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Application Note

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High-Precision Calibration of a Three-Axis Accelerometer

by Mark Pedley

1 Introduction

Three-axis accelerometers supplied for the consumer market are typically calibrated by the sensor manufacturer using a six-element linear model comprising a gain and offset in each of the three axes. This factory calibration will change slightly as a result of the thermal stresses during soldering of the accelerometer to the circuit board. Additional small errors, external to the accelerometer, including rotation of the accelerometer package relative to the circuit board and misalignment of the circuit board to the final product, will also be introduced during the soldering and final assembly process.

The original factory accelerometer calibration will still be adequate for the vast majority of consumer applications. Manufacturers of premium products looking to obtain improved accuracy from a consumer accelerometer may, however, wish to perform their own calibration either by repeating the calibration performed by the accelerometer manufacturer or by using a more sophisticated calibration model.

Contents

This application note describes, with worked examples, the mathematics and measurements required for final accelerometer recalibration using models of increasing sophistication and accuracy.

For convenience, it is assumed that the accelerometer provides a digital, rather than analog, output but the same techniques are applicable to the recalibration of analog accelerometers after signal digitization using an analog to digital converter.

Related Freescale application notes are: i) AN3461 “Tilt Sensing Using Linear Accelerometers” and ii) AN4249 “Accuracy of Angle Estimation in eCompass and 3D Pointer Applications”.

1.1 Key words

Accelerometer, Calibration, Linear, Nonlinear, Least Squares.

1.2 Summary

- The apparent gravitational acceleration on the earth's surface varies by 0.7% from minimum to maximum. The apparent gravitational acceleration at the recalibration site is irrelevant if the product is to be used to provide orientation angle estimates from ratios of accelerometer channel readings but should be known if the product is required to provide high-accuracy acceleration or gravitational measurements.
- The original six parameter (gain and offset in each channel) factory calibration can be recomputed to correct for thermal stresses introduced in the soldering process.
- A 12 parameter linear calibration model can correct for accelerometer package rotation on the circuit board and for cross-axis interference between the accelerometer’s x, y and z channels.
- The orientation angles used for the recalibration must be carefully selected to provide the best calibration accuracy from the limited number of measurement orientations available. Optimum orientation angles for a given number of measurements are listed.
- Linear least squares optimization is an efficient mathematical technique to compute the recalibration parameters from the available measurements using simple matrix algebra. Worked examples are provided throughout the text.
- These techniques can be extended to include temperature dependence by performing the recalibration at two or more temperatures and interpolating the fitted calibration parameters to the actual temperature.

2 Absolute or Relative Calibration

Accelerometers are used in applications requiring either absolute or relative acceleration measurements. Examples of absolute acceleration measurements are determining the earth's gravitational field or the acceleration forces experienced in an automobile in units of ms^{-2}. An example of using relative accelerations is the calculation of orientation angles using ratios of the readings from the *x*, *y* and *z* accelerometer channels.

Although what follows may seem an obscure point, it does need to be briefly discussed since the objective of this application note is high-precision calibration. Although the earth's gravitational field is often stated to be 9.81 ms^{-2}, in practice the apparent gravitational field measured by an accelerometer varies by 0.7% from minimum to maximum over the earth's surface as a consequence of the earth's rotation, the earth's equatorial bulge and the effects of altitude. The apparent gravitational field at sea level at the north pole is 9.832 ms^{-2} but is only 9.763 ms^{-2} at the 5895 m summit of Mount Kilimanjaro located almost on the equator.

This document assumes that the accelerometer recalibration is being undertaken for high-precision calculation of roll and pitch orientation angles from the ratios of accelerometer channel readings. In this case the precise apparent gravitational field at the recalibration site cancels in the mathematics and is simply assumed to be '1 *g*'.

If, however, the recalibration is being performed to produce an absolute estimate of gravity or linear acceleration in units of ms^{-2} then the apparent gravitational field at the recalibration site must be known and stored on the product for a simple final multiplication from '1 *g*' to the required gravity or acceleration estimate measured in ms^{-2}.

3 Original Factory Six Parameter Calibration

The standard model used for the original factory calibration of a consumer grade digital accelerometer relates the calibrated accelerometer output $N\boldsymbol{G}_f$ (in units of bit counts) to the outputs of the internal analog to digital converter using a simple linear model with a total of six calibration parameters comprising the three channel gains p_{xx}, p_{yy} and p_{zz} and the three zero-g offsets q_x, q_y, and q_z:

$$N\boldsymbol{G}_f = N\begin{pmatrix} G_{fx} \\ G_{fy} \\ G_{fz} \end{pmatrix} = \begin{pmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{pmatrix}\begin{pmatrix} ADC_x \\ ADC_y \\ ADC_z \end{pmatrix} + \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} \quad \textbf{\textit{Eqn. 1}}$$

Division of the accelerometer's output bit count $N\boldsymbol{G}_f$ by N gives an output $\boldsymbol{G}_f$ in units of 'g'. For the particular case of the Freescale MMA8451 family operating in 2 mode, the constant N has the value 4096.

Expanding Equation 1 into its x, y and z components gives:

$$NG_{fx} = p_{xx}ADC_x + q_x \quad \textbf{\textit{Eqn. 2}}$$

$$NG_{fy} = p_{yy}ADC_y + q_y \quad \textbf{\textit{Eqn. 3}}$$

$$NG_{fz} = p_{zz}ADC_z + q_z \quad \textbf{\textit{Eqn. 4}}$$

A minimum of two measurement orientations giving a total of six measured data points (two in each of the x, y, and z channels) are required to solve Equations 2, 3 and 4 for the six calibration parameters.

Conventionally, the two measurement orientations are selected to give an equal positive gravitational field of $\frac{g}{\sqrt{3}}$ in each axis followed by $\frac{-g}{\sqrt{3}}$ in each axis.

If $ADC_x[0]$ is defined as the internal x channel ADC output during the first measurement orientation using the stimulus of $\frac{g}{\sqrt{3}}$ and $ADC_x[1]$ is defined as the x channel ADC output during the second orientation using the stimulus of $\frac{-g}{\sqrt{3}}$, then Equation 2 gives:

$$p_{xx}ADC_x[0] + q_x = \frac{N}{\sqrt{3}} \qquad p_{xx}ADC_x[1] + q_x = \frac{-N}{\sqrt{3}} \quad \textbf{\textit{Eqn. 5}}$$

Equation 5 can be readily solved to give the x-channel gain and offset as:

$$p_{xx} = \frac{2N}{\sqrt{3}(ADC_x[0] - ADC_x[1])} \qquad q_x = \frac{-N(ADC_x[0] + ADC_x[1])}{\sqrt{3}(ADC_x[0] - ADC_x[1])} \quad \textbf{\textit{Eqn. 6}}$$

The y and z channel factory calibration parameters are given similarly as:

$$p_{yy} = \frac{2N}{\sqrt{3}(ADC_y[0] - ADC_y[1])} \qquad q_y = \frac{-N(ADC_y[0] + ADC_y[1])}{\sqrt{3}(ADC_y[0] - ADC_y[1])}$$

Eqn. 7

$$p_{zz} = \frac{2N}{\sqrt{3}(ADC_z[0] - ADC_z[1])} \qquad q_z = \frac{-N(ADC_z[0] + ADC_z[1])}{\sqrt{3}(ADC_z[0] - ADC_z[1])}$$

Eqn. 8

Once the six calibration parameters defined by Equations 6 to 8 are calculated, they are written to nonvolatile memory within the accelerometer and used by the accelerometer's internal arithmetic logic to automatically compute the calibrated accelerometer output in bit counts using equations 2 to 4.

Division by N gives the factory calibrated output G_{fx}, G_{fy} and G_{fz} in units of g. For simplicity, and since the scaling factor N varies between accelerometers and for different configurations of a single accelerometer, the remainder of this document assumes that the conversion from bit counts to g has taken place and refers only to the outputs G_{fx}, G_{fy} and G_{fz} with units of g.

4 Final Product Six Parameter Calibration

The original factory accelerometer calibration becomes less accurate once the accelerometer is soldered onto its circuit board as a result of thermal stresses during the soldering process. The manufacturer of the final product may therefore wish to apply a recalibration to compute the same six calibration parameters as the original factory calibration (a gain and offset in each of three channels) but which are then applied on top of the factory calibrated output $\boldsymbol{G_f}$.

The recalibrated accelerometer output $\boldsymbol{G_6}$ is then:

$$\boldsymbol{G_6} = \begin{pmatrix} G_{6x} \\ G_{6y} \\ G_{6z} \end{pmatrix} = \boldsymbol{WG_f} + \boldsymbol{V} = \begin{pmatrix} W_{xx} & 0 & 0 \\ 0 & W_{yy} & 0 \\ 0 & 0 & W_{zz} \end{pmatrix} \begin{pmatrix} G_{fx} \\ G_{fy} \\ G_{fz} \end{pmatrix} + \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \qquad \textbf{\textit{Eqn. 9}}$$

The six calibration parameters in Equation 9 are now dimensionless since the factory calibrated accelerometer output $\boldsymbol{G_f}$ is normalized to units of g.

