \tau] \quad \text{(both small)} \tag{6}$$

Σ **Partition property**

The four masks partition each day i exactly:

$$m_{ab,i} + m_{ab',i} + m_{a'b,i} + m_{a'b',i} = 1 \quad \forall i$$

This holds because $\mathbf{1}[|x| \geq \tau] + \mathbf{1}[|x| < \tau] = 1$ for any $x \neq 0$, and taking the Cartesian product over x and y gives four disjoint events that together cover the entire sample space. (Days with $x_i = 0$ or $y_i = 0$ have sign $= 0$ and are excluded.)

4.3 Justification of $\tau = 0.05$

(i) **Economic meaning.** A 5% daily move is approximately 2–3 standard deviations of typical daily equity volatility ($\sigma_{\text{daily}} \approx 1.5$–$2.5\%$). This threshold separates routine trading from genuine event-driven price changes—the kind driven by earnings surprises, geopolitical shocks, or contagion during crises.

(ii) **Non-degeneracy.** With a 20-day window, $\tau = 0.05$ typically places 1–5 observations in each regime cell, giving a non-trivial but computationally feasible split. If τ is too low, almost every day qualifies as "large" and the "small" regime degenerates (too few observations). If τ is too high, the "large" regime degenerates.

(iii) **Literature consistency.** Zarifian et al. (2025) use the same threshold and confirm qualitative robustness across $\tau \in [0.01, 0.05]$.

△ The threshold is a modelling choice, not a universal law

The Bell *bound* of 2 is derived mathematically and is independent of τ. But the *values* of S_1 depend on τ. Results should always be reported with a sensitivity analysis varying τ.

5 Step 4 — The Four Conditional Expectation Values

5.1 Rolling windows

All computations use a rolling window of $W = 20$ trading days ending on date t, containing observations $i = 1, \ldots, 20$. This window is the same length as the rolling realized volatility computation, making comparisons direct.

★ Why 20 days?

Twenty trading days is roughly one calendar month. It is long enough to populate each of the four regime cells with at least a few observations (needed for stable estimates) and short enough to respond quickly to emerging market stress. Longer windows would smooth out the crisis spikes; shorter windows would be too noisy.

5.2 Computing each expectation

Definition 5.1 (Conditional expectation)**.** For mask m_{ab}:

$$\mathbb{E}(a,\, b) = \frac{\sum_{i=1}^{W} a_i\, b_i\, m_{ab,i}}{\sum_{i=1}^{W} m_{ab,i}} \tag{7}$$

and analogously $\mathbb{E}(a,\, b')$, $\mathbb{E}(a',\, b)$, $\mathbb{E}(a',\, b')$ using masks $m_{ab'}$, $m_{a'b}$, $m_{a'b'}$ respectively. If the denominator is zero, the expectation is undefined and the pair is excluded from the violation count for date t.

Σ Unpacking Equation (7)

Numerator: $\sum_i a_i b_i m_{ab,i}$
Sums the sign products over days when *both* stocks made large moves. On days when at least one stock made a small move, $m_{ab,i} = 0$ so those days contribute nothing.
Denominator: $\sum_i m_{ab,i}$
Counts how many days in the window had both stocks making large moves. This is the effective sample size for this regime.
Result: The ratio lies in $[-1, +1]$ and equals the average direction agreement *among large-large days only*:

$$\begin{aligned}
\mathbb{E}(a, b) = +1 &\implies \text{whenever both moved a lot, they moved the same way} \\
\mathbb{E}(a, b) = -1 &\implies \text{whenever both moved a lot, they moved opposite ways} \\
\mathbb{E}(a, b) = 0 &\implies \text{no consistent pattern in large-large days}
\end{aligned}$$

▸ Numerical Example: 5 days (simplified)

Day	x_i	y_i	a_i	b_i	$a_i b_i$	$\lvert x\rvert \geq 0.05$	$\lvert y\rvert \geq 0.05$
1	$+0.07$	$+0.06$	$+1$	$+1$	$+1$	✓	✓
2	-0.08	-0.09	-1	-1	$+1$	✓	✓
3	$+0.02$	$+0.03$	$+1$	$+1$	$+1$	×	×
4	$+0.06$	-0.01	$+1$	-1	-1	✓	×
5	-0.07	$+0.08$	-1	$+1$	-1	✓	✓

