ECE4433-5433

Differentiation of decaying sinusoid

We discovered that the derivative of an eigenfunction is the function itself, except that the coefficient modifying it changes. So, for example, the exponential function is an eigenfunction under differentiation.

$$\begin{aligned} x(t) &= e^{-\alpha t} \\ &\Downarrow \tfrac{d}{dt} \\ \frac{dx(t)}{dt} &= -\alpha e^{-\alpha t} \end{aligned}$$

Similarly, it should be no surprise that sinusoidal functions are also eigenfunctions under differentiation, since Euler showed us that we could write a sinusoid as a combination of (complex) exponentials.

$$\begin{aligned} x(t) &= \cos(\omega t) \\ &\Downarrow \tfrac{d}{dt} \\ \frac{dx(t)}{dt} &= -\omega \sin \omega t \end{aligned}$$

During our bouncing toy study, we decided–somewhat heuristically–that we could write the position equation for the toy based on finding the frequency and damping rate from the accelerometer data. In general, students were not required to find the precise equations for velocity and acceleration, meaning the student could ignore the constants for initial amplitude and phase. However, having measure the initial displacement, and finding the frequency and damping rate, the position (displacement) equation can be precisely specified. Further, if we overcome our fear of math, we can precisely find the equations for velocity and acceleration. Having done that, you can determine the calibration factor of the unknown accelerometer. The differentiation is rather cumbersome, but is worth investigating. Clearly, because we have two functions of time, t, we will need to use the chain rule to find the derivative. Moreover, because we will wind up with both cosine and sine functions, I will employ Euler's relationship to combine the two. We poceed as follows:

$$
\begin{aligned}
x(t) &= Ae^{-\alpha t}\cos(\omega t) \\
\frac{dx(t)}{dt} &= A\left[-\alpha e^{-\alpha t}\cos(\omega t) - \omega e^{-\alpha t}\sin(\omega t)\right] \\
&= -Ae^{-\alpha t}\left[\alpha\cos(\omega t) + \omega\sin(\omega t)\right] \\
&= -Ae^{-\alpha t}\left[\alpha\frac{e^{j\omega t}+e^{-j\omega t}}{2} + \omega\frac{e^{j\omega t}-e^{-j\omega t}}{2j}\right] \\
&= -Ae^{-\alpha t}\left[\alpha\frac{e^{-j\frac{\pi}{2}}}{e^{-j\frac{\pi}{2}}}\frac{e^{j\omega t}+e^{-j\omega t}}{2} + \omega\frac{e^{j\omega t}-e^{-j\omega t}}{2j}\right] \\
&= -Ae^{-\alpha t}\left[\alpha\frac{1}{e^{-j\frac{\pi}{2}}}\frac{e^{j\left(\omega t-\frac{\pi}{2}\right)}+e^{-j\left(\omega t-\frac{\pi}{2}\right)}}{2} + \omega\frac{e^{j\omega t}-e^{-j\omega t}}{2j}\right] \\
&= -Ae^{-\alpha t}\left[j\alpha\cos\left(\omega t-\frac{\pi}{2}\right) + \omega\sin(\omega t)\right] \\
&= -Ae^{-\alpha t}\left[(j\alpha+\omega)\sin(\omega t)\right] \\
&= -Ae^{-\alpha t}\sqrt{\alpha^2+\omega^2}\,e^{\left(j\tan^{-1}\left(\frac{\alpha}{\omega}\right)\right)}\,\sin(\omega t) \\
&\Downarrow \quad \textit{carrying the phase change back into the sinusoid} \\
v(t) &= -A\sqrt{\alpha^2+\omega^2}\,e^{-\alpha t}\,\sin\left(\omega t+\tan^{-1}\left(\frac{\alpha}{\omega}\right)\right)
\end{aligned}
$$