If the final product is placed in a jig permitting measurements to be taken at the same two orientations used in the original factory calibration giving $\frac{g}{\sqrt{3}}$ and then $\frac{-g}{\sqrt{3}}$ in each channel then, by analogy with Equations 6 to 8, the six calibration parameters are:

$$W_{xx} = \frac{2}{\sqrt{3}(G_{fx}[0] - G_{fx}[1])} \qquad W_{yy} = \frac{2}{\sqrt{3}(G_{fy}[0] - G_{fy}[1])} \qquad W_{zz} = \frac{2}{\sqrt{3}(G_{fz}[0] - G_{fz}[1])}$$

$$V_x = \frac{-(G_{fx}[0] + G_{fx}[1])}{\sqrt{3}(G_{fx}[0] - G_{fx}[1])} \qquad V_y = \frac{-(G_{fy}[0] + G_{fy}[1])}{\sqrt{3}(G_{fy}[0] - G_{fy}[1])} \qquad V_z = \frac{-(G_{fz}[0] + G_{fz}[1])}{\sqrt{3}(G_{fz}[0] - G_{fz}[1])}$$

Eqn. 10

Worked Example 1

The MMA8451 on a production circuit board was configured to 2 g mode giving N = 4096 bits per g and oriented to apply x, y, and z gravitational fields equal to $\frac{g}{\sqrt{3}}$ and then $\frac{-g}{\sqrt{3}}$ in each axis. The factory calibrated output $\boldsymbol{G_f}$ was recorded and averaged over multiple readings to give for the two positions [0] and [1]:

$$N\begin{pmatrix} G_{fx}[0] \\ G_{fy}[0] \\ G_{fz}[0] \end{pmatrix} = \begin{pmatrix} 2470.928 \\ 2261.981 \\ 2300.661 \end{pmatrix} counts \Rightarrow \begin{pmatrix} G_{fx}[0] \\ G_{fy}[0] \\ G_{fz}[0] \end{pmatrix} = \begin{pmatrix} \frac{2470.928}{4096} \\ \frac{2261.981}{4096} \\ \frac{2300.661}{4096} \end{pmatrix} g$$

$$N\begin{pmatrix} G_{fx}[1] \\ G_{fy}[1] \\ G_{fz}[1] \end{pmatrix} = \begin{pmatrix} -2244.084 \\ -2156.898 \\ -2361.297 \end{pmatrix} counts \Rightarrow \begin{pmatrix} G_{fx}[1] \\ G_{fy}[1] \\ G_{fz}[1] \end{pmatrix} = \begin{pmatrix} \frac{-2244.084}{4096} \\ \frac{-2156.898}{4096} \\ \frac{-2361.297}{4096} \end{pmatrix} g$$

Eqn. 11

The six calibration parameters are then:

$$W_{xx} = \frac{2}{\sqrt{3}(G_{fx}[0] - G_{fx}[1])} = \frac{8192}{\sqrt{3}(2470.928 + 2244.084)} = 1.0031$$

$$W_{yy} = \frac{2}{\sqrt{3}(G_{fy}[0] - G_{fy}[1])} = \frac{8192}{\sqrt{3}(2261.981 + 2156.898)} = 1.0703$$

$$W_{zz} = \frac{2}{\sqrt{3}(G_{fz}[0] - G_{fz}[1])} = \frac{8192}{\sqrt{3}(2300.661 + 2361.297)} = 1.0145$$

Eqn. 12

$$V_x = \frac{-(G_{fx}[0] + G_{fx}[1])}{\sqrt{3}(G_{fx}[0] - G_{fx}[1])} = \frac{-(2470.928 - 2244.084)}{\sqrt{3}(2470.928 + 2244.084)}g = -0.02778g$$

$$V_y = \frac{-(G_{fy}[0] + G_{fy}[1])}{\sqrt{3}(G_{fy}[0] - G_{fy}[1])} = \frac{-(2261.981 - 2156.898)}{\sqrt{3}(2261.981 + 2156.898)}g = -0.01373g$$

$$V_z = \frac{-(G_{fz}[0] + G_{fz}[1])}{\sqrt{3}(G_{fz}[0] - G_{fz}[1])} = \frac{-(2300.661 - 2361.297)}{\sqrt{3}(2300.661 + 2361.297)}g = 0.00751g$$

Eqn. 13

The recalibration parameters computed on the final production line would typically be stored in nonvolatile memory accessible by the system processor and applied to the accelerometer output (in units of *g*) using Equation 9. If the x channel accelerometer was 3950 bit counts in 2 *g* mode (4096 bits per 1000 m*g*) then the factory calibrated output G_{fx} is 3950/4096 bits = 964.355 m*g*. The recalibrated and improved x channel accelerometer reading is then 1.0031 x 964.355 m*g* - 27.78 m*g* = 939.565 m*g*.

The approach just described has the advantage of requiring measurements at two orientations only but has the drawback of requiring a jig or other mechanism to orient the circuit board at exactly $\frac{g}{\sqrt{3}}$ in each channel and then $\frac{-g}{\sqrt{3}}$ in each channel. An alternative approach, which may be simpler if the final product has faces at right angles to each other, is to make six measurements with the circuit board aligned on its top, bottom, front, back, left and then right faces. This leads to 1 *g* or -1 *g* in each channel and zero in the other channels and provides a total of 18 measurements of which only 6 will be used.

If $G_{fx}[0]$ is the first measurement when aligned for +1 *g* in the *x* axis and $G_{fx}[1]$ is the second measurement when aligned for -1 *g* in the *x* axis, Equation 9 gives:

$$G_{6x}[0] = W_{xx}G_{fx}[0] + V_x = 1 \qquad G_{6x}[1] = W_{xx}G_{fx}[1] + V_x = -1$$

Eqn. 14

The solution for the *x*-channel calibration using measurements at +1 *g* and -1 *g* is given below alongside the solutions for the *y*-channel calibration using measurements at +1 *g* and -1 *g* in the *y* axis (taken in the third and fourth measurements) and the *z*-channel calibration using measurements at +1 *g* and -1 *g* in the *z* axis (taken in the fifth and sixth measurements).

$$W_{xx} = \frac{2}{(G_{fx}[0] - G_{fx}[1])} \qquad W_{yy} = \frac{2}{(G_{fy}[2] - G_{fy}[3])} \qquad W_{zz} = \frac{2}{(G_{fz}[4] - G_{fz}[5])}$$

$$V_x = \frac{-(G_{fx}[0] + G_{fx}[1])}{(G_{fx}[0] - G_{fx}[1])} \qquad V_y = \frac{-(G_{fy}[2] + G_{fy}[3])}{(G_{fy}[2] - G_{fy}[3])} \qquad V_z = \frac{-(G_{fz}[4] + G_{fz}[5])}{(G_{fz}[4] - G_{fz}[5])}$$

Eqn. 15

Worked Example 2

A production circuit board was placed in six different directions to apply gravitational fields equal to g and then $-g$ in the x, y and then z directions. The factory calibrated output $\boldsymbol{G_f}$ in g was recorded and averaged over multiple readings to give for the six positions:

$$\begin{pmatrix} G_{fx}[0] \\ G_{fy}[0] \\ G_{fz}[0] \end{pmatrix} = \begin{pmatrix} 1.015864 \\ \text{Not used} \\ \text{Not used} \end{pmatrix} g \quad \begin{pmatrix} G_{fx}[1] \\ G_{fy}[1] \\ G_{fz}[1] \end{pmatrix} = \begin{pmatrix} -1.009256 \\ \text{Not used} \\ \text{Not used} \end{pmatrix} g \quad \begin{pmatrix} G_{fx}[2] \\ G_{fy}[2] \\ G_{fz}[2] \end{pmatrix} = \begin{pmatrix} \text{Not used} \\ 1.008724 \\ \text{Not used} \end{pmatrix} g$$

$$\begin{pmatrix} G_{fx}[3] \\ G_{fy}[3] \\ G_{fz}[3] \end{pmatrix} = \begin{pmatrix} \text{Not used} \\ -0.991598 \\ \text{Not used} \end{pmatrix} g \quad \begin{pmatrix} G_{fx}[4] \\ G_{fy}[4] \\ G_{fz}[4] \end{pmatrix} = \begin{pmatrix} \text{Not used} \\ \text{Not used} \\ 1.016945 \end{pmatrix} g \quad \begin{pmatrix} G_{fx}[5] \\ G_{fy}[5] \\ G_{fz}[5] \end{pmatrix} = \begin{pmatrix} \text{Not used} \\ \text{Not used} \\ -1.008254 \end{pmatrix} g$$

Eqn. 16

The six calibration parameters are then:

$$W_{xx} = \frac{2}{(G_{fx}[0] - G_{fx}[1])} = \frac{2}{(1.015864 + 1.009256)} = 0.9876$$

$$W_{yy} = \frac{2}{(G_{fy}[2] - G_{fy}[3])} = \frac{2}{(1.008724 + 0.991598)} = 0.9998$$

$$W_{zz} = \frac{2}{(G_{fz}[4] - G_{fz}[5])} = \frac{2}{(1.016945 + 1.008254)} = 0.9876$$

$$V_x = \frac{-(G_{fx}[0] + G_{fx}[1])}{(G_{fx}[0] - G_{fx}[1])} = \frac{-(1.015864 - 1.009256)}{(1.015864 + 1.009256)} g = -0.00326g$$

$$V_y = \frac{-(G_{fy}[2] + G_{fy}[3])}{(G_{fy}[2] - G_{fy}[3])} = \frac{-(1.008724 - 0.991598)}{(1.008724 + 0.991598)} g = -0.00856g$$

$$V_z = \frac{-(G_{fz}[4] + G_{fz}[5])}{(G_{fz}[4] - G_{fz}[5])} = \frac{-(1.016945 - 1.008254)}{(1.016945 + 1.008254)} g = -0.00429g$$

Eqn. 17

This document has, up to this point, provided simple strategies for repeating the factory six parameter calculation on the final manufactured product. The remainder of this document provides more formal answers to the questions below:

- What are the optimal orientation angles that should be used for calibration measurements?
- What is the optimal mathematical solution which best uses additional measurements to optimize the calibration?
- How can the original factory calibration be extended and what errors can the extended models correct?

5 Optimal Measurement Orientations

In the previous section, Worked Example 1 used measurements from two orientations (giving $\frac{g}{\sqrt{3}}$ in all the three x, y, and z channels followed by $\frac{-g}{\sqrt{3}}$ in the three channels) to recalculate a six parameter calibration. Worked Example 2, in contrast, used six measurement orientations, giving +1 g and then -1 g sequentially in each of the three accelerometer channels.

The first orientation sequence is obviously superior to the second since it uses only two orientations to obtain the six measurements required to solve for the calibration whereas the second sequence uses six orientations to obtain eighteen measurements of which only six are actually used.

This section calculates the optimum orientation angles for various numbers of measurements using the two criteria that:

- The gravitational field vectors at each orientation should be maximally separated from each other
- The minimum separation between any pair of measurements in the x, y or z accelerometer channels should be maximized.