$\mathbb{E}(a, b)$ (mask m_{ab}: both large): Days 1, 2, 5.

$$\mathbb{E}(a, b) = \frac{(+1) + (+1) + (-1)}{3} = \frac{1}{3} \approx +0.33$$

$\mathbb{E}(a, b')$ (mask $m_{ab'}$: A large, B small): Day 4 only.

$$\mathbb{E}(a, b') = \frac{-1}{1} = -1$$

$\mathbb{E}(a', b)$ (mask $m_{a'b}$: A small, B large): No days. $\Rightarrow$ *undefined*; pair excluded from violation count.
$\mathbb{E}(a', b')$ (mask $m_{a'b'}$: both small): Day 3 only.

$$\mathbb{E}(a', b') = \frac{+1}{1} = +1$$

6 Step 5 — Assembling S_1

6.1 The CHSH correlator

Definition 6.1 (Bell–CHSH S_1 correlator)**.**

$$\boxed{S_1 = \mathbb{E}(a, b) + \mathbb{E}(a, b') + \mathbb{E}(a', b) - \mathbb{E}(a', b')} \tag{8}$$

★ S_1 in plain English

We compute four "direction-agreement scores," one for each regime:

- $\mathbb{E}(a, b)$: agreement when *both* stocks made big moves
- $\mathbb{E}(a, b')$: agreement when *only* A made a big move
- $\mathbb{E}(a', b)$: agreement when *only* B made a big move
- $\mathbb{E}(a', b')$: agreement when *neither* made a big move

We add the first three and *subtract* the last. The reason for the minus sign is algebraic (see Section 7): it is the combination that has a classical bound of exactly 2.

6.2 Why this particular sign pattern $(+, +, +, -)$?

There are four CHSH variants, all of which obey $|S_k| \leq 2$ classically:

$$S_1 = +\mathbb{E}(a, b) + \mathbb{E}(a, b') + \mathbb{E}(a', b) - \mathbb{E}(a', b') \tag{9}$$
$$S_2 = +\mathbb{E}(a, b) + \mathbb{E}(a, b') - \mathbb{E}(a', b) + \mathbb{E}(a', b') \tag{10}$$
$$S_3 = +\mathbb{E}(a, b) - \mathbb{E}(a, b') + \mathbb{E}(a', b) + \mathbb{E}(a', b') \tag{11}$$
$$S_4 = -\mathbb{E}(a, b) + \mathbb{E}(a, b') + \mathbb{E}(a', b) + \mathbb{E}(a', b') \tag{12}$$

Σ Why S_1 is the preferred form for financial data

Rewriting S_1:

$$S_1 = \underbrace{\mathbb{E}(a, b) + \mathbb{E}(a, b') + \mathbb{E}(a', b)}_{\text{all involve at least one large move}} - \underbrace{\mathbb{E}(a', b')}_{\text{both small}}$$

The three positive terms capture correlations in *at least one* extreme regime. The negative term is the background correlation in quiet periods. S_1 therefore measures *excess* co-movement during volatile episodes relative to calm conditions—which is exactly what a crisis indicator should capture. The other three variants (S_2, S_3, S_4) mix large-move and small-move terms in both the positive and negative sums, losing this interpretation. This is the argument of Zarifian et al. (2025) for using S_1 exclusively.

6.3 Numerical range

Each expectation term lies in $[-1, +1]$, so:

$$S_1 \in [-4,\ +4]$$

because $|\pm 1 \pm 1 \pm 1 \mp (\mp 1)| \le 4$.

7 Step 6 — The Bell Bound: Why $|S_1| > 2$ Is Special

This is the mathematical heart of the method. We prove from first principles that any classical shared-factor model must satisfy $|S_1| \le 2$.

7.1 What is a local hidden-variable model?

Definition 7.1 (Local Hidden-Variable (LHV) Model)**.** An LHV model for a stock pair assumes an unobserved random variable λ (the "hidden variable" or "common factor") such that:

(i) Stock A's outcome depends only on λ and A's own regime indicator $x \in \{0, 1\}$, not on stock B.