The result is a precise expression of the velocity equation, $v(t)$. Similarly, we can apply the same process on the equation $v(t) = -A\sqrt{\alpha^2+\omega^2}\,e^{-\alpha t}\,\sin\left(\omega t+\tan^{-1}\left(\frac{\alpha}{\omega}\right)\right)$ to solve for acceleration. If we let $\phi_v = \tan^{-1}\left(\frac{\alpha}{\omega}\right)$, then we can write

$$
\begin{aligned}
v(t) &= -A\sqrt{\alpha^2+\omega^2}\, e^{-\alpha t} \sin(\omega t + \phi_v) \quad \text{(where } \phi_v \text{ is a phase shift)} \\
\frac{dv(t)}{dt} &= -A\sqrt{\alpha^2+\omega^2}\, \left[-\alpha e^{-\alpha t} \sin(\omega t + \phi_v) + \omega e^{-\alpha t} \cos(\omega t + \phi_v)\right] \\
&= -A\sqrt{\alpha^2+\omega^2}\, e^{-\alpha t} \left[-\alpha \sin(\omega t + \phi_v) + \omega \cos(\omega t + \phi_v)\right] \\
&= -A\sqrt{\alpha^2+\omega^2}\, e^{-\alpha t} \left[-\alpha \left(\frac{e^{j\frac{\pi}{2}}}{e^{j\frac{\pi}{2}}}\right) \sin(\omega t + \phi_v) + \omega \cos(\omega t + \phi_v)\right] \\
&= -A\sqrt{\alpha^2+\omega^2}\, e^{-\alpha t} \left[-\alpha e^{-j\frac{\pi}{2}} \sin\left(\omega t + \phi_v + \frac{\pi}{2}\right) + \omega \cos(\omega t + \phi_v)\right] \\
&= -A\sqrt{\alpha^2+\omega^2}\, e^{-\alpha t} \left[j\alpha \cos(\omega t + \phi_v) + \omega \cos(\omega t + \phi_v)\right] \\
&= -A\sqrt{\alpha^2+\omega^2}\, e^{-\alpha t} (j\alpha + \omega) \cos(\omega t + \phi_v) \\
&= -A\sqrt{\alpha^2+\omega^2}\, e^{-\alpha t} \left(\sqrt{\alpha^2+\omega^2}\, e^{\tan^{-1}\left(\frac{\alpha}{\omega}\right)}\right) \cos(\omega t + \phi_v) \\
&= -A\sqrt{\alpha^2+\omega^2}\, \sqrt{\alpha^2+\omega^2}\, e^{-\alpha t} \cos\left(\omega t + \phi_v + \tan^{-1}\left(\frac{\alpha}{\omega}\right)\right) \\
&= -A\left(\alpha^2+\omega^2\right) e^{-\alpha t} \cos(\omega t + \phi_a)
\end{aligned}
$$

where $\phi_a = \phi_v + \tan^{-1}\left(\frac{\alpha}{\omega}\right) = 2\ \tan^{-1}\left(\frac{\alpha}{\omega}\right)$

Thus, we have the following equations of motion:

$$
\begin{aligned}
x(t) &= A_x e^{-\alpha t} \cos(\omega t) \\
v(t) &= A_v e^{-\alpha t}\ \sin(\omega t + \phi_v) \\
a(t) &= A_a e^{-\alpha t} \cos(\omega t + \phi_a)
\end{aligned}
$$

where

$$
\begin{aligned}
A_v &= -A_x\sqrt{\alpha^2+\omega^2} \\
\phi_v &= \tan^{-1}\left(\frac{\alpha}{\omega}\right) \\
A_a &= -A_x\left(\alpha^2+\omega^2\right) \\
& \quad \textit{and} \\
\phi_a &= \phi_v + \tan^{-1}\left(\frac{\alpha}{\omega}\right) \quad = \quad \tan^{-1}\left(\frac{\alpha}{\omega}\right) + \tan^{-1}\left(\frac{\alpha}{\omega}\right) \quad = \quad 2\tan^{-1}\left(\frac{\alpha}{\omega}\right)
\end{aligned}
$$

It is noteworthy that in our specific bouncy toy example the α and ω values are small so that the phase changes are almost negligible (which one can verify by plotting). For consistency, we could have couched the *velocity* term as a cosine as well, to get:

$$v(t) = A_v e^{-\alpha t}\cos\left(\omega t - \left(\tfrac{\pi}{2} - \phi_v\right)\right)$$

but using the alternating cosine and sine terms is more indicative of what is seen from plotting these functions.