These criteria will become clearer when the specific examples of 2, 3, 4, 6 and 8 orientations are considered and it will be shown that the previously used orientations giving $\frac{g}{\sqrt{3}}$ in all the three x, y, and z channels followed by $\frac{-g}{\sqrt{3}}$ are optimal for two measurements. But first, the mathematics defining the apparent gravitational field experienced by the accelerometer at arbitrary orientation must be developed.

For convenience, it will be assumed that the three axis accelerometer is mounted in a consumer smartphone whose orientation is modeled as resulting from successive rotations in yaw, then pitch and finally in roll angle from an initial flat starting point. The coordinate system used is right handed with the z axis pointed downwards (see Figure 1). The yaw rotation is about the positive z axis, the pitch rotation is about the positive y axis and the roll rotation is about the positive x axis. The convention will be used that the accelerometer output is positive when aligned in the direction of the earth's gravitational field.

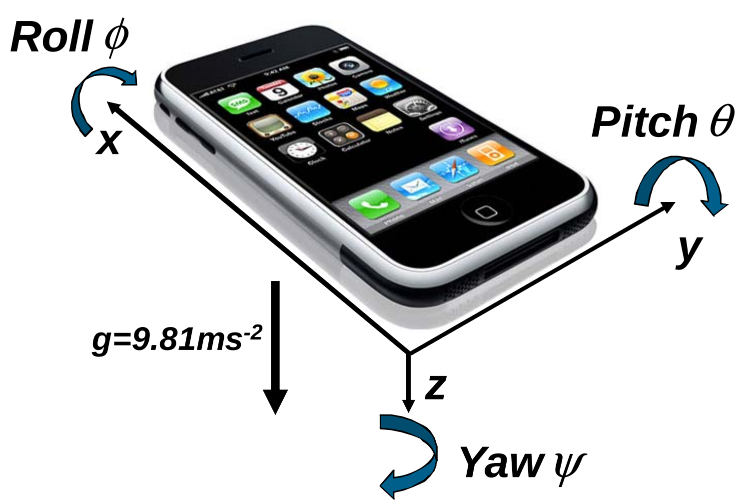

Figure 1. Coordinate System for the End Product Embedding the Accelerometer

The rotation matrices for a yaw rotation ψ about the z axis, a pitch rotation θ about the y axis and a roll rotation ϕ about the x axis are:

$$\boldsymbol{R}_z(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \qquad \boldsymbol{R}_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

Eqn. 18

The gravitational field vector $\boldsymbol{G}_p$ measured by the smartphone accelerometer is then determined by applying roll, pitch and yaw rotation matrices to the downwards pointing gravity vector of magnitude 1 g:

$$\boldsymbol{G}_p = \begin{pmatrix} G_{px} \\ G_{py} \\ G_{pz} \end{pmatrix} = \boldsymbol{R}_x(\phi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_z(\psi)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix}$$

Eqn. 19

The initial rotation in yaw angle ψ is a rotation about the earth's gravity vector and therefore has no effect on the gravitational field experienced by the smartphone. The initial yaw rotation can therefore be ignored and the smartphone orientation is defined in this document in terms of pitch and roll angles only.

Equation 19 is defined in units of g which is appropriate for consumer accelerometer sensors which provide an output in bit counts which directly translates to units of g. As Section 2 explains, the definition of '1 g' is the apparent gravitational field at the calibration test site.

5.1 Two measurement orientations

Maximizing the separation of two gravitational field vectors subject to the constraint that their modulus is 1 g results in their lying at opposite ends of a line with radius 1 g. This is the first optimization criterion discussed above.

However, simply orienting this line along the x, y or z acceleration axes would result in 0 g acceleration applied to the two other axes resulting in no useful information for the calculation of gain or offset in those two axes. The second optimization criterion requires that the minimum distance between the two measurements in the x, y, and z channels be maximized which is obviously achieved when the line is oriented to provide $\frac{1}{\sqrt{3}}$ g in all of the three x, y, and z channels followed by $\frac{-1}{\sqrt{3}}$ g in all the three channels. Using a brute force optimization algorithm or, most simply, by inspection, the separation between the two measurements in the x, y, and z channels is maximized at $\frac{2}{\sqrt{3}}$ g = 1.155 g which permits the most accurate estimate of the gain and offset.

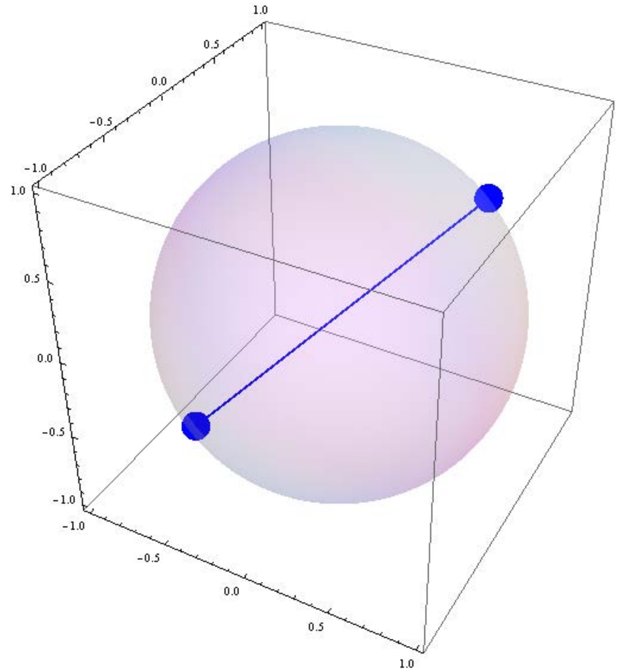

Figure 2. Optimum orientations for two measurements

Table 1 shows the optimum roll ϕ and pitch angles θ rounded to the nearest degree and the apparent gravitational field $\boldsymbol{G}_p$ experienced by the accelerometer in this example of two measurement orientations.

The vectors $\boldsymbol{Y}_x$, $\boldsymbol{Y}_y$ and $\boldsymbol{Y}_z$ simply contain the x, y, and z components of the apparent gravitational field $\boldsymbol{G}_p$ for the two orientations. The importance of these vectors will become clearer when the mathematics of least squares optimization is presented.

Table 1. Optimum orientations for two measurements

Orientation 0	**Orientation 1**	
$\theta[0] \approx -35deg$ $\phi[0] \approx 45deg$ $\boldsymbol{G}_p[0] = \begin{pmatrix} -sin(-35°) \\ cos(-35°)sin(45°) \\ cos(-35°)cos(45°) \end{pmatrix}$	$\theta[1] \approx 35deg$ $\phi[1] \approx -135deg$ $\boldsymbol{G}_p[1] = \begin{pmatrix} -sin(35°) \\ cos(35°)sin(-135°) \\ cos(35°)cos(-135°) \end{pmatrix}$	
$\boldsymbol{Y}_x = \begin{pmatrix} -sin(-35°) \\ -sin(35°) \end{pmatrix}$	$\boldsymbol{Y}_y = \begin{pmatrix} cos(-35°)sin(45°) \\ cos(35°)sin(-135°) \end{pmatrix}$	$\boldsymbol{Y}_z = \begin{pmatrix} cos(-35°)cos(45°) \\ cos(35°)cos(-135°) \end{pmatrix}$
Minimum separation: 1.155 *g*		

5.2 Three measurement orientations

The geometry which maximizes the separation of the gravitational field vectors for three measurement orientations places them at the vertices of an equilateral triangle.

The optimum orientation of this triangle, determined by a brute force optimization algorithm to maximize the minimum separation of measurements in any channel, is shown in Figure 3 and Table 2. Angles are again approximated to the nearest integer degree. The minimum separation between any two measurements in the *x*, *y* or *z* channels is 0.707 *g*.

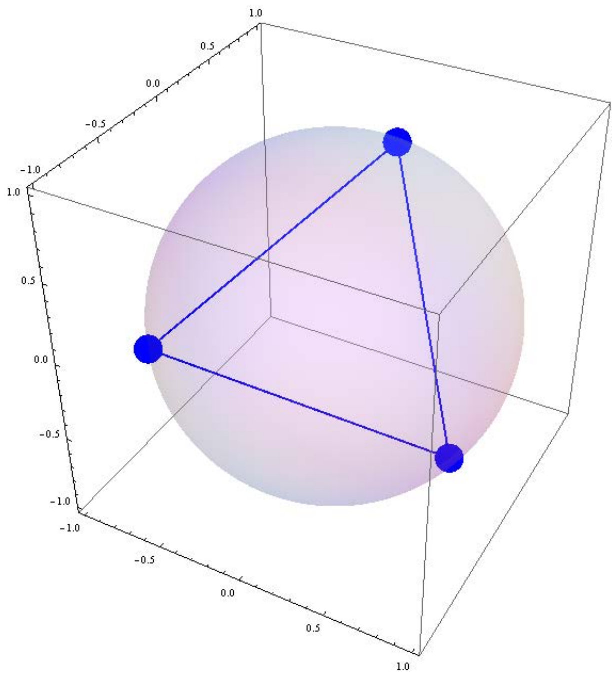

Figure 3. Optimum orientations for three measurements

Table 2. Optimum orientations for three measurements

Orientation 0	Orientation 1	Orientation 2
$\theta[0] \approx 0 deg$ $\phi[0] \approx 45 deg$ $\boldsymbol{G}[0] = \begin{pmatrix} -sin(0°) \\ cos(0°)sin(45°) \\ cos(0°)cos(45°) \end{pmatrix}$	$\theta[1] \approx -45 deg$ $\phi[1] \approx 180 deg$ $\boldsymbol{G}_p[1] = \begin{pmatrix} -sin(-45°) \\ cos(-45°)sin(180°) \\ cos(-45°)cos(180°) \end{pmatrix}$	$\theta[2] \approx 45 deg$ $\phi[2] \approx 90 deg$ $\boldsymbol{G}_p[2] = \begin{pmatrix} -sin(45°) \\ cos(45°)sin(-90°) \\ cos(45°)cos(-90°) \end{pmatrix}$

Table 2. Optimum orientations for three measurements (Continued)

$\boldsymbol{Y}_x = \begin{pmatrix} -sin(0°) \\ -sin(-45°) \\ -sin(45°) \end{pmatrix}$	$\boldsymbol{Y}_y = \begin{pmatrix} cos(0°)sin(45°) \\ cos(-45°)sin(180°) \\ cos(45°)sin(-90°) \end{pmatrix}$	$\boldsymbol{Y}_z = \begin{pmatrix} cos(0°)cos(45°) \\ cos(-45°)cos(180°) \\ cos(45°)cos(-90°) \end{pmatrix}$
Minimum separation: 0.707 *g*		

5.3 Four measurement orientations

The geometry which maximizes the separation of the gravitational field vectors for four measurement orientations places them at the vertices of a regular tetrahedron.