(ii) Stock B's outcome depends only on λ and B's own regime indicator $y \in \{0, 1\}$, not on stock A.

(iii) λ can be of any form and any dimension.

Formally, outcomes are generated by response functions $a(x, \lambda) \in \{-1, +1\}$ and $b(y, \lambda) \in \{-1, +1\}$, and the joint probability satisfies:

$$P(a, b \mid x, y) = \int P(a \mid x, \lambda)\, P(b \mid y, \lambda)\, p(\lambda)\, d\lambda \tag{13}$$

★ LHV in plain English

An LHV model says: "There is some shared cause λ—perhaps market sentiment, the Fed funds rate expectation, the dollar index, or anything else—that influences both stocks. Once you know λ, the two stocks behave *independently* of each other. All correlation between them is entirely due to this shared cause."

This is the implicit assumption behind Pearson correlation: if two stocks are correlated, they share a common driver. Bell's theorem says: if $|S_1| > 2$, no such story works—no matter how many common factors you posit, no matter how complex.

7.2 Proof that LHV models satisfy $|S_1| \leq 2$

Proposition 7.1 (Bell–CHSH inequality)**.** Under any LHV model as in Definition 7.1,

$$|S_1| \leq 2$$

Proof. For a fixed realization of λ, denote the four deterministic outcomes (one for each setting combination):

$$a_0 = a(0, \lambda), \quad a_1 = a(1, \lambda), \quad b_0 = b(0, \lambda), \quad b_1 = b(1, \lambda), \quad a_k, b_k \in \{-1, +1\}$$

The realized CHSH expression for this λ is:

$$s(\lambda) = a_0b_0 + a_0b_1 + a_1b_0 - a_1b_1 = a_0(b_0 + b_1) + a_1(b_0 - b_1) \tag{14}$$

Since $b_0, b_1 \in \{-1, +1\}$, exactly one of two cases holds:

Case 1: $b_0 = b_1$**.** Then $b_0 + b_1 = \pm 2$ and $b_0 - b_1 = 0$, giving $s(\lambda) = a_0(\pm 2) + 0 = \pm 2$.

Case 2: $b_0 = -b_1$**.** Then $b_0 + b_1 = 0$ and $b_0 - b_1 = \pm 2$, giving $s(\lambda) = 0 + a_1(\pm 2) = \pm 2$.

In both cases $|s(\lambda)| = 2$ for every realization of λ. The observed S_1 is an average over λ:

$$S_1 = \int s(\lambda)\, p(\lambda)\, d\lambda$$

By the triangle inequality:

$$|S_1| \leq \int |s(\lambda)|\, p(\lambda)\, d\lambda = 2 \int p(\lambda)\, d\lambda = 2$$

□

Σ The key algebraic insight

The proof shows that for *any single* realization of the hidden variable, the expression $a_0b_0 + a_0b_1 + a_1b_0 - a_1b_1$ always equals exactly ± 2—never more, never less. This is pure algebra, requiring only $b_0, b_1 \in \{-1, +1\}$. Averaging over λ cannot push the result beyond ± 2.

Therefore, *observing* $|S_1| > 2$ in data is a mathematical proof that the data cannot have come from any LHV model, regardless of how complex or numerous the hidden variables are.

7.3 The three classical and quantum bounds

Model class	Condition	Bound on $\lvert S_1\rvert$
Local hidden-variable (classical)	Any common-cause model	≤ 2
Quantum mechanics	Entangled quantum systems	$\leq 2\sqrt{2} \approx 2.83$
No-signaling (most general)	Any non-signaling theory	≤ 4

★ **What the three bounds mean in practice**

- $|S_1| \leq 2$: The data are compatible with (though do not prove) a classical hidden-variable explanation. Normal market periods.
- $2 < |S_1| \leq 2\sqrt{2}$: No classical hidden-variable model fits. The co-movement is structurally "more than classical." In finance, this is our crisis signal: the pair exhibits synchronized extreme behavior that cannot be attributed to any shared common factor.
- $2\sqrt{2} < |S_1| \leq 4$: Even quantum mechanics (with the standard CHSH setup) cannot explain this. In practice, this range typically reflects statistical noise from sparse regime cells (too few observations in a 20-day window). It should be interpreted cautiously.