The output of the brute force optimization algorithm, which orients the tetrahedron to maximize the minimum separation of measurements in any channel, is shown in Figure 4 and Table 3. The minimum separation between any two measurements is 0.330 *g*.

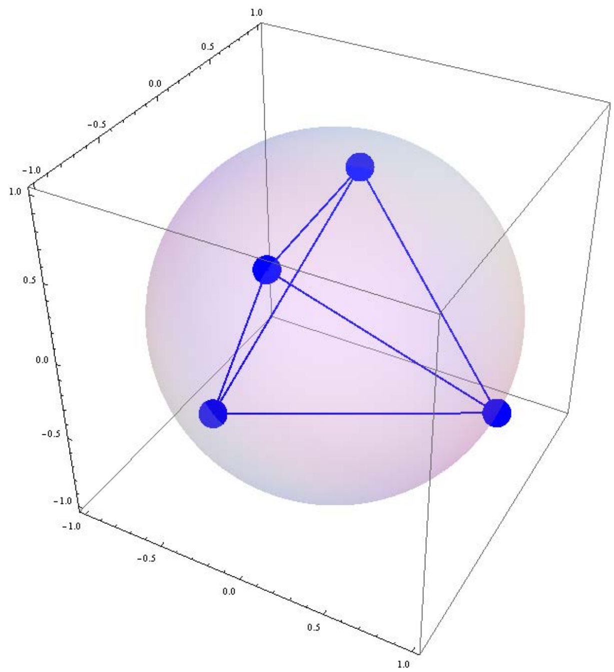

Figure 4. Optimum orientations for four measurements

Table 3. Optimum orientations for four measurements

Orientation 0	**Orientation 1**	**Orientation 2**
$\theta[0] \approx 39 deg$ $\phi[0] \approx -158 deg$ $\boldsymbol{G}[0] = \begin{pmatrix} -sin(39°) \\ cos(39°)sin(-158°) \\ cos(39°)cos(-158°) \end{pmatrix}$	$\theta[1] \approx -66 deg$ $\phi[1] \approx 164 deg$ $\boldsymbol{G}_p[1] = \begin{pmatrix} -sin(-66°) \\ cos(-66°)sin(164°) \\ cos(-66°)cos(164°) \end{pmatrix}$	$\theta[2] \approx 18 deg$ $\phi[2] \approx 66 deg$ $\boldsymbol{G}_p[2] = \begin{pmatrix} -sin(18°) \\ cos(18°)sin(66°) \\ cos(18°)cos(66°) \end{pmatrix}$
Orientation 3 $\theta[3] \approx -1 deg$ $\phi[3] \approx -44 deg$ $\boldsymbol{G}[3] = \begin{pmatrix} -sin(-1°) \\ cos(-1°)sin(-44°) \\ cos(-1°)cos(-44°) \end{pmatrix}$		
$\boldsymbol{Y}_x = \begin{pmatrix} -sin\,\theta(39°) \\ -sin(-66°) \\ -sin(18°) \\ -sin\,\theta(-1°) \end{pmatrix}$	$\boldsymbol{Y}_y = \begin{pmatrix} cos(39°)sin(-158°) \\ cos(-66°)sin(164°) \\ cos(18°)sin(66°) \\ cos(-1°)sin(-44°) \end{pmatrix}$	$\boldsymbol{Y}_z = \begin{pmatrix} cos(39°)cos(-158°) \\ cos(-66°)cos(164°) \\ cos(18°)cos(66°) \\ cos(-1°)cos(-44°) \end{pmatrix}$
Minimum separation: 0.330 *g*		

5.4 Six measurement orientations

The geometry which maximizes the separation of the gravitational field vectors for six measurement orientations places them at the vertices of a regular octahedron.

The output of the brute force optimization algorithm, which orients the octahedron to maximize the minimum separation of measurements in any channel, is shown in Figure 5 and Table 4. The minimum separation between any two measurements is 0.225 *g*.

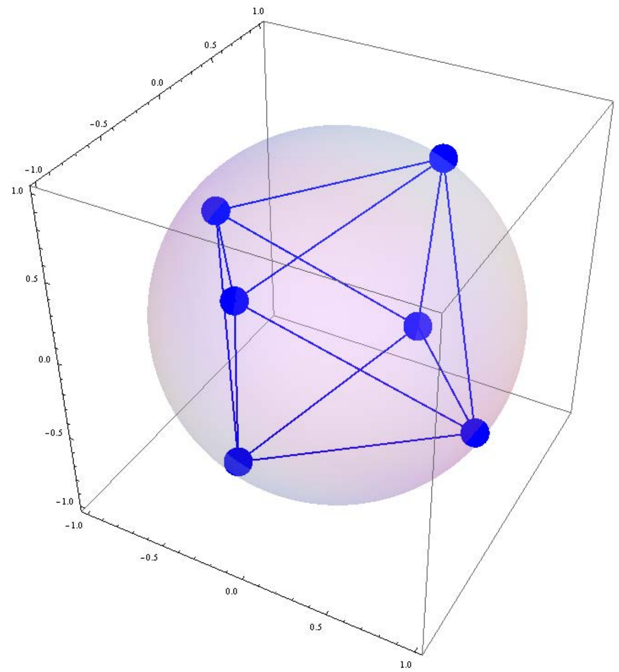

Figure 5. Optimum orientations for six measurements

Table 4. Optimum orientations for six measurements

Orientation 0	Orientation 1	Orientation 2
$\theta[0] \approx 6deg$ $\phi[0] \approx -55deg$ $\boldsymbol{G}[0] = \begin{pmatrix} -sin(6°) \\ cos(6°)sin(-55°) \\ cos(6°)cos(-55°) \end{pmatrix}$	$\theta[1] \approx -6deg$ $\phi[1] \approx 125deg$ $\boldsymbol{G}_p[1] = \begin{pmatrix} -sin(-6°) \\ cos(-6°)sin(125°) \\ cos(-6°)cos(125°) \end{pmatrix}$	$\theta[2] \approx 20deg$ $\phi[2] \approx -147deg$ $\boldsymbol{G}_p[2] = \begin{pmatrix} -sin(20°) \\ cos(20°)sin(-147°) \\ cos(20°)cos(-147°) \end{pmatrix}$
Orientation 3 $\theta[3] \approx -20deg$ $\phi[3] \approx 33deg$ $\boldsymbol{G}[3] = \begin{pmatrix} -sin(-20°) \\ cos(-20°)sin(33°) \\ cos(-20°)cos(33°) \end{pmatrix}$	**Orientation 4** $\theta[4] \approx -69deg$ $\phi[4] \approx -128deg$ $\boldsymbol{G}_p[4] = \begin{pmatrix} -sin(-69°) \\ cos(-69°)sin(-128°) \\ cos(-69°)cos(-128°) \end{pmatrix}$	**Orientation 5** $\theta[5] \approx 69deg$ $\phi[5] \approx 52deg$ $\boldsymbol{G}_p[5] = \begin{pmatrix} -sin(69°) \\ cos(69°)sin(52°) \\ cos(69°)cos(52°) \end{pmatrix}$
$\boldsymbol{Y}_x = \begin{pmatrix} -sin(6°) \\ -sin(-6°) \\ -sin(20°) \\ -sin(-20°) \\ -sin(-69°) \\ -sin(69°) \end{pmatrix}$	$\boldsymbol{Y}_y = \begin{pmatrix} cos(6°)sin(-55°) \\ cos(-6°)sin(125°) \\ cos(20°)sin(-147°) \\ cos(-20°)sin(33°) \\ cos(-69°)sin(-128°) \\ cos(69°)sin(52°) \end{pmatrix}$	$\boldsymbol{Y}_z = \begin{pmatrix} cos(6°)cos(-55°) \\ cos(-6°)cos(125°) \\ cos(20°)cos(-147°) \\ cos(-20°)cos(33°) \\ cos(-69°)cos(-128°) \\ cos(69°)cos(52°) \end{pmatrix}$
Minimum separation: 0.225 *g*		

5.5 Eight measurement orientations

The geometry which maximizes the separation of the gravitational field vectors for eight measurement orientations places them at the vertices of a cube.

The output of the brute force optimization algorithm, which orients the cube to maximize the minimum separation of measurements in any channel, is shown in Figure 6 and Table 5. The minimum separation between any two measurements is 0.188 *g*.

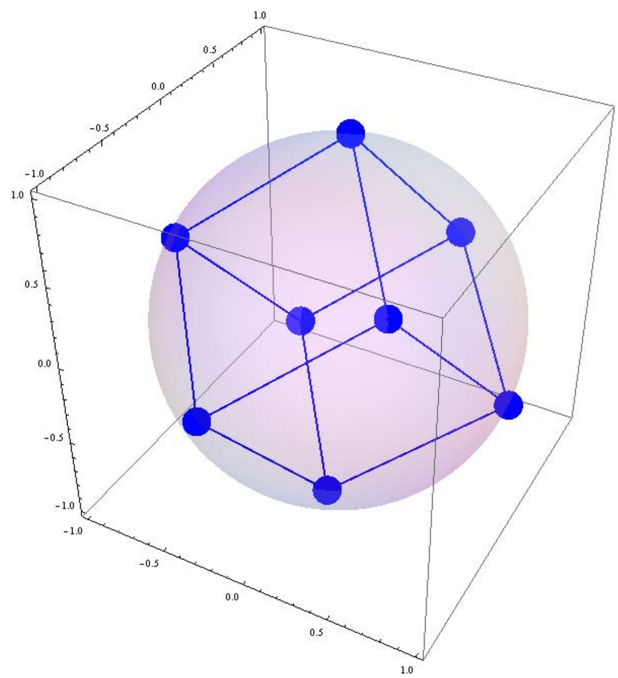

Figure 6. Optimum orientations for eight measurements

Table 5. Optimum orientations for eight measurements

Orientation 0	Orientation 1	Orientation 2
$\theta[0] \approx -35deg$ $\phi[0] \approx -45deg$ $G[0] = \begin{pmatrix} -sin(-35°) \\ cos(-35°)sin(-45°) \\ cos(-35°)cos(-45°) \end{pmatrix}$	$\theta[1] \approx -73deg$ $\phi[1] \approx 161deg$ $G_p[1] = \begin{pmatrix} -sin(-73°) \\ cos(-73°)sin(161°) \\ cos(-73°)cos(161°) \end{pmatrix}$	$\theta[2] \approx 5deg$ $\phi[2] \approx 17deg$ $G_p[2] = \begin{pmatrix} -sin(5°) \\ cos(5°)sin(17°) \\ cos(5°)cos(17°) \end{pmatrix}$
Orientation 3 $\theta[3] \approx -16deg$ $\phi[3] \approx 84deg$ $G[3] = \begin{pmatrix} -sin(-16°) \\ cos(-16°)sin(84°) \\ cos(-16°)cos(84°) \end{pmatrix}$	**Orientation 4** $\theta[4] \approx 16deg$ $\phi[4] \approx -96deg$ $G_p[4] = \begin{pmatrix} -sin(16°) \\ cos(16°)sin(-96°) \\ cos(16°)cos(-96°) \end{pmatrix}$	**Orientation 5** $\theta[5] \approx -5deg$ $\phi[5] \approx -163deg$ $G_p[5] = \begin{pmatrix} -sin(-5°) \\ cos(-5°)sin(-163°) \\ cos(-5°)cos(-163°) \end{pmatrix}$
Orientation 6 $\theta[6] \approx 73deg$ $\phi[6] \approx -18deg$ $G[6] = \begin{pmatrix} -sin(73°) \\ cos(73°)sin(-18°) \\ cos(73°)cos(-18°) \end{pmatrix}$	**Orientation 7** $\theta[7] \approx 35deg$ $\phi[7] \approx 135deg$ $G_p[7] = \begin{pmatrix} -sin(35°) \\ cos(35°)sin(135°) \\ cos(35°)cos(135°) \end{pmatrix}$	
$Y_x = \begin{pmatrix} -sin(-35°) \\ -sin(-73°) \\ -sin(5°) \\ -sin(-16°) \\ -sin(16°) \\ -sin(-5°) \\ -sin(73°) \\ -sin(35°) \end{pmatrix}$	$Y_y = \begin{pmatrix} cos(-35°)sin(-45°) \\ cos(-73°)sin(161°) \\ cos(5°)sin(17°) \\ cos(-16°)sin(84°) \\ cos(16°)sin(-96°) \\ cos(-5°)sin(-163°) \\ cos(73°)sin(-18°) \\ cos(35°)sin(135°) \end{pmatrix}$	$Y_z = \begin{pmatrix} cos(-35°)cos(-45°) \\ cos(-73°)cos(161°) \\ cos(5°)cos(17°) \\ cos(-16°)cos(84°) \\ cos(16°)cos(-96°) \\ cos(-5°)cos(-163°) \\ cos(73°)cos(-18°) \\ cos(35°)cos(135°) \end{pmatrix}$
Minimum separation: 0.188 *g*		

5.6 Distribution of measurements

Figure 7 to Figure 9 shows the distribution of measurements in the *x*, *y*, and *z* channels respectively for 2, 3, 4, 6 and 8 orientations. The measurements are maximally spread within the constraint that the magnitude of the measurement acceleration must equal *g*.

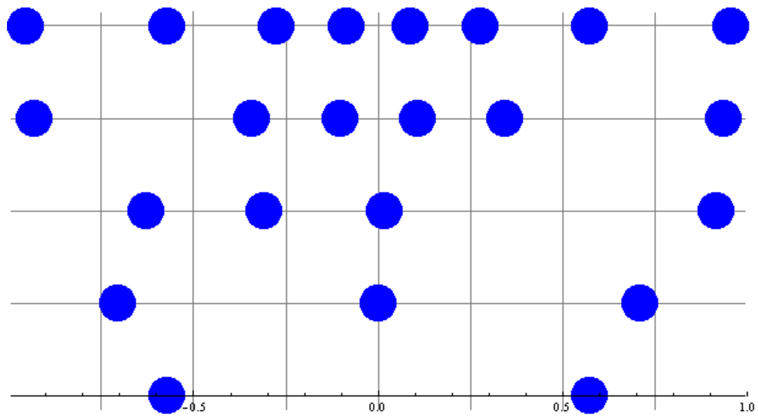

Figure 7. Distribution of x axis measurements for 2, 3, 4, 6 and 8 orientations

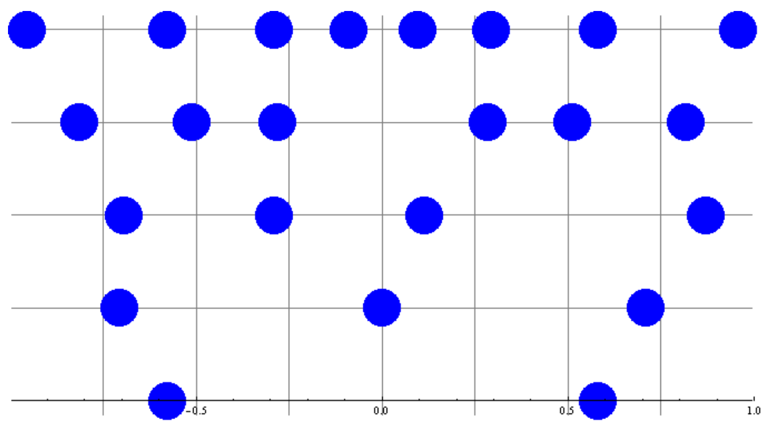

Figure 8. Distribution of y axis measurements for 2, 3, 4, 6 and 8 orientations

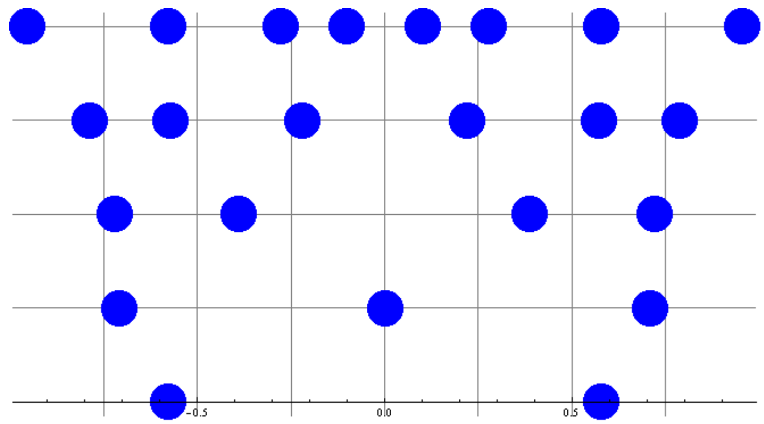

Figure 9. Distribution of z axis measurements for 2, 3, 4, 6 and 8 orientations

The next section develops the mathematics of optimum least squares estimation which is then combined with the optimum orientations defined in this section to fit increasingly sophisticated calibration models.

6 Linear Least Squares Optimization

The general linear model relating the dependent variable y to the N inputs x_0 to $x_{N\text{-}1}$ through N coefficients β_0 to $\beta_{N\text{-}1}$ is:

$$y = \beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_{N-1} x_{N-1} \qquad \text{Eqn. 20}$$

The N model parameters β_0 to $\beta_{N\text{-}1}$ are determined from M measurements labeled by $i = 0$ to $i = M - 1$ where M must be equal to or greater than N. Equation 20 can then be written in vector form for all M measurements as:

$$\begin{pmatrix} y[0] \\ y[1] \\ \ldots \\ y[M-1] \end{pmatrix} = \begin{pmatrix} x_0[0] & x_1[0] & \ldots & x_{N-1}[0] \\ x_0[1] & x_1[1] & \ldots & x_{N-1}[1] \\ \ldots & \ldots & \ldots & \ldots \\ x_0[M-1] & x_1[M-1] & \ldots & x_{N-1}[M-1] \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \ldots \\ \beta_{N-1} \end{pmatrix} \qquad \text{Eqn. 21}$$

The vector $\boldsymbol{Y}$ of the dependent variables is defined as:

$$\boldsymbol{Y} = \begin{pmatrix} y[0] \\ y[1] \\ \ldots \\ y[M-1] \end{pmatrix} \qquad \text{Eqn. 22}$$

The matrix $\boldsymbol{X}$ of measurements of the independent variables is defined as:

$$\boldsymbol{X} = \begin{pmatrix} x_0[0] & x_1[0] & \ldots & x_{N-1}[0] \\ x_0[1] & x_1[1] & \ldots & x_{N-1}[1] \\ \ldots & \ldots & \ldots & \ldots \\ x_0[M-1] & x_1[M-1] & \ldots & x_{N-1}[M-1] \end{pmatrix} \qquad \text{Eqn. 23}$$

The solution vector of the N model parameters β is defined as:

$$\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \ldots \\ \beta_{N-1} \end{pmatrix} \qquad \text{Eqn. 24}$$