△ **We are NOT claiming stocks are quantum systems**

When we say "Bell violation," we do *not* claim quantum entanglement in financial markets. The Bell inequality is a constraint on *classical probability theory*, not only on quantum mechanics. The claim is:

> *The conditional joint return distribution of this stock pair, across these four magnitude regimes, cannot be reproduced by any model in which the two stocks respond independently to a shared common factor.*

This is a statement about probability structure. The Bell framework is used because it provides the mathematical tool to detect this structural property rigorously, with a threshold (the bound of 2) that follows from first principles rather than being a tuning parameter.

8 Step 7 — Rolling Windows: S_1 as a Time Series

8.1 Computing $S_1(t)$ for each date t

For each pair (A, B) and each date t, we apply the 20-day rolling window ending on t and compute $S_1^{(A,B)}(t)$ as in Definition 6.1. This yields:

$$S_1^{(A,B)}(t) \in [-4, +4] \quad \text{for each pair } (A, B) \text{ and date } t$$

△ **Statistical precision in a 20-day window**

With $W = 20$ observations split across 4 regime cells, each cell contains on average only 5 observations. Individual S_1 values will be noisy. This is acceptable because:

(a) We aggregate over 1,378 pairs—noise at the pair level averages out in the violation percentage.

(b) Short windows are deliberately chosen to be *responsive*: a 60-day window would be more statistically stable but would detect crises weeks later, reducing practical utility as an early-warning indicator.

(c) Noise is symmetric around the true S_1, so pairs should not *systematically* exceed 2 by chance. This can be verified by comparing observed violation rates to a permutation-test null.

8.2 Handling undefined expectations

If any regime cell contains zero observations (denominator = 0), S_1 is undefined for that pair on that date. Such pairs are:

- Excluded from the numerator of violation percentage (not counted as either violating or non-violating)
- Excluded from the denominator (do not inflate TotalPairs)

This ensures the violation percentage is always computed over valid pairs only.

9 Step 8 — The Violation Percentage

9.1 Definition

Definition 9.1 (Daily violation percentage)**.** Let N_t^{viol} be the number of pairs with $|S_1^{(A,B)}(t)| > 2$ and valid data. Let TotalPairs_t be the number of pairs with valid S_1. Then:

$$\text{ViolationPct}(t) = 100 \times \frac{N_t^{\text{viol}}}{\text{TotalPairs}_t} \tag{15}$$

★ **Violation percentage in plain English**

On any given day we look at all ∼1,378 stock pairs. For each pair, we ask: is $|S_1| > 2$? We count the fraction that say "yes."

During calm periods: few pairs violate; the percentage is low.

During crises (2008, COVID-19, Ukraine War): many pairs simultaneously exhibit non-classical co-movement; the percentage spikes sharply.

Plotting this one number against time gives a crisis indicator that can be compared directly with VIX, GARCH volatility, and CDS spreads.

9.2 From scalar to network

The violation percentage is a single scalar per day. Our paper's key extension is to treat the pairs (A, B) with $|S_1| > 2$ as *edges* in a daily network:

$$G_t = (\mathcal{V}, \mathcal{E}_t), \qquad \mathcal{E}_t = \{(A, B) : \left|S_1^{(A,B)}(t)\right| > 2\}$$

The topology of G_t—its density, clustering, giant component size, community structure, hub structure—characterizes the *architecture* of the crisis, not just its intensity.

★ The network picture in plain English

Think of the 53 stocks as cities. An edge (road) exists between two cities when their Bell statistic exceeds 2—meaning their co-movement is too tightly synchronized to be explained by any common factor.

On a quiet day: few roads; cities are mostly isolated.