Equation 21 can then be written as:

$$\boldsymbol{Y} = \boldsymbol{X}\beta \qquad \text{Eqn. 25}$$

Equation 25 will not, in general, permit an exact fit to all M measurements for $M > N$. If a vector $\boldsymbol{r}$ of residuals to the fit is defined as:

$$\boldsymbol{r} = \boldsymbol{Y} - \boldsymbol{X}\beta = \begin{pmatrix} r[0] \\ r[1] \\ \dots \\ r[M-1] \end{pmatrix} = \begin{pmatrix} y[0] \\ y[1] \\ \dots \\ y[M-1] \end{pmatrix} - \begin{pmatrix} x_0[0] & x_1[0] & \dots & x_{N-1}[0] \\ x_0[1] & x_1[1] & \dots & x_{N-1}[1] \\ \dots & \dots & \dots & \dots \\ x_0[M-1] & x_1[M-1] & \dots & x_{N-1}[M-1] \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \dots \\ \beta_{N-1} \end{pmatrix} \qquad \textbf{Eqn. 26}$$

then the optimal least squares fit for the solution vector β is that which minimizes the performance function P defined as the modulus squared of the residuals vector:

$$P = \|\boldsymbol{r}\|^2 = \|\boldsymbol{Y} - \boldsymbol{X}\beta\|^2 = \boldsymbol{r}^T\boldsymbol{r} = (\boldsymbol{Y} - \boldsymbol{X}\beta)^T(\boldsymbol{Y} - \boldsymbol{X}\beta) \qquad \textbf{Eqn. 27}$$

The optimal solution vector β occurs at the global minimum of the performance function P where, to first order, P is stationary for arbitrary small perturbations $\delta\beta$ about the optimum solution vector β:

$$(\delta\beta)^T \nabla P = P(\beta + \delta\beta) - P(\beta) = 0 \qquad \textbf{Eqn. 28}$$

Substituting Equation 27 into Equation 28 and ignoring second order terms gives:

$$(\boldsymbol{Y} - \boldsymbol{X}(\beta + \delta\beta))^T(\boldsymbol{Y} - \boldsymbol{X}(\beta + \delta\beta)) - (\boldsymbol{Y} - \boldsymbol{X}\beta)^T(\boldsymbol{Y} - \boldsymbol{X}\beta) = 0 \qquad \textbf{Eqn. 29}$$

$$\Rightarrow -\boldsymbol{Y}^T\boldsymbol{X}\delta\beta + (\boldsymbol{X}\beta)^T\boldsymbol{X}\delta\beta - (\boldsymbol{X}\delta\beta)^T(\boldsymbol{Y} - \boldsymbol{X}\beta) = 0 \qquad \textbf{Eqn. 30}$$

$$\Rightarrow -\boldsymbol{Y}^T\boldsymbol{X}\delta\beta + (\boldsymbol{X}\beta)^T\boldsymbol{X}\delta\beta - \delta\beta^T\boldsymbol{X}^T\boldsymbol{Y} + \delta\beta^T\boldsymbol{X}^T\boldsymbol{X}\beta = 0 \qquad \textbf{Eqn. 31}$$

Since $\delta\beta^T\boldsymbol{X}^T\boldsymbol{Y}$ and $\delta\beta^T\boldsymbol{X}^T\boldsymbol{X}\beta$ are scalars, their values are unchanged by the transpose operation and Equation 31 can be rewritten as:

$$(-\boldsymbol{Y}^T\boldsymbol{X} + \beta^T\boldsymbol{X}^T\boldsymbol{X})\delta\beta = 0 \text{ for all } \delta\beta \qquad \textbf{Eqn. 32}$$

$$\Rightarrow \beta^T\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{Y}^T\boldsymbol{X} \Rightarrow \boldsymbol{X}^T\boldsymbol{X}\beta = \boldsymbol{X}^T\boldsymbol{Y} \qquad \textbf{Eqn. 33}$$

$$\Rightarrow \beta = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} \qquad \textbf{Eqn. 34}$$

Equation 34 is commonly referred to as the Normal Equations for least squares optimization.

The next two sections extend the calibration model to a general linear model and then to include a cubic nonlinearity component. These are then solved in worked examples using the framework developed in the previous two sections.

7 Final Product 12 Parameter Calibration

The most general linear calibration of the accelerometer fits a total of 12 parameters. The recalibrated accelerometer output $\boldsymbol{G}_{12}$ is defined in terms of the factory calibration $\boldsymbol{G}_f$ by:

$$\boldsymbol{G}_{12} = \begin{pmatrix} G_{12x} \\ G_{12y} \\ G_{12z} \end{pmatrix} = \boldsymbol{W}\boldsymbol{G}_f + \boldsymbol{V} = \begin{pmatrix} W_{xx} & W_{xy} & W_{xz} \\ W_{yx} & W_{yy} & W_{yz} \\ W_{zx} & W_{zy} & W_{zz} \end{pmatrix} \begin{pmatrix} G_{fx} \\ G_{fy} \\ G_{fz} \end{pmatrix} + \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \qquad \textbf{\textit{Eqn. 35}}$$

The gain matrix $\boldsymbol{W}$ now has nine independent elements and therefore extends the factory calibration model by including all possible cross-axis interactions and any rotation of the sensor package on the circuit board. The 12 calibration parameters are calculated in order to best approximate the applied gravitational field resulting from the circuit board orientation.

The *i-th* measurement at orientation angles $\theta[i]$ and $\phi[i]$ can be written as:

$$\begin{pmatrix} G_{12x}[i] \\ G_{12y}[i] \\ G_{12z}[i] \end{pmatrix} = \begin{pmatrix} W_{xx} & W_{xy} & W_{xz} \\ W_{yx} & W_{yy} & W_{yz} \\ W_{zx} & W_{zy} & W_{zz} \end{pmatrix} \begin{pmatrix} G_{fx}[i] \\ G_{fy}[i] \\ G_{fz}[i] \end{pmatrix} + \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \approx \begin{pmatrix} -\sin\theta[i] \\ \cos\theta[i]\sin\phi[i] \\ \cos\theta[i]\cos\phi[i] \end{pmatrix} \qquad \textbf{\textit{Eqn. 36}}$$

Equation 36 conveniently decomposes into three independent equations for the four calibration parameters in each of the three accelerometer channels.

$$W_{xx}G_{fx}[i] + W_{xy}G_{fy}[i] + W_{xz}G_{fz}[i] + V_x = -\sin\theta[i]$$

$$W_{yx}G_{fx}[i] + W_{yy}G_{fy}[i] + W_{yz}G_{fz}[i] + V_y = \cos\theta[i]\sin\phi[i]$$

$$W_{zx}G_{fx}[i] + W_{zy}G_{fy}[i] + W_{zz}G_{fz}[i] + V_z = \cos\theta[i]\cos\phi[i]$$

Eqn. 37

The residuals for the *i-th* measurement are then:

$$r_x[i] = -\sin\theta[i] - W_{xx}G_{fx}[i] - W_{xy}G_{fy}[i] - W_{xz}G_{fz}[i] - V_x$$

$$r_y[i] = \cos\theta[i]\sin\phi[i] - W_{yx}G_{fx}[i] - W_{yy}G_{fy}[i] - W_{yz}G_{fz}[i] - V_y$$

$$r_z[i] = \cos\theta[i]\cos\phi[i] - W_{zx}G_{fx}[i] - W_{zy}G_{fy}[i] - W_{zz}G_{fz}[i] - V_z$$

Eqn. 38

The x component of Equation 38 can be written as:

$$r_x[i] = -\sin\theta[i] - (G_{fx}[i] \;\; G_{fy}[i] \;\; G_{fz}[i] \;\; 1)\begin{pmatrix} W_{xx} \\ W_{xy} \\ W_{xz} \\ V_x \end{pmatrix} \quad \text{Eqn. 39}$$

If M measurements are used for the 12-element calibration then Equation 39 can be written in the form:

$$\begin{pmatrix} r_x[0] \\ r_x[1] \\ \dots \\ r_x[M-1] \end{pmatrix} = \begin{pmatrix} -\sin\theta[0] \\ -\sin\theta[1] \\ \dots \\ -\sin\theta[M-1] \end{pmatrix} - \begin{pmatrix} G_{fx}[0] & G_{fy}[0] & G_{fz}[0] & 1 \\ G_{fx}[1] & G_{fy}[1] & G_{fz}[1] & 1 \\ \dots & \dots & \dots & \dots \\ G_{fx}[M-1] & G_{fy}[M-1] & G_{fz}[M-1] & 1 \end{pmatrix}\begin{pmatrix} W_{xx} \\ W_{xy} \\ W_{xz} \\ V_x \end{pmatrix} \quad \text{Eqn. 40}$$

Equation 40 can be written as:

$$\boldsymbol{r}_x = \boldsymbol{Y}_x - \boldsymbol{X}\beta_x \quad \text{Eqn. 41}$$

with the following matrix definitions.

$\boldsymbol{X}$ is the matrix of M accelerometer measurements defined as:

$$\boldsymbol{X} = \begin{pmatrix} G_{fx}[0] & G_{fy}[0] & G_{fz}[0] & 1 \\ G_{fx}[1] & G_{fy}[1] & G_{fz}[1] & 1 \\ \dots & \dots & \dots & \dots \\ G_{fx}[M-1] & G_{fy}[M-1] & G_{fz}[M-1] & 1 \end{pmatrix} \quad \text{Eqn. 42}$$

The vector $\boldsymbol{r}_x$ is the array of M residuals for the x-channel calibration fit:

$$\boldsymbol{r}_x = \begin{pmatrix} r_x[0] \\ r_x[1] \\ \dots \\ r_x[M-1] \end{pmatrix} \quad \text{Eqn. 43}$$

$\boldsymbol{Y}_x$ is the vector of the true x components of the gravitational field for the M measurement orientations:

$$\boldsymbol{Y}_x = \begin{pmatrix} -sin\,\theta[0] \\ -sin\,\theta[1] \\ \ldots \\ -sin\,\theta[M-1] \end{pmatrix} \qquad \textbf{\textit{Eqn. 44}}$$

β_x is the solution vector for four of the calibration parameters:

$$\beta_x = \begin{pmatrix} W_{xx} \\ W_{xy} \\ W_{xz} \\ V_x \end{pmatrix} \qquad \textbf{\textit{Eqn. 45}}$$

Using Equation 34, the optimum least squares solution for β_x is:

$$\beta_x = \begin{pmatrix} W_{xx} \\ W_{xy} \\ W_{xz} \\ V_x \end{pmatrix} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}_x = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\begin{pmatrix} -sin\,\theta[0] \\ -sin\,\theta[1] \\ \ldots \\ -sin\,\theta[M-1] \end{pmatrix} \qquad \textbf{\textit{Eqn. 46}}$$

Similarly, the optimum least squares solution for the remaining calibration parameters is:

$$\beta_y = \begin{pmatrix} W_{yx} \\ W_{yy} \\ W_{yz} \\ V_y \end{pmatrix} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}_y = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\begin{pmatrix} cos\,\theta[0]\,sin\,\phi[0] \\ cos\,\theta[1]\,sin\,\phi[1] \\ \ldots \\ cos\,\theta[M-1]\,sin\,\phi[M-1] \end{pmatrix} \qquad \textbf{\textit{Eqn. 47}}$$

$$\beta_z = \begin{pmatrix} W_{zx} \\ W_{zy} \\ W_{zz} \\ V_z \end{pmatrix} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}_z = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\begin{pmatrix} cos\,\theta[0]\,cos\,\phi[0] \\ cos\,\theta[1]\,cos\,\phi[1] \\ \ldots \\ cos\,\theta[M-1]\,cos\,\phi[M-1] \end{pmatrix} \qquad \textbf{\textit{Eqn. 48}}$$

For this particular calibration model, the measurement matrix $\boldsymbol{X}$ is common to all three of Equations 46 to 48.

To apply Equations 46 to 48, it is simply necessary to decide how many measurement orientations to use which in turn determines the vectors $\boldsymbol{Y}_x$, $\boldsymbol{Y}_y$ and $\boldsymbol{Y}_z$. The minimum number of measurement orientations M

to solve for the 12 parameters of this model is 4 although more measurements may give a more robust calibration.

Worked Example 3

An MMA8451 production circuit board was successively placed in the four orientations defined by the tetrahedron of Figure 4 and Table 3 and the factory calibrated output $\boldsymbol{G_f}$ in bits and g was recorded for each channel and each orientation.

Substituting the vectors $\boldsymbol{Y_x}$, $\boldsymbol{Y_y}$ and $\boldsymbol{Y_z}$ from Table 4 into Equations 46 to 48 gives:

$$\begin{pmatrix} W_{xx} \\ W_{xy} \\ W_{xz} \\ V_x \end{pmatrix} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \begin{pmatrix} -sin(39°) \\ -sin(-66°) \\ -sin(18°) \\ -sin(-1°) \end{pmatrix} \quad \textbf{Eqn. 49}$$

$$\begin{pmatrix} W_{yx} \\ W_{yy} \\ W_{yz} \\ V_y \end{pmatrix} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \begin{pmatrix} cos(39°)sin(-158°) \\ cos(-66°)sin(164°) \\ cos(18°)sin(66°) \\ cos(-1°)sin(-44°) \end{pmatrix} \quad \textbf{Eqn. 50}$$

$$\begin{pmatrix} W_{zx} \\ W_{zy} \\ W_{zz} \\ V_z \end{pmatrix} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \begin{pmatrix} cos(39°)cos(-158°) \\ cos(-66°)cos(164°) \\ cos(18°)cos(66°) \\ cos(-1°)cos(-44°) \end{pmatrix} \quad \textbf{Eqn. 51}$$

The factory calibrated measurements output from the accelerometer in bit counts and g at the four orientations were measured to be:

$$\boldsymbol{G_f}[0] = \begin{pmatrix} -2434.354 \\ -977.106 \\ -2937.402 \end{pmatrix} bits \quad \boldsymbol{G_f}[1] = \begin{pmatrix} 3783.543 \\ 397.825 \\ -1485.275 \end{pmatrix} bits \quad \boldsymbol{G_f}[2] = \begin{pmatrix} -1009.383 \\ 3574.526 \\ 1527.156 \end{pmatrix} bits \quad \boldsymbol{G_f}[3] = \begin{pmatrix} 126.391 \\ -2745.795 \\ 2844.153 \end{pmatrix} bits$$

$$\boldsymbol{G_f}[0] = \begin{pmatrix} -0.5943247 \\ -0.2385511 \\ -0.7171391 \end{pmatrix} g \quad \boldsymbol{G_f}[1] = \begin{pmatrix} 0.923716 \\ 0.0971252 \\ -0.362616 \end{pmatrix} g \quad \boldsymbol{G_f}[2] = \begin{pmatrix} -0.2464314 \\ 0.8726869 \\ 0.3728409 \end{pmatrix} g \quad \boldsymbol{G_f}[3] = \begin{pmatrix} 0.0308571 \\ -0.6703601 \\ 0.6943732 \end{pmatrix} g$$

Eqn. 52

The measurement matrix $\boldsymbol{X}$ in units of g is then:

$$\boldsymbol{X} = \begin{pmatrix} -0.5943247 & -0.2385511 & -0.7171391 & 1 \\ 0.923716 & 0.0971252 & -0.362616 & 1 \\ -0.2464314 & 0.8726869 & 0.3728409 & 1 \\ 0.0308571 & -0.6703601 & 0.6943732 & 1 \end{pmatrix} \quad \textbf{Eqn. 53}$$

Using Equations 46 to 48, the solution vectors are:

$$\begin{pmatrix} W_{xx} \\ W_{xy} \\ W_{xz} \\ V_x \end{pmatrix} = \begin{pmatrix} 1.02354 \\ -0.02844 \\ -0.00383 \\ -0.03054 \end{pmatrix} \quad \textbf{Eqn. 54}$$

$$\begin{pmatrix} W_{yx} \\ W_{yy} \\ W_{yz} \\ V_y \end{pmatrix} = \begin{pmatrix} 0.03721 \\ 1.02203 \\ 0.01036 \\ -0.01777 \end{pmatrix} \quad \textbf{Eqn. 55}$$

$$\begin{pmatrix} W_{zx} \\ W_{zy} \\ W_{zz} \\ V_z \end{pmatrix} = \begin{pmatrix} -0.02188 \\ -0.00511 \\ 1.02816 \\ 0.00255 \end{pmatrix} \quad \textbf{Eqn. 56}$$

The accelerometer output $\boldsymbol{G}_{12}$ recalibrated with a 12-element model is then given by:

$$\begin{pmatrix} G_{12x} \\ G_{12y} \\ G_{12z} \end{pmatrix} = \begin{pmatrix} 1.02354 & -0.02844 & -0.00383 \\ 0.03721 & 1.02203 & 0.01036 \\ -0.02188 & -0.00511 & 1.02816 \end{pmatrix} \begin{pmatrix} G_{fx} \\ G_{fy} \\ G_{fz} \end{pmatrix} + \begin{pmatrix} -0.03054 \\ -0.01777 \\ 0.00255 \end{pmatrix} \quad \textbf{Eqn. 57}$$

In this example, the number of measurements (12 results from 3 channels in 4 circuit board orientations) equals the number of model parameters to be fitted and the performance function of squared residuals will be zero.

$P_x = (\boldsymbol{Y}_x - \boldsymbol{X}\beta_x)^T(\boldsymbol{Y}_x - \boldsymbol{X}\beta_x) = 0$	$P_y = (\boldsymbol{Y}_y - \boldsymbol{X}\beta_y)^T(\boldsymbol{Y}_y - \boldsymbol{X}\beta_y) = 0$	$P_z = (\boldsymbol{Y}_z - \boldsymbol{X}\beta_z)^T(\boldsymbol{Y}_z - \boldsymbol{X}\beta_z) = 0$

Eqn. 58

Worked Example 4

An MMA8451 production circuit board was successively placed in the six orientations defined by the octahedron of Figure 5 and Table 4 and the factory calibrated output $\boldsymbol{G_f}$ was recorded for each channel and each orientation.

Substituting the vectors $\boldsymbol{Y_x}$, $\boldsymbol{Y_y}$ and $\boldsymbol{Y_z}$ from Table 4 into Equations 46 to 48 gives:

$$\begin{pmatrix} W_{xx} \\ W_{xy} \\ W_{xz} \\ V_x \end{pmatrix} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \begin{pmatrix} -sin(6°) \\ -sin(-6°) \\ -sin(20°) \\ -sin(-20°) \\ -sin(-69°) \\ -sin(69°) \end{pmatrix} \quad \text{Eqn. 59}$$

$$\begin{pmatrix} W_{yx} \\ W_{yy} \\ W_{yz} \\ V_y \end{pmatrix} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \begin{pmatrix} cos(6°)sin(-55°) \\ cos(-6°)sin(125°) \\ cos(20°)sin(-147°) \\ cos(-20°)sin(33°) \\ cos(-69°)sin(-128°) \\ \cos(69°)sin(52°) \end{pmatrix} \quad \text{Eqn. 60}$$

$$\begin{pmatrix} W_{zx} \\ W_{zy} \\ W_{zz} \\ V_z \end{pmatrix} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \begin{pmatrix} cos(6°)cos(-55°) \\ cos(-6°)cos(125°) \\ cos(20°)cos(-147°) \\ cos(-20°)cos(33°) \\ cos(-69°)cos(-128°) \\ cos(69°)cos(52°) \end{pmatrix} \quad \text{Eqn. 61}$$