During a crisis: roads appear everywhere simultaneously. The resulting "crisis network" can be analysed:

- **High density**: almost all pairs are synchronized
- **High clustering**: violations cluster within sectors
- **Few communities**: the market consolidates into fewer, larger synchronized blocs
- **Hub emergence**: certain stocks become central connectors

These topological features carry information that the scalar violation percentage alone cannot reveal.

10 Complete Summary

The Complete S_1 Computation at a Glance

Input: Adjusted close prices for stocks A and B, rolling 20-day window ending on date t.

Step 1: Daily returns.

$$r_i = \frac{P_i - P_{i-1}}{P_{i-1}}, \quad i = 1, \ldots, 20$$

Step 2: Binary signs.

$$a_i = \text{sign}(r_{A,i}), \qquad b_i = \text{sign}(r_{B,i})$$

Step 3: Threshold masks ($\tau = 0.05$):

$$\begin{aligned} m_{ab,i} &= \mathbf{1}[|r_{A,i}| \geq \tau] \cdot \mathbf{1}[|r_{B,i}| \geq \tau] \\ m_{ab',i} &= \mathbf{1}[|r_{A,i}| \geq \tau] \cdot \mathbf{1}[|r_{B,i}| < \tau] \\ m_{a'b,i} &= \mathbf{1}[|r_{A,i}| < \tau] \cdot \mathbf{1}[|r_{B,i}| \geq \tau] \\ m_{a'b',i} &= \mathbf{1}[|r_{A,i}| < \tau] \cdot \mathbf{1}[|r_{B,i}| < \tau] \end{aligned}$$

Step 4: Conditional expectations.

$$\mathbb{E}(a,\, b) = \frac{\sum_i a_i b_i m_{ab,i}}{\sum_i m_{ab,i}}, \quad \mathbb{E}(a,\, b') = \frac{\sum_i a_i b_i m_{ab',i}}{\sum_i m_{ab',i}}, \quad \text{etc.}$$

Step 5: CHSH correlator.

$$\boxed{S_1 = \mathbb{E}(a,\, b) + \mathbb{E}(a,\, b') + \mathbb{E}(a',\, b) - \mathbb{E}(a',\, b')}$$

Step 6: Classify.

$$\text{Violation} = \mathbf{1}[|S_1| > 2], \quad \text{Classical bound: } |S_1| \leq 2 \text{ (proven)}$$

Step 7: Aggregate over all pairs.

$$\text{ViolationPct}(t) = 100 \times \frac{\#\{\text{pairs with } |S_1| > 2\}}{\#\{\text{pairs with valid } S_1\}}$$

11 Addressing Common Objections

11.1 "This is just Pearson correlation in disguise"

No. Pearson correlation is an *unconditional* average of co-movement across all days. S_1 computes *conditional* averages within each of four magnitude regimes and combines them with a specific sign pattern. Even if two stocks have high Pearson correlation, their S_1 may remain within $[-2, +2]$. Conversely, two stocks with low Pearson correlation could still violate if the regime-conditional structure is non-classical. The algebraic bound of 2 has no analogue for Pearson correlation; it is specific to the CHSH combination.

11.2 "Shared market shocks explain all the correlation"

This is the LHV claim, and Bell's theorem refutes it when $|S_1| > 2$. Any model in which both stocks respond (however complexly) to a shared factor λ—a Fed announcement, a crop report, a geopolitical event—is by definition an LHV model and therefore satisfies $|S_1| \leq 2$. Observation of $|S_1| > 2$ is mathematical proof that no such model fits.

11.3 “The 5% threshold is arbitrary”

The value $\tau = 0.05$ is a design choice. The Bell bound of 2 is not—it is derived algebraically and holds for any threshold. Sensitivity analyses confirm robustness across $\tau \in [0.01, 0.05]$.

11.4 “Violations could arise from statistical noise in small windows”

True: with ∼5 observations per regime cell, individual pair S_1 values are noisy. But the aggregate violation percentage over 1,378 pairs is far more stable. And crucially, noise should not produce *systematic* bias toward $|S_1| > 2$; this can be tested with a permutation null (randomly shuffling return time series within pairs destroys true temporal structure while preserving marginal distributions). The persistence of elevated violation percentages specifically during known crisis periods, and not during calm periods, is the key empirical validation.

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