The factory calibrated measurements output from the accelerometer in bit counts and g at the four orientations were measured to be:

$$\boldsymbol{G}_f[0] = \begin{pmatrix} -378.804 \\ -3204.493 \\ 2235.182 \end{pmatrix} bits \quad \boldsymbol{G}_f[1] = \begin{pmatrix} 648.067 \\ 3341.771 \\ -2264.316 \end{pmatrix} bits \quad \boldsymbol{G}_f[2] = \begin{pmatrix} -1298.636 \\ -1889.573 \\ -3191.104 \end{pmatrix} bits$$

$$\boldsymbol{G}_f[3] = \begin{pmatrix} 1523.352 \\ 2012.605 \\ 3133.065 \end{pmatrix} bits \quad \boldsymbol{G}_f[4] = \begin{pmatrix} 3815.935 \\ -1148.782 \\ -817.115 \end{pmatrix} bits \quad \boldsymbol{G}_f[5] = \begin{pmatrix} -3624.643 \\ 1324.928 \\ 811.351 \end{pmatrix} bits$$

$$\boldsymbol{G}_f[0] = \begin{pmatrix} -0.092481 \\ -0.782347 \\ 0.545699 \end{pmatrix} g \quad \boldsymbol{G}_f[1] = \begin{pmatrix} 0.158219 \\ 0.815862 \\ -0.552812 \end{pmatrix} g \quad \boldsymbol{G}_f[2] = \begin{pmatrix} -0.317050 \\ -0.461321 \\ -0.779078 \end{pmatrix} g$$

$$\boldsymbol{G}_f[3] = \begin{pmatrix} 0.371912 \\ 0.491359 \\ 0.764908 \end{pmatrix} g \quad \boldsymbol{G}_f[4] = \begin{pmatrix} 0.931625 \\ -0.280464 \\ -0.199491 \end{pmatrix} g \quad \boldsymbol{G}_f[5] = \begin{pmatrix} -0.884923 \\ 0.323469 \\ 0.198084 \end{pmatrix} g$$

The measurement matrix in units of g is then:

$$\boldsymbol{X} = \begin{pmatrix} -0.092481 & -0.782347 & 0.545699 & 1 \\ 0.158219 & 0.815862 & -0.552812 & 1 \\ -0.317050 & -0.461321 & -0.779078 & 1 \\ 0.371912 & 0.491359 & 0.764908 & 1 \\ 0.931625 & -0.280464 & -0.199491 & 1 \\ -0.884923 & 0.323469 & 0.198084 & 1 \end{pmatrix} \quad \textbf{\textit{Eqn. 62}}$$

Computations of Equations 59 to 61 gives the calibration parameters to be:

$$\begin{pmatrix} W_{xx} \\ W_{xy} \\ W_{xz} \\ V_x \end{pmatrix} = \begin{pmatrix} 1.01996 \\ -0.02633 \\ 0.00415 \\ -0.02796 \end{pmatrix} \quad \textbf{\textit{Eqn. 63}}$$

$$\begin{pmatrix} W_{yx} \\ W_{yy} \\ W_{yz} \\ V_y \end{pmatrix} = \begin{pmatrix} 0.03330 \\ 1.02498 \\ 0.01562 \\ -0.01907 \end{pmatrix} \quad \text{Eqn. 64}$$

$$\begin{pmatrix} W_{zx} \\ W_{zy} \\ W_{zz} \\ V_z \end{pmatrix} = \begin{pmatrix} -0.01821 \\ -0.00264 \\ 1.03058 \\ 0.00445 \end{pmatrix} \quad \text{Eqn. 65}$$

In this example, there are four model parameters per channel but six orientations so the fit error will not therefore, in general, be zero.

$\boldsymbol{P}_x = (\boldsymbol{Y}_x - \boldsymbol{X}\beta_x)^T(\boldsymbol{Y}_x - \boldsymbol{X}\beta_x)$ $= 96.95 \times 10^{-6}$	$\boldsymbol{P}_y = (\boldsymbol{Y}_y - \boldsymbol{X}\beta_y)^T(\boldsymbol{Y}_y - \boldsymbol{X}\beta_y)$ $= 45.89 \times 10^{-6}$	$\boldsymbol{P}_z = (\boldsymbol{Y}_z - \boldsymbol{X}\beta_z)^T(\boldsymbol{Y}_z - \boldsymbol{X}\beta_z)$ $= 44.07 \times 10^{-6}$

Eqn. 66

8 Including Temperature Dependence

Up to this point, it has been implicitly assumed that the accelerometer will be used at the same temperature as the final calibration so that temperature dependence can be ignored. However, the recalibration approach can be very simply extended to include temperature dependence by performing the recalibration at two or more different temperatures.

If the 12 element recalibration model of Equation 35 is computed from measurements at the two temperatures T_1 and T_2 then the calibration parameters at any other temperature T can be approximated using a linear interpolation as:

$$\begin{pmatrix} W_{xx}(T) & W_{xy}(T) & W_{xz}(T) \\ W_{yx}(T) & W_{yy}(T) & W_{yz}(T) \\ W_{zx}(T) & W_{zy}(T) & W_{zz}(T) \end{pmatrix} = \left(\frac{T_2 - T}{T_2 - T_1}\right) \begin{pmatrix} W_{xx}(T_1) & W_{xy}(T_1) & W_{xz}(T_1) \\ W_{yx}(T_1) & W_{yy}(T_1) & W_{yz}(T_1) \\ W_{zx}(T_1) & W_{zy}(T_1) & W_{zz}(T_1) \end{pmatrix} + \left(\frac{T - T_1}{T_2 - T_1}\right) \begin{pmatrix} W_{xx}(T_2) & W_{xy}(T_2) & W_{xz}(T_2) \\ W_{yx}(T_2) & W_{yy}(T_2) & W_{yz}(T_2) \\ W_{zx}(T_2) & W_{zy}(T_2) & W_{zz}(T_2) \end{pmatrix} \qquad \textbf{Eqn. 67}$$

$$\begin{pmatrix} V_x(T) \\ V_y(T) \\ V_z(T) \end{pmatrix} = \left(\frac{T_2 - T}{T_2 - T_1}\right) \begin{pmatrix} V_x(T_1) \\ V_y(T_1) \\ V_z(T_1) \end{pmatrix} + \left(\frac{T - T_1}{T_2 - T_1}\right) \begin{pmatrix} V_x(T_2) \\ V_y(T_2) \\ V_z(T_2) \end{pmatrix} \qquad \textbf{Eqn. 68}$$

The calibration parameters for both temperatures would now be stored in the final product along with the temperatures T_1 and T_2. A temperature sensor is required to provide the current temperature T.

The two calibration temperatures T_1 and T_2 should be well separated and close to the lower and upper expected operating temperatures. If the temperatures are too close together then the estimate of the gradient of the calibration curve will become susceptible to error as a result of measurement noise.

A three temperature calibration may be suitable if the product has a typical operating temperature (perhaps room temperature) but must remain in good calibration over a wide range of temperature (perhaps from -40°C to +85°C). In this case, T_1 might be -40°C, T_2 + 20°C and T_3 + 85°C.

One approach for the interpolation to an arbitrary temperature T is to use a piecewise linear interpolation. Here, the calibration at temperature T is interpolated (using Equations 67 and 68) between the calibration at temperatures T_1 and T_2 (if T lies between T_1 and T_2) or between the calibration at temperatures T_2 and T_3 (if T lies between and T_3).

Alternatively, a quadratic curve could be fitted through the three points. Taking the single parameter W_{xx} as an example, its temperature dependence would be modelled by the three values α, β and γ as:

$$W_{xx}(T) = \alpha + \beta T + \gamma T^2 \qquad \textbf{Eqn. 69}$$

Evaluating Equation 69 at the three calibration temperatures gives:

$$W_{xx}(T_1) = \alpha + \beta T_1 + \gamma T_1^2 \qquad \textbf{Eqn. 70}$$

$$W_{xx}(T_2) = \alpha + \beta T_2 + \gamma T_2^{\ 2} \quad \textbf{\textit{Eqn. 71}}$$

$$W_{xx}(T_3) = \alpha + \beta T_3 + \gamma T_3^{\ 2} \quad \textbf{\textit{Eqn. 72}}$$

Equation 70 through Equation 72 can be written in matrix form as:

$$\begin{pmatrix} W_{xx}(T_1) \\ W_{xx}(T_2) \\ W_{xx}(T_3) \end{pmatrix} = \begin{pmatrix} 1 & T_1 & T_1^{\ 2} \\ 1 & T_2 & T_2^{\ 2} \\ 1 & T_3 & T_3^{\ 2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \quad \textbf{\textit{Eqn. 73}}$$

with solution:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & T_1 & T_1^{\ 2} \\ 1 & T_2 & T_2^{\ 2} \\ 1 & T_3 & T_3^{\ 2} \end{pmatrix}^{-1} \begin{pmatrix} W_{xx}(T_1) \\ W_{xx}(T_2) \\ W_{xx}(T_3) \end{pmatrix} \quad \textbf{\textit{Eqn. 74}}$$

Worked Example 6

The calibration parameter W_{xx} (the x-axis accelerometer gain term) was measured to be 0.989160 at -40°C, 0.995040 at +20°C and 1.002223 at +85°C.The three coefficients of the temperature curve for the parameter W_{xx} are then given by:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & -40 & 1600 \\ 1 & 20 & 400 \\ 1 & 85 & 7225 \end{pmatrix}^{-1} \begin{pmatrix} 0.989160 \\ 0.995040 \\ 1.002223 \end{pmatrix} = \left(\frac{1}{97500}\right) \begin{pmatrix} 22100 & 85000 & -9600 \\ -1365 & 1125 & 240 \\ 13 & -25 & 12 \end{pmatrix} \begin{pmatrix} 0.989160 \\ 0.995040 \\ 1.002223 \end{pmatrix} = \begin{pmatrix} 0.993000 \\ 0.0001 \\ 1.0006 \times 10^{-7} \end{pmatrix} \quad \textbf{\textit{Eqn. 75}}$$

If the temperature sensor on the PCB indicates +32°C then the value of W_{xx} at this temperature using the quadratic model is:

$$W_{xx}(+32°\mathrm{C}) = 0.993000 + 0.0001*32 + 1.0006\mathrm{x}10^{-7} * 32^2 = 0.996302 \quad \textbf{\textit{Eqn. 76}}$$

The calculation of the quadratic temperature coefficients (Equation 75) would be performed on the production line for each calibration parameter and stored in nonvolatile memory in the product. Instead of 12 coefficients being stored for the 12 parameter model, 36 coefficients would be stored representing the three terms α, β and γ for each of the 12 model parameters. Equation 76 would then be computed at run time to compute the 12 calibration parameters at the current temperature.

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